Linear maps respecting unitary conjugation

B. V. Rajarama Bhat
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B. V. RAJARAMA BHAT

ABSTRACT. We characterize linear maps on von Neumann algebras which leave every unital subalgebra invariant. We use this characterization to determine linear maps which respect unitary conjugation, answering a question of M. S. Moslehian.

1. Introduction

Let $\mathcal{H}$ be a complex, separable Hilbert space and let $B(\mathcal{H})$ be the algebra of all bounded operators on $\mathcal{H}$. It was asked by M.S. Moslehian (private communication) as to what are linear maps $\alpha$ on $B(\mathcal{H})$ which satisfy

$$\alpha(UXU^*) = U\alpha(X)U^* \quad \forall X \in B(\mathcal{H}),$$

for every unitary $U$ on $\mathcal{H}$. We answer this question by first proving a theorem characterizing linear maps on von Neumann algebras which leave all subalgebras invariant.

2. Maps leaving subalgebras invariant

Theorem 2.1. Let $\mathcal{A}$ be a von Neumann algebra and let $I$ denote the identity element in $\mathcal{A}$. Let $\alpha : \mathcal{A} \to \mathcal{A}$ be a norm continuous linear map. Then the following are equivalent:

(i) $\alpha(B) \subseteq B$ for every von Neumann subalgebra $B$ of $\mathcal{A}$ with $I \in B$.

(ii) $\alpha(B) \subseteq B$ for every abelian von Neumann subalgebra $B$ of $\mathcal{A}$ with $I \in B$.

(iii) $\alpha(x) = cx + \psi(x)I$ for some $c \in \mathbb{C}$ and some norm continuous linear functional $\psi : \mathcal{A} \to \mathbb{C}$.

Before we prove this Theorem in general, we prove a special case as a Lemma and recall Halmos decomposition for pairs of generic projections. In the following for any projection $p$, $p^\perp$ denotes the projection $(I - p)$.

Lemma 2.2. Let $\mathcal{A}$ be the algebra $M_2(\mathbb{C})$ of $2 \times 2$ complex matrices. Suppose $\alpha : \mathcal{A} \to \mathcal{A}$ is a linear map which leaves every unital $*$-subalgebra of $M_2(\mathbb{C})$ invariant. Then $\alpha(x) = cx + \psi(x)I$ for some $c \in \mathbb{C}$ and some linear functional $\psi$ on $M_2(\mathbb{C})$.

Proof. To begin with we assume that $\text{trace}(\alpha(X)) = 0$ for all $X$. As $\{cI : c \in \mathbb{C}\}$ is a unital commutative $*$-subalgebra of $M_2(\mathbb{C})$, $\alpha(I) = bI$ for some $b \in \mathbb{C}$. Combined with the trace assumption made now, $\alpha(I) = 0$.
Similarly since any rank one projection \( p \) generates a unital commutative algebra consisting of linear combinations of \( p, p^\perp \), we get \( \alpha(p) = c_p(p - p^\perp) \) for some scalar \( c_p \), for every rank one projection \( p \). Hence \( \alpha(p - p^\perp) = 2c_p(p - p^\perp) \). Equivalently, every self-adjoint trace zero element of \( M_2(\mathbb{C}) \) is an eigenvector for \( \alpha \). In particular, there exist constants \( c_1, c_2, \ldots, c_5 \) such that \( \alpha(A_i) = c_iA_i, 1 \leq i \leq 5 \) where matrices \( A_i \)'s are

\[
\begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix},
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix},
\begin{bmatrix}
0 & i \\
-i & 0
\end{bmatrix},
\begin{bmatrix}
1 & 1 \\
1 & -1
\end{bmatrix},
\begin{bmatrix}
1 & i \\
-i & -1
\end{bmatrix}
\]

respectively. Observing \( A_1 + A_2 = A_4 \) and \( A_1 + A_3 = A_5 \), linearity of \( \alpha \) yields \( c_1 = c_2 = c_4 \) and \( c_1 = c_3 = c_5 \). Writing these matrices in the form \( p - p^\perp \), we get a basis for \( M_2(\mathbb{C}) \) consisting of rank one projections and \( \alpha(X) = c_1(X - \frac{1}{2}\text{trace}(X)I) \) for all \( X \).

If \( \alpha \) does not satisfy the assumption made above, consider \( \beta \) where,

\[ \beta(X) = \alpha(X) - \frac{1}{2}\text{trace}(\alpha(X))I. \]

Proving the result for \( \beta \) is as good as proving the result for \( \alpha \).

\[ \square \]

For any two projections \( p, q \), denote the largest projection smaller than both \( p \) and \( q \) by \( p \wedge q \). Recall that two projections \( p, q \) are said to be a generic pair if \( p \wedge q = p \wedge q^\perp = p^\perp \wedge q = p^\perp \wedge q^\perp = 0 \). The following result is well-Known as Halmos decomposition ([1], [3]). If a pair of projections \( p, q \) on a Hilbert space \( \mathcal{H} \) are generic, then \( p(\mathcal{H}) \) and \( p^\perp(\mathcal{H}) \) are isomorphic as Hilbert spaces and making use of this isomorphism, with respect to the decomposition \( \mathcal{H} = p(\mathcal{H}) \oplus p^\perp(\mathcal{H}) \), \( p \) and \( q \) have the form:

\[
p = \begin{bmatrix}
I & 0 \\
0 & 0
\end{bmatrix},
q = \begin{bmatrix}
c^2 & cs \\
cs & s^2
\end{bmatrix}
\]

with \( 0 < c, s < I, s = (I - c^2)^{\frac{1}{2}} \).

**Proof of Theorem 2.1**: (i) \( \Rightarrow \) (ii) and (iii) \( \Rightarrow \) (i) are obvious. Now we show (ii) \( \Rightarrow \) (iii).

If \( \mathcal{A} \) has no non-trivial projection then \( \mathcal{A} \) is isomorphic to \( \mathbb{C} \) and there is nothing to show. Suppose \( p \) is a non-trivial projection in \( \mathcal{A} \) and if only other non-trivial projection \( \mathcal{A} \) has is \( (I - p) \), then \( \mathcal{A} \) is isomorphic to \( \mathbb{C}^2 \) and once again the result is obvious. In the following we exclude these two trivial cases.

Suppose \( p \) is a projection in \( \mathcal{A} \) then the von Neumann algebra generated by \( p \) and \( I \) is \( \{ ap + bI : a, b \in \mathbb{C} \} \). It is abelian and hence left invariant by \( \alpha \). This shows that for any projection \( p \) in \( \mathcal{A} \),

\[
\alpha(p) = c_pp + d_pI \tag{2.1}
\]

for some \( c_p, d_p \in \mathbb{C} \). Note that scalars \( c_p, d_p \) are uniquely defined for non-trivial projections \( p \). We wish to show that \( c_p = c_q \) for any two non-trivial projections \( p, q \in \mathcal{A} \).

Now suppose \( p_1, p_2, p_3 \) are three mutually orthogonal non-trivial projections in \( \mathcal{A} \) such that \( p_1 + p_2 + p_3 = I \). We have \( \alpha(p_i) = c_{p_i}p_i + d_{p_i}I \) for \( i = 1, 2, 3 \). We
also have $\alpha(p_1 + p_2) = c_{p_1+p_2} (p_1 + p_2) + d_{p_1+p_2} I$. But by linearity $\alpha(p_1 + p_2) = \alpha(p_1) + \alpha(p_2)$. So we get,

$$c_{p_1}p_1 + d_{p_1} I + c_{p_2}p_2 + d_{p_2} I = c_{p_1+p_2} (p_1 + p_2) + d_{p_1+p_2} I.$$  \hspace{1cm} (2.2)

Multiplying this by $p_3$, yields, $d_{p_1}p_3 + d_{p_2}p_3 = d_{p_1+p_2}p_3$ or $d_{p_1} + d_{p_2} = d_{p_1+p_2}$. Substituting this back in (2.2) yields $c_{p_1}p_1 + c_{p_2}p_2 = c_{p_1+p_2}(p_1 + p_2)$, and then multiplications by $p_1, p_2$ show us $c_{p_1} = c_{p_2} = c_{p_1+p_2}$.

If $p, q$ are two non-trivial projections in $\mathcal{A}$, such that $p \wedge q \neq 0$. Considering the triple $p \wedge q, p \ominus (p \wedge q), p^\perp$ we get $c_{p \wedge q} = c_p$, similarly $c_{p \ominus q} = c_q$, so $c_p = c_q$. It follows, that if $p, q$ are non-trivial projections in $\mathcal{A}$, which are not in generic position and $q \neq p^\perp$, then $c_p = c_q$.

Suppose $p, q$ are projections in $\mathcal{A}$ and are in generic position. If $pqp$ is not a scalar multiple of $p$, then considering a non-trivial spectral projection $p'$ of $pqp$, from the Halmos decomposition, we see that $p', q$ are not in generic position as $(p')^\perp \wedge q \neq 0$. Hence $c_p = c_{p'} = c_q$. On the other hand, if $p, q$ are in generic position and $pqp$ is a scalar multiple of $p$, then by the Halmos decomposition it is clear that the algebra generated by $p, q$ is $M_2(\mathbb{C})$ and we can apply Lemma 2.2 to get $c_p = c_q$.

Finally if $q = p^\perp$, on the one hand if there is a third non-trivial projection $r$ different from $p, q$, we get $c_p = c_r = c_q$, and on the other hand if there is no such third projection then clearly $\mathcal{A}$ is isomorphic to $\mathbb{C}^2$ and we have already excluded this case.

This proves that for any two non-trivial projections in $\mathcal{A}$ we have $c_p = c_q$ (call this constant as $c$). Now if $p_1, p_2, \ldots, p_k$ are mutually orthogonal projections in $\mathcal{A}$ then for $x = \sum_{i=1}^{k} a_ip_i$ with scalars $a_1, a_2, \ldots, a_k$, $\alpha (x) = \sum_{i} a_i \alpha(p_i) = \sum_{i} a_i (c_{p_i} + d_{p_i} I) = cx + d_x I$, for some scalar $d_x$. By spectral theorem every self-adjoint element of $\mathcal{A}$ can be approximated in norm by elements of the form $\sum_{i} a_i p_i$. It follows that, for every self-adjoint element $x \in \mathcal{A}$, $\alpha$ has the form,

$$\alpha(x) = cx + \psi(x) I,$$

for some $\psi(x) \in \mathbb{C}$. By continuity and linearity of $\alpha$, it is clear that $\psi$ is a continuous linear functional. \hfill \Box

**Remark 2.3.** No continuity assumption is needed in Theorem 2.1 in certain situations. For instance if the algebra $\mathcal{A} = \mathcal{B}(\mathcal{H})$, then as every bounded operator is a finite linear combination of projections (See [2], [4]), Theorem 2.1 follows without any continuity assumption (of course, then the functional $\psi$ also need not be continuous).

**Remark 2.4.** It is a natural question as to whether in Theorem 2.1 (ii), we can replace ‘abelian’ by ‘maximal abelian’. Clearly the answer is no, if the algebra $\mathcal{A}$ itself is abelian, as in this case every map $\alpha$ would satisfy (ii). However, this can be done if the algebra $\mathcal{A}$ is $\mathcal{B}(\mathcal{H})$. To see this consider any rank one projection $p$ in $\mathcal{B}(\mathcal{H})$. Looking at maximal abelian subalgebras of $\mathcal{B}(p^\perp(\mathcal{H}))$, one has $\alpha(p) = c_p p + \beta_p$, where $c_p \in \mathbb{C}$ and $\beta_p$ is in every maximal abelian subalgebra of $\mathcal{B}(p^\perp(\mathcal{H}))$. This of course, means that $\alpha(p)$ has the form (2.1). Now one can continue as in the proofs of Lemma 2.2 and Theorem 2.1 to get $c_p = c_q$ for
every rank one projections \( p, q \) and that suffices to obtain (iii), under continuity assumption on \( \alpha \).

3. Unitary Conjugation

Finally, we have the result we were looking for.

**Theorem 3.1.** Let \( \alpha : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) \) be a linear map. Then the following are equivalent.

1. \( \alpha(U X U^*) = U \alpha(X) U^* \forall X, U \) in \( \mathcal{B}(\mathcal{H}) \) with \( UU^* = U^*U = I \).
2. \( \alpha(X) = cX + d \text{ trace } (X)I \) for some \( c, d \in \mathbb{C} \) if \( \mathcal{H} \) is finite dimensional, \( \alpha(X) = cX \) for some \( c \in \mathbb{C} \) if \( \mathcal{H} \) is infinite dimensional.

**Proof.** Clearly (ii) \( \Rightarrow \) (i). Now suppose \( \mathcal{A} \) is a von Neumann subalgebra of \( \mathcal{B}(\mathcal{H}) \) with \( I \in \mathcal{A} \). If \( U \) is a unitary in the commutant von Neumann algebra \( \mathcal{A}' \) we get \( \alpha(X) = \alpha(U X U^*) = U \alpha(X) U^* \) for \( X \in \mathcal{A} \). So

\[
\alpha(X)U = U \alpha(X).
\]

However every element in a unital \( C^* \) is algebra is a linear combination of at most four unitaries. Hence

\[
\alpha(X)Y = Y \alpha(X) \forall X \in \mathcal{A}, Y \in \mathcal{A}'.
\]

Then by von Neumann’s double commutant theorem \( \alpha(X) \in \mathcal{A} \). Now with Remark 2.3, Theorem 2.1 is applicable, and we have \( \alpha(X) = cX + \psi(X)I \) for some \( c \in \mathbb{C} \) and some linear functional \( \psi \). Further, by (i), for every unitary \( U \),

\[
cUXU^* + \psi(U X U^*)I = U[cX + \psi(X)I]U^*.
\]

So, \( \psi(U X U^*) = \psi(X) \) for all \( X \). Taking \( X = Y U \), we get \( \psi(U Y) = \psi(Y U) \) for every \( Y \). Once again, since every operator is a linear combination of at most four unitaries, \( \psi(XY) = \psi(Y X) \). So \( \psi \) is a trace. It is well-known that if \( \mathcal{H} \) is infinite dimensional \( \mathcal{B}(\mathcal{H}) \) does not admit a non-trivial finite trace. \( \square \)

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**References**


1 **Indian Statistical Institute, R. V. College Post, Bangalore-560059, India.**

E-mail address: bhat@isibang.ac.in