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Abstract

Erdős and Selfridge [2] proved that a product of consecutive integers can never be a perfect power. That is, the equation $x(x+1)(x+2)...(x+(m-1)) = y^n$ has no solutions in positive integers x, m, n where $m, n > 1$. and $y \in \mathbf{Q}$. We consider the equation

$$(x-a_1)(x-a_2)\cdots(x-a_k)+r=y^n$$

where $0 \leq a_1 < a_2 < \cdots < a_k$ are integers and, with $r \in \mathbf{Q}$, $n \geq 3$ and we prove a finiteness theorem for the number of solutions x in $\mathbf{Z}$, y in $\mathbf{Q}$. Following that, we show that, more interestingly, for every nonzero integer

$n > 2$ and for any nonzero integer r which is not a perfect n^{th} power for which the equation admits solutions, k is bounded by an effective bound.

Introduction

Erdős and Selfridge [2] proved that a product of consecutive integers can never be a perfect power. That is, the equation $x(x+1)(x+2)...(x+(m-1)) = y^n$ has no solutions in positive integers x, y, m, n where $m, n > 1$. After this, a natural question is to study $x(x+1)(x+2)...(x+(m-1)) + r = y^n$ with a nonzero integral or rational parameter r. However, this equation is not symmetric like the Erdős-Selfridge equation and requires different methods. In [1], we have proved that in this case there are effective finiteness results for $x, m, n \in \mathbf{Z}$ and $y \in \mathbf{Q}$. We shall also prove finiteness results if we delete many terms from the product involving consecutive integers. We consider the equation

$$(x - a_1)(x - a_2) \cdots (x - a_k) + r = y^n$$

where $0 \leq a_1 < a_2 < \cdots < a_k$ are integers and, with $r \in \mathbf{Q}$, $n \geq 3$. Our first aim is to prove a finiteness theorem for the number of solutions x in $\mathbf{Z}$, y in $\mathbf{Q}$. Following that, we show that, more interestingly, for every nonzero integer $n > 2$ and for any nonzero integer r which is not a perfect n^{th} power for which the equation admits solutions, k is bounded by an effective bound. We recall that the height $H(\alpha)$ of an algebraic number α is the maximum of the absolute values of the integer coefficients in its minimal defining polynomial. In particular, if α is a rational integer, then $H(\alpha) = |\alpha|$ and if α is a rational number $\frac{p}{q} \neq 0$ then $H(\alpha) = \max(|p|, |q|)$.

Our first result is the following one.

Theorem 1.
Let $r \in \mathbf{Q}$, let $0 \leq a_1 < a_2 < \cdots < a_k$ be integers where $k > 2$. Further, let $n > 2$ and assume that we are not in the case when $n = k = 4$. Then, there are only finitely many solutions $x \in \mathbf{Z}$, $y \in \mathbf{Q}$ to the equation

$$(x - a_1)(x - a_2) \cdots (x - a_k) + r = y^n$$

and, all the solutions satisfy

$$max\{H(x), H(y)\} < C$$

where C is an effectively computable constant depending only on n, r and the a_i's.

When r is an integer and not a perfect n^{th} power, we bound k in the following result.

Theorem 2.
Let n be a fixed positive integer > 2 and let r be a nonzero integer which is not a perfect n^{th} power. Let $\{t_m\}_m$ be a sequence of positive integers such that $m/t_m \to \infty$ as $m \to \infty$. There exists an effectively computable number C depending only on n and r such that if $(x-a_1)(x-a_2)\cdots(x-a_{m-t_m})+r = y^n$ with $0 \leq a_1 < a_2 < \cdots < a_{m-t_m}$ has a solution, then $m/(t_m+1) < C$.

To prove theorem 1, we use a beautiful result of Brindza [3].
Let K be an algebraic number field, $R \subset K$ be a finitely generated subring and $g \in R[X]$. Write $g = a\prod_{i=1}^{s}(X-\beta_i)^{r_i}$ over an extension of K, where $a \neq 0$ and $\beta_i \neq \beta_j$ for $i \neq j$. Let R_1 be the ring generated by R along with the denominators of the β_i's. For an integer $n > 1$, consider the equation $g(x) = y^n$ with $x, y \in R_1$. Then, Brindza's theorem asserts :

Theorem (Brindza) [3] :
With the above notations, put $t_i = \frac{n}{(n,r_i)}, i = 1, 2\ldots\cdots, s$. Suppose that $(t_1, t_2, \ldots\cdots, t_s)$ is not a permutation of any of the s-tuples :
(i) $(t, 1, 1, 1, \ldots.\cdots, 1)$ for some t, or
(ii) $(2, 2, 1, 1, 1, \ldots.\cdots, 1)$.
Then, all the solutions of the equation $g(x) = y^n$ with $x, y \in R_1$ satisfy

$$max\{H(x), H(y)\} < C$$

where C is an effectively computable constant depending on K, n and g.

Let us prove theorem 1 using this now.

Proof of Theorem 1.
Let us write f for the polynomial $(X-a_1)(X-a_2)\cdots(X-a_k)$. Suppose $f + r = a\prod_{i=1}^{s}(X-\beta_i)^{r_i}$ with $a \neq 0$ and $\beta_i \neq \beta_j$ for $i \neq j$ algebraic integers. We take R to be the subring $\mathbf{Z}[r]$ of $\mathbf{Q}$ and $K = Q(\beta_1, \beta_2, \cdots, \beta_s)$.
We consider solutions $x, y \in O_K[r]$. We show that $r_i = 1$ or 2 and then use Brindza's theorem to get the result.

Claim : The multiplicity of a root of $f(x) + r$ is at most 2.
Proof. Note that $f + r$ is a polynomial of degree k and hence, its derivative

f' is a polynomial of degree $k-1$. Now, by Rolle's theorem, it has zeroes in the intervals $(a_1, a_2), (a_2, a_3), \cdots, (a_{k-1}, a_k)$. Thus, the roots of f' are distinct. Therefore, if $f + r$ has a multiple root then its multiplicity can be at most two which proves the claim.

Thus,

$$f + r = a \prod_{i=1}^{s} (X - \alpha_i)^{r_i}$$

where each $r_i = 1$ or 2. Also note that $s > 1$ since $k > 2$.
Let $t_i = \frac{n}{(n,r_i)} = \frac{n}{(n,1)}$ or $\frac{n}{(n,2)}$. This implies $t_i = n$ or $\frac{n}{2}$. As $n > 2$, note that $t_i > 1$ for each i. So, the s-tuple $(t_1, t_2, \cdots, \cdots, t_s)$ can never look like $(t, 1, 1, \cdots, 1)$ for any t. If this s-tuple looks like $(2, 2, 1, \cdots, 1)$, then it must be $(2, 2)$ which gives $k = 4 = n$ which is excluded by assumption. So, by Brindza's theorem we get the result.

Proof of Theorem 2.
Since r is not a perfect n^{th} power we can write r as
$r = p_1^{h_1 n + r_1} p_2^{h_2 n + r_2} \ldots\ldots p_t^{h_t n + r_t}$ where p_i's are primes in Z and r_i's are such that not all of them are zeroes. Choose the smallest p_i for which r_i is not zero; so, the exact power of p_i dividing r is $h_i n + r_i$. Take $C = (h_i n + r_i + 1)p_i$ and suppose, if possible, $m/(t_m + 1) \geq C$. Let us write $k := m - t_m$ for simplicity. Then we claim that $(x - a_1)(x - a_2) \cdots (x - a_k)$ is divisible by $p_i^{h_i n + r_i + 1}$. Indeed, look at the number of terms of the product $(x-1)(x-2)\cdots(x-m)$ which are missing in the product $(x - a_1)(x - a_2) \cdots (x - a_k)$; this number is $m - k = t_m$. We claim that there is a string of consecutive integers of length at least $(h_i n + r_i + 1)p_i$ in the product $(x - a_1)(x - a_2) \cdots (x - a_k)$. Indeed, if each consecutive string of integers occurring in the last product is of length at the most $(h_i n + r_i + 1)p_i - 1$, then we would have $k = m - t_m < (t_m + 1)((h_i n + r_i + 1)p_i - 1)$ which means $m < (t_m + 1)(h_i n + r_i + 1)p_i$. Thus, $m/(t_m + 1) < C$. In other words, if m is so large that $m/(t_m + 1) \geq C$, then there is a string of consecutive integers of length at least $(h_i n + r_i + 1)p_i$ in the product $(x - a_1)(x - a_2) \cdots (x - a_k)$. Hence the power of p_i in $(x - a_1)(x - a_2) \cdots (x - a_k)$ is at least $h_i n + r_i + 1$. Thus the power of p_i in $(x - a_1)(x - a_2) \cdots (x - a_k) + r$ is exactly $h_i n + r_i \not\equiv 0 (\mathrm{mod} n)$ since $0 < r_i < n$. This is a contradiction to the equation under consideration.

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