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Strong relative property (T) and spectral gap of random walks

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Abstract

We consider strong relative property (T) for pairs (Γ, G) where Γ acts on G. If N is a connected nilpotent Lie group and Γ is a group of automorphisms of N, we choose a finite index subgroup Γ^0 of Γ and obtain that $(\Gamma, [\Gamma^0, N])$ has strong relative property (T) provided Zariski-closure of Γ has no compact factor of positive dimension. We apply this to obtain the following: G is a connected Lie group with solvable radical R and a semisimple Levi subgroup S. If S_{nc} denotes the product of noncompact simple factors of S and S_T denotes the product of simple factors in S_{nc} that have property (T), then we show that (Γ, R) has strong relative property (T) for a Zariski-dense closed subgroup Γ of S_{nc} if and only if $R = [S_{nc}, R]$. The case when N is a vector group is discussed separately and some interesting results are proved. We also consider actions on solenoids K and proved that if Γ acts on a solenoid K, then (Γ, K) has strong relative property (T) under certain conditions on Γ. For actions on solenoids we provide some alternatives in terms of amenability and strong relative property (T). We also provide some applications to the spectral gap of $\pi(\mu) = \int \pi(g) d\mu(g)$ where π is a certain unitary representation and μ is a probability measure.

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Key words. unitary representations, strong relative property (T), Lie groups, probability measure, semisimple Levi subgroup, solvable radical, spectral radius.

1 Introduction

Let G be a topological (Hausdorff) group and π be a strongly continuous unitary representation of G. We first recall the following weak containment notion: we say that π weakly contains the trivial representation or we write $I \prec \pi$ if for each compact set K and for each $\epsilon > 0$, there is a vector v such that $\sup_{g\in K} ||\pi(g)v - v|| < \epsilon ||v||$. It is easy to see that existence of nontrivial invariant vector implies weak containment. We would be looking at the existence of nontrivial invariant vectors for various forms of weak containment. One such well known condition is property (T): we say that G has property T if $I \prec \pi$ implies π has nontrivial invariant vectors. We mainly consider relative property (T) for triples (G, H, N) and strong relative property (T) for pairs (H, N).

A triple (G, H, N) consisting of a topological group G and its subgroups H and N is said to have relative property (T) if for any unitary representation π of G such that restriction of π to H weakly contains the trivial representation of H, we have $\pi(N)$ has nontrivial invariant vectors.

A pair (H, N) consisting of topological groups H and N with H acting on N by automorphisms is said to have strong relative property (T) if (G, H, N) has relative property (T) for $G = H \ltimes N$: if Γ is a topological group acting on a topological group N by automorphisms, then the semidirect product $\Gamma \ltimes N$ is the product space $\Gamma \times N$ with multiplication given by $(\alpha, x)(\beta, y) = (\alpha\beta, x\alpha(y))$ and N (resp. Γ) is identified with the closed subgroup $\{(e, x) \mid x \in N\}$ (resp. $\{(\alpha, e) \mid \alpha \in \Gamma\}$) of $\Gamma \ltimes N$ under the map $x \mapsto (e, x)$ (resp. $\alpha \mapsto (\alpha, e)$).

Strong relative property (T) was considered by [Sh-99] to obtain a characterization of algebraic groups with property (T) and related results. Relative property (T) was crucial in determining property (T) for various type of groups (see [BHV-08]).

If G is a topological group and H, N are subgroups of G, then $[H, N]$ denotes the closed subgroup generated by $\{aba^{-1}b^{-1} \mid a \in H \text{ and } b \in N\}$.

We will be looking at strong relative property (T) for actions on connected Lie groups and for actions on solenoids: compact connected abelian group of finite dimension will be called a solenoid.

We first recall that a group of automorphisms of a connected Lie group G may also be viewed as a group of linear transformation on the Lie algebra $\mathcal{G}$ of G by identifying each automorphism with its differential. Let Γ be a group of automorphisms of G. Then for any Γ-invariant subgroup N of G, let $[\Gamma, N]$ be the closed subgroup generated by $\{\alpha(g)g^{-1} \mid g \in N, \alpha \in \Gamma\}$. Then $[\Gamma, G]$ is a closed connected normal subgroup of G invariant under any automorphism α normalizing Γ. It can be proved that if $[\Gamma', G] \neq G$ for a subgroup Γ' of finite index in Γ, then (Γ, G) will not have strong relative property (T) (cf. Lemma 2.1). Thus, we would have to look at $[\Gamma', G]$ for any finite index subgroup Γ' of Γ. But there could possibly be plenty of finite index subgroups of Γ. Considering Zariski-closure of Γ we now choose a "smallest" finite index subgroup of Γ with which we obtain our result.

Let Γ_Z be the Zariski-closure of Γ in $GL(\mathcal{G})$ and Γ_Z^0 be the Zariski-connected component of identity in Γ_Z. Then Γ_Z^0 is a closed normal subgroup of finite index in Γ_Z. Let $\Gamma^0 = \Gamma \cap \Gamma_Z^0$. Then Γ^0 is a closed normal subgroup of finite index in Γ. So, $[\Gamma^0, G]$ is Γ-invariant. The action of Γ^0 satisfies an interesting alternative that any

Γ^0-orbit is either singleton or infinite and it can be easily seen that this alternative holds good for connected subgroups of $GL(\mathcal{G})$, but Γ^0 need not be connected.

Remark 1.1 In general, Γ_Z and Γ_Z^0 need not be subgroups of automorphisms of the Lie group G, however if G is a simply connected group, then Γ_Z consists of automorphisms of G. As we will be working with action of Γ on $\mathcal{G}$, Γ_Z as well as Γ_Z^0 is more convenient.

We will now state our result for actions on connected nilpotent Lie groups: proof of the following uses the well known criterion of relative property (T) in terms of invariant measures on projective spaces and the results on invariant groups of measures on projective spaces (cf. [Da-82] and [Ra-04]).

Theorem 1.1 *Let N be a connected nilpotent Lie group and Γ be a group of automorphisms of N. Suppose Γ_Z has no compact factor of positive dimension. Then $[\Gamma_0, N]$ is the maximal subgroup of N such that $(\Gamma, [\Gamma^0, N])$ has strong relative property (T).*

Let G be a connected Lie group with solvable radical R and a semisimple Levi subgroup S. Let S_{nc} denote the product of noncompact simple factors in S and S_T be the product of simple factors in S that have property (T). Cornulier [Co-06] introduced T-radical R_T defined by $R_T = \overline{S_T[S_{nc}, R]}$ and proved that (S_{nc}, R_T) has strong relative property (T) (cf. Remark 3.3.7 of [Co-06]). We now obtain a Zariski-dense subgroup version of this result in the following form. Let X be the Lie algebra of $[S_{nc}, R]$ and $\rho\colon S_{nc} \to GL(X)$ be defined by for $g \in S_{nc}$, $\rho(g)$ is the differential of the restriction to $[S_{nc}, R]$ of the innerautomorphism given by g.

Theorem 1.2 *Suppose Γ is a closed subgroup of S_{nc} such that $\rho(\Gamma)$ is Zariski-dense in $\rho(S_{nc})$. Then the following are equivalent*

(1) (Γ, R) *and* $(\Gamma S_T, \overline{S_T R})$ *have strong relative property* (T)*;*

(2) (G, Γ, R) *and* $(G, \Gamma S_T, \overline{S_T R})$ *have relative property* (T)*;*

(3) $R = [S_{nc}, R]$.

In general, $(\Gamma, [S_{nc}, R])$ *and* $(\Gamma S_T, R_T)$ *(resp.* $(G, \Gamma, [S_{nc}, R])$ *and* $(G, \Gamma S_T, R_T)$*) have strong relative property* (T) *(resp. have relative property* (T)*).*

One of the important step in the proof of [Co-06] is strong relative property (T) for certain semisimple Lie group actions on simply connected nilpotent Lie groups (cf. Proposition 1.5 of [Co-06]) but our Theorem 1.1 is more general. As we shall see condition (3) in 1.2 implies that solvable group has to be nilpotent (see Corollary 4.1). So, we pay more attention to actions on connected nilpotent Lie groups.

To extend Theorem 1.1 to actions on general Lie groups, it is sufficient to consider the remaining case of actions on semisimple Lie groups. This is considered in the following (see Proposition 4.1) which shows that actions on connected nilpotent Lie groups is the crucial case.

Our techniques used in the proof of Theorem 1.1 can also be used to prove strong relative property (T) for actions on solenoids: any finite-dimensional compact connected abelian group will be called a solenoid. Let K be a solenoid and Γ be a group of automorphisms of K. Denote by $\hat{K}$, the dual group of characters on K. Then $\hat{K}$ is a torsion free abelian group of finite rank and hence for some n, $\mathbb{Z}^n \subset \hat{K} \subset \hat{K} \otimes \mathbb{Q} \simeq \mathbb{Q}^n$: n is the dimension of K. Any automorphism α of K, gives an automorphism $\hat{\alpha}$ of $\hat{K}$. Since $\mathbb{Z}^n \subset \hat{K}$, $\hat{\alpha}$ extends to a linear map of $\mathbb{Q}^n$. Thus, any group of automorphisms of K may be realized as a subgroup of $GL_n(\mathbb{Q})$.

For a prime number p, let $\mathbb{Q}_p$ be the p-adic field and $\mathbb{Q}_\infty = \mathbb{R}$. Let Γ_p be the p-adic-closure of Γ in $GL_n(\mathbb{Q}_p)$ for finite p. Let Γ_∞ be the group of $\mathbb{R}$-points of the Zariski-closure of Γ in $GL_n(\mathbb{C})$. We prove the following for actions on solenoids.

Theorem 1.3 *Let K be a solenoid and Γ be a group of automorphisms of K such that the dual action of Γ on $\hat{K}$ has no nontrivial invariant characters. Suppose one of the following condition is satisfied:*

(c_p) *Γ_p is topologically generated by p-adic one-parameter subgroups for some finite prime p;*

(c_∞) *the real connected component of Γ_∞ has no nontrivial compact factor and the dual action of Γ has no finite orbit.*

Then (Γ, K) has strong relative property (T)

Example 1.1 We now give examples of subgroups of $GL_n(\mathbb{Q})$ that satisfies (c_p) for some p.

For $p = \infty$, take $\Gamma \subset GL_n(\mathbb{Z})$ and Γ is Zariski-dense in (a finite extension of) $SL_n(\mathbb{C})$ (or more generally in a noncompact simple Lie group). Then Γ satisfies (c_∞) but Γ does not satisfy (c_p) for any prime p as Γ_p is contained in the compact group $GL_n(\mathbb{Z}_p)$.

Fix a prime number p. For simplicity we will restrict our attention to $n = 2$. Consider the group Γ generated by

$$\{\begin{pmatrix} 1 & \frac{k}{p^i} \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \frac{l}{p^j} & 1 \end{pmatrix} \mid k, l, i, j \in \mathbb{Z}\}.$$

Then Γ_p is the closed subgroup generated by the two p-adic one-parameter unipotent groups

$$\{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{Q}_p\}, \quad \{\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} \mid y \in \mathbb{Q}_p\}.$$

Thus, Γ verifies C_p. It is easy to see that Γ_q is a compact subgroup of $GL_2(\mathbb{Z}_q)$ for any prime $q \neq p$. Thus, Γ does not verify (c_q) for any prime $q \neq p$.

We now look at application of strong relative property (T) to ergodic theory of random walks. Let μ be a regular Borel probability measure on G and π be any unitary representation of G. Consider the μ-average $\pi(\mu)$ of π defined by

$$\pi(\mu)(v) = \int \pi(g) v d\mu(g)$$

for any vector v. Since μ is a probability measure, $||\pi_\mu|| \leq 1$. Let $\mathrm{Spr}(\pi(\mu))$ be the spectral radius of $\pi(\mu)$. $\mathrm{Spr}(\pi(\mu)) < 1$ has many interesting consequences to the ergodic properties of the contraction $\pi(\mu)$ and its iterates $\pi(\mu^n)$ (see [JRT-94] and [LW-95]). It is easy to see that $\mathrm{Spr}(\pi(\mu)) < 1$ implies π does not weakly contains the trivial representation. Results in [Sh-00] and Corollary 8 of [BG-06] prove that this necessary condition is also sufficient for any adapted probability measure μ (that is, the closed subgroup generated by the support of μ is the whole group) on connected semisimple Lie groups with finite center that have no nontrivial compact factors. We use relative property (T) to extend these results to a large class of connected Lie groups.

Corollary 1.1 *Let S be a connected semisimple Lie group having no nontrivial compact factors and R be a locally compact group such that R acts on S and (S, R) has strong relative property (T). Let G be a factor group of $S \ltimes R$. Then*

1. *for any unitary representation π of G with $I \not\prec \pi$ and for any adapted probability measure μ on G,* $\mathrm{Spr}(\pi(\mu)) < 1$.
2. *In particular, if G is a connected Lie group satisfying* (3) *of Theorem 1.2 and $S = S_{nc}$, then* $\mathrm{Spr}(\pi(\mu)) < 1$ *for any adapted probability measure μ on G and for any unitary representation π of G with $I \not\prec \pi$.*

Interesting and important kind of unitary representations are obtained by considering G-spaces. Suppose X is a G-space with G-invariant measure m. Then there is a unitary representation corresponding to the G-space X given by

$$\pi(g)(f)(x) = f(g^{-1}x)$$

for $g \in G$, $x \in X$ and $f \in L^2(X,m)$. We are mainly interested in the unitary representation π_0 of G obtained by restricting $\pi(g)$ to the subspace $L^2_0(X,m) = \{f \in L^2(X,m) \mid \int f dm = 0\}$.

In this situation of representations arising from G-spaces, the point-wise convergence of $\pi(\mu^n)(f)$ is also an interesting problem to consider not just for $f \in L^2$ but also for $f \in L^p$ for all $p \geq 1$. Using methods in [JRT-94], we derive the following on point-wise convergence for adapted and strictly aperiodic μ (that is, the smallest closed normal subgroup a coset of which contains the support of μ is G) and last part uses the result of [BC-09].

Corollary 1.2 *Let G be a connected Lie group satisfying* (3) *of Theorem 1.2 with $S = S_{nc}$ and μ be an adapted and strictly aperiodic probability measure on G. Let X be a G-space with G-invariant probability measure m and π_0 be the associated unitary representation of G on $L^2_0(X,m)$. Suppose $I \not\prec \pi_0$. Then*

$$\pi(\mu^n)f(x) = \int f(g^{-1}x)d\mu^n(g)$$

converges m-a.e. for any $f \in L^p(X,m)$ and $1 < p < \infty$ and the norm convergence holds for $p = 1$ also.

In particular, if Δ is a lattice in G, then for $f \in L^p(G/\Delta)$ and $1 < p < \infty$,

$$\pi(\mu^n)(f)(x) = \int f(g^{-1}x)d\mu^n(g)$$

converges a.e.

2 preliminaries

Let G be a topological (Hausdorff) group and H be a subgroup of G. For a unitary representation π of G, $\mathcal{H}_\pi$ denotes the Hilbert space on which π is defined and $\pi|_H$ denotes the restriction of π to H.

We first recall the following well-known weak-containment properties.

Definition 1 We say that a unitary representation π of a topological group G weakly contains the trivial representation and we write $I \prec \pi$ if for each compact set K and for each $\epsilon > 0$, there is a vector $v \in \mathcal{H}_\pi$ such that $\sup_{g \in K} ||\pi(g)v - v|| < \epsilon ||v||$.

Remark 2.1 Suppose G is a locally compact σ-compact group. Then using the theory of positive definite functions, one can easily see that $I \prec \pi$ if and only if there is a sequence (v_n) of unit vectors such that $||\pi(g)v_n - v_n|| \to 0$ for all g in G.

Definition 2 We say that a locally compact group G has property (T) if any unitary representation of G that weakly contains the trivial representation has nontrivial invariant vectors.

Structure of Lie groups and algebraic groups having property (T) is well understood resulting in rich class of groups having property (T) (cf. [Sh-99] and [Wa-82]). We refer to [BHV-08] for details on groups having property (T). We now look at relativized versions of property (T).

Definition 3 Let G be a topological group with subgroups H and N. We say that (G, H, N) has relative property (T) if for any unitary representation π of G, $I \prec \pi|_H$ implies $\pi(N)$ has nontrivial invariant vectors.

Definition 4 Let H be a topological group acting on a topological group N by automorphisms. We say that the pair (H, N) has strong relative property (T) if $(H \ltimes N, H, N)$ has relative property (T).

Relative versions of property (T) are used in constructing new examples of groups with property (T) and in obtaining spectral radius of random walks (cf. [BG-06]). We first prove the following useful elementary result on strong relative property (T).

Lemma 2.1 *Let H be a locally compact group acting on a locally compact group N by automorphisms and M be a closed subgroup of finite index in H.*

1. *If (H, N) has strong relative property (T), then (M, N) also has strong relative property (T).*
2. *If $[M, N] \neq N$, then (H, N) does not have strong relative property (T).*

Proof Let π be a unitary representation of $M \ltimes N$ such that $I \prec \pi|_M$. By considering a subgroup of M we may assume that M is normal in H. Consider the induced representation σ from π to $H \ltimes N$. Since H/M is finite, the space on which σ is defined consists of functions $f\colon H \ltimes N \to \mathcal{H}_\pi$ satisfying

$$f(hx) = \pi(h)f(x), \quad h \in M \ltimes N, \quad x \in H \ltimes N$$

and σ is defined by

$$\sigma(g)f(x) = f(xg)$$

for all $x, g \in H \ltimes N$.

Take a system of coset representatives $\{Mx_i\}$ of M in H. Let E be a compact subset of H and $\epsilon > 0$. Then there is a compact subset F of M such that $E \subset \cup_i F x_i$. Since $I \prec \pi_M$, there is a vector v such that $\sup_{h \in K} ||\pi(h)v - v|| < \epsilon ||v||$ where

$K = \cup x_i F x_i^{-1} F_1$ and F_1 is a finite subset of M such that $\{x_i x_j\} \subset \cup F_1 x_k$. Define $f\colon H \ltimes N \to \mathcal{H}_\pi$ by $f(hx_i) = \pi(h)v$ for all $h \in M \ltimes N$ and all i. Then it can easily be seen that $f \in \mathcal{H}_\sigma$.

Take $g \in F$ and $h \in M \ltimes N$, if $x_i x_j = h_{ij} x_k$ for some k with $h_{ij} \in F_1$, then $\sigma(gx_j)f(hx_i) = \pi(hx_i g x_i^{-1} h_{ij})v$. This shows that

$$||\sigma(gx_j)f - f|| \leq \sup_{a \in K} ||\pi(a)v - v||$$

and hence

$$\sup_{g \in E} ||\sigma(g)f - f|| < \epsilon ||v|| = \epsilon ||f||$$

Thus, $I \prec \sigma|_H$. If (H, N) has strong relative property (T), $\sigma(N)$ has nontrivial invariant vectors. Thus, there is a non-zero function $f\colon H \ltimes N \to \mathcal{H}_\pi$ such that $f(hx) = \pi(h)f(x)$ and $f(xg) = f(x)$ for all $h \in M \ltimes N$, $x \in H \ltimes N$ and $g \in N$. Since f is non-zero, $f(x_i)$ is a non-zero vector in $\mathcal{H}_\pi$ for some i. Thus, we get $f(x_i) = f(x_i g) = \pi(x_i g x_i^{-1}) f(x_i)$ for all $g \in N$. This implies that $f(x_i)$ is a non-zero vector invariant under $\pi(N)$. This proves that (M, N) has strong relative property (T).

If $[M, N] \neq N$, then in view of the first part, we may assume that $[H, N] \neq N$. Replacing N by $N/[H, N]$ we may assume that H is trivial on N. This shows that H is a normal subgroup $H \ltimes N$, that is $H \ltimes N = H \times N$. Let ρ be a nontrivial irreducible unitary representation of N. Define π on $H \ltimes N = H \times N$ by $\pi(h, x) = \rho(x)$. Then $I \prec \pi|_H = I$ and $\pi(N)$ has no nontrivial invariant vectors as $\pi|_N = \rho$ is nontrivial and irreducible.

We recall the following nondegeneracy conditions on measures.

Definition 5 A probability measure μ on a locally compact group G is called adapted (resp. strictly aperiodic) if the support of μ is not contained in a proper closed subgroup (resp. if the support of μ is not contained in a coset of a proper closed normal subgroup).

2.1 Groups without finite-dimensional representations

Let μ be an adapted and strictly aperiodic probability measure on a locally compact group G and $\hat{G}$ be the equivalent classes of irreducible unitary representations of G. Considering the unitary representation $\oplus_{\pi \in \hat{G} \setminus \{I\}} \pi$, we see that $\sup_{\pi \in \hat{G} \setminus \{I\}} ||\pi_\mu|| < 1$ if $||\rho(\mu)|| < 1$ for unitary representations ρ without invariant vectors. It follows from Theorem 1 of [BG-06] that for groups having property (T) and without nontrivial finite-dimensional unitary representations, $||\pi(\mu)|| < 1$ for any unitary representation π without invariant vectors. Thus for such groups, $\sup_{\pi \in \hat{G} \setminus \{I\}} ||\pi_\mu|| < 1$ and hence point-wise convergence holds (see also [JRT-94]). Such groups are useful in building new classes of groups on which the strong and point-wise convergences hold (see [BG-06]). We now attempt to study this type of groups.

We say that a Hausdorff topological group G has (NCF) (resp. (NFU)) if G has no nontrivial compact factor (resp. G has no nontrivial finite-dimensional unitary representation). We first observe the following:

Lemma 2.2 *Let G be a locally compact group having (NFU). Then we have $[G, G] = G$.*

Proof Since nontrivial locally compact abelian groups have nontrivial finite-dimensional unitary representations, we have $[G, G] = G$.

As our main results are concerned about connected Lie groups, we study Lie groups having (NFU) or (NCF).

As irreducible unitary representations of compact groups are finite-dimensional, G has (NCF) if G has (NFU) but the converse need not be true: the abelian group $\mathbb{Q}_p$ of p-adic numbers has (NCF) but all its irreducible unitary representations are one-dimensional. We now prove the converse for connected Lie groups.

Lemma 2.3 *Let G be a connected Lie group and S be a semisimple Levi subgroup of G. Then the following are equivalent:*

1. *G has (NCF);*
2. *G has (NFU);*
3. *$[G, G] = G$, S has (NCF) and G is the smallest closed normal subgroup containing the semisimple Levi subgroup S.*

Moreover, if G has (NCF), then G is either trivial or nonamenable.

Proof Suppose G has (NCF). Let ρ be a finite-dimensional unitary representation of G. Since $[G, G] = G$, we have $[\rho(G), \rho(G)] = \rho(G)$. Let V be the real Lie subalgebra of $\mathrm{End}(\mathcal{H}_\rho)$ such that V is the Lie algebra of the Lie subgroup $\rho(G)$ of $GL(\mathcal{H}_\rho)$. Then V coincides with its commutator subalgebra and hence V is an algebraic subalgebra (see Corollary 3, section 6.2, Chapter 1 of [GOV-94] or Theorem 15, Chapter III of [Ch-51]). Since $\rho(G)$ is a connected Lie subgroup of $GL(\mathcal{H}_\rho)$, $\rho(G)$ is a subgroup of finite index in a real algebraic group. This implies that $\rho(G)$ is a closed subgroup of $GL(\mathcal{H}_\rho)$. Since ρ is unitary, $\rho(G)$ is compact. Since G has (NCF), ρ is the trivial representation.

Assume that G has (NFU). Since compact groups have nontrivial finite-dimensional representations, and S is a factor of G, S has (NCF). Let M_G be the smallest closed normal subgroup of G containing S. Let R be the solvable radical of G. Then $G = SR$ and so $G/M_G \simeq R/R \cap M_G$. This implies that G/M_G is a solvable group but by Lemma 2.2 we get that $[G, G] = G$, hence $G = M_G$.

Assume that $[G, G] = G$, S has (NCF) and G is the smallest closed normal subgroup containing S. Let H be a closed normal subgroup of G such that G/H is compact and let $\phi: G \to G/H$ be the canonical projection. Then $\phi(S)$ is a semisimple Levi subgroup of G/H. Since G/H is compact, semisimple Levi subgroup $\phi(S)$ of G/H is also compact. Since S has no compact factors, $\phi(S)$ is trivial. Thus, H is a closed normal subgroup containing S, hence $H = G$. This proves that G has (NCF).

We now prove the second part. Suppose G has (NCF) and amenable. Then S has (NCF) and amenable, hence S is trivial. This shows that G is solvable. Since $[G, G] = G$, G is trivial.

3 Actions on vector spaces

We first prove the following lemma characterizing strong relative property (T) for linear actions, a part of the proof uses projection valued measure method of Furstenberg as in [Bu-91], [HV-89] and [Sh-99].

Lemma 3.1 *Let V be a finite-dimensional vector space over a local field of characteristic zero and Γ be a locally compact σ-compact group of linear transformations on V. Then the following are equivalent:*

(1) (Γ, V) has strong relative property (T);

(2) for any Γ-invariant subspace W of V such that the action of Γ on V/W is contained in a compact extension of a diagonalizable group, we have $W = V$.

Before we proceed to the proof we fix the following notation: for a measure λ on a locally compact group G and an automorphism α of G, define the measure $\alpha(\lambda)$ by $\alpha(\lambda)(B) = \lambda(\alpha^{-1}(B))$ for any Borel subset B of G.

Proof It is easy to see that (1) implies (2). We now prove the converse. Suppose (Γ, V) does not have strong relative property (T). Then there is a unitary representation π of $\Gamma \ltimes V$ on a Hilbert space $\mathcal{H}$ such that $I \prec \pi|_\Gamma$ and $\pi(V)$ has no nontrivial invariant vectors. Let (v_n) be a sequence of unit vectors such that $||\pi(h)v_n - v_n|| \to 0$ for all $h \in \Gamma$. Let $\hat{V}$ be the dual of V. Since dual of quotient subspaces of V correspond to subspaces of $\hat{V}$, it is sufficient to find a nontrivial Γ-invariant subspace U of $\hat{V}$ such that dual action of Γ on U is contained in a compact extension of a diagonalizable group.

Let P be the projection valued measure associated to the direct sum decomposition of $\pi|_V$. For any vector $v \in \mathcal{H}$, let $\mu_v(B) = ||P(B)v||^2$ for any Borel subset B of $\hat{V}$. Then μ_v is a non-negative measure on $\hat{V}$. It is easy to verify that $h\mu_v = \mu_{\pi(h)v}$ for $h \in \Gamma$ and $v \in \mathcal{H}$. Since $\pi(V)$ has no nontrivial invariant vectors, $P(\{0\})$ is the trivial projection, hence all μ_v have full measure in $\hat{V} \setminus \{0\}$. Let $\mathbb{P}(\hat{V})$ be the projective space associated to $\hat{V}$ and $\varphi\colon \hat{V} \setminus \{0\} \to \mathbb{P}(\hat{V})$ be the canonical quotient map. Then any $\alpha \in GL(\hat{V})$ defines a transformation $\overline{\alpha}$ on $\mathbb{P}(\hat{V})$ by $\overline{\alpha}(\varphi(v)) = \varphi(\alpha(v))$ for all $v \in \hat{V} \setminus \{0\}$. For simplicity we denote $\overline{\alpha}$ also by α. Then we have $\alpha\varphi = \varphi\alpha$. Now, take $\lambda_n = \varphi(\mu_{v_n})$. Then (λ_n) is a sequence of probability measures on $\mathbb{P}(\hat{V})$. Since $\mathbb{P}(\hat{V})$ is compact, by passing to a subsequence, we may assume that $\lambda_n \to \lambda$ in the weak* topology for some probability measure λ on $\mathbb{P}(\hat{V})$. For $h \in \Gamma$, since $||\pi(h)v_n - v_n|| \to 0$, $h\lambda_n - \lambda_n \to 0$ in the total variation norm. This implies since $\lambda_n \to \lambda$ that $h\lambda = \lambda$ for all $h \in \Gamma$.

Let L be the smallest quasi-linear variety (that is, a finite union of subspaces) of $\hat{V}$ such that $\varphi(L \setminus \{0\})$ contains the support of λ. Define

$$N_L = \{g \in GL(\hat{V}) \mid g(L) = L\}$$

and

$$I_L = \{g \in GL(\hat{V}) \mid g(x) = x \text{ for all } x \in \pi(L \setminus \{0\})\}.$$

Then I_L and N_L are algebraic groups and I_L is a normal subgroup of N_L (see [Da-82] and [Fu-76] for further details). Let $G_\lambda = \{g \in N_L \mid g\lambda = \lambda\}$. Then since $\pi(L \setminus \{0\})$ contains the support of λ, $I_L \subset G_\lambda$. By Corollary 2.5 of [Da-82] in the archimedean case and Proposition 1 of [Ra-04] in the non-archimedean case, G_λ/I_L is compact (see also [Fu-76]): in the real case this implies that G_λ is a group of real points of an algebraic group defined over reals. Let Γ' be the image of Γ in $GL(\hat{V})$ under the dual action. Since $h\lambda = \lambda$ for all $h \in \Gamma$, $\Gamma' \subset G_\lambda$. Let U be the subspace spanned by L. Since L is Γ-invariant, U is Γ-invariant. It follows from the definition of I_L that I_L restricted to U is a group of diagonalizable transformations. Since G_λ/I_L is compact and $\Gamma' \subset G_\lambda$, we get that the dual action of Γ on U is contained in a compact extension of a diagonalizable group.

As a consequence of this criterion we have the following alternative for closed irreducible subgroups of $GL(V)$.

Corollary 3.1 *Let Γ be a closed subgroup of $GL(V)$.*

1. *If V is Γ-irreducible, then (Γ, V) has strong relative property (T) or Γ has polynomial growth.*

2. *If dimension of V is two, then (Γ, V) has strong relative property (T) or Γ is amenable.*

Proof Assume V is Γ-irreducible. If (Γ, V) does not have strong relative property (T), then by Lemma 3.1, there is a proper Γ-invariant subspace W of V such that Γ action on V/W is contained in a compact extension of an abelian group. Since V is Γ-irreducible, $W = \{0\}$. Thus, Γ is contained in a compact extension of an abelian group. Since Γ is closed in $GL(V)$, Γ has polynomial growth.

Assume dimension of V is two. If V is Γ-irreducible, then the result follows from the first part as groups of polynomial growth are amenable. If V is not Γ-irreducible, then since dimension of V is two, Γ is solvable, hence amenable.

We will now put some sufficient conditions on linear actions to obtain the strong relative property (T) of the corresponding pair.

Proposition 3.1 *Let V be a finite-dimensional vector space over $\mathbb{R}$ and Γ be a locally compact σ-compact group of linear transformations on V such that the dual action of Γ on $\hat{V}$ has no nonzero finite orbit.*

(1) Suppose Γ_Z has no compact factor of positive dimension. Then (Γ, V) has strong relative property (T).

(2) Suppose Γ has no finite dimensional unitary representation. Then (Γ, V) has strong relative property (T).

Remark 3.1 It is easy to see that conditions on Γ as in (1) and (2) are not a necessity for strong relative property (T) but it can easily be seen that having no nonzero finite orbit for the dual action is a necessity.

Remark 3.2 If $\Gamma \subset GL(V)$ is finitely generated, then Γ is residually finite, hence Γ has nontrivial finite-dimensional unitary representations. Thus, (2) is not applicable for finitely generated Γ but there are many finitely generated Γ having Γ_Z with no compact factor of positive dimension (for instance $\Gamma = SL(n, \mathbb{Z})$), hence (1) is applicable (see the following Example 3.1).

Proof Let W be a Γ-invariant subspace of V such that the action of Γ on V/W is contained in a compact extension of a diagonalizable group.

We now prove (1). Suppose Γ_Z has no compact factor of positive dimension and the dual action of Γ has no nonzero finite orbit. Let Γ_Z be the group of $\mathbb{R}$-points of the Zariski-closure of Γ. Then W is invariant under Γ_Z and Γ_Z-action on V/W is contained in a compact extension of a diagonalizable group. Since Γ_Z has no compact factor of positive dimension and nontrivial connected abelian Lie groups have nontrivial compact factor, we get that Γ_Z is finite on V/W. If $W \neq V$, then this implies that the dual action of Γ_Z has a nonzero finite orbit in $\hat{V}$. Since $\Gamma \subset \Gamma_Z$, the dual action of Γ has a nonzero finite orbit. This is a contradiction to the assumption that the dual action of Γ on $\hat{V}$ has no nonzero finite orbit. Thus, $W = V$, hence (1) follows from Lemma 3.1.

We now prove (2). Suppose Γ has no nontrivial finite-dimensional unitary representation and the dual action of Γ has no nonzero finite orbit. Since nontrivial subgroup of compact extension of abelian groups have nontrivial finite-dimensional unitary representations, Γ is trivial on V/W. If $W \neq V$, then this implies that the dual action of Γ has nontrivial invariant vectors in $\hat{V}$. This is a contradiction to the assumption that the dual action of Γ on $\hat{V}$ has no nonzero finite orbit. Thus, $W = V$ and hence Lemma 3.1 implies that (Γ, V) has strong relative property (T).

Example 3.1 (1) If $\Gamma \subset GL_n(\mathbb{R})$ is a Zariski-dense subgroup in a connected non-compact simple Lie group, then Γ_Z has no compact factor of positive dimension: for instance, Γ is a lattice in a non-compact simple Lie group such as $SU(n,1), SO(p,q)$.

(2) If $\Gamma \subset GL_n(\mathbb{R})$ is such that no subgroup of finite index in the group of $\mathbb{R}$-points of the Zarkiski-closure of Γ has invariant vectors, then it may be shown that Γ has no finite orbit: for instance if Γ is a lattice in $SL_n(\mathbb{R})$ or $Sp_{2n}(\mathbb{R})$, then Γ has no nonzero finite orbit.

(3) In particular, if Γ is a lattice in a connected non-compact simple Lie subgroup of $GL_n(\mathbb{R})$ that has no invariant vectors in $\mathbb{R}^n$, then Γ_Z has no compact factor of positive dimension and Γ has no finite orbit.

4 Actions on connected Lie groups

We first consider actions on connected nilpotent Lie groups. Let N be a connected nilpotent Lie group. Then denote $N_i = [N, N_{i-1}]$ for $i \geq 1$ and $N_0 = N$. It can easily be seen that each N_i is a closed connected characteristic subgroup of N and N_{i-1}/N_i is abelian.

We first prove a Lemma on actions of Lie groups.

Lemma 4.1 *Let G be a connected Lie group and Γ be a group of automorphisms of G such that Γ_Z has no compact factors of positive dimension. Suppose H and N are a closed connected Γ-invariant subgroup of G such that Γ^0 is trivial on G/H and on H/N. Then Γ^0 is trivial on G/N.*

Proof Let $\mathcal{G}, \mathcal{H}, \mathcal{N}$ be Lie algebras of G, H, N respectively. Since Γ^0 acts trivially on G/H and on H/N, Γ_0 on $\mathcal{G}/\mathcal{N}$ is contained in a unipotent subgroup. Since Γ^0 is a subgroup of finite index in Γ, Γ on $\mathcal{G}/\mathcal{N}$ is contained in a finite extension of a unipotent group. Since Γ_Z has no compact factor of positive dimension, Γ_Z^0 is trivial on $\mathcal{G}/\mathcal{N}$. This implies that Γ^0 is trivial on $\mathcal{G}/\mathcal{N}$.

Proof of Theorem 1.1 Let $H = [\Gamma^0, N]$. Since N is connected, H is a closed connected normal subgroup of N, hence H is a Lie group. Let $M = [\Gamma^0, H]$. Then M is a closed connected normal subgroup of H. Then Γ^0 is trivial on N/H and on H/M. By Lemma 4.1, Γ^0 is trivial on N/M. Since $H = [\Gamma^0, N]$, $H = M$. Replacing N by H we may assume that $N = [\Gamma^0, N]$.

Let $\mathcal{N}$ be the Lie algebra of N and $\exp\colon \mathcal{N} \to N$ be the exponential map. Since N is nilpotent, there is a $k \geq 0$ such that $N_k \neq \{e\}$ and $N_{k+1} = \{e\}$. We prove the result by induction on the dimension of N. Suppose N is abelian. Then exp is a homomorphism. Let π be a unitary representation of $\Gamma \ltimes N$ such that $I \prec \pi|_\Gamma$. We now claim that $\pi(N)$ has nontrivial invariant vectors. Let $\pi_1\colon \Gamma \ltimes \mathcal{N}$ be given by $\pi_1(\alpha, v) = \pi(\alpha, \exp(v))$. Then π_1 is a unitary representation such that $I \prec \pi_1|_\Gamma$. Let $V_0 = [\Gamma^0, \mathcal{N}]$. Then V_0 is a Γ-invariant subspace of $\mathcal{N}$. If Γ_1 is a normal subgroup of finite index in Γ, let $W = [\Gamma_1, V_0]$. Then W is a Γ-invariant vector space and Γ is finite on V_0/W. This implies that Γ^0_Z is trivial on V_0/W, hence Γ^0 is trivial on V_0/W. Since Γ^0 is trivial on $\mathcal{N}/V_0$, by Lemma 4.1, Γ^0 is trivial on $\mathcal{N}/W$. Since $V_0 = [\Gamma^0, \mathcal{N}]$, $W = V_0$. This implies that the no finite index subgroup of Γ acts trivially on a nontrivial quotient of V_0. By Proposition 3.1, we get that (Γ, V_0) has strong relative property (T). Thus, $\pi_1(V_0) = \pi(\exp(V_0))$ has nontrivial invariant vectors. It is easy to see that Γ^0 acts trivially on $N/\overline{\exp(V_0)}$. This implies that $N = [\Gamma^0, N] = \overline{\exp(V_0)}$. Since $\pi(\exp(V_0))$ has nontrivial invariant vectors, $\pi(N)$ has nontrivial invariant vectors. This proves that (Γ, N) has strong relative property (T).

Suppose N is not abelian. Then N_k is a nontrivial closed connected Γ-invariant central subgroup of N. Let $\mathcal{N}_k$ be the Lie algebra of N_k and $\exp_k$ is the exponential map of $\mathcal{N}_k$ onto N_k. Let V_2 be a nontrivial Γ-irreducible subspace of $\mathcal{N}_k$ and $A = \overline{\exp_k(V_2)}$. Since $\exp_k$ is a local diffeomorphism, A is a nontrivial closed connected Γ-invariant central subgroup of N. Applying induction hypothesis to Γ-action on N/A, $(\Gamma, N/A)$ has strong relative property (T).

If a finite index normal subgroup Γ_2 of Γ acts trivially on V_2, then A is in the center of $\Gamma_2 \ltimes N$. Since $(\Gamma, N/A)$ has strong relative property (T), by Lemma 2.1, $(\Gamma_2, N/A)$ has strong relative property (T). Since $A \subset [N, N]$, Proposition 3.1.3 of [Co-06] implies that (Γ_2, N) has strong relative property (T) (see also Remark 3.1.7 of [Co-06]). This implies that (Γ, N) has strong relative property (T).

If no finite index subgroup of Γ acts trivially on V_2. Since V_2 is Γ-irreducible, the dual action of Γ on the dual of V_2 has no finite orbit. Hence by (1) of Proposition 3.1, (Γ, V_2) has strong relative property (T). This implies that (Γ, A) has strong relative property (T). Since $(\Gamma, N/A)$ has strong relative property (T), we get that (Γ, N) has strong relative property (T).

Example 4.1 We now look at few examples of (Γ, N) that are relevant to Theorem 1.1.

(i) Take $N = \mathbb{R}^n$ and $\Gamma = SL_n(R)$ where R is a subring of $\mathbb{R}$ (for instance, $R = \mathbb{Z}, \mathbb{Q}, \mathbb{R}$). Then it is easy to see that $\Gamma_Z = SL_n(\mathbb{R})$ and $\Gamma^0 = \Gamma$. Here $[\Gamma, \mathbb{R}^n] = \mathbb{R}^n$. By Theorem 1.1, (Γ, N) has strong relative property (T).

(ii) Take $N = \{(t, s, r \mid t, s \in \mathbb{R}^n,\ r \in \mathbb{R}\}$ to be the $(2n+1)$-dimensional Heisenberg group with multiplication given by

$$(t, s, r)(t, s', r') = (t + t', s + s', r + r' + < t, s' >)$$

for $t, s, t', s', \in \mathbb{R}^n$, $r, r' \in \mathbb{R}$ and Γ be any Zariski-dense subgroup of $SL_{2n}(\mathbb{R})$. Then $[\Gamma, N] = N$ and hence by Theorem 1.1, (Γ, N) has strong relative property (T).

Using general theory of connected semisimple Lie groups, we have the following which extends Proposition 1.5 of [Co-06].

Corollary 4.1 *Let N be a connected solvable Lie group and S be a connected semisimple Lie subgroup of automorphisms of N. If S has (NCF) and $[S, N] = N$, then N is a nilpotent group with its maximal compact subgroup contained in $[N, N]$ and (Γ, N) has strong relative property (T) for any Zariski-dense closed subgroup Γ of S.*

Proof We first claim that N is nilpotent. Let $G = S \ltimes N$. Then G is a connected Lie group and N is its solvable radical. Let $\rho: G \to GL(\mathcal{G})$ be the adjoint representation of G where $\mathcal{G}$ is the Lie algebra of G. Let $\tilde{G}$ be the algebraic closure of $\rho(G)$ in $GL(\mathcal{G})$. Then Chevalley decomposition implies that $\tilde{G} = \tilde{S}TU$ where $\tilde{S}$ is a semisimple Levi subgroup, T is an abelian group consisting of semisimple elements and U is the unipotent radical. Also, solvable radical of $\tilde{G}$ is TU and $[\tilde{S}, T] = \{e\}$. This implies that $[\tilde{S}, TU] \subset U$. Since $\rho(S)$ is a semisimple subgroup of $\tilde{G}$, replacing $\tilde{S}$ by a conjugate we may assume that $\rho(S) \subset \tilde{S}$. Since $\rho(N)$ is a connected solvable normal subgroup of $\rho(G)$, $\rho(N) \subset TU$. Hence $[\rho(S), \rho(N)] \subset U$. Since $[S, N] = N$, $\rho(N) \subset U$, hence $\rho(N)$ is nilpotent. Since kernel of ρ is the center of G, N is a nilpotent group.

Let Γ be a Zariski-dense subgroup of S. Then since S is connected semisimple Lie group, the connected component of Γ_Z is S, hence $\Gamma^0 = \Gamma$. Since S has (NCF), Γ_Z has no compact factor of positive dimension. Let $H = [\Gamma, N]$. Then H is a closed connected Γ-invariant normal subgroup of N. Since H is connected and Γ is Zariski-dense in S, H is S-invariant. Since $H = [\Gamma, N]$, Γ is trivial on N/H and hence S is trivial on N/H. Since $[S, N] = N$, $H = [\Gamma, N] = N$. By Theorem 1.1, (Γ, N) has strong relative property (T).

In order to prove maximal compact subgroup of N is contained in $[N, N]$. We assume that N is abelian and prove that N has no compact subgroup. Let L be a maximal compact subgroup of N. Then L is S-invariant. Since L is a torus and S is connected, S acts trivially on L. Let $\mathcal{N}$ and $\mathcal{L}$ be the Lie algebras of N and L

respectively. Since S is semisimple and $\mathcal{L}$ is S-invariant, there is a S-invariant vector space V such that $V \oplus \mathcal{L} = \mathcal{N}$. Let exp be the exponential map of $\mathcal{N}$ onto N. If $\{\exp(tv) \mid t \in \mathbb{R}\}$ is relatively compact for some $v \in V$, then $\exp(tv) \in L$ for all $t \in \mathbb{R}$. This implies that $v \in \mathcal{L}$. Since $V \cap \mathcal{L} = \{0\}$, $v = 0$. Thus, $\exp(V)$ is closed. Since $V \oplus \mathcal{L} = \mathcal{N}$, S is trivial on $N/\exp(V)$, hence $[S, N] \subset \exp(V)$. Since $[S, N] = N$, we get that $N = \exp(V)$, hence L is trivial.

Example 4.2 We now give an example to show that N in Corollary 4.1 may have nontrivial compact subgroup. Take $N = \{(t, s, r + \mathbb{Z}) \mid t, s, r \in \mathbb{R}\}$ to be the 3-dimensional reduced Heisenberg group with multiplication given by

$$(t, s, r + \mathbb{Z})(t, s', r' + \mathbb{Z}) = (t + t', s + s', r + r' + ts' + \mathbb{Z})$$

for $t, s, r, t', s', r' \in \mathbb{R}$ and $S = SL_2(\mathbb{R})$. Then $[S, N] = N$ and $\{(0, 0, r + \mathbb{Z}) \mid r \in \mathbb{R}\}$ is a compact central subgroup of N of dimension one.

We now prove Theorem 1.2.

Proof of Theorem 1.2 Using the surjective homomorphism $S \ltimes R \to G$ given by $(x, g) \mapsto gx$ we get that (1) implies (2). If $[S_{nc}, R] \neq R$, then let $N = [S_{nc}, R]$. Then N is a closed normal subgroup of G and S_{nc} acts trivially on R/N. This implies that (Γ, R) does not have strong relative property (T). Since $G/S_{nc}R$ is compact, (G, Γ, R) does not have relative property (T). This proves that (2) implies (3). Assume $[S_{nc}, R] = R$. It follows from Corollary 4.1 that (Γ, R) has strong relative property (T). Since $S_T \subset S_{nc}$ and S_T has property (T), (1) follows.

The second part may be proved from the first part by considering the Lie group $S \ltimes R_T$ (instead of $S \ltimes R$) as S_{nc} has no nontrivial compact factor implies $[S_{nc}, R_T] = R_T$.

Remark 4.1 We would like to remark that Theorem 1.2 may be proved for any locally compact σ-compact Zariski-dense (not necessarily closed) subgroup Γ of S_{nc}.

We now discuss extension of Theorem 1.1 for actions on connected Lie groups. It is quite clear that one needs to consider the two distinct cases of actions on connected solvable Lie groups and actions on connected semisimple Lie groups. We now discuss the case of actions on connected semisimple Lie groups.

Proposition 4.1 *Let Γ be a group of automorphisms of a connected semisimple Lie group G. If (Γ, G) has strong relative property (T), then G has property (T).*

Proof Suppose (Γ, G) has strong relative property (T). Let Z be the center of G. Then Z is Γ-invariant. It can easily be seen that $(\Gamma, G/Z)$ also has strong relative property (T). Using Theorem 1.7.11 of [BHV-08] and replacing G by G/Z, we may assume that G has no center and hence $\mathrm{Aut}(G)$ is an almost algebraic group (see [Da-92]). This implies that the connected component of $\mathrm{Aut}(G)$ has finite index in $\mathrm{Aut}(G)$. Since G is a connected semisimple Lie group, the group of inner automorphisms of G is the connected component of $\mathrm{Aut}(G)$ (see Chapter III, Section 10.2, Corollary 2 of [Bo-98]). Thus, there is a subgroup Γ_1 of finite index in Γ such that Γ_1 is a group of inner-automorphisms on G, hence since G has no center, Γ_1 is isomorphic to a subgroup G_1 of G. We may identify Γ_1 with the subgroup G_1 of G.

Let π be a unitary representation of G such that $I \prec \pi$. Define a unitary representation σ of $\Gamma_1 \ltimes G$ by $\sigma(x, g) = \pi(gx)$ for all $(x, g) \in \Gamma_1 \ltimes G$. Then $I \prec \sigma|_{\Gamma_1}$. Since Γ_1 is a subgroup of finite index in Γ, by Lemma 2.1 we get that (Γ_1, G) has strong relative property (T) and hence $\sigma|_G = \pi$ has nontrivial invariant vector. Thus, G has property (T).

We now prove the spectral gap result.

Proof of Corollary 1.1 Let $\varphi\colon S \ltimes R \to G$ be the canonical projection and π be any unitary representation of G. If μ is any adapted probability measure on G such that $\mathrm{Spr}(\pi(\mu)) = 1$. It follows from [BG-06] that $I \prec \pi \otimes \overline{\pi}$. Let $\rho = \pi \circ \varphi$. Then ρ is a unitary representation of $S \ltimes R$ and $I \prec \rho \otimes \overline{\rho}$. Since S is a connected semisimple Lie group having no nontrivial compact factors, Lemma 4 of [Be-98] implies that $I \prec \rho|_S$. Since (S, R) has strong relative property (T), $\rho(R)$ has nontrivial invariant vectors. Since R is normal, the space of $\rho(S)$-invariant vectors is invariant and hence we may assume that $\rho(S)$ is trivial. This proves that $I \prec \rho$ and hence $I \prec \pi$.

5 Solenoids

We now look at actions on solenoids: recall that a compact connected finite-dimensional abelian group is called solenoid. Recall that if K is a solenoid (of dimension n), then $\mathbb{Q}_p^n$ may be realized as a dense subgroup of K and any group Γ of automorphisms can be realized as a group of linear transformations on $\mathbb{Q}_p^n$.

Proposition 5.1 *Let K be a solenoid and Γ be a group of automorphisms of K. Suppose (Γ, K) does not have strong relative property (T). Then we have the following:*

(1) for each p, there is a proper Γ-invariant subspace V_p of $\mathbb{Q}_p^n$ such that the action of Γ on $\mathbb{Q}_p^n/V_p$ is contained in a compact extension of a diagonalizable group over $\mathbb{Q}_p$;

(2) In addition if Γ is a finitely generated group and for each p either Γ_p is compact or the action of Γ is irreducible on $\mathbb{Q}_p^n$, then there is an abelian subgroup Γ_1 of finite index in Γ.

Proof Let p be a prime number or $p = \infty$. Since $\hat{K} \subset \mathbb{Q}^n \subset \mathbb{Q}_p^n$ and the dual of $\mathbb{Q}_p^n$ is itself, we get that there is a continuous homomorphisms $f_p: \mathbb{Q}_p^n \to K$ such that $f_p(\mathbb{Q}_p^n)$ is dense in K and $f_p(\alpha(x)) = \alpha(f_p(x))$ for all $\alpha \in \Gamma$ and $x \in K$.

Suppose (Γ, K) does not have strong relative property (T). Since $f_p(\mathbb{Q}_p^n)$ is dense in K, $(\Gamma, \mathbb{Q}_p^n)$ also does not have strong relative property (T). By Lemma 3.1, there is a proper Γ-invariant subspace W of $\mathbb{Q}_p^n$ such that the action of Γ on $\mathbb{Q}_p^n/W$ is contained in a compact extension of a diagonalizable group. This proves (1).

We now prove (2). Let B be the dual of $\mathbb{Q}^n$. Then K is a quotient of B and the Γ action on K lifts to an action of B. Suppose (Γ, K) does not have strong relative property (T). Then (Γ, B) also does not have strong relative property (T). Let I be the set of all p such that Γ_p is not compact. Since Γ action is irreducible on $\mathbb{Q}_p^n$ or Γ_p is compact, by Proposition 3.1 we get that Γ action on $\mathbb{Q}_p^n$ is contained in a compact extension of a diagonalizable group, say L_p for $p \in I$. Since Γ is a finitely generated group of matrices over $\mathbb{Q}$, I is finite. Since L_p is a compact extension of a diagonalizable group, L_p contains a subgroup L_p' of finite index such that L_p' is a central group, in particular derived group of L_p' is compact. Let $\Gamma_1 = \cap_{p \in I}(L_p' \cap \Gamma)$. Since I is finite, Γ_1 is a subgroup of finite index in Γ. Since L_p' has compact derived group for any $p \in I$ and Γ_1 is contained in L_p', we get that Γ_1 has finite derived group. Since Γ_1 has finite index in Γ, Γ_1 is also finitely generated, hence Γ_1 contains a subgroup Γ_2 of finite index such that Γ_2 is torsion free. Since $[\Gamma_1, \Gamma_1]$ is finite, Γ_2 is abelian.

The next result shows the effectiveness of (2) of Proposition 5.1 second result of which generalizes a result of Burger [Bu-91]: recall that action of a group Γ on a vector space is called totally irreducible if the action of every subgroup of finite index is irreducible.

Corollary 5.1 *Let K be a solenoid and Γ be a finitely generated group of automorphisms of K.*

1. *If for each p either Γ_p is compact or the action of Γ is irreducible on $\mathbb{Q}_p^n$, then (Γ, K) has strong relative property (T) or Γ is amenable.*

2. *If dimension of K is 2, then Γ is amenable or (Γ, K) has strong relative property (T).*

3. *If $\Gamma \subset GL_n(\mathbb{Z})$ and Γ is totally irreducible on $\mathbb{R}^n$, then (Γ, K) has strong relative property (T) or K has dimension at most two.*

Remark 5.1 If K is the n-dimensional torus (more generally $\Gamma \subset GL_n(\mathbb{Z})$), then Γ_p is compact for any finite prime. Thus, in this case, irreducibility conditions in Corollary 5.1 needed only for the action of Γ on $\mathbb{R}^n$.

Proof Assume that (Γ, K) does not have strong relative property (T). Suppose Γ is irreducible on $\mathbb{Q}_p^n$ or Γ_p is compact, (2) of Proposition 5.1 implies that Γ contains an abelian subgroup Γ_1 of finite index. This proves the first part.

If dimension of K is 2 and (Γ, K) does not have relative property (T). If Γ is not irreducible on some $\mathbb{Q}_p^2$, then Γ is contained in the group of upper triangular matrices which is solvable, hence Γ is amenable. So we may assume that Γ is irreducible on all $\mathbb{Q}_p^2$. By the first part, Γ is amenable. This proves the second part.

If $\Gamma \subset GL_n(\mathbb{Z})$ is totally irreducible on $\mathbb{R}^n$ and (Γ, K) does not have relative property (T). Then $\Gamma \subset GL_n(\mathbb{Z})$ implies that Γ_p is compact for any finite prime p. By the first part Γ is amenable. By [Ti-70], Γ has a solvable subgroup Γ_1 of finite index. Since Γ is totally irreducible on $\mathbb{R}^n$, Γ_1 is irreducible on $\mathbb{R}^n$. Since Γ_1 is solvable, $n \leq 2$.

Proof of Theorem 1.3 Suppose Γ_p is topologically generated by p-adic one-parameter subgroups. If Γ is contained in a compact extension of a diagonalizable group over $\mathbb{Q}_p$, then let H be a closed linear group over $\mathbb{Q}_p$ and D be a diagonalizable normal subgroup of H such that H/D is compact and Γ is contained in H. Since there are no continuous homomorphism from a p-adic one-parameter group into a compact p-adic Lie group, we get that any p-adic one-parameter subgroup of H is contained in D. Since D is diagonalizable, H has no p-adic one-parameter subgroup. Since Γ_p is generated by one-parameter subgroups and $\Gamma \subset H$, Γ is trivial. Since dual action of Γ has no nontrivial invariant characters, (1) of Proposition 5.1 is not satisfied. Thus, (Γ, K) has strong relative property (T).

Suppose Γ satisfies (c_∞). Let Γ_∞^0 be the connected component of identity in Γ_∞. Then Γ_∞^0 is a connected Lie group. Let S be the semisimple Levi subgroup of Γ_∞^0. Since S is a factor of Γ_∞^0, S is noncompact. This implies by Lemma 2.3 that Γ_∞^0 is either trivial or nonamenable. If Γ is a contained in a compact extension of a diagonalizable group over $\mathbb{R}$. Since a compact extension of full diagonalizable group over $\mathbb{R}$ is a $\mathbb{R}$-algebraic group, Γ_∞ is contained in a compact extension of a diagonalizable group. Since compact extension of diagonalizable groups are amenable, we get that Γ_∞ is amenable. Since Γ_∞^0 is either trivial or nonamenable, we get that Γ_∞^0 is trivial. Since dual action of Γ has no finite orbits, (c_∞) violates (1) of Proposition 5.1. Thus, (Γ, K) has strong relative property (T).

We now prove spectral gap for actions on solenoids.

Corollary 5.2 *Let K be a solenoid and Γ be a group of automorphisms of K. Let π_0 be the unitary representation of Γ on $L_0^2(K) = \{f \in L^2(K) \mid \int f = 0\}$ given by $\pi_0(\alpha)f(x) = f(\alpha^{-1}(x))$ for $\alpha \in \Gamma$ and $x \in K$. Suppose (Γ, K) has strong relative property (T). Then $\mathrm{Spr}(\pi_0(\mu)) < 1$ for any adapted probability measure μ on Γ.*

Proof Let ρ be the unitary representation of $\Gamma \ltimes K$ given by $\rho(\alpha, a)(f)(x) = f(\alpha^{-1}(x)a)$ for all $(\alpha, a) \in \Gamma \ltimes K$, $x \in K$ and $f \in L_0^2(K)$. Then $\rho|_\Gamma = \pi_0$. Also, any $f \in L_0^2(K)$ is $\rho(K)$-invariant implies $f = 0$. Thus, $\rho(K)$ has no nontrivial invariant vectors in $L_0^2(K)$.

Suppose (Γ, K) has strong relative property (T). Then $I \not\prec \rho|_\Gamma = \pi_0$ as $\rho(K)$ has no nontrivial invariant vectors. This implies since Γ is countable that $\mathrm{Spr}(\pi_0(\mu)) < 1$ for any adapted probability measure μ on Γ (cf. [LW-95]).

6 G-spaces

Proof of Corollary 1.2 Let μ be any adapted and strictly aperiodic probability measure. Then define $\check{\mu}$ to be the probability measure defined by $\check{\mu}(B) = \mu(B^{-1})$ for any Borel subset B of G. Let $\lambda = \sum_{n=1}^{\infty} \frac{1}{2^{n+1}}(\mu^n * \check{\mu}^n + \check{\mu}^n * \mu^n)$. Then λ is a symmetric adapted probability measure on G.

Let π be a unitary representation of G such that $I \not\prec \pi$. Then it follows from Corollary 1.1 that $\mathrm{Spr}(\pi(\lambda)) < 1$. Since $\pi(\lambda)$ is a self-adjoint positive operator, $||\pi(\lambda)|| < 1$. This implies that $||\pi(\mu)|| < 1$.

Let X be any G-space with G-invariant measure m. For any $1 \leq p < \infty$, let π_p be the representation of G defined on $L_0^p(X, m) = \{f \in L^p(X, m) \mid \int f(x)dm(x) = 0\}$ given by $\pi_p(g)f(x) = f(g^{-1}x)$ for $g \in G$, $f \in L_0^p(X, m)$ and $x \in X$. Let $||\pi_p(\mu)||_p$ be the norm of the operator $\pi_p(\mu)$ on $L_0^p(X, m)$.

Since $I \not\prec \pi_0 = \pi_2$, we get that $||\pi_2(\mu)||_2 < 1$. Since $||\pi_1(\mu)||_1 \leq 1$ and $||\pi_\infty(\mu)(f)||_\infty \leq 1$ where π_∞ is similarly defined on L^∞, we have by interpolation $||\pi_p(\mu)||_p < 1$ for $1 < p < \infty$ (cf. [Ro-86]). This implies for $f \in L_0^p(X, m)$ $(1 < p < \infty)$, that $|| \sum \pi_p(\mu^n)f||_p \leq \sum ||\pi_p(\mu^n)f||_p < \infty$, hence $\lim \pi(\mu^n)f(x) = 0$ a.e. Now the second part of the result follows from Theorem 1 of [BC-09] and from the first part.

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References

[BHV-08] B. Bekka, P. de la Harpe, and A. Valette, Kazhdan's property (T), New Mathematical Monographs, 11. Cambridge University Press, Cambridge, 2008.

[Be-98] B. Bekka, On uniqueness of invariant means, Proc. Amer. Math. Soc. 126 (1998), 507–514.

[BC-09] B. Bekka and Y. Cornulier, A spectral gap property for subgroups of finite covolume in Lie groups, Colloq. Math. 118 (2010), no. 1, 175-182.

[BG-06] B. Bekka and Y. Guivarc'h, A spectral gap property for random walks under unitary representations, Geom. Dedicata 118 (2006), 141–155.

[Bo-98] N. Bourbaki, Lie groups and Lie algebras. Chapters 13. Translated from the French. Reprint of the 1989 English translation. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 1998.

[Bu-91] M. Burger, Kazhdan constants for SL$(3, \mathbf{Z})$, J. Reine Angew. Math. 413 (1991), 36-67.

[Co-06] Y. de Cornulier, Relative Kazhdan property, Ann. Sci. cole Norm. Sup. (4) 39 (2006), 301333.

[Ch-51] C. Chevalley, Théorie des groupes de Lie. Tome II. Groupes algébriques. Actualités Sci. Ind. no. 1152. Hermann & Cie., Paris, 1951.

[Da-82] S. G. Dani, On ergodic quasi-invariant measures of group automorphism, Israel J. Math. 43 (1982), 62-74.

[Da-92] S. G. Dani, On automorphism groups of connected Lie groups, Manuscripta Math. 74 (1992), 445-452.

[Fu-76] H. Furstenberg, A note on Borel's density theorem, Proc. Amer. Math. Soc. 55 (1976), 209-212.

[GOV-94] V. V. Gorbatsevich, A. L. Onishchik, and E. B. Vinberg, Lie Groups and Lie Algebras III, Encyclopaedia of Mathematical Sciences 41. Berlin: Springer, 1994.

[HV-89] P. de la Harpe and A. Valette, La propriété (T) de Kazhdan pour les groupes localement compacts (avec un appendice de Marc Burger). Astérisque No. 175 (1989).

[JRT-94] R. Jones, J. Rosenblatt and A. Tempelman, Ergodic theorems for convolutions of a measure on a group, Illinois J. Math. 38 (1994), no. 4, 521–553.

[LW-95] M. Lin and R. Wittmann, Averages of unitary representations and weak mixing of random walks, Studia Math. 114 (1995), 127–145.

[Ra-04] C. R. E. Raja, Ergodic amenable actions of algebraic groups, Glasg. Math. J. 46 (2004), 97–100.

[Ro-86] J. Rosenblatt, Ergodic group actions, Arch. Math. (Basel) 47 (1986), 263–269.

[Sh-99] Y. Shalom, Invariant measures for algebraic actions, Zariski dense subgroups and Kazhdan's property (T), Trans. Amer. Math. Soc. 351 (1999), 3387–3412.

[Sh-00] Y. Shalom, Explicit Kazhdan constants for representations of semisimple and arithmetic groups, Ann. Inst. Fourier (Grenoble) 50 (2000), 833–863.

[Ti-70] J. Tits, Free subgroups in linear groups, Journal of Algebra 20 (1972), 250–270.

[Wa-82] S. P. Wang, On the Mautner phenomenon and groups with property (T), Amer. J. Math. 104 (1982), 1191–1210.

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