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# A stochastic difference equation with stationary noise on groups

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**Abstract**

We consider the stochastic difference equation

$$\eta_k = \xi_k \phi(\eta_{k-1}), \quad k \in \mathbb{Z}$$

on a locally compact group $G$ where $\xi_k$ are given $G$-valued random variables, $\eta_k$ are unknown $G$-valued random variables and $\phi$ is an automorphism of $G$. This equation was considered by Tsirelson and Yor on one-dimensional torus. We consider the case when $\xi_k$ have a common law $\mu$ and prove that if $G$ is a pointwise distal group and $\phi$ is a distal automorphism of $G$ and if the equation has a solution, then extremal solutions of the equation are in one-one correspondence with points on the coset space $K\backslash G$ for some compact subgroup $K$ of $G$ such that $\mu$ is supported on $Kz = z\phi(K)$ for any $z$ in the support of $\mu$. We also provide a necessary and sufficient condition for the existence of solutions to the equation.

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*Keywords:* dissipating probability measures, distal automorphisms, probability measures, pointwise distal groups, shifted convolution powers.

## 1 Introduction

Let $G$ be a locally compact group. Consider the stochastic difference equation on $G$

$$\eta_k = \xi_k \phi(\eta_{k-1}), \quad k \in -\mathbb{N} \tag{$Y$}$$

where $\eta_k$ and $\xi_k$ are $G$-valued random variables and $\phi$ is an automorphims of $G$. The random variables $(\xi_k)$ are given and are called the noise process of the equation $(Y)$. We are interested in finding the law of the unknown process $(\eta_k)$. We further assume that for $k \in \mathbb{Z}$, the random variable $\xi_k$ is independent of $\eta_j$ for $j < k$ and this assumption will be enforced whenever equation of type $(Y)$ is considered.

B. Tsirelson [Ts75] considered the following stochastic difference equation on the real line

$$\eta_k = \xi_k + \mathrm{frac}(\eta_{k-1}) \qquad k \in -\mathbb{N} \tag{$T_1$}$$

where $\mathrm{frac}(x)$ is the fractional part of $x \in \mathbb{R}$ and $(\xi_k)$ is a given stationary Gaussian noise process in order to obtain his celebrated example of the stochastic differential equation

$$dX_t = dB_t + b^+(t, X)dt, \quad X(0) = 0 \tag{$T_2$}$$

which has unique solution that is not strong where $(B_t)$ is the one-dimensional Brownian motion. It was also noted that under some conditions solution of stochastic difference equation $(T_1)$ determines solution of Tsirelson's stochastic differential equation $(T_2)$ (see [Ts75] for more details).

It is easy to see that the set of all solutions $(\eta_k)$ of equation $(Y)$ is a convex set, hence by extremal solution we mean the extreme point of the convex set of all solutions.

M. Yor [Yo92] formulated equation $(T_1)$ in the form of equation $(Y)$ on the one-dimensional torus $\mathbb{R}/\mathbb{Z}$ when $\phi$ is the identity automorphism and $(\xi_k)$ is a general noise process. In particular, [Yo92] proved that extremal solutions of the equation $(Y)$ are in one-one correspondence with points on the coset space $(\mathbb{R}/\mathbb{Z})/M$ where $M$ is a closed subgroup of $\mathbb{R}/\mathbb{Z}$. When $\phi$ is the identity automorphism, equation $(Y)$ was considered on general compact groups [AkUY08] and when the noise law $(\xi_k)$ is stationary, equation $(Y)$ was considered on abelian groups [Ta09]. The main results of [AkUY08] amd [Ta09] extend the result of [Yo92] and proved that the extremal solutions can be identified with $G/H$ where $H$ is a certain compact subgroup of $G$. Assuming that the noise process $(\xi_k)$ is stationary, we extend the result of [Yo92] to a larger class of groups called pointwise distal groups (that is, $e$ is not a limit point of $\{g^nxg^{-n} \mid n \in \mathbb{Z}\}$ for any $g, x \in G$ and $x \neq e$) that includes nilpotent groups, compact groups, discrete groups and connected groups of polynomial growth and when the automorphism $\phi$ is distal (that is, $e$ is not the limit of $\{\phi^n(x) \mid n \in \mathbb{Z}\}$ for any $x \in G$ and $x \neq e$).

**Theorem 1.1** *Let $G$ be a locally compact pointwise distal group and $\phi$ be a distal automorphism of $G$. Let $(\xi_k)_{k\in\mathbb{Z}}$ be $G$-valued random variables with common law $\mu$. Suppose the equation*

$$\eta_k = \xi_k \phi(\eta_{k-1}), \qquad k \in \mathbb{Z} \tag{1}$$

*has a solution. Then there exists a compact subgroup $K_\mu$ such that for any $z$ in the support of $\mu$, $\mu z^{-1}$ is supported on $K_\mu = z\phi(K_\mu)z^{-1}$ and we have a one-one correspondence between left $K_\mu$-invariant probability measures $\lambda$ on $G$ and the laws*

$(\lambda_k)$ *of the solutions* $(\eta_k)$ *of the equation* (1) *given by*

$$\lambda_k = \begin{cases} z\phi(z)\cdots\phi^{k-1}(z)\phi^k(\lambda) & k \geq 1 \\ \lambda & k = 0 \\ \phi^{-1}(z^{-1})\cdots\phi^k(z^{-1})\phi^k(\lambda) & k < 0. \end{cases}$$

*Moreover, extremal solutions* $(\eta_k)$ *of the equation (1) are in one-one correspondence with the elements of the coset space* $K_\mu\backslash G$*.*

# 2 Preliminaries

Let $G$ be a locally compact group and $\phi$ be an automorphism of $G$. For a (regular Borel) probability measure $\mu$ on $G$, we define a probability measures $\check{\mu}$ and $\phi(\mu)$ on $G$ by $\check{\mu}(E) = \mu(E^{-1})$ and $\phi(\mu)(E) = \mu(\phi^{-1}(E))$ for any Borel subset $E$ of $G$.

For any two probability measures $\mu$ and $\lambda$, the convolution of $\mu$ and $\lambda$ is denoted by $\mu * \lambda$ and is defined by

$$\mu * \lambda(E) = \int \mu(Ex^{-1})d\lambda(x)$$

for any Borel subset $E$ of $G$. For $n \geq 1$ and for a probability measure $\mu$ on $G$, $\mu^n$ denotes the $n$-th convolution power of $\mu$.

For $x \in G$ and a probability measure $\mu$ on $G$, $x\mu$ (resp., $\mu x$) denotes $\delta_x * \mu$ (resp., $\mu * \delta_x$).

For a compact subgroup $K$ of $G$, $\omega_K$ denotes the normalized Haar measure on $K$ and a probability measure $\lambda$ on $G$ is called left $K$-invariant if $x\lambda = \lambda$ for all $x \in K$ (which is equivalent to $\omega_K * \lambda = \lambda$, by Theorem 1.2.7 of [He77]).

We say that a sequence $(\lambda_n)$ of probability measures on $G$ converges (in the weak topology) to a probability measure $\lambda$ on $G$ if $\int f d\lambda_n \to \int f d\lambda$ for all continuous bounded functions on $G$.

A set $\mathcal{F}$ of probability measures on $G$ is said to be uniformly tight if for each $\epsilon > 0$ there is a compact set $C_\epsilon$ of $G$ such that $\rho(C_\epsilon) > 1 - \epsilon$ for all $\rho \in \mathcal{F}$. It follows from Prohorov's Theorem that $\mathcal{F}$ is uniformly tight if and only if $\mathcal{F}$ is relatively compact in the space of probability measures on $G$ equipped with weak topology (cf. Theorem 1.1.11 of [He77]).

Let $(\xi_k)_{k\in\mathbb{Z}}$ be $G$-valued random variables. We are interested in investigating the laws of random variables $(\eta_k)$ that satisfies the stochastic difference equation

$$\eta_k = \xi_k\phi(\eta_{k-1}), \qquad k \in \mathbb{Z} \tag{1}$$

and $\xi_k$ is independent of $\eta_j$ for all $j < k$: [AkUY08] and [Yo92] considered only negative $k$, we also could have considered $k \in -\mathbb{N}$ but that would not have made any difference. Since we are interested in the law of the solutions of equation (1), we would be studying the corresponding convolution equation

$$\lambda_k = \mu_k * \phi(\lambda_{k-1}) \tag{1'}$$

for all $k \in \mathbb{Z}$ where $\mu_k$ and $\lambda_k$ are the laws of $\xi_k$ and $\eta_k$ respectively.

We consider equation (1) when $\xi_k$ is stationary on groups of the following type:

**Definition 1** A locally compact group $G$ is called pointwise distal if for any $x, g \in G \setminus (e)$, $e$ is not in the closure of $\{g^n x g^{-n} \mid n \in \mathbb{Z}\}$.

It is easy to see that discrete groups, in general groups having small invariant neighborhoods are pointwise distal. The class of pointwise distal groups includes all compact extensions of nilpotent groups, connected groups of polynomial growth (cf. [Ro86]), p-adic Lie groups of type $R$ (cf. [Ra05]). The class of pointwise distal groups appear in the study of shifted convolution powers which is crucial to our study (cf. [RaS10]).

The following type of automorphisms will be considered.

**Definition 2** An automorphism $\phi$ of a locally compact group $G$ is called distal if $e$ is not in the closure of $\{\phi^n(x) \mid n \in \mathbb{Z}\}$ for any $x \in G \setminus (e)$.

It is easy to see that if $\phi$ preserves a metric on $G$, then $\phi$ is distal on $G$. All unipotent matrices on finite-dimensional vector spaces are distal. It can easily be verified that inner automorphisms of pointwise distal groups are distal. In the final section we give examples of compact groups in which all automorphisms are distal.

Given a automorphism $\phi$ on a locally compact group $G$, the following type of group is useful in our approach to the equation (1): the semidirect product of $\mathbb{Z}$ and $G$, denoted by $\mathbb{Z} \ltimes_\phi G$ is defined by

$$(n, g)(m, h) = (n + m, g\phi^n(h))$$

for all $n, m \in \mathbb{Z}$ and $g, h \in G$. It is easy to see that $\mathbb{Z} \ltimes_\phi G$ is a locally compact group with $G$ as an open subgroup.

Given a probability measure $\mu$ on a locally compact group $G$ and an automorphisms $\phi$ of $G$, we will also be studying the probability measure $1 \otimes \mu$ on $\mathbb{Z} \ltimes_\phi G$ defined by

$$1 \otimes \mu(A \times B) = \delta_1(A)\mu(B)$$

for any subset $A$ of $\mathbb{Z}$ and any Borel subset $B$ of $G$.

## 3 Dissipating measures

A probability measure $\mu$ on a locally compact group $G$ is called dissipating if for any compact set $C$ in $G$, $\sup_{x\in G}\mu^n(xC)\to 0$.

In the study of dissipating measures the smallest closed normal subgroup a coset of which contains the support of $\mu$ plays a crucial role, let $N_\mu$ denote this normal subgroup of $G$. Let $G_\mu$ be the closed subgroup generated by the support of $\mu$. If $G_\mu$ is non-compact and $G_\mu/N_\mu$ is compact, then [JaRW96] showed that $\mu$ is dissipating. [Ja96] and [Ja07] provided many sufficient conditions (on $G_\mu$ or on $\mu$) for $\mu$ to be dissipating, for instance if $G_\mu$ is not amenable, then $\mu$ is dissipating (Corollary 1.6 of [Ja96] or Corollary 3.6 of [Ja07]).

We will now provide a necessary and sufficient condition so that the equation (1) has a solution.

**Proposition 3.1** *Let $G$ be a locally compact group and $(\xi_k)_{k\in\mathbb{Z}}$ be $G$-valued random variables with common law $\mu$. Let $\phi$ be an automorphism of $G$. Then there is a solution $(\eta_k)$ of the equation*

$$\eta_k=\xi_k\phi(\eta_{k-1}),\qquad k\in\mathbb{Z}$$

*if and only if the probability measure $\rho=1\otimes\mu$ on $\tilde G=\mathbb{Z}\ltimes_\phi G$ is not dissipating.*

**Remark 3.1** If $\phi$ is trivial in Proposition 3.1, then $\rho^n = n\otimes\mu^n$, hence for any compact set $C$ of $G$, $\sup_{a\in\tilde G}\rho^n(Ca)=\sup_{x\in G}\mu^n(Cx)$. Since $G$ is open in $\tilde G$, $\rho$ is dissipating if and only if $\mu$ is dissipating. Thus, when $\phi$ is trivial, the equation in Proposition 3.1 has a solution if and only if $\mu$ is dissipating.

**Proof** If there is a sequence $(\eta_k)$ of $G$-valued random variables such that

$$\eta_k=\xi_k\phi(\eta_{k-1})$$

for all $k\in\mathbb{Z}$. Let $\lambda_k$ be the law of $\eta_k$. Then the corresponding convolution equation is

$$\lambda_k=\mu*\phi(\lambda_{k-1})$$

for all $k\in\mathbb{Z}$. Iterating the convolution equation we get that

$$\lambda_k=\mu*\phi(\mu)*\cdots*\phi^{j-1}(\mu)*\phi^j(\lambda_{k-j})$$

for all $k\in Z$ and all $j\geq 1$. It follows from Theorems 1.2.21 (iii) of [He77] that there is a sequence $(g_j)$ in $G$ such that the sequence $(\mu*\phi(\mu)*\cdots*\phi^{j-1}(\mu)g_j^{-1})$ is uniformly tight.

Consider the group $\tilde{G} = \mathbb{Z} \ltimes_\phi G$ and the measure $\rho = 1 \otimes \mu$ on $\tilde{G}$. Then $\rho^j = j \otimes \mu * \phi(\mu) * \cdots * \phi^{j-1}(\mu)$, hence we get that $\rho^j(j, g_j)^{-1} = j \otimes \mu * \phi(\mu) * \cdots * \phi^{j-1}(\mu)(-j, \phi^{-j}(g_j^{-1})) = \mu * \phi(\mu) * \cdots * \phi^{j-1}(\mu)g_j^{-1}$. This implies that $(\rho^j(j, g_j)^{-1})$ is uniformly tight. Then there is a compact set $C$ of $\tilde{G}$ such that $\rho^j(C(j, g_j)) > \frac{1}{2}$ for all $j \geq 1$. This proves that $\rho$ is not dissipating.

For the converse, suppose $\rho$ on $\tilde{G} = \mathbb{Z} \ltimes_\phi G$ is not dissipating. We first assume that $G$ is separable. Then by Theorem 3.1 of [Cs66], there is a sequence $(n_j, g_j)$ in $\tilde{G}$ such that $(\rho^j(n_j, g_j)^{-1})$ converges (see also [To65]). Now since $\rho^j = j \otimes \mu * \phi(\mu) * \cdots * \phi^{j-1}(\mu)$, we get that $\rho^j(n_j, g_j)^{-1} = (j - n_j) \otimes \mu * \phi(\mu) * \cdots * \phi^{j-1}(\mu)\phi^{j-n_j}(g_j^{-1})$, hence $(\mu * \phi(\mu) * \cdots * \phi^{j-1}(\mu)x_j)$ converges for $x_j = \phi^{j-n_j}(g_j^{-1})$. Let $\gamma = \lim \mu * \phi(\mu) * \cdots * \phi^{j-1}(\mu)x_j$. Then $\gamma = \mu * \phi(\lim(\mu * \phi(\mu) * \cdots * \phi^{j-2}(\mu)x_{j-1}) * \phi(x_{j-1}^{-1})x_j$, hence by Theorems 1.2.21 and 1.1.11 of [He77], we get that $(\phi(x_{j-1}^{-1})x_j)$ is relatively compact. If $x$ is a limit point of $(\phi(x_{j-1}^{-1})x_j)$, then $\gamma = \mu * \phi(\gamma)x$.

Let $\lambda_0 = \gamma$ and for $k \geq 1$, let $\lambda_k = \gamma x^{-1}\phi(x^{-1}) \cdots \phi^{k-1}(x^{-1})$. Then for $k \geq 0$, $\lambda_{k+1} = \lambda_k \phi^k(x^{-1})$. If $\lambda_k = \mu * \phi(\lambda_{k-1})$ for $k \geq 1$, then $\lambda_{k+1} = \mu * \phi(\lambda_{k-1})\phi^k(x^{-1}) = \mu * \phi(\lambda_{k-1}\phi^{k-1}(x^{-1})) = \mu * \phi(\lambda_k)$. Since $\lambda_1 = \gamma x^{-1} = \mu * \phi(\lambda_0)$, it follows from induction that $\lambda_k = \mu * \phi(\lambda_{k-1})$ for all $k \geq 1$.

For $k < 0$, let $\lambda_k = \gamma\phi^{-1}(x) \cdots \phi^k(x)$. Then for $k < 0$, $\lambda_k = \lambda_{k+1}\phi^k(x)$. If $\lambda_k = \mu * \phi(\lambda_{k-1})$ for $k \leq 0$, then $\lambda_{k-1} = \lambda_k \phi^{k-1}(x) = \mu * \phi(\lambda_{k-1}\phi^{k-2}(x)) = \mu * \phi(\lambda_{k-2})$. Since $\lambda_0 = \gamma = \mu * \phi(\gamma)x = \mu * \phi(\gamma\phi^{-1}(x)) = \mu * \phi(\lambda_{-1})$, it follows from induction that $\lambda_k = \mu * \phi(\lambda_{k-1})$ for all $k \leq 0$. Thus, proving $\lambda_k = \mu * \phi(\lambda_{k-1})$ for all $k \in \mathbb{Z}$.

In general, if $G$ is any locally compact group, then replacing $G$ by the smallest $\phi$-invariant closed subgroup of $G$ containing the support of $\mu$ we may assume that $G$ is $\sigma$-compact. This implies that $\mathbb{Z} \ltimes_\phi G$ is $\sigma$-compact. Then by Theorem 8.7 of [HeR79], there exists a $\phi$-invariant compact normal subgroup $K$ of $G$ such that $G/K$ is separable. Let $\mu'$ be the image of $\mu$ on $G/K$. Since $K$ is compact, $1 \otimes \mu'$ is not dissipating on $\mathbb{Z} \ltimes_\phi G/K$. By the previous case, there exists probability measures $\lambda'_k$ on $G/K$ such that $\lambda'_k = \mu' * \phi(\lambda'_{k-1})$ for all $k \in \mathbb{Z}$. It follows from 1.2.15 (iii) of [He77] that there exists probability measures $\lambda_k$ on $G$ such that $\lambda_k * \omega_K = \lambda_k$ and image of $\lambda_k$ on $G/K$ is $\lambda'_k$. Since $K$ is $\phi$-invariant, $\mu * \phi(\lambda_{k-1}) * \omega_K = \mu * \phi(\lambda_{k-1})$ for all $k \in \mathbb{Z}$. Since both $\lambda_k$ and $\mu * \phi(\lambda_{k-1})$ are projected onto the same probability measure $\lambda'_k$ on $G/K$, by Theorem 1.2.15 (iii) of [He77] we get that $\lambda_k = \mu * \phi(\lambda_{k-1})$ for all $k \in \mathbb{Z}$.

# 4 Shifted convolution property

A probability measure $\mu$ on a locally compact group $G$ is said to have shifted convolution property if either $\mu$ is dissipating or there is a compact subgroup $K$ of $G$ and a $g \in G$ such that $\mu^n g^{-n} \to \omega_K$ and $gKg^{-1} = K$. Shifted convolution property was studied in details in [RaS10], it is shown that all probability measures on a locally compact group $G$ have shifted convolution property if and only if the group $G$ is pointwise distal (see Theorem 6.1 of [RaS10]). We first prove the following result which provides a sufficient condition for the existence of large collection of solutions.

**Proposition 4.1** *Let $G$ be a locally compact group and $\mu$ be a probability measure on $G$. If there is a compact subgroup $K$ of $G$ such that for any $z$ in the support of $\mu$, $\mu z^{-1}$ is supported on $K = z\phi(K)z^{-1}$, then for any left $K$-invariant probability measure $\lambda$, $(\lambda_k)$ defined by*

$$\lambda_k = \begin{cases} z\phi(z)\cdots\phi^{k-1}(z)\phi^k(\lambda) & k \geq 1 \\ \lambda & k = 0 \\ \phi^{-1}(z^{-1})\cdots\phi^k(z^{-1})\phi^k(\lambda) & k < 0 \end{cases}$$

*is a solution to equation*

$$\lambda_k = \mu * \phi(\lambda_{k-1}), \qquad k \in \mathbb{Z} \tag{1'}$$

*for any $z$ in the support of $\mu$.*

**Proof** Assume that there is a compact subgroup $K$ of $G$ such that for any $x$ in the support of $\mu$, $\mu x^{-1}$ is supported on $K = x\phi(K)x^{-1}$. Suppose $z$ is in the support of $\mu$. Let $\lambda$ be a left $K$-invariant probability measure on $G$ and define $\lambda_0 = \lambda$. For $k \geq 1$, let $\lambda_k = z\phi(z)\cdots\phi^{k-1}(z)\phi^k(\lambda)$. Then $\lambda_k = z\phi(\lambda_{k-1})$ for all $k \geq 1$.

We first claim that $\lambda_k$ is left $K$-invariant for all $k \geq 0$. Our claim is based on induction. For $k \geq 1$, if $\lambda_{k-1}$ is left $K$-invariant, then $\phi(\lambda_{k-1})$ is left $\phi(K)$-invariant, hence for $x \in K$, $x\lambda_k = xz\phi(\lambda_{k-1}) = z\phi(\lambda_{k-1}) = \lambda_k$ as $\phi(K) = z^{-1}Kz$ implies $z^{-1}xz \in \phi(K)$. This proves that $\lambda_k$ is left $K$-invariant, if $\lambda_{k-1}$ is left $K$-invariant. Since $\lambda_0 = \lambda$ is left $K$-invariant, induction implies that $\lambda_k$ is left $K$-invariant for all $k \geq 0$.

Since $\mu z^{-1}$ is supported on $K = z\phi(K)z^{-1}$, we get that $z^{-1}\mu$ is supported on $\phi(K)$. Since $\lambda_{k-1}$ is left $K$-invariant, $\mu * \phi(\lambda_{k-1}) = z\phi(\lambda_{k-1}) = \lambda_k$ for all $k \geq 1$.

For $k < 0$, let $\lambda_k = \phi^{-1}(z^{-1})\cdots\phi^k(z^{-1})\phi^k(\lambda)$. Then $\lambda_k = \phi^{-1}(z^{-1})\phi^{-1}(\lambda_{k+1})$ for all $k < 0$.

We now claim by induction that $\lambda_k$ is left $K$-invariant for all $k \leq 0$. If for $k < 0$, $\lambda_{k+1}$ is left $K$-invariant, then since $z^{-1}Kz = \phi(K)$, we have $\phi^{-1}(K) = \phi^{-1}(z)K\phi^{-1}(z^{-1})$, hence for $x \in K$, $x\lambda_k = x\phi^{-1}(z^{-1})\phi^{-1}(\lambda_{k+1}) = \phi^{-1}(z^{-1})\phi^{-1}(\lambda_{k+1}) = \lambda_k$. This proves that $\lambda_k$ is left $K$-invariant for all $k \leq 0$.

For $k < 0$, we have $\lambda_k = \phi^{-1}(z^{-1})\phi^{-1}(\lambda_{k+1})$, hence $\phi(\lambda_k) = z^{-1}\lambda_{k+1}$. This implies that for $k \leq 0$, $\mu * \phi(\lambda_{k-1}) = \mu * z^{-1}\lambda_k = \lambda_k$ as $\mu z^{-1}$ is supported on $K$. Thus, $(\lambda_k)$ is a solution to equation $(1')$.

**Proposition 4.2** *Let $G$ be a locally compact group and $\phi$ be a automorphism of $G$. Let $(\xi_k)$ be a sequence of $G$-valued random variables with common law $\mu$. Suppose the measure $1 \otimes \mu$ has shifted convolution property on $\mathbb{Z} \ltimes_\phi G$. Then for any solution $(\eta_k)$ of the equation*

$$\eta_k = \xi_k \phi(\eta_{k-1}), \qquad k \in \mathbb{Z} \tag{2}$$

*and for any $z$ in the support of $\mu$, we have*

*(1) there is a compact subgroup $K_\mu$ such that $\mu$ is supported on $K_\mu z = z\phi(K_\mu)$,*

*(2) $\eta_k = z\phi(\eta_{k-1})$ in law,*

*and when equation (2) has a solution*

*(3) there is a one-one correspondence between left $K_\mu$-invariant probability measures $\lambda$ on $G$ and the laws $(\lambda_k)$ of the solutions $(\eta_k)$ of the equation (2) given by*

$$\lambda_k = \begin{cases} z\phi(z)\cdots\phi^{k-1}(z)\phi^k(\lambda) & k \geq 1 \\ \lambda & k = 0 \\ \phi^{-1}(z^{-1})\cdots\phi^k(z^{-1})\phi^k(\lambda) & k < 0. \end{cases}$$

**Proof** Let $\tilde{G} = \mathbb{Z} \ltimes_\phi G$ be the semidirect product of $\mathbb{Z}$ and $G$ where the $\mathbb{Z}$-action is given $\phi$ and $\rho = \delta_1 \otimes \mu$. Suppose $(\eta_k)$ is a solution to equation (2). Then by Proposition 3.1, $\rho$ is not dissipating. Since $\rho$ has shifted convolution property, there is a compact subgroup $K$ of $\tilde{G}$ and $g \in \tilde{G}$ such that $\rho^i g^{-i} \to \omega_K$ and $gKg^{-1} = K$. By Theorem 4.3 of [Ei92] we get that for any $a$ in the support of $\rho$, $\rho^i a^{-i} \to \omega_K$ and $aKa^{-1} = K$ (cf. Remark 1.2 [RaS10]).

Since $\tilde{G}/G \simeq \mathbb{Z}$, we get that $K \subset G$. Let $z$ be in the support of $\mu$ and $a = (1, z)$. Then $a$ is in the support of $\rho$. Let $z_i = \phi^{-1}(z^{-1})\cdots\phi^{-i+1}(z^{-1})\phi^{-i}(z^{-1})$ for $i > 1$. Then we get that $\rho^i a^{-i} = \mu * \phi(\mu) * \cdots * \phi^{i-1}(\mu)\phi^i(z_i) \to \omega_K$.

Let $\lambda_k$ be the law of $\eta_k$, $k \in \mathbb{Z}$. We now claim that for $k \in \mathbb{Z}$, $\lambda_k$ is left $K$-invariant. For $k \in \mathbb{Z}$,

$$\lambda_k = \mu * \phi(\lambda_{k-1}) = \mu * \phi(\mu) * \cdots * \phi^{i-1}(\mu)\phi^i(z_i) * \phi^i(z_i^{-1})\phi^i(\lambda_{k-i}), \quad i \geq 1$$

and hence by Theorems 1.2.21 (ii) and 1.1.11 of [He77], $(\phi^i(z_i^{-1})\phi^i(\lambda_{k-i}))_{i\geq 1}$ is relatively compact. Thus, for any limit point $\nu$ of $(\phi^i(z_i^{-1})\phi^i(\lambda_{k-i}))$, we get that $\lambda_k = \omega_K \nu$. Thus, $\lambda_k$ is left $K$-invariant.

For any $a$ in the support of $\rho$, we have $\rho^i a^{-i} \to \omega_K$, hence

$$\rho\omega_K a^{-1} = \lim \rho\rho^{i-1}a^{-i+1}a^{-1} = \lim \rho^i a^{-i} = \omega_K.$$

This shows that $\rho$ is supported on $aK = Ka$. Now for $a = (1, z)$, $a^{-1}\rho = \phi^{-1}(z^{-1})\phi^{-1}(\mu)$ is supported on $K$, hence we get that $z^{-1}\mu$ is supported on $\phi(K)$. This implies that for $k \in \mathbb{Z}$,

$$\lambda_k = \mu * \phi(\lambda_{k-1}) = zz^{-1}\mu * \phi(\lambda_{k-1}) = z\phi(\lambda_{k-1})$$

as $\lambda_{k-1}$ is left $K$-invariant. This proves (2).

Let $\lambda = \lambda_0$. Then $\lambda$ is left $K$-invariant and for $k \geq 1$, $\lambda_k = z\phi(\lambda_{k-1}) = z\phi(z) \cdots \phi^{k-1}(z)\phi^k(\lambda)$. For $k < 0$, $\lambda = z\phi(\lambda_{-1}) = \cdots = z\phi(z) \cdots \phi^{-k-1}(z)\phi^{-k}(\lambda_k)$, hence $\lambda_k = \phi^{-1}(z^{-1}) \cdots \phi^k(z^{-1})\phi^k(\lambda)$.

For any $z$ in the support of $\mu$, $a = (1, z)$ is in the support of $\rho$, hence $aKa^{-1} = K$. This implies that $z\phi(K)z^{-1} = K$, hence $\phi(K) = z^{-1}Kz$. Since $z^{-1}\mu$ is supported on $\phi(K)$, we get that $\mu z^{-1} = z(z^{-1}\mu)z^{-1}$ is supported on $z\phi(K)z^{-1} = K$. This shows that the conditions of Proposition 4.1 are satisfied. Thus, for a left $K$-invariant measure $\lambda$ if we define $(\lambda_k)$ as in the proposition, we get that $\lambda_k$ satisfies $\lambda_k = \mu * \phi(\lambda_{k-1})$ for all $k \in \mathbb{Z}$.

**Corollary 4.1** *Let $G$, $\phi$ and $(\xi_k)$, $\mu$ be as in Proposition 4.2. Suppose $1 \otimes \mu$ has shifted convolution property and $1 \otimes \mu$ is not dissipating. Then there exists a compact subgroup $K$ such that extremal solutions $(\eta_k)$ of the equation*

$$\eta_k = \xi_k \phi(\eta_{k-1}), \qquad k \in \mathbb{Z} \tag{2}$$

*are in one-one correspondence with the elements of the coset space $K \backslash G$.*

**Proof** Let $K$ be as in Proposition 4.2. Then it follows from Proposition 4.2 that left $K$-invariant measures and laws of solutions to the equation (2) are in one-one correspondence.

If $\lambda$ is a left $K$-invariant measure and $z$ is in the support of $\lambda$. Let $a \in K$ and $U$ be a neighborhood of $az$. Then $\lambda(U) = \lambda(a^{-1}U) > 0$. This implies that $az$ is in the support of $\lambda$. Thus, support of $\lambda$ is a union of cosets of $K$. If the support of $\lambda$ contains more than one coset, then the corresponding solution to the equation (2) is not extremal. This proves the corollary.

We have the following converse to Proposition 4.2.

**Proposition 4.3** *Let $G$ be a locally compact and $\phi$ be an automorphism. Let $(\xi_k)$ be $G$-valued random variables with common law $\mu$. Suppose the laws of the solutions $(\eta_k)$ to the equation*

$$\eta_k = \xi_k \phi(\eta_{k-1}), \qquad k \in \mathbb{Z}$$

*are left $K$-invariant for some compact subgroup $K$ of $G$ such that $\mu z^{-1}$ is supported on $K = z\phi(K)z^{-1}$ for any $z$ in the support of $\mu$. Then $1 \otimes \mu$ on $\mathbb{Z} \ltimes_\phi G$ has shifted convolution property.*

**Proof** Let $\rho = 1 \otimes \mu$. We first assume that $G$ is separable. Since the equation has solutions, by Proposition 3.1, $1 \otimes \mu$ on $\mathbb{Z} \ltimes_\phi G$ is dissipating. As in Proposition 3.1, there is a $x_j \in G$ and a probability measure $\gamma$ on $G$ such that $\mu * \phi(\mu) * \cdots * \phi^j(\mu) x_j \to \gamma$ and a solution $(\lambda_k)$ with $\lambda_0 = \gamma$. Assumption implies that $\gamma$ is left $K$-invariant.

We now claim by induction that $\mu * \phi(\mu) * \cdots * \phi^j(\mu)$ is supported on $Kz\phi(z) \cdots \phi^j(z)$ for all $j \geq 0$. If $\mu * \phi(\mu) * \cdots * \phi^j(\mu)$ is supported on $Kz\phi(z) \cdots \phi^j(z)$ for some $j \geq 0$, then $\mu * \phi(\mu) * \cdots * \phi^j(\mu) * \phi^{j+1}(\mu)$ is supported on $Kz\phi(z) \cdots \phi^j(z)\phi^{j+1}(K)\phi^{j+1}(z)$. Since $Kz = z\phi(K)$, we get that $\phi^k(z)\phi^{k+1}(K) = \phi^k(K)\phi^k(z)$ for all $k \geq 0$. This shows that $\mu * \phi(\mu) * \cdots * \phi^j(\mu) * \phi^{j+1}(\mu)$ is supported on $Kz\phi(z) \cdots \phi^j(z)\phi^{j+1}(z)$. Since $\mu$ is supported on $Kz$, claim follows from induction.

For $j \geq 1$, let $\sigma_j = \mu * \phi(\mu) * \cdots * \phi^j(\mu) x_j$. Then $\sigma_j * \check{\sigma}_j \to \gamma * \check{\gamma}$ and $\sigma_j x_j^{-1}$ is supported on $Kz\phi(z) \cdots \phi^j(z)$. This implies that $\sigma_j * \check{\sigma}_j$ is supported on $K$, hence $\gamma * \check{\gamma}$ is supported on $K$. Since $\gamma$ is left $K$-invariant, $\gamma * \check{\gamma}$ is also left $K$-invariant and hence $\gamma * \check{\gamma} = \omega_K$. Now $\rho^j = j \otimes \sigma_{j-1} x_{j-1}^{-1}$, hence $\rho^j * \check{\rho}^j = \sigma_{j-1} * \check{\sigma}_{j-1}$ for all $j > 1$. This implies that $\rho^j * \check{\rho}^j \to \omega_K$. By Theorem 4.3 of [Ei92], for any $g$ in the support of $\rho$, $\rho^j g^{-j} \to \omega_K$. For any $g$ in the support of $\rho = 1 \otimes \mu$, there is a $z$ in the support of $\mu$ such that $g = (1, z)$ and hence $gKg^{-1} = z\phi(K)z^{-1} = K$. This proves that $\rho$ has shifted convolution property.

If $G$ is any locally compact group, replacing $G$ by the smallest $\phi$-invariant closed subgroup of $G$ containing the support of $\mu$, we may assume that $G$ is $\sigma$-compact and hence $\mathbb{Z} \ltimes_\phi G$ is also $\sigma$-compact. Then by Theorem 8.7 of [HeR79], each neighborhood $U$ of $e$ contains a $\phi$-invariant compact normal subgroup $K_U$ such that $G/K_U$ is separable. It can easily be verified that the assumptions in the proposition are valid for $G/K_U$ with $KK_U/K_U$ and the image of $\mu$ on $G/K_U$. It follows from the previous case that image of $\mu$ on $G/K_U$ has shifted convolution property. By Proposition 2.3 of [RaS10] we get that $\mu$ itself has shifted convolution property.

We now extend the results of [AkUY08], [Ta09] and [Yo92] to all pointwise distal groups when $\xi_k$ is stationary and $\phi$ is distal on $G$.

**Proof of Theorem 1.1** If $G$ is pointwise distal and $\phi$ is distal, then the group $\mathbb{Z} \ltimes_\phi G$ is a pointwise distal group. By Theorem 6.1 of [RaS10], we get that $1 \otimes \mu$ has shifted convolution property. Now the result follows from Proposition 4.2 and Corollary 4.1.

**Remark 4.1** We would like to remark that if $1 \otimes \mu$ does not have shifted convolution property, then the conclusion on extreme points of the solutions in Theorem 1.1 may not be true even on compact groups. Consider the two dimensional torus $K = (\mathbb{R}/\mathbb{Z})^2$ and $\phi$ be an automorphism of $K$ such that $C(\phi) = \{x \in K \mid \phi^n(x) \to e \ \ as \ \ n \to \infty\}$ is dense in $K$, for instance if we take $\phi$ to be $\phi(x, y) = (x + y + \mathbb{Z}, x + 2y + \mathbb{Z})$ for all $x, y \in \mathbb{R}$, then $C(\phi) = \{(t + \mathbb{Z}, (\frac{1-\sqrt{5}}{2})t + \mathbb{Z}) \mid t \in \mathbb{R}\} \simeq \mathbb{R}$ is a vector (nonclosed) subgroup of $K$ and is dense in $K$. Take $\mu$ to be a probability measure on

$K$ such that support of $\mu$ is a compact subset contained in $C(\phi)$. Since $\phi$ on $C(\phi)$ is multiplication by $\frac{3-\sqrt{5}}{2}$, [Za96] implies that there is a probability measure $\rho$ on $C(\phi)$ such that $\mu * \phi(\mu) * \cdots * \phi^i(\mu) \to \rho$ in the space of probability measures on $C(\phi)$. This implies that $\rho = \lim \mu * \phi(\mu) * \cdots * \phi^i(\mu) = \mu * \phi(\lim(\mu * \cdots * \phi^{i-1}(\mu)) = \mu * \phi(\rho)$. Taking $\lambda_k = \rho$ for all $k \in \mathbb{Z}$, we get a stationary solution to the equation (1). Further, assume that $\mu \neq \delta_x$ for any $x \in K$. Then $\lambda_k = \rho$ are also not dirac measures. If $x\lambda_k = \lambda_k$ for some $x \in K$, then since $\lambda_k(C(\phi)) = 1$, we get that $\lambda_k(C(\phi)) = 1 = x\lambda_k(C(\phi))$, hence $x \in C(\phi)$ as $C(\phi)$ is a group. By Theorem 1.2.4 of [He77], $\{g \in C(\phi) \mid g\rho = \rho\}$ is a compact subgroup of $C(\phi)$. Since $C(\phi)$ is a vector group, $C(\phi)$ has no non-trivial compact subgroup and hence $x = e$. Thus, $\lambda_k$ is not left invariant for any nontrivial compact subgroup of $K$. Also in this case $K_\mu = K$. Hence, the conclusion on the extreme points of solutions in Theorem 1.1 does not hold.

# 5 Examples

We first provide examples of groups for which the group of automorphisms is compact.

(i) **Compact p-adic Lie groups:** Let $K$ be a compact $p$-adic Lie group. Then $\mathrm{Aut}(K)$ is a compact group (see Corollary 8.35 of [DidMS99] or [Ra04]). The following are examples of compact $p$-adic Lie groups.

(a) If $\mathbb{Q}_p$ is the field of $p$-adic numbers with valuation $|\cdot|_p$, then $\mathbb{Z}_{p^n} = \{x \in \mathbb{Q}_p \mid |x|_p \leq p^{n-1}\}$ is a compact $p$-adic Lie group.

(b) The group $GL_k(\mathbb{Z}_p)$ of all invertible $k \times k$-matrices over $\mathbb{Z}_p$.

(c) Pro-p group of finite rank, that is a totally disconnected group of finite rank in which every open normal subgroup has index equal to some power of $p$.

(ii) **A Compact abelian group:** For $n \geq 1$, let $A_n$ be the group of all $n$-th roots of unity and $A = \cup A_n$. Then $A$ is a countable abelian group. Equip $A$ with discrete topology. Let $K$ be the dual of $A$. Then $K$ is a compact (totally disconnected) metrizable group with dual $A$ (see 24.15 of [HeR79]).

Let $K_n$ be the closed subgroup of $K$ such that $K/K_n$ is the dual of $A_n$. Since $A_n$ is finite, $K/K_n$ is finite. Then $K_n$ is an open subgroup of $K$. Now, if $x \in \cap K_n$, then $x \in K_n$ for all $n \geq 1$. This implies that $a(x) = 1$ for all $a \in A_n$ and for all $n \geq 1$. Since $A = \cup A_n$, $a(x) = 1$ for all $a \in A$. Since $A$ is the dual of $K$, $x = e$. Thus, $\cap K_n = e$.

Let $\alpha$ be an automorphism of $K$ and $\hat{\alpha}$ be the corresponding dual automorphism on $A$. Then it is easy to see that $\hat{\alpha}(A_n) = A_n$ for all $n \geq 1$. This implies that

$\alpha(K_n) = K_n$. This proves that $(K_n)$ is a sequence of arbitrarily small characteristic open subgroups, hence the group of automorphisms of $K$ is compact.

(iii) **All automorphisms are distal but group of automorphsims is not compact:** Let $\mathbb{R}/\mathbb{Z}$ be the one-dimensional torus and $K$ be the compact group in (i) or in (ii). Let $G = \mathbb{R}/\mathbb{Z} \times K$ be the direct product of $\mathbb{R}/\mathbb{Z}$ and $K$. Then $G$ is a compact group.

Let $\tau$ be an automorphism of $G$. We now claim that there is an automorphism $\alpha$ of $K$ and a character $\chi$ of $K$ such that $\tau(z,x) = (z^{\pm 1}\chi(x), \alpha(x))$ for all $(z,x) \in G$. Since the connected component of identity in $G$ is $\mathbb{R}/\mathbb{Z}$, we get that $\tau(\mathbb{R}/\mathbb{Z}) = \mathbb{R}/Z$, hence $\tau(z,e) = (z^{\pm 1}, e)$ for all $z \in \mathbb{R}/\mathbb{Z}$. Let $\alpha\colon K \to K$ be defined by $\alpha(x) = p(\tau(1,x))$ where $p\colon G \to K$ is the canonical projection of $G$ onto $K$. It is easy see that $\alpha$ is a continuous homomorphism. If $\alpha(x) = e$, then $p(\tau(1,x)) = e$ and hence $\tau(1,x) = (z,e)$ for some $z \in \mathbb{R}/\mathbb{Z}$. But $\tau(z',e) = (z,e)$ for $z' = z$ or $z' = z^{-1}$. Since $\tau$ is an automorphism, $(z',e) = (1,x)$, hence $x = e$. This shows that $\alpha$ is injective. For $x \in K$, let $y \in K$ and $z \in \mathbb{R}/\mathbb{Z}$ be such that $\tau(z,y) = (1,x)$ as $\tau$ is onto. This implies that $\alpha(y) = p(\tau(1,y)) = p(\tau(z,y)) = x$. This proves that $\alpha$ is bijective. Continuity of $\alpha^{-1}$ follows from open mapping theorem as $K$ is a compact metrizable group (cf. 5.29 of [HeR79]). Thus, $\alpha$ is an automorphism of $K$. Let $\chi\colon K \to \mathbb{R}/\mathbb{Z}$ be defined by $\chi(x) = q(\tau(1,x))$ where $q\colon G \to \mathbb{R}/\mathbb{Z}$ is the canonical projection of $G$ onto $\mathbb{R}/\mathbb{Z}$. Then $\chi$ is a continuous homomorphism. For $z \in \mathbb{R}/\mathbb{Z}$ and $x \in K$, $\tau(z,x) = (z^{\pm 1}, e)\tau(1,x) = (z^{\pm 1}, e)(\chi(x), \alpha(x)) = (z^{\pm 1}\chi(x), \alpha(x))$. This proves the claim.

We now claim that $\tau$ is distal. Suppose $(1,e)$ is in the closure of $\{\tau^n(z,x) \mid n \in \mathbb{Z}\}$. Then $e$ is in the closure of $\{\alpha^n(x) \mid n \in \mathbb{Z}\}$. Since the group of automorphisms of $K$ is compact, $x = e$. This implies that $\tau^n(z,e) = (z^{\pm 1}, e)$ and hence $z = 1$. Thus, $\tau$ is distal. In fact, one can show that each $\tau$ generates a compact subgroup.

If $K$ is not a finite group, then the group of automorphisms of $G$ is not a compact group as it is homeomorphic to $\{\pm 1\} \times \hat{K} \times \mathrm{Aut}(K)$ where $\mathrm{Aut}(K)$ is the group of automorphisms of $K$.

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