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Note on the characteristic rank of vector bundles

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NOTE ON THE CHARACTERISTIC RANK OF VECTOR BUNDLES

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Abstract. We define the notion of the characteristic rank, $\mathrm{charrank}_\xi(X)$, of a real vector bundle ξ over a finite CW-complex X. This is a bundle dependent version of the notion of characteristic rank introduced by Julius Korbas [9]. We obtain bounds for the cup length of manifolds in terms of the characteristic rank of vector bundles and compute the characteristic rank of vector bundles over the Dold manifolds, the Moore spaces and the stunted projective spaces amongst others.

1. Introduction

Recently, J Korbăs [9] has introduced a new homotopy invariant, called the characteristic rank, of a connected, closed, smooth manifold X. The characteristic rank of a connected, closed smooth d-manifold X, denoted by $\mathrm{charrank}(X)$, is the largest integer k, $0 \leq k \leq d$, such that every cohomology class $x \in H^j(X;\mathbb{Z}_2)$, $0 \leq j \leq k$ is a polynomial in the Stiefel-Whitney classes of (the tangent bundle of) X.

An important question is understanding the characteristic rank of manifolds. Apart from being an interesting question in its own right, a part of the motivation for computing the characteristic rank comes from a theorem of Korbăs ([9], Theorem 1.1), where the author has described a bound for the $\mathbb{Z}_2$-cup-length of (unorientedly) null cobordant, closed, smooth manifolds in terms of its charateristic rank. More specifically, J Korbăs has proved the following.

Theorem 1.1. ([9], Theorem 1.1) *Let X be a closed smooth connected d-dimensional manifold unorientedly cobordant to zero. Let $\widetilde{H}^r(X;\mathbb{Z}_2)$, $r < d$, be the first nonzero reduced cohomology group of X. Let z ($z < d-1$) be an integer such that for $j \leq z$ each element of $H^j(X;\mathbb{Z}_2)$ can be expressed as a polynomial in the Stiefel-Whitney classes of the manifold X. Then we have that*

$$\mathrm{cup}(X) \leq 1 + \frac{d-z-1}{r}.$$

Recall that the $\mathbb{Z}_2$-cup-length, denoted by $\mathrm{cup}(X)$, of a space X is the largest integer t such that there exist classes $x_i \in H^*(X;\mathbb{Z}_2)$, $\deg(x_i) \geq 1$, such that the cup product $x_1 \cdot x_2 \cdots x_t \neq 0$. We mention in passing that the $\mathbb{Z}_2$-cup-length is well known to have connections with the Lyusternik-Shnirel'man category of the space.

With the computation of the characteristic rank in mind, Balko-Korbăs [2] obtain bounds for the characteristic rank of manifolds X which occur as total spaces of smooth fiber bundles with fibers totally non-homologous to zero, and also in the situation where, additionally, X itself is null cobordant (see [2], Theorem 2.1 and Theorem 2.2).

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It is useful to think of the characteristic rank of a manifold as the characteristic rank "with respect to the tangent bundle" and introduce bundle dependency in its definition. Bundle dependency can be introduced as in the definition below.

Definition 1.2. Let X be a connected, finite CW-complex and ξ a real vector bundle over X. The characteristic rank of X with respect to the bundle ξ, denoted by $\operatorname{charrank}_\xi(X)$, is by definition the largest integer k, $0 \leq k \leq \dim(X)$, such that every cohomology class $x \in H^j(X;\mathbb{Z}_2)$, $0 \leq j \leq k$, is a polynomial in the Stiefel-Whitney classes $w_i(\xi)$ of ξ. The upper characteristic rank of X, denoted by $\operatorname{ucharrank}(X)$, is the maximum of $\operatorname{charrank}_\xi(X)$ as ξ varies over all vector bundles over X.

Thus, if X is a connected, closed, smooth d-manifold, then $\operatorname{charrank}_{TX}(X) = \operatorname{charrank}(X)$ where TX is the tangent bundle of X.

In this note we discuss some general properties of $\operatorname{charrank}_\xi$ and give a complete description (sometimes in terms of bounds) of $\operatorname{charrank}_\xi(X)$ of vector bundles ξ over X when X is: a product of spheres, the real and complex projective spaces, the Dold manifold $P(m,n)$, the Moore space $M(\mathbb{Z}_2,n)$ and the stunted projective spaces $\mathbb{RP}^n/\mathbb{RP}^m$. We now briefly describe the contents of this note.

For a space X, let r_X denote the smallest positive integer such that $\widetilde{H}^{r_X}(X;\mathbb{Z}_2) \neq 0$. In the case that such an integer does not exist, that is, all the reduced cohomology groups $\widetilde{H}^i(X;\mathbb{Z}_2) = 0$, $1 \leq i \leq \dim(X)$, then we set $r_X = \dim(X)+1$. In any case, $r_X \geq 1$.

Making the definition of the characteristic rank bundle dependent almost immediately shows that a more general version of Theorem 1.1 is true which yields sharper bounds on the cup-length in certain cases. We shall prove the following.

Theorem 1.3. *Let X be a connected, closed, smooth d-manifold. Let ξ be a vector bundle over X satisfying the following:*

- *there exists k, $k \leq \operatorname{charrank}_\xi(X)$, such that every monomial $w_{i_1}(\xi)\cdots w_{i_r}(\xi)$, $i_t \leq k$, of total degree d is zero.*

Then,

$$\operatorname{cup}(X) \leq 1 + \frac{d-k-1}{r_X}.$$

We note that if X is an unoriented boundary, then $\xi = TX$ satisfies the conditions of the above theorem with $k = \operatorname{charrank}_{TX}(X)$. In the above theorem we do not assume that X is an unoriented boundary. We also show that the assumption in Theorem 1.1 that $z < d-1$ is always satisfied (see Corollary 2.11 and Remark 2.12).

If X is an unoriented boundary and there exists a vector bundle ξ over X and k satisfying the conditions of the above theorem, such that

$$\operatorname{charrank}(X) = \operatorname{charrank}_{TX}(X) < k \leq \operatorname{charrank}_\xi(X),$$

then the bound for $\operatorname{cup}(X)$ using $\operatorname{charrank}_\xi(X)$ is much sharper than that obtained by Theorem 1.1. We note that over the null cobordant manifold $S^d \times S^m$, $d = 2,4,8$ and $m \neq 2,4,8$ there exists a vector bundle ξ satisfying the conditions of Theorem 1.3 (see, for example, Example 4.5, Example 4.6 below).

If X is a space with $\operatorname{ucharrank}(X) = \dim(X)$, it turns out that the cup-length $\operatorname{cup}(X)$ of X can be computed as the maximal length of a non-zero product of the Stiefel-Whitney classes of a suitable bundle over X. We prove the following.

Theorem 1.4. *Let X be a connected, closed, smooth d-manifold. If* $\mathrm{ucharrank}(X) = \dim(X)$*, then there exists a vector bundle ξ over X such that*

$$\mathrm{cup}(X) = \max\{k \mid \textit{there exist } i_1, \dots i_k \geq 1 \textit{ with } w_{i_1}(\xi) \cdots w_{i_k}(\xi) \neq 0\}.$$

Making the definition of charateristic rank bundle dependent allows us, under certain conditions, to construct homomorphisms on the group $KO(X)$. It is clear from the definition that $\mathrm{charrank}_\xi(X) = \mathrm{charrank}_\eta(X)$ if ξ and η are isomorphic. Let $\mathrm{Vect}_{\mathbb{R}}(X)$ denote the semi-ring of isomorphism classes of real vector bundles over X. We then have a function

$$f : \mathrm{Vect}_{\mathbb{R}}(X) \longrightarrow \mathbb{Z}_2$$

defined by $f(\xi) = \mathrm{charrank}_\xi(X) \pmod 2$. We observe that under certain restrictions on the values of $\mathrm{charrank}_\xi(X)$ the function f is actually a semi-group homomorphism. More precisely we prove the following.

Theorem 1.5. *Let X be a connected finite CW-complex with $r_X = 1$. Assume that for any vector bundle ξ over X,* $\mathrm{charrank}_\xi(X)$ *is either $r_X - 1 = 0$ or is an odd integer. Assume that* $\mathrm{ucharrank}_\xi(X) \geq r_X$*. Then the function*

$$f : \mathrm{Vect}_{\mathbb{R}}(X) \longrightarrow \mathbb{Z}_2$$

defined by $f(\xi) = \mathrm{charrank}_\xi(X)$ *(mod 2) is surjective a semi-group homomorphism and hence gives rise to a group homomorphism $\widetilde{f} : KO(X) \longrightarrow \mathbb{Z}_2$.*

The function f defined in the above theorem is in general not a semi-ring homomorphism (see Remark 4.3). There is a large class of spaces that satisfy the conditions of the above theorem. We prove the following.

Theorem 1.6.
1. *Let $X = \mathbb{RP}^n$. Then* $\mathrm{ucharrank}(X) \geq 1$ *and for any vector bundle ξ over X, the characteristic rank* $\mathrm{charrank}_\xi(X)$ *is either $r_X - 1 = 0$ or is n.*
2. *Let $X = S^1 \times \mathbb{CP}^n$. Then* $\mathrm{ucharrank}(X) \geq 1$ *and for any vector bundle ξ over X, the characteristic rank* $\mathrm{charrank}_\xi(X)$ *is either $r_X - 1 = 0$, 1 or $(2n+1)$.*
3. *Let X be the Dold manifold $X = P(m, n)$. Then* $\mathrm{ucharrank}(X) \geq 1$ *and for any vector bundle ξ over X, the characteristic rank* $\mathrm{charrank}_\xi(X)$ *is either $r_X - 1 = 0$, 1 or $(2n + m)$.*

Recall that the Dold manifold $P(m, n)$ is the quotient of $S^m \times \mathbb{CP}^n$ by the fixed point free involution $(x, z) \mapsto (-x, \bar{z})$.

In this note we concentrate on the computational part of $\mathrm{charrank}_\xi$. We compute the characteristic rank of vector bundles over products of two spheres $S^d \times S^m$, the real and complex projective spaces, the spaces $S^1 \times \mathbb{CP}^n$, the Dold manifold $P(m, n)$ and the Moore space $M(\mathbb{Z}_2, n)$. We also prove some general facts about $\mathrm{charrank}_\xi(\xi)$.

The paper is organized as follows. In Section 2 we prove some general facts about $\mathrm{charrank}_\xi$. In Section 3 we prove Theorem 1.3, Theorem 1.4 and Theorem 1.5. Finally, in Section 4, we compute $\mathrm{charrank}_\xi(X)$ where X is either the product of spheres $S^d \times S^m$, the real and complex projective spaces, the product $S^1 \times \mathbb{CP}^n$, the Dold manifold $P(m, n)$, the Moore space $M(\mathbb{Z}_2, n)$ and the stunted projective space and ξ a vector bundle over X.

Convention. By a space we shall mean a connected, finite CW-complex. All vector bundles are real unless otherwise stated.

2. Generalities

In this section we make some general observations about $\mathrm{charrank}_\xi$. Recall that, for a space X, r_X denotes the smallest positive integer for which the reduced cohomology $\widetilde{H}^{r_X}(X;\mathbb{Z}_2) \neq 0$ and if such a r_X does not exist, then we set $r_X = \dim(X)+1$. Then for any vector bundle ξ over X we have

$$r_X - 1 \leq \mathrm{charrank}_\xi(X) \leq \mathrm{ucharrank}(X).$$

We begin with some easy observations.

Lemma 2.1. *Let ξ and η be any two vector bundles over a space X.*

1. *If $w_{r_X}(\xi) = 0$, then* $\mathrm{charrank}_\xi(X) = r_X - 1$*;*
2. *If $w(\xi) = 1$, then* $\mathrm{charrank}_\xi(X) = r_X - 1$.
3. *If $w(\eta) = 1$, then* $\mathrm{charrank}_{\xi\oplus\eta}(X) = \mathrm{charrank}_\xi(X)$*. Hence if $\widetilde{KO}(X) = 0$, then* $\mathrm{charrank}_\xi(X) = r_X - 1$ *for any vector bundle over X;*
4. *If ξ and η are stably isomorphic, then* $\mathrm{charrank}_\xi(X) = \mathrm{charrank}_\eta(X)$*;*
5. *There exists a vector bundle θ over X such that* $\mathrm{charrank}_{\xi\oplus\theta}(X) = r_X - 1$.

Proof. (1) follows from the definition. Clearly, (2) follows from (1). To prove (3) we note that since $w(\xi \oplus \eta) = w(\xi)$ we have $\mathrm{charrank}_{\xi\oplus\eta}(X) = \mathrm{charrank}_\xi(X)$. As $\widetilde{KO}(X) = 0$, we have $\xi \oplus \varepsilon \cong \varepsilon'$. Hence

$$\mathrm{charrank}_\xi = \mathrm{charrank}_{\xi\oplus\varepsilon}(X) = \mathrm{charrank}_{\varepsilon'}(X) = r_X - 1.$$

This completes the proof of (3). Next, if ξ and η are stably isomorphic we have trivial bundles ε and ε' such that $\xi \oplus \varepsilon \cong \eta \oplus \varepsilon'$. Hence (4) follows from (3). Finally, as X is compact, given ξ we can find a vector bundle θ such that $\xi \oplus \theta \cong \varepsilon$. Hence (5) follows from (4) and (2). □

Lemma 2.2. *Let X be a space.*

1. *If* $\mathrm{ucharrank}(X) \geq r_X$*, then* $\dim_{\mathbb{Z}_2} H^{r_X}(X;\mathbb{Z}_2) = 1$.
2. *If r_X is not a power of 2, then* $\mathrm{ucharrank}(X) = r_X - 1$.

Proof. If ξ is a vector bundle over X with $\mathrm{charrank}_\xi(X) \geq r_X$, then by Lemma 2.1 (1), $w_{r_X}(\xi) \neq 0$. This forces the equality $\dim_{\mathbb{Z}_2} H^{r_X}(X;\mathbb{Z}_2) = 1$ and proves (1). It is known that for any vector bundle ξ, the smallest integer k such that $w_k(\xi) \neq 0$ is always a power of 2 (see, for example, [11]). Lemma 2.1 (1) now completes the proof of (2). □

Let Y be a space and and let $X = \Sigma Y$ be the suspension of Y. Then cup product of elements of positive degree in $H^*(X;\mathbb{Z}_2)$ is zero. The following lemma is an easy consequence of this fact and we omit the proof.

Lemma 2.3. *Let Y be a space and $X = \Sigma Y$. Let k_X be the integer defined by*

$$k_X = \max\{k \mid \dim_{\mathbb{Z}_2} H^j(X;\mathbb{Z}_2) \leq 1, 0 \leq j \leq k\}.$$

Let ξ be any vector bundle over X. Then, $\mathrm{charrank}_\xi(X) \leq k_X$*. In particular,* $\mathrm{ucharrank}(X) \leq k_X$. □

Lemma 2.4. *Let $f : X \longrightarrow Y$ be a map between spaces. If $f^* : H^*(Y;\mathbb{Z}_2) \longrightarrow H^*(X;\mathbb{Z}_2)$ is surjective, then*

$$\mathrm{charrank}_{f^*\xi}(X) \geq \min\{\mathrm{charrank}_\xi(Y), \dim(X)\}$$

for any vector bundle ξ over Y.

Proof. As $w_i(f^*\xi) = f^*(w_i(\xi))$, the surjectivity of f^* implies that every cohomology class in $H^*(X;\mathbb{Z}_2)$ of degree at most $\mathrm{charrank}_\xi(Y)$ is a polynomial in the Stiefel-Whitney classes of $f^*\xi$. If $\mathrm{charrank}_\xi(Y) \geq \dim(X)$, then $\mathrm{charrank}_{f^*\xi}(X) = \dim(X)$. If $\mathrm{charrank}_\xi(Y) \leq \dim(X)$, then $\mathrm{charrank}_\xi(Y) \leq \mathrm{charrank}_{f^*\xi}(X) \leq \dim(X)$. □

Before mentioning further general properties of the characteristic rank we record the characteristic rank of vector bundles over the sphere. The description of the characteristic rank of vector bundles over the spheres is an easy consequence of the following theorem due to Atiyah-Hirzebruch ([1], Theorem 1), (see also [10]).

Theorem 2.5. ([1], Theorem 1) *There exists a real vector bundle ξ over the sphere S^d with $w_d(\xi) \neq 0$ only for $d = 1, 2, 4,$ or 8.* □

For the Hopf bundle ν_d over S^d ($d = 1, 2, 4, 8$), the Stiefel-Whitney class $w_d(\nu_d)$ is not zero. Thus,

$$\mathrm{ucharrank}(S^d) = \begin{cases} d & \text{if } d = 1, 2, 4, \text{ or } 8 \\ d-1 & \text{otherwise} \end{cases}$$

Note that $\mathrm{charrank}(S^d) = d - 1$. We shall use the above description of characteristic rank of vector bundles over the spheres in the sequel without explicit reference.

There are conditions under which one can obtain a natural upper bound on the upper characteristic rank of a space. One such condition is the existence of a spherical class. Recall that a cohomology class $x \in H^k(X;\mathbb{Z}_2)$ is spherical if there exists a map $f : S^k \longrightarrow X$ with $f^*(x) \neq 0$. Note that a spherical class $x \in H^k(X;\mathbb{Z}_2)$ is indecomposable as an element of the cohomology ring. We shall show that the upper characteristic rank of a space is bounded above by the degree of a spherical class.

Proposition 2.6. *Let X be a space and assume that $x \in H^k(X;\mathbb{Z}_2)$ is spherical, $k \neq 1, 2, 4, 8$.*

1. *There does not exist a vector bundle ξ over X with $w_k(\xi) = x$.*
2. *Then for any covering $\pi : E \longrightarrow X$, we have* $\mathrm{ucharrank}(E) < k$. *In particular,* $\mathrm{ucharrank}(X) < k$.

Proof. Assume that ξ is vector bundle over X with $w_k(\xi) \neq 0$. Let $f : S^k \longrightarrow X$ be a map with $f^*(x) \neq 0$. Then $f^*\xi$ is a bundle over S^k with $w_k(f^*\xi) = f^*(w_k(\xi)) \neq 0$. This contradiction completes the proof of (1).

To prove (2), we first show that $\mathrm{ucharrank}(X) < k$. Assume that ξ is a vector bundle over X with $\mathrm{charrank}_\xi(X) \geq k$. Let $f : S^k \longrightarrow X$ be a map with $f^*(x) \neq 0$. By Lemma 2.4, we have

$$\mathrm{charrank}_{f^*\xi}(S^k) = k.$$

This is a contradiction as $k \neq 1, 2, 4, 8$. Hence $\mathrm{ucharrank}(X) < k$. Now f factors through the covering $\pi : E \longrightarrow X$. Let $f = \pi \circ g$. As $g^* \circ \pi^*(x) = f^*(x) \neq 0$, we conclude that $\pi^*(x)$ is spherical. This completes the proof of (2). □

We remark that the case (2) in the above proposition also follows from (1).

Suppose that $\pi : S^d \longrightarrow X$ is a k-sheeted covering with k odd. If X is orientable, then $\pi_* : H_d(S^d;\mathbb{Z}) \longrightarrow H_d(X;\mathbb{Z})$ is well known to be multiplication by k. Since k is odd the homomorphism $\pi^* : H^d(X;\mathbb{Z}_2) \longrightarrow H^d(S^d;\mathbb{Z}_2)$ is an isomorphism. The following is now a consequence of the above proposition.

Corollary 2.7. *Assume that $\pi : S^d \longrightarrow X$ is a k-sheeted covering with k odd. If X is orientable and $d \neq 1, 2, 4, 8$, then* $\mathrm{ucharrank}(X) < d$. □

Let $L = L_m(\ell_1, \ldots, \ell_n)$ denote the Lens space which is a quotient of S^{2n-1} by a free action of the cyclic group $\mathbb{Z}_m$ by orientation preserving maps (see [7], pp. 144). Then, we have a m-sheeted covering $\pi : S^{2n-1} \longrightarrow L$ with L orientable. We have the following corollary.

Corollary 2.8. *Let L be the Lens space as above of dimension greater than 1. If m is odd, then for any vector bundle ξ over L, the total Stiefel-Whitney class $w(\xi) = 1$.*

Proof. As L has dimension $2n-1 \neq 1$, by Corollary 2.7 we have that $\mathrm{ucharrank}(L) < 2n-1$. The description of integral homology groups of L ([7], pp. 144) readily implies that L is actually a $\mathbb{Z}_2$-cohomology $(2n-1)$-sphere. As $\mathrm{ucharrank}(L) < 2n-1$, we must have $w_{2n-1}(\xi) = 0$ for any vector bundle ξ over L. This completes the proof. □

When a spherical class has degree $k = 1, 2, 4$, or 8, there can exist vector bundles of characteristic rank greater than or equal to the degree of the spherical class. For example, the sphere S^k with $k = 1, 2, 4$, or 8 has upper characteristic rank equal to k. The complex projective space $\mathbb{CP}^n$ has a spherical class in degree 2, however $\mathrm{ucharrank}(\mathbb{CP}^n) = 2n$ (see Example 4.4). When a spherical class exists in degree $1, 2, 4$ or 8 we have the following.

Lemma 2.9. *Let X be a space and assume that $x \in H^k(X; \mathbb{Z}_2)$ is spherical where $k = 1, 2, 4, 8$. Then for any vector bundle ξ over X with $\mathrm{charrank}_\xi(X) \geq k$, we have $w_k(\xi) \neq 0$.*

Proof. Let $f : S^k \longrightarrow X$ be a map with $f^*(x) \neq 0$. By Lemma 2.4, $\mathrm{charrank}_{f^*\xi}(S^k) = k$. This implies that $w_k(f^*\xi) \neq 0$. Hence $w_k(\xi) \neq 0$. □

When X is a connected, closed, smooth d-manifold the characteristic rank $\mathrm{charrank}_\xi(X)$ of X takes values in a certain specific range. We prove the following.

Theorem 2.10. *Let X be a connected, closed, smooth d-manifold. Assume that $2r_X \leq d$. Then, for any vector bundle ξ over X, $\mathrm{charrank}_\xi(X)$ is either d or less than $(d - r_X)$.*

Proof. Let ξ be a vector bundle over X with $\mathrm{charrank}_\xi(X) \geq d - r_X$. We shall show that $\mathrm{charrank}_\xi(X) = d$. Since the groups $H^j(X; \mathbb{Z}_2) = 0$ for $d - r_X < j < d$, the proof will be complete if the non-zero element in $H^d(X; \mathbb{Z}_2)$ is a polynomial in the Stiefel-Whitney classes of ξ. As $\mathrm{charrank}_\xi(X) \geq d - r_X \geq r_X$, then by Lemma 2.2, $H^{r_X}(X; \mathbb{Z}_2) \cong \mathbb{Z}_2$. Hence, by Poincare duality, $H^{d-r_X}(X; \mathbb{Z}_2) \cong \mathbb{Z}_2$. Let a, b, x denote the non-zero cohomology classes in degrees r_X, $d - r_X$ and d respectively. The non-degeneracy of the pairing

$$H^{r_X}(X; \mathbb{Z}_2) \otimes H^{d-r_X}(X; \mathbb{Z}_2) \longrightarrow H^d(X; \mathbb{Z}_2)$$

implies that $a \cdot b = x$. As $\mathrm{charrank}_\xi(X) \geq d - r_X \geq r_X$ we have, by Lemma 2.1 (1), $w_{r_X}(\xi) \neq 0$ and hence $w_{r_X}(\xi) = a$ and $b = p(w_1(\xi), w_2(\xi), \ldots)$ is a polynomial in the Stiefel-Whitney classes of ξ. This shows that

$$x = w_{r_X}(\xi) \cdot p(w_1(\xi), w_2(\xi), \ldots)$$

is a polynomial in the Stiefel-Whitney classes of ξ. This completes the proof of the theorem. □

Let X be a connected, closed, smooth d-manifold. If X is an unoriented boundary, then any monomial in the Stiefel-Whitney classes of X of total degree d is zero. Hence the non-zero element in $H^d(X; \mathbb{Z}_2)$ is never a polynomial in the Stiefel-Whitney classes of X. We thus have the following corollary.

Corollary 2.11. *Let X be a connected, closed, smooth d-manifold. Assume that $2r_X \leq d$. If X is an unoriented boundary, then $\mathrm{charrank}_{TX}(X) < d - r_X$.* □

Remark 2.12. If X is as in the above corollary, then as $r_X \geq 1$, we always have $\text{charrank}_{TX}(X) < d-1$. Thus, the assumption in Theorem 1.1 that $z < d-1$ is always satisfied.

3. Proof of Theorem 1.3, Theorem 1.4 and Theorem 1.5

In this section we prove Theorem 1.3,Theorem 1.4 and Theorem 1.5. The proof of Theorem 1.3 is essentially the same as the proof of Theorem 1.1. We reproduce it here for completeness.

Proof of Theorem 1.3 Let $x = x_1 \cdot x_2 \cdots x_s \neq 0$ be a non-zero product of cohomology classes of positive degree and of maximal length. Then $x \in H^d(X;\mathbb{Z}_2)$. If not, then by Poincare duality one can find a y in complimentary dimension such that $x \cdot y \neq 0$ contradicting the maximality of s. By rearranging, we write

$$x = \alpha_1 \cdots \alpha_m \cdot \beta_1 \cdots \beta_n$$

where $\deg(\alpha_i) \leq k$ and $\deg(\beta_j) \geq k+1$. We note that $n \neq 0$. For otherwise the product $\alpha = \alpha_1 \cdots \alpha_m$ which is now polynomial in $w_1(\xi), \ldots, w_k(\xi)$, will be a non-zero element of total degree d contradicting the assumption on ξ. Therefore, if $\beta = \beta_1 \cdots \beta_n$, then $\deg(\beta) \geq k+1$. Thus $\deg(\alpha) \leq d-(k+1)$. Thus

$$\begin{aligned}
\text{cup}(X) &= m+n \\
&\leq \deg(\alpha)/r_X + \deg(\beta)/(k+1) \\
&= \deg(\alpha)/r_X + (d-\deg(\alpha))/(k+1)) \\
&= ((k+1-r_X)\deg(\alpha) + dr))/r_X(k+1) \\
&\leq ((k+1-r_X)(d-(k+1)) + dr_X)/r_X(k+1) \\
&= 1 + \tfrac{d-k-1}{r_X}.
\end{aligned}$$

This completes the proof. □

Remark 3.1. In the case that X is an unoriented boundary and assuming that ξ and k are as in the above theorem and satisfy

$$\text{charrank}_{TX}(X) < k \leq \text{charrank}_\xi(X)$$

then the bound for $\text{cup}(X)$ using the integer k is sharper than the bound in Theorem 1.1 (see also Example 4.5 and Example 4.6 below).

Proof of Theorem 1.4. Let ξ be any vector bundle over X with

$$\text{charrank}_\xi(X) = \text{ucharrank}(X) = \dim(X).$$

Let $\text{cup}(X) = k$. We shall show that some product of the Stiefel-Whitney classes of ξ of length k is non-zero. Let

$$x = x_1 \cdot x_2 \cdots x_k \neq 0$$

be a non-zero product of cohomology classes $x_i \in H^*(X;\mathbb{Z}_2)$ with $\deg(x_i) \geq 1$. As $\text{charrank}_\xi(X) = \dim(X)$, each x_i can be written as a sum of monomials in the Stiefel-Whitney classes of ξ. Thus x can be written as a sum of monomials in the Stiefel-Whitney classes of ξ each of length at least k. As $x \neq 0$, it follows that some monomial in the Stiefel-Whitney classes of ξ of length k is non-zero. This completes the proof of the theorem. □

Remark 3.2. (1) The proof of the above Theorem 1.4 actually shows that in the case $\text{ucharrank}(X) < \dim(X)$ and

$$x = x_1 \cdots x_k \neq 0$$

with $1 \leq \deg(x_i) \leq \operatorname{ucharrank}(X)$, then for any vector bundle ξ over X with $\operatorname{charrank}_\xi(X) = \operatorname{ucharrank}(X)$ some product of the Stiefel-Whitney classes of ξ of length k is non-zero.

(2) The conclusion of the Theorem 1.4 is not true if $\operatorname{ucharrank}(X) < \dim(X)$. If $X = S^k$, $k \neq 1, 2, 4, 8$, then $\operatorname{ucharrank}(X) = k - 1 < k$, $\operatorname{cup}(X) = 1$ however $w(\xi) = 1$ for any vector bundle ξ over X.

Proof of Theorem 1.5. First note that the assumptions that $r_X = 1$ and that $\operatorname{ucharrank}(X) \geq r_X$ implies that the function

$$f : \operatorname{Vect}_{\mathbb{R}}(X) \longrightarrow \mathbb{Z}_2$$

defined by

$$f(\xi) = \operatorname{charrank}_\xi(X) \pmod 2$$

is surjective. We shall now check that f is a semi-group homomorphism. To see this let ξ and η be two bundles over X. We have the following cases.

If ξ and η are both orientable, then so is $\xi \oplus \eta$. Hence $w_1(\xi \oplus \eta) = 0$. As $r_X = 1$, it follows that $\operatorname{charrank}_{\xi\oplus\eta}(X) = 0$. The same argument shows that $\operatorname{charrank}_\xi(X) = 0 = \operatorname{charrank}_\eta(X)$. Thus in this case we have $f(\xi \oplus \eta) = f(\xi) + f(\eta)$.

Next suppose that both ξ and η are non-orientable. Then $\xi \oplus \eta$ is orientable and hence $f(\xi \oplus \eta) = \operatorname{charrank}_{\xi\oplus\eta}(X) = 0$ as $r_X = 1$. On the other hand as ξ and η are non-orientable we have

$$f(\xi) = \operatorname{charrank}_\xi(X) = n = \operatorname{charrank}_\eta(X) = f(\eta).$$

Thus, as n is odd, we have the equality $f(\xi \oplus \eta) = f(\xi) + f(\eta)$.

Finally, assume that ξ is orientable and η is not. Then $\xi \oplus \eta$ is not orientable and hence $f(\xi \oplus \eta) = 1$, $f(\xi) = 0$ and $f(\eta) = 1$. So in this case we have $f(\xi \oplus \eta) = f(\xi) + f(\eta)$. This completes the proof that f is a semi-group homomorphism.

This gives rise to a surjective homomorphism

$$f : KO(X) \longrightarrow \mathbb{Z}_2$$

defined by $f(\xi - \eta) = f(\xi) - f(\eta)$. This completes the proof. □

Remark 3.3. The (group) homomorphism $f : KO(X) \longrightarrow \mathbb{Z}_2$ restricts to a surjective homomorphism $g : \widetilde{KO}(X) \longrightarrow \mathbb{Z}_2$. It is easy to see that

$$\ker(f) = \{\xi - \eta \mid \xi \text{ and } \eta \text{ are both orientable or both non-orientable}\}$$

and $\ker(g) = \{\xi - rk(\xi) \mid \xi \text{ is orientable}\}$.

4. Computations and examples

In this section we give a proof of Theorem 1.6 and compute the characteristic rank of vector bundles over X where X is: the product of spheres $S^d \times S^m$, the real or complex projective space, the product space $S^1 \times \mathbb{CP}^n$, the Moore space $M(\mathbb{Z}_2, n)$ and the stunted projective space $\mathbb{RP}^n/\mathbb{RP}^m$.

We begin by describing the characteristic rank of vector bundles over $X = S^d \times S^m$. First note that if $d = m$, then as $r_X = d$ and $\dim_{\mathbb{Z}_2} H^d(X; \mathbb{Z}_2) = 2$, it follows from Lemma 2.2 (1) that $\operatorname{ucharrank}(X) = r_X - 1 = d - 1$.

Lemma 4.1. *Let $X = S^d \times S^m$ with $d < m$. Then,*
$$\mathrm{ucharrank}(X) = \begin{cases} d-1 & \text{if } d \neq 1,2,4,8, \\ m-1 & \text{if } d = 1,2,4,8,\ m \neq 2,4,8 \\ d+m & \text{if } d,m = 1,2,4,8. \end{cases}$$

Proof. The lemma follows from the obeservations made after Theorem 2.5. We note that $r_X = d$ and consider the maps
$$S^d \xrightarrow{i} S^d \times S^m \xrightarrow{\pi_1} S^d,$$
$$S^m \xrightarrow{j} S^d \times S^m \xrightarrow{\pi_2} S^m,$$
where i is the map $x \mapsto (x, y)$ for a fixed $y \in S^m$ and π_1 is the projection to the first factor. The map j is similarly defined. The homomorphisms i^* and j^* are isomorphisms (with inverses π_1^* and π_2^* respectively) in degree d and m respectively.

Assume that $d \neq 1,2,4,8$ and let ξ be a vector bundle over X. Then as $w_d(i^*\xi) = 0$, it follows that $w_d(\xi) = 0$. Thus by Lemma 2.1 (1) we have $\mathrm{charrank}_\xi(X) = r_X - 1 = d - 1$.

Next assume that $d = 1,2,4,8$ and $m \neq 2,4,8$. Let ν_d denote the Hopf bundle over S^d. As $w_d(\nu_d) \neq 0$ it follows that $w_d(\pi_1^*\nu_d) \neq 0$. Thus $\mathrm{charrank}_{\pi_1^*\nu_d}(X) \geq m-1$. Since $m \neq 1,2,4,8$, for any vector bundle ξ over X we must have $w_m(\xi) = 0$. This completes the proof that $\mathrm{charrank}_{\pi_1^*\nu_d}(X) = m-1$ and that $\mathrm{ucharramk}(X) = m-1$.

Finally, let $d = 1,2,4,8$ and $m = 1,2,4,8$. Let ν_d and ν_m denote the Hopf bundles over S^d and S^m. Then, clearly $w_d(\pi_1^*\nu_d \oplus \pi_2^*\nu_m) \neq 0$, $w_m(\pi_1^*\nu_d \oplus \pi_2^*\nu_m) \neq 0$ and $w_{d+m}(\pi_1^*\nu_d \oplus \pi_2^*\nu_m) \neq 0$. This shows that in this case $\mathrm{charrank}(X) = d + m$. This completes the proof of the lemma. □

We shall now prove Theorem 1.6. Recall that the Dold manifold $P(m, n)$ is a $(m + 2n)$-dimensional manifold defined as the quotient of $S^m \times \mathbb{CP}^n$ by the fixed point free involution $(x, z) \mapsto (-x, \bar{z})$. This gives rise to a two-fold covering
$$\mathbb{Z}_2 \hookrightarrow S^m \times \mathbb{CP}^n \longrightarrow P(m, n),$$
and via the projection $S^m \times \mathbb{CP}^n \longrightarrow S^m$, a fiber bundle
$$\mathbb{CP}^n \hookrightarrow P(m, n) \longrightarrow \mathbb{RP}^m$$
with fiber $\mathbb{CP}^n$ and structure group $\mathbb{Z}_2$. In particular, for $n = 1$, we have a fiber bundle
$$S^2 \hookrightarrow P(m, 1) \longrightarrow \mathbb{RP}^m.$$

The mod-2 cohomology ring of $P(m, n)$ is given by [6]
$$H^*(P(m, n); \mathbb{Z}_2) = \mathbb{Z}_2[c, d]/(c^{m+1} = d^{n+1} = 0)$$
where $c \in H^1(P(m, n); \mathbb{Z}_2)$ and $d \in H^2(P(m, n); \mathbb{Z}_2)$.

We shall make use of the following result which shows the existence of certain bundles with suitable Stiefel-Whitney classes.

Proposition 4.2. ([13], p. 86) ([15], Proposition 1.4) *Over $P(m, n)$,*

(1) *there exists a line bundle ξ with total Stiefel-Whitney class $w(\xi) = 1 + c$;*
(2) *there exists a 2-plane bundle η with total Stiefel-Whitney class $w(\eta) = 1 + c + d$.* □

Proof of Theorem 1.6 Let $X = \mathbb{RP}^n$ be the real projective space. Then $r_X = 1$. Let ξ be a vector bundle over X. If ξ is orientable, then $w_1(\xi) = 0$ and hence, by Lemma 2.1 (1), $\text{charrank}_\xi(X) = 0$. On the other hand if if ξ is non orientable, then $w_1(\xi) \neq 0$ and hence $\text{charrank}_\xi(X) = n$ as $H^*(X; \mathbb{Z}_2)$ is polynomially generated by the non zero element in $H^1(X; \mathbb{Z}_2)$. This proves (1).

Next let $X = S^1 \times \mathbb{CP}^n$, then $r_X = 1$. The $\mathbb{Z}_2$ cohomology ring of X is given by

$$H^*(X; \mathbb{Z}_2) = H^*(S^1; \mathbb{Z}_2) \otimes H^*(\mathbb{CP}^n; \mathbb{Z}_2) \cong \mathbb{Z}_2[a, b]/(a^2, b^{n+1}),$$

where a is of degree one and b is of degree two. Let ξ be a vector bundle over X. Evidently, $\text{charrank}_\xi(X)$ is completely determined by the first two Stiefel-Whitney classes of ξ.

We look at several cases. If $w_1(\xi)$ and $w_2(\xi)$ are both non zero, then the description of the cohomology ring $H^*(X; \mathbb{Z}_2)$ forces $\text{charrank}_\xi(X) = 2n + 1$. If $w_1(\xi) = 0$, we have $\text{charrank}_\xi(X) = 0$. If $w_1(\xi) \neq 0$ and $w_2(\xi) = 0$, then $\text{charrank}_\xi(X) = 1$. This completes the proof of (2).

Finally, the proof of (3) is similar to the case (2) above in view of Proposition 4.2. Indeed, as $w_1(\eta) \neq 0$ and $w_2(\eta) \neq 0$ we have $\text{charrank}_\eta(X) = 2n + m$. As $w_1(\xi) = c \neq 0$ and $w_2(\xi) = 0$ we have $\text{charrank}_\xi(X) = 1$ as $c^2 = 0$. This completes the proof of (3) and the theorem. $\square$

Remark 4.3. (1) We remark that, in the case (2) of the above theorem, there exists a line bundle γ over X such that $w_1(\xi) \neq 0$. Thus, $\text{charrank}_\xi(X) = 1$. We can also find a 2-plane bundle η over X such that $w_1(\eta) = 0$ and $w_2(\eta) \neq 0$. Thus $\text{charrank}_\eta(X) = 0$. Then for the Whitney sum $\gamma \oplus \eta$ we have $w_1(\gamma \oplus \eta) = w_1(\gamma) \neq 0$ and $w_2(\gamma \oplus \eta) = w_2(\eta) \neq 0$ and hence $\text{charrank}_{\gamma\oplus\eta}(X) = 2n+1$. The bundles γ and η can be obtained as the pull backs of suitable canonical bundles over $S^1 = \mathbb{RP}^1$ and $\mathbb{CP}^n$ via the projections. Thus, over $X = S^1 \times \mathbb{CP}^n$, there exist vector bundles having all the three possible characteristic ranks.

(2) The function $f : \text{Vect}_{\mathbb{R}}(X) \longrightarrow \mathbb{Z}_2$ constructed in the proof of Theorem 1.5 is in general not a semi ring homomorphism. For example, let γ denote the canonical line bundle over $X = \mathbb{RP}^n$ (n odd). Then $w_1(\gamma) \neq 0$ and hence $f(\gamma) = 1 \in \mathbb{Z}_2$. Now, $w_1(\gamma \otimes \gamma) = w_1(\gamma) + w_1(\gamma) = 0$ and therefore, $f(\gamma \otimes \gamma) = 0 \in \mathbb{Z}_2$. Clearly, $0 = f(\gamma \otimes \gamma) \neq f(\gamma) \cdot f(\gamma) = 1$.

Example 4.4. Let $X = \mathbb{CP}^n$ be the complex projective space. Then $r_X = 2$. Let ξ be a vector bundle over X. Then $\text{charrank}_\xi(X) = 1$ if $w_2(\xi) = 0$ and $\text{charrank}_\xi(X) = 2n$ if $w_2(X) \neq 0$. For the canonical (complex) line bundle γ over X we have $\text{charrank}_\gamma(X) = 2n$.

We now give some examples where the bound for the cup length given by Theorem 1.3 is sharper than that given by Theorem 1.1.

Example 4.5. Let $X = S^2 \times S^6$ and let $\pi_1 : X \longrightarrow S^2$ be the projection. Let, as usual, ν_2 denote the Hopf bundle over S^2. Then, $\text{charrank}_{TX}(X) = 1$, and $\text{charrank}_\xi(X) = 5$ where $\xi = \pi_1^* \nu_2$. The bundle ξ satisfies the condition of Theorem 1.3 with $k = 5$. Then the bound for the cup length, $\text{cup}(X)$, of X given by Theorem 1.1 is 4 and that given by Theorem 1.3 is 2.

Example 4.6. Let $X = S^4 \times S^8$. Let $\xi = \pi_1^* \nu_4 \oplus \pi_2^* \nu_8$. Then, $\text{charrank}_{TX}(X) = 3$ and $\text{charrank}_\xi(X) = 12$. Then ξ satisfies the condition of Theorem 1.3 with $k = 7$. Then the bound for the cup length, $\text{cup}(X)$, of X given by Theorem 1.1 is 3 and that given by Theorem 1.3 is 2.

Example 4.7. Let $X = \widetilde{G}_3(\mathbb{R}^5)$ be the oriented Grassmannian consisting of oriented 3-dimensional vector subspaces in $\mathbb{R}^5$. Let $\xi := \widetilde{\gamma}_{5,3}$ be the canonical oriented 3-plane bundle over X. In this case, $H^1(X;\mathbb{Z}_2) = 0$ and $w_2(\xi)$ generates the vector space $H^2(X;\mathbb{Z}_2)$, and hence, $\mathrm{charrank}_\xi(X) \geq 2$. Now let $\gamma_{5,2}$ and $\gamma_{5,3}$ be the canonical bundle over the Grassmannian $G_2(\mathbb{R}^5)$ and $G_3(\mathbb{R}^5)$ respectively. By fixing an inner product on $\mathbb{R}^5$, let $f : G_3(\mathbb{R}^5) \to G_2(\mathbb{R}^5)$ be the diffeomorphism which takes a 3-dimensional subspace to the orthogonal 2-dimensional subspace. Since, $w_1(\gamma_{5,3})$ and $w_1(\gamma_{5,2})$ generates the vector spaces $H^1(G_3(\mathbb{R}^5)$ and $H^1(G_2(\mathbb{R}^5)$ respectively, we have $f^*(w_1(\gamma_{5,2})) = w_1(\gamma_{5,3})$, where f^* is the induced isomorphism of cohomology rings. The height, $\mathrm{ht}(w_1(\gamma_{5,2}))$, of the first Stiefel-Whitney class $w_1(\gamma_{5,2}) \in H^*(G_2(\mathbb{R}^5);\mathbb{Z}_2)$ of the bundle $\gamma_{5,2}$ is equal to $\dim(G_2(\mathbb{R}^5)) = 6$ ([14, pp. 103, Proposition]) and hence, $\mathrm{ht}(w_1(\gamma_{5,3})) = 6$. Hence, by [9, Lemma 2.1], the cohomology class $w_2(\xi)^3 \in H^6(X;\mathbb{Z}_2)$ is zero. Then the bundle ξ satisfies the condition of Theorem 1.3 with $k = 2$. Hence, the bound for the cuplength, $\mathrm{cup}(X)$, given by Theorem 1.3 is 2.

In the following example we shall compute a bound for the cuplength of a manifold which is not cobordant to zero. This manifold does not statisfy the hypothesis of the Theorem 1.1.

Example 4.8. Let $X = G_3(\mathbb{R}^{11})$ be the Grassmannian consists of 3-dimensional vector subspaces in $\mathbb{R}^{11}$. By [12, Theorem 1.1(ii)], the Grassmannian $G_3(\mathbb{R}^{11})$ does not bound. Let $\xi := \gamma_{11,3}$ be the canonical 3-plane bundle over X. Since the $\mathbb{Z}_2$-cohomology ring $H^*(X;\mathbb{Z}_2)$ is generated by the Stiefel-Whitney classes $w_0(\xi), w_1(\xi), w_2(\xi), w_3(\xi)$, the $\mathrm{charrank}_\xi(X) = \dim X = 24$. By [14, pp. 111 , Corollary], the cohomology element $w_1(\xi)^{15} \cdot w_2(\xi)^3 \cdot w_3(\xi)$ is the longest nonzero cupproduct in $H^*(X,\mathbb{Z}_2)$. Hence, any monomial in $w_1(\xi)$ and $w_2(\xi)$ of total degree 24 is zero. Then the bundle ξ satisfies the condition of Theorem 1.3 with $k = 2$. Hence, the bound for the cuplength, $\mathrm{cup}(X)$, given by Theorem 1.3 is 22.

We now compute $\mathrm{charrank}_\xi(X)$ where X is the Moore space $M(\mathbb{Z}_2, n)$, $n > 1$, and ξ a vector bundle over X. We recall that X is a $(n-1)$-connected $(n+1)$-dimensional CW-complex. We refer the reader to [7] for the basic properties of Moore spaces. We prove the following.

Proposition 4.9. *Let X denote the Moore space $M(\mathbb{Z}_2, n)$ with $n > 1$. Then,*

$$\mathrm{ucharrank}(X) = \begin{cases} = & n-1 & \text{if } n \neq 2,4,8 \\ \geq & 2 & \text{if } n = 2 \\ = & 3 & \text{if } n = 4 \\ = & 7 & \text{if } n = 8 \end{cases}$$

Proof. The Moore space X is a $(n+1)$-dimensional CW-complex with n-skeleton S^n. Let $i : S^n \hookrightarrow X$ denote the inclusion map. Using the cellular chain complex, for example, it is easy to see that the homomorphism

$$i^* : H^n(X;\mathbb{Z}_2) \longrightarrow H^n(S^n;\mathbb{Z}_2)$$

in degree n is an isomorphism.

Assume that $n \neq 2, 4, 8$. It follows from what we know about the characteristic rank of vector bundles over the sphere that if ξ is a vector bundle over X, then $w_n(\xi) = 0$. Hence, $\mathrm{charrank}_\xi(X) = n - 1$. This proves the first equality.

Next let $X = M(\mathbb{Z}_2, 2)$. Then X is a simply connected 3-dimensional CW-complex. We shall make use of Proposition 1 [4] to show that there exists a bundle ξ with $w_2(\xi) \neq 0$. This will prove the second inequality. To see this consider the ordered element

$$(a, 0, 0, 0) \in H^2(X;\mathbb{Z}_2) \oplus H^4(X;\mathbb{Z}_2) \oplus H^4(X;\mathbb{Z}) \oplus H^6(X;\mathbb{Z})$$

where $a \in H^2(X;\mathbb{Z}_2) \cong \mathbb{Z}_2$ is the non-zero element. Using the criteria in Proposition 1 [4], it is straightforward to verify that $(a,0,0,0)$ is in the image of the map

$$\gamma : [X, BSO(6)] \longrightarrow H^2(X;\mathbb{Z}_2) \oplus H^4(X;\mathbb{Z}_2) \oplus H^4(X;\mathbb{Z}) \oplus H^6(X;\mathbb{Z})$$

defined by $\gamma(\xi) = (w_2(\xi), w_4(\xi), p_1(\xi), e(\xi))$. This completes the proof of the second inequality.

Next let $X = M(\mathbb{Z}_2, 4)$. Then X is a 3-connected, 5-dimensional CW-complex. We now use Proposition 1 [4] once again to show that for any vector bundle ξ over X, we have $w_4(\xi) = 0$. This will complete the proof of the third equality. Consider the ordered element

$$(0, b, 0, 0) \in H^2(X;\mathbb{Z}_2) \oplus H^4(X;\mathbb{Z}_2) \oplus H^4(X;\mathbb{Z}) \oplus H^6(X;\mathbb{Z}).$$

where $b \in H^4(X;\mathbb{Z}_2) \cong \mathbb{Z}_2$ is the non-zero element. We note that if $i_* : H^4(X;\mathbb{Z}_2) \longrightarrow H^4(X;\mathbb{Z}_4)$ is induced by the inclusion of the coefficient groups, then i_* is an isomorphism as can be easily checked. It is then immediate, from the conditions in Proposition 1 [4], that $(0, b, 0, 0)$ cannot be in the image of γ.

Finally let $X = M(\mathbb{Z}_2, 8)$. Then X is a 7-connected, 9-dimensional CW-complex. Methods similar to the second case, and using Theorem 3 [5], show that there does not exist a vector bundle ξ over X with $w_8(\xi) \neq 0$. This completes the proof of the proposition. $\square$

Proposition 4.10. *Let X denote the stunted projective space $\mathbb{RP}^n/\mathbb{RP}^m$ with $1 \leq m \leq n-2$. Then*

$$\operatorname{charrank}(X) \begin{cases} = & m \quad \text{if } m+1 \neq 2,4,8 \\ \leq & 2 \quad \text{if } m+1 = 2 \\ \leq & 4 \quad \text{if } m+1 = 4 \\ \leq & 8 \quad \text{if } m+1 = 8 \end{cases}$$

Proof. The stunted projective space X is m-connected with $(m+1)$-skeleton $X^{(m+1)} = S^{m+1}$. If $f : S^{m+1} = X^{(m+1)} \longrightarrow X$ denotes the inclusion map, then it is easy to check that the homomorphism $f^* : H^{m+1}(X;\mathbb{Z}_2) \longrightarrow H^{m+1}(S^{m+1};\mathbb{Z}_2)$ is an isomorphism. Thus, the non-zero element in $H^{m+1}(X;\mathbb{Z}_2)$ is spherical. The first equality of the proposition now follows from Proposition 2.6.

We note that if $j : \mathbb{RP}^n/\mathbb{RP}^m \hookrightarrow \mathbb{RP}^{n+1}/\mathbb{RP}^m$ denotes the obvious inclusion, then j^* is an isomorphism in degrees $i \leq n$ in cohomology with $\mathbb{Z}_2$-coefficients. We claim that $\mathbb{RP}^n/\mathbb{RP}^m$, has a spherical class in degree $m+2$ when $m+1 = 2,4,8$. This will complete the proof of the proposition. Since the map j^* is an isomorphism in suitable degrees, the claim will follow once we show that $\mathbb{RP}^3/\mathbb{RP}^1$, $\mathbb{RP}^5/\mathbb{RP}^3$ and $\mathbb{RP}^9/\mathbb{RP}^7$ have spherical classes in degrees $3, 5, 9$ respectively. We shall show the existence of a spherical class of degree 3 in $\mathbb{RP}^3/\mathbb{RP}^1$, the other cases are dealt with similarly.

Let $X = \mathbb{RP}^3/\mathbb{RP}^1$. The integral homology groups of X can be computed to be

$$H_i(X;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = 0,2,3 \\ 0 & \text{otherwise.} \end{cases}$$

Since X is simply connected the Hurewicz homomorphism

$$\varphi_k : \pi_k(X) \longrightarrow H_k(X;\mathbb{Z})$$

is an isomorphism for $k \leq 2$ and $\varphi_3 : \pi_3(X) \longrightarrow H_3(X;\mathbb{Z})$ is surjective. We shall show the existence of a map $f : S^3 \longrightarrow X$ such that $f_* : H_3(S^3;\mathbb{Z}_2) \longrightarrow H_3(X;\mathbb{Z}_2) \cong \mathbb{Z}_2$ is an isomorphism. This will imply that f^* is an isomorphism in degree 3 (with $\mathbb{Z}_2$-coefficients) and complete the proof. First observe that, since the integral homology of X is torsion free, the mod-2 reduction homomoprhism $\alpha : H_3(X;\mathbb{Z}) \longrightarrow H_3(X;\mathbb{Z}_2)$ is surjective. Let

$f \in \pi_3(X)$ be such that $\alpha\varphi_3(f) \neq 0$. We claim that $f : S^3 \longrightarrow X$ is the required map. To see this note that if $\sigma \in \pi_3(S^3)$ is a generator, then $\alpha\varphi_3 f_*(\sigma) = \alpha\varphi_3(f) \neq 0$ and hence the composition

$$\pi_3(S^3) \xrightarrow{f_*} \pi_3(X) \xrightarrow{\varphi_3} H_3(X;\mathbb{Z}) \xrightarrow{\alpha} H_3(X;\mathbb{Z}_2)$$

is surjective. Using the naturality of the Hurewicz and the mod-2 reduction homomorphisms, we see that the composition

$$\pi_3(S^3) \xrightarrow{\varphi_3} H_3(S^3;\mathbb{Z}) \xrightarrow{\alpha} H_3(S^3;\mathbb{Z}_2) \xrightarrow{f_*} H_3(X;\mathbb{Z}_2)$$

is also surjective. This shows that $f_* : H_3(S^3;\mathbb{Z}_2) \longrightarrow H_3(X;\mathbb{Z}_2)$ is an isomorphism. A similar proof works for the other cases. This completes the proof of the proposition. $\square$

In some cases of the stunted projective spaces we can precisely compute the upper characteristic rank. We just give one example.

Example 4.11. Let $X = \mathbb{RP}^3/\mathbb{RP}^1$. Then X is a simply connected 3-dimensional CW-complex with $r_X = 2$. By the discussion before Theorem 4.1 (2), [16], it follows that there exists a vector bundle ξ over X with $w_1(\xi) = w_3(\xi) = 0$ and $w_2(\xi) \neq 0$. It follows that $\mathrm{charrank}_\xi(X) = 2$ and hence $\mathrm{ucharrank}(X) = 2$.

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