

isibang/ms/2009/7
July 31st, 2009
<http://www.isibang.ac.in/~statmath/eprints>

Recurrence and ergodicity of random walks on linear groups and on homogeneous spaces

Y. GUIVARC'H AND C. R. E. RAJA

Indian Statistical Institute, Bangalore Centre
8th Mile Mysore Road, Bangalore, 560059 India

Recurrence and ergodicity of random walks on linear groups and on homogeneous spaces

Y. Guivarc'h and C. R. E. Raja

Abstract

We discuss recurrence and ergodicity properties of random walks and associated skew products for large classes of locally compact groups and homogeneous spaces. In particular we show that a closed subgroup of a product of finitely many linear groups over local fields supports a recurrent random walk if and only if it has at most quadratic growth. We give also a detailed analysis of ergodicity properties for special classes of random walks on homogeneous spaces. The structure of closed subgroups of linear groups over local fields and the properties of group actions with respect to stationary measures play an important role in the proofs.

2000 Mathematics Subject Classification: 22D40, 37A50, 37A30, 60G50.

Key words: ergodic, homogeneous spaces, linear groups, Markov operators, polynomial growth, recurrent random walk, skew product.

1 Introduction

Let G be a locally compact second countable group and μ be a Borel probability measure on G . We denote by \mathbb{P} the product measure $\mathbb{P} = \mu^{\mathbb{N}}$ on $\Omega = G^{\mathbb{N}}$. Let $Y_i(\omega)$ be the i -th coordinate of $\omega \in \Omega$ in G ($i \in \mathbb{N}$). Then the left random walk on G of law μ , starting from $g \in G$ is the sequence of G -valued random variables X_n^g defined by $X_n^g(\omega) = Y_n(\omega) \cdots Y_1(\omega)g$, $X_0^g(\omega) = g$. Given a G -space E and $x \in E$, we write $X_n(\omega, x) = X_n^e(\omega)x$. The sequence $X_n(\omega, x)$ is called the random walk of law μ on E , starting from x and its properties will play an essential role in the study of $X_n^g(\omega)$ if E is chosen as a homogeneous space of G . We say that X_n^g is *recurrent* if for every neighborhood W of the

identity e of G \mathbb{P} -a.e $X_n(\omega) = X_n^e(\omega) \in W$ infinitely often. If X_n^g is not recurrent, then \mathbb{P} -a.e $X_n^g(\omega)$ escapes to infinity and the random walk X_n^g is said to be transient. We can also define the right random walk ${}^gX_n = gY_1 \cdots Y_n$. Also recurrence of gX_n is defined in a similar fashion. From the above we see that recurrence of gX_n is equivalent to recurrence of X_n^g . Hence, in this case we say that μ is recurrent. We denote by μ^k the k -th convolution power of μ . Then recurrence of μ is equivalent to the condition that $\sum_0^\infty \mu^k(U) = \infty$ for some neighborhood U of e in G . We denote by G_μ the smallest closed subgroup of G containing the support of μ and we say that μ is adapted if $G_\mu = G$. Then the group G is said to be *recurrent* if there exists an adapted probability μ on G such that μ is recurrent.

A stronger notion of recurrence where Haar measure enters explicitly in the definition is H -recurrence. We will say that G is *H-recurrent* if there exists a probability measure μ on G such that for every Borel set B with positive Haar measure $X_n(\omega) \in B$, \mathbb{P} -a.e infinitely often.

It is a classical result that $\mathbb{R}^p \times \mathbb{Z}^q$ is recurrent if and only if $p + q \leq 2$ (cf. [43]). Also a countable abelian group is recurrent if and only if it has rank at most two (see [12]). These groups are also H -recurrent. We denote by λ_G a left Haar measure on G . We recall that G is said to have polynomial growth of degree at most $d \in \mathbb{N}$ if for each compact neighborhood W of $e \in G$ there exists a constant c such that $\lambda_G(W^n) \leq cn^d$ for every $n \in \mathbb{N}$; when $d = 2$ we say that G has at most quadratic growth. We observe that if G has polynomial growth, then G as well as its closed subgroups is unimodular.

We give now some reference from previous works. The idea of relating growth and recurrence appeared in [30] for the case of countable groups: non-exponential growth of finitely generated recurrent groups was conjectured. The case of locally compact groups and especially real Lie groups, was considered in [22] where it was shown that recurrent groups are amenable and unimodular. Probabilistic ideas and calculations were developed in order to show transience or recurrence in various situations, particularly for connected Lie groups and their countable subgroups. For example, it was shown there that the group \mathbb{G}_2 of euclidean motion of \mathbb{R}^2 is recurrent while its universal cover as well as the 3-dimensional Heisenberg groups (continuous or discrete) is not. Also the affine group of the line was shown to be transient. In [21] the following "quadratic growth conjecture" was stated: G is recurrent if and only if it has polynomial growth of degree at most two. This conjecture has been settled for various classes of groups: compactly generated nilpotent groups (cf. [21]), connected groups (cf. [2]), finitely generated groups

(cf. [48]), quasi-transitive groups of automorphisms of graphs (cf. [50]), p-adic Lie groups and totally disconnected groups of polynomial growth (cf. [41]). Powerful analytical techniques from real analysis were developed in [48] which allows to obtain precise asymptotics for μ^n if μ has nice density and G is compactly generated unimodular. In particular, a direct relation between polynomial growth and recurrence of such a μ was obtained. See also [15], [26] and [50] for surveys on random walks and recurrence properties.

Here one of our main results proves the validity of the conjecture for the class of closed subgroups of products of full linear groups over a finite number of local fields. We show also the validity of the above conjecture in the following situations:

- (a) G is a real Lie group;
- (b) G is H-recurrent.

In fact, we show that G is compactly generated and has quadratic growth is equivalent to the following: G has a compact normal subgroup K and a finite index subgroup G_1 such that $K \subset G_1$ and G_1/K is isomorphic to a closed subgroup of the motion group of the plane. In the situations considered here, if G is compactly generated and totally disconnected we get that up to finite index and to a compact normal subgroup, G is isomorphic to a subgroup of \mathbb{Z}^2 . Concerning the above conjecture, our main result is the following.

Theorem 1 *Assume G is a closed group of a product of finitely many linear groups over local fields: $G \subset \prod_{i \in I} GL(d_i, \mathbb{F}_i)$. If G is recurrent, then G has at most quadratic growth and in addition if G is compactly generated, then G contains a compact normal subgroup K and a finite index subgroup H such that $K \subset H$ and H/K is isomorphic to a closed subgroup of the group \mathbb{G}_2 of euclidean motions of the plane.*

To our knowledge this result is new even in case of one field \mathbb{F} , in particular for the real field. For the proof we use in particular the following previously-known results. The analysis developed in [2] and [22] for recurrence in locally compact groups and real Lie groups. The structure of finitely generated recurrent groups given in [48] which is based on [19]. The existence of invariant Radon measures for a class of random walks in homogeneous spaces as follows from Theorem 5.1 of [33]; this result leads to the unimodularity of closed

subgroups of recurrent groups. The structural results for automorphisms of totally disconnected groups in [4] and [29] allows us to use efficiently the facts above in case of linear groups over non-archimedean fields.

In this paper we give also a detailed analysis of recurrence and ergodicity properties for some special classes of homogeneous spaces and random walks which occur naturally in other studies. Let $E = G/H$ be such a G -space and $\mu \in M^1(G)$. As shown below and in contrast to the group case, recurrence properties are valid for $X_n(\omega)x$ in a much wider setting not related to polynomial growth of G and the asymptotic properties of $X_n(\omega)x$ depend strongly on x , in general. Then it is convenient to discuss recurrence properties in terms of a fixed Radon measure λ such that the class of λ is μ -invariant. For a positive Radon measure η on E , we define the convolution of μ and η by $\mu * \eta = \int g\eta d\mu(g)$ where $g\eta$ is the push-forward of η by $g \in G$. Let λ be a fixed μ -invariant measure on E ($\mu * \lambda = \lambda$), $\hat{\Omega} = G^{\mathbb{Z}} = G^{-\mathbb{N}} \times G \times \Omega$ be the product space, θ be the shift on $\hat{\Omega}$ and $\omega \in \Omega$ the projection of $\hat{\omega} \in \hat{\Omega}$ on Ω . We consider the skew product $(\Omega \times E, \tilde{\theta}, \mathbb{P} \otimes \lambda)$ where $\tilde{\theta}$ is defined by $\tilde{\theta}(\omega, x) = (\theta\omega, Y_1(\omega)x)$ for $(\omega, x) \in \Omega \times E$. We consider also the map $\hat{\theta}$ of $\hat{\Omega} \times E$ into itself defined by $\hat{\theta}(\hat{\omega}, x) = (\theta\hat{\omega}, Y_1(\omega)x)$ for $(\hat{\omega}, x) \in \hat{\Omega} \times E$. The second coordinate of $\hat{\theta}^n(\hat{\omega}, x)$ for $n \in \mathbb{Z}$ defines an extension of $X_n(\omega, x)$ to negative time, i.e a bilateral random walk on E . We observe that $\tilde{\lambda} = \mathbb{P} \otimes \lambda$ is $\tilde{\theta}$ -invariant and we will show that there exists a unique $\hat{\theta}$ -invariant measure $\hat{\lambda}$ on $\hat{\Omega} \times E$ which has projection $\tilde{\lambda}$ on $\Omega \times E$. The system $(\hat{\Omega} \times E, \hat{\theta}, \hat{\lambda})$ can be considered as the natural extension of the system $(\Omega \times E, \tilde{\theta}, \tilde{\lambda})$. In section 2.3 we study recurrence and ergodicity properties of such systems from a general point of view. As a result of this discussion in Section 5 we prove the following

Theorem 2 *In the situation of examples 2, 4, 5 of Section 5, the skew product $(\hat{\Omega} \times E, \hat{\theta}, \hat{\lambda})$ is ergodic with respect to the infinite $\hat{\theta}$ -invariant measure $\hat{\lambda}$.*

This result provides large classes of invertible transformations with stochastic properties which are ergodic with respect to natural infinite invariant measures. Examples 4 and 5 can be considered as fibered dynamical systems with fiber \mathbb{R} or \mathbb{Z} , which have properties similar to those of the skew products considered in [24].

As an illustration of recurrence and ergodicity properties on homogeneous spaces we show the singularity of stationary measure for the action of

$SL(2, \mathbb{Z})$ and its cofinite subgroups on the projective line.

Acknowledgement The authors wish to thank School of Mathematics, TIFR, Mumbai India for its hospitality during the special trimester Lie Group-Ergodic theory and Probability measures. The authors would also like to thank B. Bekka, J.P. Conze and G. Willis for useful remarks. Also, the first named author would like to thank ESI (Vienna) for hospitality during the special semester on Random walks and amenability (2007).

2 Recurrence of random walks on groups and G -spaces

2.1 Some basic facts

A left-random walk X_n^g on G is said to be transient (resp. recurrent) if for any compact neighborhood W of $e \in G$ we have \mathbb{P} -a.e. $X_n^e(\omega) \notin W$ for all large n (resp. $X_n^e(\omega) \in W$, infinitely often): see [15], [26] [43] and [50] for further detailed information. A random walk is either transient or recurrent. Transience of X_n^g is equivalent to the fact that $\sum_0^\infty \mu^k$ is finite on compact sets.

Given a Borel probability measure μ on G , a Markov operator $P = P_\mu$ on G is defined by $P_\mu \psi(x) = \int \psi(gx) d\mu(g)$, where ψ is a bounded Borel function. This operator allows to express various quantities of probabilistic significance for the left random walk X_n^g . For example, if B is a Borel subset of G :

$$\mathbb{P}\{X_n^g \in B\} = P_\mu^n 1_B(g)$$

for any $g \in G$. In particular, if μ is adapted and recurrent then for any positive continuous function ϕ on G and any $g \in G$: $\sum_0^\infty P_\mu^k \phi(g) = +\infty$. For the sake of completeness we will give proofs for certain general properties used below and some of which have been considered in the context of Markov operators on measured spaces (see [14]). We will also use the framework of skew products in the context of G -spaces (see [15], [16] and [24]).

A positive Borel function f on G is said to be left μ -harmonic (resp. left μ -superharmonic) if $P_\mu f = f$ (resp. $P_\mu f \leq f$). If μ is adapted and recurrent then continuous positive superharmonic functions are constant (see below).

A useful concept when dealing with recurrent random walks is that of induced random walk. If μ is recurrent and U is an open subgroup the return

time T to U is defined by $T = \inf\{n \geq 1 \mid X_n^e \in U\}$ and we denote by μ^U the law of X_T^e . Then μ^U is clearly a recurrent probability on U . In particular, if G is recurrent, then any open subgroup of G is also recurrent. Also, if H is a closed normal subgroup of G and μ is recurrent, then the projection of μ on G/H is also recurrent. In particular, if G is recurrent, then G/H is also recurrent. In dealing with the structure of recurrent groups one can reduce the study to the case of compactly generated groups. This is because, if W is a compact symmetric neighborhood of $e \in G$, then the subgroup generated by W , that is, $U = \cup_{n \geq 0} W^n$ is open, hence one can consider the induced random walk defined by μ which is also recurrent. Conversely, if G is a union of compactly generated subgroups and on each of these compactly generated subgroups, symmetric random walks with compactly supported density are recurrent, then one can use the method in [41] to construct recurrent random walks on G . In particular, it suffices to prove the quadratic growth conjecture for compactly generated groups.

A basic fact proved here is that every closed subgroup of a recurrent group is unimodular. This extends the known fact that a recurrent group is unimodular, strongly used in [22] for the early classification of recurrent locally compact groups. It extends also Theorem 3.26 of [50].

We now give a few typical examples of recurrent groups. The euclidean motion group $\mathbb{G}_2 = O(2) \times \mathbb{R}^2$ of the euclidean plane was already mentioned above. If C is a compact group, then the groups of the form $C \times H$ with H a closed subgroup of \mathbb{R}^2 are recurrent. These two classes of groups arise geometrically as similarity groups or products of similarity groups relative to a euclidean or ultrametric norm. It follows from [41] that any p -adic unipotent algebraic group is also recurrent.

If A is any hyperbolic automorphism of a compact abelian group C , then one can form the semidirect product $G = \mathbb{Z} \ltimes C$ where the action of \mathbb{Z} on C is given by A^n ($n \in \mathbb{Z}$). For example, one can take $C = \mathbb{T}^2$ and A is given by the matrix $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. These groups are not isomorphic to closed subgroups of linear groups but are recurrent.

2.2 A class of Markov operators

Let E be a locally compact second countable space. We denote by $C_b(E)$ the space of bounded continuous functions on E , by $C_c(E) \subset C_b(E)$, the subset of compactly supported functions and $C_b^+(E)$ (respectively, $C_c^+(E)$)

the set of positive elements in $C_b(E)$ (respectively, $C_c(E)$). Let $M^1(E)$ be the set of Borel probability measures on E . Here, by a Markov operator on E , we mean a positive operator P on $C_b(E)$ such that $P1 = 1$. Then P defines a transition probability $P(x, \cdot)$. Clearly P acts on $M^1(E)$. We will also consider its action on some positive Radon measures. If η is such a measure and if for any $\phi \in C_c^+(E)$, $\eta(P\phi)$ is finite, then $\phi \mapsto \eta(P\phi)$ defines a Radon measure which we will denote by $P\eta$. In particular η will be said to be P -invariant if $P\eta$ is defined and $P\eta = \eta$.

As a special case, we will consider a G -space E and P defined by

$$P\psi(x) = \int \psi(gx) d\mu(g)$$

where $\mu \in M^1(G)$ and $\psi \in C_b(E)$. In this case the trajectories, starting from $x \in E$, for the associated Markov chain can be written as

$$X_n(\omega)x = Y_n(\omega) \cdots Y_1(\omega)x \quad (n > 0), \quad X_0(\omega)x = x$$

with $\omega \in \Omega$. Then for $u \in C_b^+(E)$, we have

$$\sum_0^\infty P^k u(x) = \int \sum_0^\infty u(X_n(\omega)x) d\mathbb{P}(\omega).$$

The first part of the following was proved in [33]. In view of the role of this result here and in other contexts we provide a proof different from [33], we give examples and complements; see also [40] for other examples of probabilistic significance.

Proposition 1 *Let E be a locally compact second countable space and P be a Markov operator on E . Assume that there exists $u \in C_c^+(E)$ such that for any $x \in E$, $\lim_{n \rightarrow \infty} \sum_0^n P^k u(x) = \infty$. Then there exists a P -invariant positive Radon measure ν on E with $\nu(u) > 0$. If ν is unique up to normalization, then we have the following convergence:*

$$\lim_{n \rightarrow \infty} \frac{\sum_0^n P^k \phi(x)}{\sum_0^n P^k u(x)} = \frac{\nu(\phi)}{\nu(u)}$$

for all $x \in E$ and all $\phi \in C_c(E)$.

Proof Let $\phi \in C_c^+(E)$. Since the sequence $\sum_0^n P^k u$ is increasing tending to $+\infty$ and the support of ϕ is compact, there exists $r \in \mathbb{N}$ such that $\phi \leq \sum_0^r P^k u$. It follows that for any $n \in \mathbb{N}$, $\sum_0^n P^k \phi \leq r \sum_0^n P^k u + r^2 \|u\|_\infty$. In particular, if $x \in E$ is fixed, then

$$\left(\sum_0^n P^k \delta_x\right)(\phi) \leq r \sum_0^n (P^k \delta_x)(u) + r^2 \|u\|_\infty.$$

It follows that for any $\rho \in M^1(E)$, if $\rho_n = \sum_0^n P^k \rho$, then

$$\rho_n(\phi) \leq r \rho_n(u) + r^2 \|u\|_\infty.$$

Since $\lim_{n \rightarrow \infty} \rho_n(u) = +\infty$, for n large we have $\rho_n(u) > 0$, hence the sequence $\frac{\rho_n(\phi)}{\rho_n(u)} = \eta_n(\phi)$, say, is bounded. This implies that the sequence of Radon measures η_n is relatively compact in the weak topology. On the other hand $P\eta_n = \eta_n + \frac{P^n \rho - \rho}{\rho_n(u)}$. Since for any $\phi \in C_c^+(E)$, $0 \leq P^n \rho(\phi) \leq \|\phi\|_\infty$, the sequence $\epsilon_n = \frac{P^n \rho - \rho}{\rho_n(u)}$ converges to zero weakly. By weak compactness we can extract a convergent subsequence η_{n_k} of η_n such that $\lim_{k \rightarrow \infty} \eta_{n_k} = \nu$ and $\nu(u) = 1$. Then for $\phi \in C_c^+(E)$, the relation $P\eta_n(\phi) = \eta_n(\phi) + \epsilon_n(\phi)$ gives $\lim_{k \rightarrow \infty} \eta_{n_k}(P\phi) = \nu(\phi)$. Let $\phi, \psi \in C_c^+(E)$ with $\psi \leq P\phi$. Then the relation $\eta_n(P\phi) = \eta_n(\phi) + \epsilon_n(\phi)$ implies $\eta_n(\psi) \leq \eta_n(\phi) + \epsilon_n(\phi)$. Hence, in the limit $\nu(\psi) \leq \nu(\phi)$. Since $P\phi$ is an increasing limit of elements of $C_c^+(E)$, we get that $P\phi$ is ν -integrable and $\nu(P\phi) \leq \nu(\phi)$. Since ϕ is arbitrary, $P\nu \leq \nu$. In order to show $P\nu = \nu$, we consider the positive Radon measure $\eta = \nu - P\nu$ and we observe that $\sum_0^n P^k \eta = \nu - P^{n+1}\nu$. If $\eta \neq 0$, then the condition $\lim_{n \rightarrow \infty} \sum_0^n P^k u = +\infty$ implies that $\lim_{n \rightarrow \infty} (\sum_0^n P^k \eta)(u) = +\infty$. This contradicts the fact that $\sum_0^n P^k \eta(u)$ is bounded by $\nu(u)$. Hence $P\nu = \nu$.

If the Radon measure ν is uniquely defined by $P\nu = \nu$, $\nu(u) = 1$, we can improve the above considerations: every limit point η of η_n satisfies $P\eta = \eta$ with $\eta(u) = 1$, hence $\eta = \nu$ and so by compactness, the convergence to ν of $\eta_n = \frac{\rho_n}{\rho_n(u)}$ follows.

Remark 1 (a) The condition $\sum_0^\infty P^k u = +\infty$ on E is satisfied for some $u \in C_c^+(E)$ if there exists a relatively compact open set $U \subset E$ such that for any $x \in E$ the trajectories of the Markov chain defined by P visit U infinitely often with positive probability. In particular, if $E = G$ and $P_\mu = P$ is associated to an adapted recurrent $\mu \in M^1(G)$, then the condition that $\sum_0^\infty P^k u = +\infty$ is satisfied. Also in this case ν is unique up to normalization and equal to a Haar measure on G .

(b) We will use the above result in the following situation. E is a G -space, $\mu \in M^1(G)$ is recurrent and adapted and P is defined by $P\psi(x) = \int \psi(gx)d\mu(g)$ where $\psi \in C_b(E)$. The condition that $\sum_0^\infty P^k u = +\infty$ is satisfied if there exists a compact subset C of E with $GC = E$. However the proposition can be used in various situations not related to recurrence of G , where E is a non-compact homogeneous space of G . Some classes of examples are discussed in section 5.

The next result is well-known if E is a countable discrete space.

Proposition 2 *Let E be a locally compact second countable space and P be a Markov operator on E which satisfies*

$$\sum_0^\infty P^k u = +\infty \quad \text{on } E$$

for all $u \in C_c^+(E)$. If f is a positive continuous function on E with satisfies $Pf \leq f$, then f is constant. Any non-zero P -invariant measure ν satisfies $\nu(\phi) > 0$ for any $\phi \in C_c^+(E)$.

Proof The function $\psi = f - Pf$ is a positive continuous function such that $\sum_0^n P^k \psi = f - P^{n+1} f \leq f$. If ψ is not zero, then the assumptions on P implies $\sum_0^\infty P^k \psi = +\infty$ on E which contradicts the boundedness of $\sum_0^n P^k \psi$. Hence we have $Pf = f$.

Let $r \geq 0$. Then function $f_r = \inf(r, f)$ satisfies also $Pf_r \leq f_r$. The above result applied to f_r gives $Pf_r = f_r$. But for any $r < \sup_{x \in E} f(x)$, the functions f_r has maximum r . We consider the closed set $E_r = \{x \in E \mid f_r(x) = r\}$. Then if $x \in E_r$, the equation $Pf_r = f_r$ implies that $P(x, E_r) = 1$. In other words, E_r is P -invariant. If f takes two distinct values $r \neq s$, we can find $u \in C_c^+(E)$ with $u = 0$ on E_s . Then for any $n \in \mathbb{N}$, $P^n u(x) = 0$ if $x \in E_s$ and hence $\sum_0^\infty P^k u(x) = 0$ on E_s . This contradicts the assumption that $\sum_0^\infty P^k u = +\infty$ on E . Hence f is constant on E .

Let $\phi \in C_c^+(E)$. Then since $\sum_0^\infty P^k \phi = +\infty$ on E , given any $u \in C_c^+(E)$, there exists $m \in \mathbb{N}$ such that $u \leq \sum_0^m P^k \phi$. Then $\nu(u) \leq \sum_0^m \nu(P^k \phi) = (m+1)\nu(\phi)$. Now choose $u \in C_c^+(E)$ so that $\nu(u) = 1$, hence we have $\nu(\phi) \geq \frac{1}{m+1} > 0$.

2.3 Recurrence and ergodicity for stationary measures

Here E is a locally compact second countable G -space, and P is defined by

$$P\psi(x) = \int \psi(gx)d\mu(g)$$

where $\mu \in M^1(G)$. We consider also a positive Radon measure λ on E which is P -invariant. In general λ will be of infinite mass but we will deal here with conservativity conditions which will allow to reduce the situations to the finite mass case. For $(\omega, x) \in \Omega \times E$ we write $X_n(\omega)x = Y_n(\omega) \cdots Y_1(\omega)x$ and we endow $\Omega \times E$ with the measure $\mathbb{P} \otimes \lambda$.

Definition 1 Let (E, P, λ) be as above. We say that (E, P, λ) has property R if for every open relatively compact set $U \subset E$, $\mathbb{P} \otimes \lambda$ -a.e., $(\omega, x) \in \Omega \times U$, there exists $n = n(\omega, x) \in \mathbb{N}$ such that $X_n(\omega)x \in U$.

For example if μ is a left random walk on a locally compact group G and $\lambda = \lambda_G$, then the triple (G, P, λ) has property R if and only if μ is recurrent.

Remark 2 In the definition of Property R we could have restricted U to belong to a family of relatively compact open sets U_k ($k \in \mathbb{N}$) covering the support of λ , which may be seen as follows. Replacing U_k by $\cup_{i \leq k} U_k$, we may assume that (U_k) is increasing. Let U be any relatively compact open set. Then there exists a $k \in \mathbb{N}$ such that $\lambda(U \setminus U_k) = 0$, hence for $\mathbb{P} \otimes \lambda$ -a.e. $(\omega, x) \in \Omega \times U$, there exists $n \in \mathbb{N}$ such that $X_n(\omega)x \in U$.

We consider the skew product $(\Omega \times E, \tilde{\theta}, \tilde{\lambda})$ where $\tilde{\lambda} = \mathbb{P} \otimes \lambda$ and the map $\tilde{\theta}$ is defined by $\tilde{\theta}(\omega, x) = (\theta\omega, Y_1(\omega)x)$. We observe that $\tilde{\lambda}$ is $\tilde{\theta}$ -invariant since for any $\phi \in C_c(E)$ and a Borel function ψ on Ω , we have

$$\begin{aligned} \tilde{\theta}\tilde{\lambda}(\psi \otimes \phi) &= \int \psi(\theta\omega)\phi(Y_1(\omega)x)d\mathbb{P}(\omega)d\lambda(x) \\ &= \int \psi(\omega')\phi(gx)d\mathbb{P}(\omega')d\mu(g)d\lambda(x) \\ &= \int \psi(\omega')\phi(y)d\mathbb{P}(\omega')d\lambda(y) \\ &= \tilde{\lambda}(\psi \otimes \phi). \end{aligned}$$

Property R allows us to define a return time τ and an induced transformation $\tilde{\theta}_{\Omega \times U}$ on $\Omega \times U$ where U is an open relatively compact set with $\lambda(U) > 0$ by

$$\tilde{\theta}_{\Omega \times U}(\omega, x) = (\theta^{\tau(\omega, x)}(\omega), X_{\tau(\omega, x)}x) = \tilde{\theta}^{\tau(\omega, x)}(\omega, x)$$

where $\tau(\omega, x) = \text{Inf}\{n \geq 1 \mid X_n(\omega)x \in U\}$.

We recall that, for a dynamical system (X, S, η) a measurable subset D of X is said to be wandering if $S^{-k}D \cap D = \emptyset$ for any $k \geq 1$. Then using the Poincaré recurrence theorem for $\tilde{\theta}_{\Omega \times U}$ with $\lambda(U) < +\infty$ and standard arguments we obtain the following:

Proposition 3 *Let (E, P, λ) and $(\Omega \times E, \tilde{\theta}, \tilde{\lambda})$ be as above. Then the following are equivalent:*

- (a) *Property R is valid for (E, P, λ) ;*
- (b) *For any measurable $A \subset E$ with $0 < \lambda(A) < +\infty$, $\sum_0^\infty 1_{\Omega \times A}(\tilde{\theta}^k(\omega, x)) = +\infty$ $\tilde{\lambda}$ -a.e on $\Omega \times A$;*
- (c) *$(\Omega \times E, \tilde{\theta}, \tilde{\lambda})$ has no wandering set of positive measure.*

Using Proposition 3, we see that property R for (E, P, λ) is equivalent to conservativity of the system $(\Omega \times E, \tilde{\theta}, \tilde{\lambda})$ (see [14]).

Also, property R is valid in the setting of Proposition 1 as shown below.

Corollary 1 *Assume that for some $u \in C_c^+(E)$, $\sum_0^\infty P^k u = +\infty$ on E , and let λ be any P -invariant Radon measure. Then (E, P, λ) has property R .*

Proof We observe that P acts on $\mathbb{L}^\infty(\lambda)$ and has an adjoint operator P^* on $\mathbb{L}^1(\lambda)$ defined by

$$\langle \psi, P^* \phi \rangle = \lambda(\psi P^* \phi) = \lambda(\phi P \psi) = \langle \phi, P \psi \rangle .$$

Since λ is P -invariant, $P^*1 = 1$. By definition, since $u \in \mathbb{L}_+^1(\lambda)$, P^* is conservative. Then it follows from Hopf's maximal ergodic Lemma (see [14] pp. 11) that for any $\phi \in \mathbb{L}_+^1(\lambda)$, $\phi > 0$ λ -a.e, $\sum_0^\infty P^k \phi = +\infty$, λ -a.e. As a consequence $\sum_0^\infty (P^*)^k \phi = +\infty$ λ -a.e. We show that $(\Omega \times E, \tilde{\theta}, \mathbb{P} \otimes \lambda)$ has no wandering set of positive measure. Assume $D \subset \Omega \times E$ is such a set. Then $\sum_0^\infty 1_D \circ \theta^k \leq 1$. For two non-negative measurable functions f and f' on $\Omega \times E$, we write

$$\langle f, f' \rangle = (\mathbb{P} \otimes \lambda)(ff').$$

Also, if v is a non-negative measurable function on E , we write $\tilde{v}(\omega, x) = v(x)$. Then we observe for $k \in \mathbb{N} \cup \{0\}$, that

$$\langle \tilde{v}, 1_D \circ \theta^k \rangle = \langle 1_D, (P^*)^k v \rangle .$$

Assume $v > 0$ λ -a.e, hence from above $\sum_0^\infty (P^*)^k v = +\infty$ λ -a.e. In particular, since $(\mathbb{P} \otimes \lambda)(D) > 0$, $\langle \tilde{v}, \sum_0^\infty 1_D \circ \theta^k \rangle = +\infty$. On the other hand, since D is wandering

$$\langle \tilde{v}, \sum_0^\infty 1_D \circ \theta^k \rangle \leq (\mathbb{P} \otimes \lambda)(\tilde{v}) = \lambda(v) < +\infty$$

if $v \in \mathbb{L}^1(\lambda)$. This gives the required contradiction.

Remark 3 In case $E = G$, it follows that the condition $\sum_0^\infty \mu^k(U) = +\infty$ for some (hence any) neighborhood U of identity is necessary and sufficient for recurrence of μ .

One also considers the product space $\hat{\Omega} = G^{\mathbb{Z}}$ and the corresponding shift θ . We denote $\Omega^- = G^{-\mathbb{N}} \times G$ and for $\hat{\omega} \in \hat{\Omega} = \Omega^- \times \Omega$, we write $\hat{\omega} = (\omega^-, \omega)$ with $\omega^- \in \Omega^-$, $\omega \in \Omega$. For $k \in \mathbb{Z}$ we denote $Y_k(\hat{\omega})$ the k -th component of $\hat{\omega}$. We consider the transformation $\hat{\theta}$ on $\hat{\Omega} \times E$ defined by $\hat{\theta}(\hat{\omega}, x) = (\theta\hat{\omega}, Y_1(\omega)x)$ and we observe that the system $(\Omega \times E, \tilde{\theta})$ is a factor of the invertible system $(\hat{\Omega} \times E, \hat{\theta})$ relative to the map $(\hat{\omega}, x) \mapsto (\omega, x)$.

The following result will play a basic role in the study of random walks on G -spaces.

Proposition 4 *Let E be a locally compact second countable G -space, $\mu \in M^1(G)$ and λ be a μ -stationary measure on E . Then with the above notations, there exists a unique $\hat{\theta}$ -invariant measure on $\hat{\Omega} \times E$ which has projection $\tilde{\lambda} = \mathbb{P} \otimes \lambda$ on $\Omega \times E$.*

Proof For the existence result we will use weak convergence of Radon measures. Let $\overline{G} = G \cup \{\infty\}$ be the Alexandrov compactification of G and let $\hat{\Omega}^\infty$ be the compact metric space $\overline{G}^{\mathbb{Z}}$. Then $\hat{\Omega}^\infty \times E$ is locally compact and we have a well defined continuous projection of $\hat{\Omega}^\infty \times E$ on $\overline{G}^{\mathbb{N}-n} \times E$ for every $n \in \mathbb{N}$.

Let $\Omega^e = \{\hat{\omega} \in \hat{\Omega} \mid \omega_k = e \text{ if } k \leq 0\}$, hence $\theta^n(\Omega^e) = \{\hat{\omega} \in \hat{\Omega} \mid \omega_k = e \text{ if } k \leq -n\}$. For each $\omega \in \Omega$, we define $\omega^e \in \Omega^e$ by $\omega_k^e = e$ if $k \leq 0$ and $\omega_k^e = \omega_k$ if $k > 0$. This allows to identify $\Omega \times E$ with $\Omega^e \times E \subset \hat{\Omega} \times E$ using the map $(\omega, x) \mapsto (\omega^e, x)$. The corresponding push-forward of $\tilde{\lambda}$ is denoted by $\tilde{\lambda}^e$ and we consider the sequence $\hat{\theta}^n(\tilde{\lambda}^e)$. Then $\hat{\theta}^n(\Omega^e \times E)$ is a Borel subset of the locally compact space $\hat{\Omega}^\infty \times E$ and $\hat{\theta}^n(\tilde{\lambda}^e)$ can be considered as a Radon

measure on $\hat{\Omega}^\infty \times E$ which gives full measure to $\theta^n(\Omega^e) \times E$. By definition:

$$\hat{\theta}^n(\tilde{\lambda}^e) = \int \delta_{\hat{\theta}^n \omega^e} \otimes X_n(\omega) \lambda d\mathbb{P}(\omega),$$

hence the projection $\hat{\theta}^n(\tilde{\lambda}^e)$ on $\Omega^e \times E$ is $\int \delta_{\omega^e} \otimes X_n(\omega) \lambda d\mathbb{P}(\omega) = \mathbb{P}^e \otimes \lambda$. Then for any function of the form $\psi \otimes \phi$ with $\psi \in C^+(\hat{\Omega}^\infty)$, $\phi \in C_c^+(E)$, $\hat{\theta}^n(\tilde{\lambda}^e)(\psi \otimes \phi)$ is bounded by $(\sup_{\hat{\omega} \in \hat{\Omega}^\infty} \psi(\hat{\omega})) \lambda(\phi)$. It follows that the sequence of Radon measures $\hat{\theta}^n(\tilde{\lambda}^e)$ on $\hat{\Omega}^\infty \times E$ is weakly relatively compact. If $\hat{\lambda}$ is a limit of a subsequence, then the projection of $\hat{\lambda}$ in $\prod_{i > -n} \overline{G} \times E \subset \overline{G}^{\mathbb{Z}} \times E$ is equal to $\hat{\theta}^n(\tilde{\lambda}^e)$ for every $n \in \mathbb{N} \cup \{0\}$. Hence $\hat{\lambda}$ gives full measure to $\hat{\Omega} \times E$. It follows that $\hat{\lambda}$ is $\hat{\theta}$ -invariant and $\hat{\theta}^n(\tilde{\lambda}^e)$ converges weakly to $\hat{\lambda}$. The uniqueness of $\hat{\lambda}$ as in the proposition follows.

Remark 4 In some cases we can get a more explicit form of $\hat{\lambda}$. We denote by $\hat{\omega}^e$, the projection of $\hat{\omega} \in \hat{\Omega}$ on $\hat{\Omega}^e$, hence:

$$\tilde{\lambda}^e = \int \delta_{\hat{\omega}^e} \otimes \lambda d\hat{\mathbb{P}}(\hat{\omega})$$

$$\hat{\theta}^n(\tilde{\lambda}^e) = \int \delta_{\hat{\theta}^n \hat{\omega}^e} \otimes X_n(\hat{\omega}) \lambda d\hat{\mathbb{P}}(\hat{\omega}) = \int \delta_{\hat{\theta}^n(\hat{\theta}^{-n} \hat{\omega}^e)} \otimes Y_0(\hat{\omega}) \cdots Y_{-n+1}(\hat{\omega}) \lambda d\hat{\mathbb{P}}(\hat{\omega}).$$

Since $\lim_{n \rightarrow \infty} \hat{\theta}^n(\hat{\theta}^{-n} \hat{\omega}^e) = \hat{\omega}$ in $\hat{\Omega}^\infty$, it follows that for any $\psi \in C^+(\hat{\Omega}^\infty)$, the sequence of measures $\int Y_0(\hat{\omega}) \cdots Y_{-n}(\hat{\omega}) \lambda \psi(\hat{\omega}) d\hat{\mathbb{P}}(\hat{\omega})$ converges weakly to the Radon measure λ_ψ defined by $\lambda_\psi(\phi) = \hat{\lambda}(\psi \otimes \phi)$. This fact is well known in the situations of μ -boundaries (see [16]) where E is compact and $\lambda \in M^1(E)$. Then $Y_0(\hat{\omega}) \cdots Y_{-n}(\hat{\omega}) \lambda$ converges $\hat{\mathbb{P}}$ -a.e. to $\lambda_{\hat{\omega}} \in M^1(E)$ and $\hat{\lambda} = \int \delta_{\hat{\omega}} \otimes \lambda_{\hat{\omega}} d\hat{\mathbb{P}}(\hat{\omega})$. Also if λ is G -invariant $Y_0(\hat{\omega}) \cdots Y_{-n}(\hat{\omega}) \lambda = \lambda$, hence $\hat{\lambda} = \hat{\mathbb{P}} \otimes \lambda$. If E is a μ -boundary $\lambda_{\hat{\omega}}$ is a Dirac measure and hence $\hat{\lambda} \neq \hat{\mathbb{P}} \otimes \lambda$. Another situation where $\hat{\lambda}$ can be calculated will be considered in Section 5, example 3.

Proposition 5 *With the above notations, assume that (E, P, λ) satisfies the conditions:*

- (a) *property R is valid;*
- (b) *the condition $Pf = f$, $f \in \mathbb{L}^\infty(\lambda)$ implies f is constant.*

Then the system $(\hat{\Omega} \times E, \hat{\theta}, \hat{\lambda})$ is ergodic. The converse is valid, except if $E = \mathbb{Z}$ and $\mu = \delta_g$ acts by translations on \mathbb{Z} .

The proof depends on the following lemma:

Lemma 1 *Let \mathcal{H} be a Hilbert space and \mathcal{L} be a closed subspace and S be a contraction of \mathcal{H} such that*

$$S(\mathcal{L}) \subset \mathcal{L}, \quad \overline{\cup_{n \geq 0} S^{-n}(\mathcal{L})} = \mathcal{H}.$$

Assume that the restriction of S to \mathcal{L} has a unique fixed point $\phi \in \mathcal{L}$. Then ϕ is the unique fixed point of S in \mathcal{H} .

Proof Assume $f \in \mathcal{H}$ satisfies $Sf = f$. Let f_n be the orthogonal projection of f onto $\mathcal{H}_n = S^{-n}(\mathcal{L})$. Since $Sf_n \in \mathcal{H}_n$ and $Sf = f$, we get that $\|f - f_n\| \leq \|f - Sf_n\| = \|S(f - f_n)\| \leq \|f - f_n\|$ as S is a contraction. Thus, $\|f - f_n\| = \|f - Sf_n\|$. This implies from the definition of the orthogonal projection that $f_n = Sf_n$. Now, $f_n = S^n f_n \in \mathcal{L}$. It follows from the uniqueness of ϕ that $f_n = \phi$. Since $\overline{\cup_{n \geq 0} S^{-n}(\mathcal{L})} = \mathcal{H}$, we have $\phi = \lim_{n \rightarrow \infty} f_n = f$.

For any Polish space X we denote by $\mathcal{B}(X)$ the σ -algebra of its Borel subsets.

Proof of Proposition 5 To begin with we show that condition (b) implies ergodicity of the non-invertible system $(\Omega \times E, \tilde{\theta}, \tilde{\lambda})$. Let $\mathcal{A} \subset \mathcal{B}(\Omega \times E)$ be the σ -algebra of sets of form $\Omega \times B$ with $B \in \mathcal{B}(E)$ and $\mathcal{A}_n \subset \mathcal{B}(\Omega \times E)$ be the σ -algebra defined by the coordinates Y_k ($1 \leq k \leq n$). Assume $F \in \mathbb{L}^\infty(\mathbb{P} \otimes \lambda)$ is $\tilde{\theta}$ -invariant and denote $f = \mathbb{E}(F|\mathcal{A})$. Then $f(x) = \int F(\omega, x) d\mathbb{P}(\omega) \in \mathbb{L}^\infty(\lambda)$ and $Pf = f$. Also, $\mathbb{E}(f|\mathcal{A} \vee \mathcal{A}_n)(\omega, x) = f(X_n(\omega)x)$ and the martingale convergence theorem implies that $F(\omega, x) = \lim_{n \rightarrow \infty} f(X_n(\omega)x)$, $\tilde{\lambda}$ -a.e. Condition (b) implies that f is constant, hence F is also constant.

Let $U \subset E$ be open relatively compact set, $A = \Omega \times U$ and $\hat{A} = \Omega^- \times A \subset \hat{\Omega} \times E$. Condition (a) implies that for $(\Omega \times E, \tilde{\theta}, \tilde{\lambda})$, the return time $\tau(\omega, x)$ from A to A is defined a.e on A . Let $\tilde{\theta}_A$ be the induced transformation on A . Then the induced transformation on \hat{A} is well defined by $\hat{\theta}_A(\hat{\omega}, x) = \tilde{\theta}^{\tau(\omega, x)}(\hat{\omega}, x)$. The restriction m_A (resp. $m_{\hat{A}}$) of $\tilde{\lambda}$ (resp. $\hat{\lambda}$) to A (resp. \hat{A}) is $\tilde{\theta}_A$ -invariant (resp. $\hat{\theta}_A$ -invariant) and (A, θ_A, m_A) is a factor of $(\hat{A}, \hat{\theta}_A, m_{\hat{A}})$. We show ergodicity of the corresponding systems as follows. Let $C \in \mathcal{B}(A)$ with $\tilde{\theta}_A^{-1}(C) = C$, $\mathbb{P} \otimes \lambda(C) > 0$, and $C' = \cup_{k \geq 0} \tilde{\theta}^{-k}(C)$. Since $\tilde{\theta}^{-1}(C') \subset C'$ and $(\Omega \times E, \tilde{\theta}, \tilde{\lambda})$ has no wandering set of positive measure, one has $\tilde{\theta}^{-1}(C') =$

$C' \bmod \tilde{\lambda}$, hence $C' = \Omega \times E$ by ergodicity of $(\Omega \times E, \tilde{\theta}, \tilde{\lambda})$. Then $C = C' \cap A = A \bmod \tilde{\lambda}$. Now Lemma 1 gives the ergodicity of $\hat{\theta}_{\hat{A}}$ which may be seen as follows. Since m_A is the projection of $m_{\hat{A}}$ on \hat{A} one can take $\mathcal{H} = \mathbb{L}^2(m_{\hat{A}})$, $\mathcal{L} = \mathbb{L}^2(m_A)$, $S = \hat{\theta}_{\hat{A}}$. Since $\bigvee_{-\infty}^{+\infty} \hat{\theta}^k(\mathcal{A}) = \mathcal{B}(\hat{\Omega} \times E)$, one has $\bigcup_0^{+\infty} \hat{\theta}_{\hat{A}}^{-k}(\mathcal{A} \cap \hat{A}) = \mathcal{B}(\hat{A})$, hence $\bigcup_0^{\infty} S^{-k}(\mathcal{L}) = \mathcal{H}$. Also the restriction of S to $\mathbb{L}^2(m_A)$ is $\tilde{\theta}_A$. Since $\tilde{\theta}_A$ is m_A -ergodic, Lemma 1 implies $S = \hat{\theta}_{\hat{A}}$ is also $m_{\hat{A}}$ -ergodic. Finally the ergodicity of $\hat{\theta}$ is obtained as follows. Let $\hat{C} \in \mathcal{B}(\hat{\Omega} \times E)$ with $\hat{\theta}^{-1}(\hat{C}) = \hat{C}$, $(\hat{\lambda})(\hat{C}) > 0$. Then $\hat{C} \cap \hat{A}$ is $\hat{\theta}_{\hat{A}}$ -invariant, hence by ergodicity $\hat{C} \cap \hat{A} = \hat{A}$, $\hat{C} \supset \hat{A}$. Since E is a union of relatively compact open sets U_n with $0 < \lambda(U_n)$, one gets $\hat{C} = \hat{\Omega} \times E \bmod \hat{\mathbb{P}} \otimes \lambda$.

For the converse, we observe that if μ is not a Dirac measure, then $(\hat{\Omega}, \hat{\lambda})$ is non-atomic. Furthermore, if $\mu = \delta_g$, ergodicity of $\hat{\theta}$ on $\hat{\Omega} \times E$ is equivalent to ergodicity of the action of g on E . Also, in this case (E, λ) atomic is equivalent to $(\hat{\Omega} \times E, \hat{\lambda})$ atomic. Furthermore, (E, λ) atomic and g ergodic implies that (E, g, λ) reduces to the translation on \mathbb{Z} . In the opposite case, ergodicity of $\hat{\theta}$ implies that $(\hat{\Omega} \times E, \hat{\theta}, \hat{\lambda})$ has no wandering set. Then by Proposition 3 property R is valid for (E, P, λ) . If $f \in \mathbb{L}^{\infty}(\lambda)$ satisfies $Pf = f$, one takes a Borel version of f again denoted by f , which satisfies $Pf = f$. Then for every $x \in E$, the sequence $(f(X_n(\omega)x))$ is a bounded martingale with respect to \mathbb{P} and the natural filtration on Ω . By the martingale convergence theorem we have

$$F(\omega, x) = \lim_{n \rightarrow \infty} f(X_n(\omega)x)$$

\mathbb{P} -a.e and $f(x) = \int F(\omega, x) d\mathbb{P}(\omega)$. Also $F(\omega, x)$ is $\tilde{\theta}$ -invariant mod $\mathbb{P} \otimes \lambda$. Then the ergodicity of $(\hat{\Omega} \times E, \hat{\theta}, \hat{\lambda})$ implies that F is constant $\mathbb{P} \otimes \lambda$ -a.e, hence f is constant λ -a.e.

Using the above remark 4 and Proposition 2 we get the following well-known result.

Corollary 2 *Assume that if $\mu \in M^1(G)$ is adapted and recurrent. Then the skew-product $(\hat{\Omega} \times G, \hat{\theta}, \hat{\mathbb{P}} \otimes \lambda_G)$ is ergodic.*

We now relate ergodicity of $(\hat{\Omega} \times E, \hat{\theta}, \hat{\lambda})$ to extremality of λ .

Corollary 3 *Assume λ is a P -invariant Radon measure on E and (E, P, λ) satisfies property R . Then $(\hat{\Omega} \times E, \hat{\theta}, \hat{\lambda})$ is ergodic if and only if λ is extremal.*

Proof Assume λ is extremal and let U be a relatively compact open subset of E with $\lambda(U) > 0$. In order to prove ergodicity of $\hat{\theta}$, we consider the return time of U defined by

$$H(\omega, x) = \inf\{n \in \mathbb{N} \mid X_n(\omega)x \in U\}, \quad (\omega, x) \in \Omega \times U.$$

Also, we can define the induced Markov operator P^H of P on U . Let f be a bounded Borel function such that $Pf = f$. Since $H(\omega, x)$ is a stopping time we have $f(x) = \mathbb{E}(f(X_{H(x,\omega)}(\omega)x)) = P^H f(x)$. Let $\lambda_U = 1_U \lambda$ be the restriction of λ to U , hence $P^H \lambda_U = \lambda_U$. Since λ_U is finite and $f1_U$ is P^H -invariant, we have $P^H(f\lambda_U) = f\lambda_U$. Then by the classical regeneration method we can construct a P -invariant measure ρ_f on E such that the restriction of ρ_f to U is $f\lambda_U$. We recall the construction: let $\rho(\omega, x)$ be the random measure given by $\rho(\omega, x) = \sum_0^{H(\omega,x)-1} \delta_{X_k(\omega)x}$ and observe that if

$$Y_1(\omega)\rho(\omega, x) = \rho(\omega, x) + \delta_{X_{H(\omega,x)}x} - \delta_x$$

and ρ_f is defined by $\rho_f = \mathbb{E}(\int \rho(\omega, x)f(x)d\lambda_U(x))$, its P -invariance follows from the P^H -invariance of $f\lambda_U$. If $f = 1$, one may verify that this procedure gives $\rho_f = 1_{U'}\lambda$ with $U' = T_\mu U$ where T_μ is the closed semigroup generated by the support of μ . Since $|f|$ is bounded by $c > 0$, it follows that ρ_f is a Radon measure with $\rho_f \leq c\lambda$ and $P\rho_f = \rho_f$. The extremality property of λ gives that ρ_f is proportional to λ , in particular $f\lambda_U$ is proportional to λ_U , that is, $f1_U$ is constant λ_U -a.e. Since U was arbitrary with $\lambda(U) > 0$, we can conclude f is constant λ -a.e. Now applying Proposition 5 to (E, P, λ) , we get the ergodicity of $(\hat{\Omega} \times E, \hat{\theta}, \hat{\lambda})$.

Conversely, let λ' be another positive Radon measure with $P\lambda' = \lambda'$, $\lambda' = f\lambda$ where $f \in \mathbb{L}^\infty(\lambda)$. We will now claim that f is constant. As above we write $\lambda_U = 1_U \lambda$, $\lambda'_U = 1_U \lambda'$. We observe that since property R is valid $H(\omega, x)$ is well defined for $\mathbb{P} \otimes \lambda$ -a.e $(\omega, x) \in \Omega \times U$. Then $P^H \lambda'_U = \lambda'_U$, $P^H \lambda_U = \lambda_U$ and $\lambda'_U = (f1_U)\lambda_U$. It follows that $\mathbb{P} \otimes \lambda_U$ is $\tilde{\theta}^H$ -invariant. By ergodicity of $(\Omega \times U, \tilde{\theta}^H, \mathbb{P} \otimes \lambda_U)$: $f1_U$ is constant λ_U -a.e. Since U is arbitrary with $\lambda(U) > 0$, we get f is constant λ -a.e and λ' is proportional to λ .

2.4 Recurrent groups and contraction subgroups

Let G be a locally compact group. For $g \in G$, we define the contraction subgroup C_g of g by

$$C_g = \{x \in G \mid \lim_{n \rightarrow \infty} g^n x g^{-n} = e\}.$$

Then C_g is a subgroup of G normalized by g . In general, C_g is not closed. However, if $G = GL(n, \mathbb{F})$ for some local field \mathbb{F} , then C_g is an algebraic \mathbb{F} -subgroup of G , hence C_g is closed. If G is a closed subgroup of $GL(n, \mathbb{F})$, then C_g is closed in G . As a simple consequence we obtain that if G is a closed subgroup of $\prod_{i \in I} GL(d_i, \mathbb{F}_i)$ (finite I), C_g is closed. We now prove the following basic lemma on C_g .

Lemma 2 *Let G be a locally compact group. Assume that $g \in G$ is such that C_g is closed and $C_g \neq \{e\}$. Then the closed subgroup L generated by g and C_g is not unimodular.*

Proof Let $H = C_g$. Then H is a normal subgroup of L and hence L/H is abelian. For a locally compact group M , we recall that λ_M is a left Haar measure on M . Let $\phi \in C_c(L)$. We observe that the formula

$$\lambda_L(\phi) = \int d\lambda_{L/H}(\bar{u}) \int \phi(uh) d\lambda_H(h)$$

defines a left Haar measure on L . For any $x \in L$, the map $h \mapsto xhx^{-1}$ is an automorphism of H and we denote by $\Delta(x)$ its module. Furthermore, for $\phi \in C_c(L)$, we define $\phi^x \in C_c(L)$ by $\phi^x(u) = \phi(ux)$ for all $x, u \in L$. Since L/H is unimodular, we see that $\lambda_L(\phi^x) = \Delta(x^{-1})\lambda_L(\phi)$ for all $x \in L$ and $\phi \in C_c(L)$. Since C_g is closed, for any compact neighborhood W of e in G , $g^n W g^{-n} \rightarrow e$ (cf. [49]). This implies that $\lim_{n \rightarrow \infty} \lambda_H(g^n W g^{-n}) = 0$, hence $\Delta(g) < 1$. It follows that $\lambda_L(\phi^g) > \lambda_L(\phi)$ for any $\phi \in C_c(L)$, hence L is not unimodular.

Proposition 6 *Let G be a locally compact group and E be a locally compact second countable G -space. Suppose there exists a compact subset C of E such that $E = GC$ and G is recurrent. Then there exists a G -invariant Radon measure on E .*

Proof Let $\mu \in M^1(G)$ be recurrent. Let P be the Markov operator on G defined by $P\phi(x) = \int \phi(gx) d\mu(g)$ for all $x \in E$. Let $u \in C_c^+(E)$ be such that $u > 0$ on C . Now for any $x \in E$, there exists $h \in G$ such that $hx \in C$, hence there exists $\delta > 0$ and an open neighborhood V_x of h in G such that $u(gx) > \delta > 0$ for all $g \in V_x$. Then $P^k u(x) = \int u(gx) d\mu^k(g) \geq \delta \mu^k(V_x)$ for all $k \geq 1$. Since μ is recurrent, $\sum_0^\infty P^k u(x) = +\infty$. Using Proposition 1, we get the existence of a P -invariant Radon measure ν on E . For any $\phi \in C_c^+(E)$,

we define h_ϕ on G by $h_\phi(g) = g\nu(\phi) = \int \phi(gx)d\nu(x)$. Since ν is P -invariant, h_ϕ is right μ -harmonic, that is, $\int h_\phi(gg')d\mu(g') = \int \phi(gg'x)d\mu(x)d\mu(g') = \int \phi(gy)d\nu(y) = h_\phi(g)$. Since h_ϕ is continuous, positive right μ -harmonic and μ is recurrent, using Proposition 2 for $E = G$, we get that h_ϕ is constant. It follows that $g\nu(\phi) = \nu(\phi)$ for any $g \in G$ and any $\phi \in C_c^+(G)$. Hence ν is G -invariant.

Corollary 4 *Suppose G is a locally compact recurrent group. Then any closed subgroup of G is unimodular. In particular, if $g \in G$ and C_g is closed, then $C_g = \{e\}$.*

Proof We first recall that for a closed subgroup H of G , the quotient space G/H carries a G -invariant measure if and only if the restriction to H of the modular function of G is equal to the modular function of H . We now claim that G is unimodular. For any closed subgroup H , let Δ_H be the modular function of H . Let $g \in G$ and $H(g)$ be the closed subgroup generated by g in G . Then by Proposition 6, $G/H(g)$ carries a G -invariant measure. This implies that $\Delta_G(g) = \Delta_{H(g)}(g) = 1$ as $H(g)$ is abelian. Thus, G is unimodular. Now, let H be any closed subgroup of G . Then by Proposition 6, G/H carries a G -invariant measure and hence $\Delta_H(h) = \Delta_G(h) = 1$ for any $h \in H$, as G is unimodular.

Suppose $g \in G$ is such that C_g is closed. Let L be the closed subgroup generated by C_g and g . Then from above L is unimodular and hence by Lemma 2, $C_g = \{e\}$.

In our effort to understand the structure of totally disconnected recurrent groups, we obtain the following.

Corollary 5 *If G is a locally compact totally disconnected recurrent group. Then G is uniscalar, that is, for each $g \in G$, there is a compact open subgroup K_g of G such that $gK_gg^{-1} = K_g$.*

Proof It follows from Corollary 4, that closed subgroups of G are unimodular and hence the result follows from Proposition 3.21 of [4].

3 Proof of the theorem 1

3.1 Recurrent real Lie groups

Theorem 3 *Let G be a locally compact group such that G has a continuous injection into a real Lie group. Suppose G is compactly generated and recurrent. Then G has a compact normal subgroup K and a finite index subgroup H such that $K \subset H$ and H/K is isomorphic to a closed subgroup of the group \mathbb{G}_2 of euclidean motions of the plane.*

Proof We first claim that G is a real Lie group. Let G' be a real Lie group and $\phi: G \rightarrow G'$ be a continuous injection. Since G' is a real Lie group, by [36], there is a neighborhood U of identity in G' such that $\{e\}$ is the only subgroup contained in U . Let $V = \phi^{-1}(U)$. Then V is a neighborhood of identity in G . If N is a subgroup contained in V , then $\phi(N)$ is a subgroup contained in U , hence $\phi(N)$ is trivial. Since ϕ is injective, N is trivial. Now it follows from [36] that G is a real Lie group.

Now, G is a real Lie group implies that its connected component G_0 is open, hence G/G_0 is a finitely generated recurrent group. Thus, by [48], G/G_0 has a normal subgroup of finite index isomorphic to \mathbb{Z}^k for $k \leq 2$. Thus, we may assume that $G/G_0 \simeq \mathbb{Z}^k$. Since G_0 is open, G_0 is a connected recurrent Lie group. By Theorem 0.1 of [2], G_0 contains a compact subgroup K and a normal vector subgroup \mathbb{R}^l for $l \leq 2$ such that $G_0 \simeq K \times \mathbb{R}^l$.

If $l = 0$, we get that G_0 is compact and up to finite index G/G_0 is a subgroup of \mathbb{Z}^2 . We consider the case when $l = 1$. Since K is connected and the only compact connected group of automorphisms of \mathbb{R} is trivial, $G_0 \simeq K \times \mathbb{R}$. Since G_0 is normal in G , K is also normal in G . Replacing G by G/K , if necessary, we may assume that $G_0 \simeq \mathbb{R}$. Let $\phi: G \rightarrow \text{Aut}(G_0) \simeq \mathbb{R}^*$ be

$$\phi(g)(v) = gvg^{-1}$$

for $g \in G$ and $v \in G_0$. By Corollary 4, G is unimodular and since G_0 is open, Haar measure on G_0 is $\phi(g)$ -invariant for any $g \in G$. This implies that $|\phi(g)| = 1$. Thus, G contains a normal subgroup G_1 of finite index such that G_0 is contained in the center of G_1 and $G_1/G_0 \simeq \mathbb{Z}^k$. Thus, G_1 is a compactly generated nilpotent recurrent group. By Corollary 13 of Chapter III of [22], $G_1 \simeq \mathbb{R} \times \mathbb{Z}$.

We next consider the case $l = 2$. The only connected compact groups of automorphisms of \mathbb{R}^2 are the trivial group and $SO_2(\mathbb{R})$. Hence, since K is

connected, its conjugation action on \mathbb{R}^2 factors through one of these groups. Therefore, modulo a compact normal subgroup, G_0 may be assumed to be either \mathbb{R}^2 or \mathbb{G}_2 , the motion group of the plane. If $G_0 = \mathbb{R}^2$, then for any $g \in G$, g acts linearly on \mathbb{R}^2 by \tilde{g} with eigenvalues of modulus one which may be seen as follows. If $g \notin G_0$, $\langle g \rangle$ is isomorphic to \mathbb{Z} and $\langle g \rangle \times \mathbb{R}^2$ is closed, hence unimodular by Corollary 4. It follows that $|\det \tilde{g}| = 1$. If \tilde{g} has an invariant line in \mathbb{R}^2 , the same argument gives that the corresponding eigenvalue has modulus one. Since $|\det \tilde{g}| = 1$, the same is true of the second eigenvalue. If \tilde{g} has no real eigenvalue, the condition $|\det \tilde{g}| = 1$ implies that \tilde{g} has conjugate pair of eigenvalues of absolute value one. Now there are two situations depending on the fact that G/G_0 acts on \mathbb{R}^2 irreducibly or not. In the first case the fact that elements \tilde{g} have eigenvalues of modulus one, implies that G/G_0 acts as a subgroup of $O(2)$. Since $G/G_0 \simeq \mathbb{Z}^k$, G is a two-step solvable group with relatively compact action on $[G, G] \subset G_0$ hence by Theorem 11, Chapter IV of [22], we get that $G = \mathbb{R}^2$. In the second situation, G has a finite index and hence recurrent nilpotent subgroup $G_1 \supset \mathbb{R}^2$. Hence as above, $G = G_1 = \mathbb{R}^2$. If $G_0 = SO(2) \times \mathbb{R}^2$, since the square of every automorphism of G_0 is interior, we can assume by passing to a finite index subgroup that for any $g \in G$, the inner automorphism of g restricted to G^0 is uniquely defined by $g' \in G_0$. Then the homomorphism $g \mapsto g'$ is a projection of G onto G_0 with kernel $N \simeq G/G_0$. Hence $G = G_0 \times N$ with $N \simeq \mathbb{Z}^i$, $0 \leq i \leq 2$. Then since G is two-step solvable and recurrent and G acts on \mathbb{R}^2 as a compact subgroup, Theorem 11, Chapter IV of [22], gives N is trivial.

Corollary 6 *Assume G is a compactly generated group of polynomial growth and is recurrent. Then G has at most quadratic growth and has a compact normal subgroup K and a finite index subgroup H such that $K \subset H$ and H/K is isomorphic to a closed subgroup of the motion group of the plane.*

The corollary is a simple consequence of Theorem 3 and of the fact that a compactly generated group G of polynomial growth contains a compact normal subgroup K such that G/K is a real Lie group (cf. [34]).

3.2 Recurrent linear groups

We now consider closed subgroups of linear groups over local fields and prove the quadratic growth conjecture for such groups. We also prove Theorem 1.

Lemma 3 *Let V be a vector space over a non-archimedean local field \mathbb{F} and G be a subgroup of $GL(V)$. Suppose for any $g \in G$ and $v \in V$, $(g^n(v))_{n \in \mathbb{Z}}$ is relatively compact. Then G is contained in a compact extension of a unipotent subgroup of $GL(V)$. In addition if G is compactly generated, then G is relatively compact.*

Remark 5 This result is also proved in [39], we provide a proof as our proof is simpler.

Proof Let V_1 be an G -irreducible subspace of V . Then using Burnside density Theorem as in [10], we conclude that the restriction of G to V_1 is relatively compact in $GL(V_1)$. Using a Jordan-Hölder sequence we get that there is a compact subgroup K and an unipotent subgroup U normalized by K such that $G \subset K \times U \subset GL(V)$. Let A be a compact generating set in G . Then there is a compact set $C \subset U$ with $kCk^{-1} = C$ for all $k \in K$ (this is possible as K is a compact group normalizing U) such that $A \subset KC$. Since a compactly generated subgroup of any unipotent group is relatively compact (as the base field is non-archimedean) and C is K -invariant, the closed subgroup $L \subset U$ generated by C is compact and is normalized by K . Thus, $G \subset K \times L$ which is compact and hence G is relatively compact.

Proposition 7 *Let G be a closed compactly generated subgroup of a linear group $GL(n, \mathbb{F})$ over a non-archimedean local field \mathbb{F} . Suppose $C_g = \{e\}$ for any $g \in G$. Then G has a basis of compact open normal subgroups.*

Proof Let $\Phi: G \rightarrow M_n(\mathbb{F})$ be given by $\Phi(x) = x - I$ for all $x \in G$. Then Φ is a homeomorphism onto $\Phi(G)$ endowed with the topology induced from $M_n(\mathbb{F})$ and $\Phi(gxg^{-1}) = g\Phi(x)g^{-1}$ for all $x, g \in G$. Let V be the smallest subspace of $M_n(\mathbb{F})$ such that $V \cap \Phi(G)$ is a neighborhood of 0 in $\Phi(G)$. We now claim that for any $g \in G$, V is g -invariant and $(g^n v g^{-n})_{n \in \mathbb{Z}}$ is relatively compact for all $v \in V$. Since \mathbb{F} is non-archimedean, G is a totally disconnected locally compact group. Then by Proposition 2.1 of [29], there is a basis of compact open subgroups (K_i) at e in G such that $gK_i g^{-1} = K_i$ for all $i \geq 1$. Since $V \cap \Phi(G)$ is a neighborhood of 0 in $\Phi(G)$, there exists a K_i such that $\Phi(K_i) \subset V$. Let W be the subspace of V spanned by $\Phi(K_i)$. Then $\Phi(K_i) \subset W \cap \Phi(G)$ is a neighborhood of 0 in $\Phi(G)$. Since V is the smallest such subspace $V = W$. For any $v \in \Phi(K_i)$, $g^n v g^{-n} \in \Phi(K_i)$ for any $n \in \mathbb{Z}$ and hence $(g^n v g^{-n})_{n \in \mathbb{Z}}$ is relatively compact as $\Phi(K_i)$ is compact in V . Since $V = W$ is spanned by $\Phi(K_i)$, we get that $(g^n v g^{-n})_{n \in \mathbb{Z}}$ is relatively

compact for all $v \in V$. Since $g\Phi(K_i)g^{-1} = \Phi(gK_i g^{-1}) = \Phi(K_i)$ and $W = V$ is spanned by $\Phi(K_i)$, $gVg^{-1} = V$. Thus, V is G -invariant and $(g^n v g^{-n})_{n \in \mathbb{Z}}$ is relatively compact for any $g \in G$ and $v \in V$.

Now, by Lemma 3, V contains a basis of open neighborhoods at 0 invariant under conjugation by elements of G and so $V \cap \Phi(G)$ has small G -invariant neighborhoods of 0 in $V \cap \Phi(G)$. Since $V \cap \Phi(G)$ is a neighborhood of 0 in $\Phi(G)$, we get that G contains a basis of open invariant neighborhoods at e . Since G is totally disconnected, G contains a basis of compact open normal subgroups.

Corollary 7 *Let G be a compactly generated closed subgroup of a linear group $GL(n, \mathbb{F})$ over a non-archimedean local field \mathbb{F} . Suppose every closed subgroup of G is unimodular. Then G has a basis (K_n) of compact open normal subgroups. In particular, if G has polynomial growth, then furthermore, G/K_n has a finite index subgroup which is finitely generated and nilpotent.*

Proof Since G is a closed subgroup of $GL(n, \mathbb{F})$, for any $g \in G$, C_g is a closed subgroup of a unipotent algebraic group, hence C_g is closed. Then by Lemma 2, $C_g = \{e\}$ and first part of the corollary follows from Proposition 7. If G has polynomial growth, then its closed subgroups are unimodular (see [20]). So, as above we obtain using Lemma 2 that there is a basis (K_n) of compact open normal subgroups. Since K_n is open, G/K_n is a finitely generated group of polynomial growth and hence the result follows from [19].

We now prove the quadratic growth conjecture for closed subgroups of linear groups over local fields.

Theorem 4 *Let G be a closed subgroup of a linear group $GL(n, \mathbb{F})$ over a local field \mathbb{F} . Then G is a recurrent group if and only if G has at most quadratic growth. More precisely, if G is compactly generated and recurrent, then we have the following two cases*

- a) *when \mathbb{F} is archimedean, up to finite index and modulo a compact normal subgroup G is isomorphic to a closed subgroup of the euclidean motion group of the plane.*
- b) *when \mathbb{F} is non-archimedean, up to finite index and modulo a compact open normal subgroup G is isomorphic to a subgroup of \mathbb{Z}^2 .*

Proof The implication that G has at most quadratic growth implies G is a recurrent group can be proved as in Proposition 3.1 of [41] as explained in section 2.1. Here we prove the implication that G is a recurrent group implies G has at most quadratic growth.

Suppose G is recurrent. In order to prove G has at most quadratic growth we may assume that G is a compactly generated group. Now we prove the stronger statement in the second part of the Theorem. Since G is a linear group, C_g is closed in G for any $g \in G$. Then by Corollary 4, C_g is trivial.

Non-archimedean case: Assume that G is a linear group over a non-archimedean field. Then by Proposition 7, G contains a compact open normal subgroup, say K . Now G/K is a finitely generated recurrent group and hence up to finite index G/K is isomorphic to a subgroup of \mathbb{Z}^2 (cf. [48]). Since K is compact, G itself has at most quadratic growth.

Archimedean Case: Suppose \mathbb{F} is archimedean. Then \mathbb{F} is \mathbb{R} or \mathbb{C} . So, we may assume that G is a closed subgroup of a linear group over \mathbb{R} . Then it follows from Cartan's Theorem that G is a real Lie group. Now the result is a direct consequence of Theorem 3.

Proof of Theorem 1 Suppose G is a compactly generated recurrent group. For $i \in I$, let G_i be the closure of the projection of G into $GL(n_i, \mathbb{F}_i)$. Then each G_i is a closed subgroup of $GL(n_i, \mathbb{F}_i)$ that is compactly generated and recurrent.

Let J be the set of indices i such that \mathbb{F}_i is non-archimedean. Then for $i \in J$, it follows from Theorem 4 that G_i has a compact open normal subgroup, say K_i . Now $\prod_{i \in I} G_i$ is a closed subgroup of $\prod_{i \in I} GL(n_i, \mathbb{F}_i)$ and hence G is a closed subgroup of $\prod_{i \in I} G_i$. Let $K = \prod_{i \in I} K_i$ with $K_i = (e)$ for $i \notin J$. Then K is a compact normal subgroup of $\prod_{i \in I} G_i$ and $(\prod_{i \in I} G_i)/K$ is a real Lie group. Since K is compact and G is closed in $\prod_{i \in I} G_i$, $G/G \cap K$ is a closed subgroup of $(\prod_{i \in I} G_i)/K$. By Cartan's Theorem $G/G \cap K$ is a real Lie group. Since $G/G \cap K$ is also recurrent, the result follows from Theorem 3.

Remark 6 More generally we can ask if G is a compactly generated locally compact group with a continuous embedding in $\prod_{i \in I} GL(d_i, \mathbb{F}_i)$ and is recurrent, then G has at most quadratic growth. This property is valid if all \mathbb{F}_i ($i \in I$) are archimedean. Answer to this hinges on the following: if the image of G is contained in a compact linear group over a non-archimedean local field, is it true that G is a projective limit of real Lie groups.

4 Harris-recurrence

We now consider the notion of recurrence known as Harris recurrence (abbr. as H-recurrence) widely use in the context of Markov operators on measured spaces (see [43]). We will say that μ is H -recurrent if for any Borel subset B of G of positive Haar measure, \mathbb{P} -a.e $X_n(\omega) \in B$ infinitely often. If G supports a H -recurrent μ , we say that G is H -recurrent. It is known that μ is H-recurrent if and only if some power of μ is non-singular and recurrent (Theorem 4.11, Chapter 3 of [43]). The methods of [48] allows to get very strong results on the asymptotics of iterates μ if μ is symmetric and has a bounded density with compact support. By combining our methods with the results of [2], [22] and [48], we can show that H -recurrent groups have quadratic growth. More precisely we have

Proposition 8 *If G is a compactly generated, locally compact second countable H -recurrent group, then there exists closed normal subgroups H and K of G such that K is compact, G/H is finite and H/K is isomorphic to a closed subgroup of \mathbb{G}_2 . In particular, any locally compact H -recurrent group has quadratic growth.*

We need the following

Lemma 4 *Assume that G is as in Proposition 8. Then there exists a compactly supported recurrent $\nu \in M^1(G)$ such that ν is symmetric with a bounded density that is bounded from below on an open neighborhood of generating G .*

Proof Using Lemma 2.2 of [2] and replacing μ by $\frac{1}{2}(\mu + \check{\mu})$, we may assume that μ is symmetric. Since μ is H-recurrent, we can write for some $k > 0$ and $\rho, \sigma \in M^1(G)$, $\mu^k = r\rho + (1-r)\sigma$ for some $r \in (0, 1]$ and ρ has symmetric density. Then, if $\mu' = \sum_0^\infty \frac{\mu^n}{2^{n+1}}$ we have for some $\epsilon > 0$ $\mu' \geq \frac{\epsilon}{3}(\rho + \rho\sigma + \sigma\rho) = \epsilon\nu'$ where $\nu' = \frac{1}{3}(\rho + \rho\sigma + \sigma\rho)$. Also using Lemma 2.3 of [2] we know that μ' and $\mu'' = \sum_0^\infty \frac{(\mu')^n}{2^{n+1}}$ are H-recurrent. From, above we get $\mu'' \geq \sum_0^\infty \frac{\epsilon^n (\nu')^n}{2^{n+1}} = \nu''$ say. Clearly ν'' has a symmetric density and since the support of ν' generates G , we get that the support of ν'' is G . We can in the above inequality replace ν'' by ν''' with a bounded symmetric density and the support of ν''' is G . Then $(\mu'')^2 \geq (\nu''')^2$ and $(\nu''')^2$ has positive continuous density. Then we can restrict $(\nu''')^2$ to a compact neighborhood of identity which generates

G . Using Lemma 2.3 of [2], the normalization of $(\nu''')^2$ is recurrent, hence H -recurrent.

In order to make explicit the property of quadratic growth we prove the following.

Proposition 9 *Let G be a compactly generated group of at most quadratic growth. Then there exists closed normal subgroups H and K of G such that K is compact, G/H is finite and H/K is isomorphic to a closed subgroup of \mathbb{G}_2 , the motion group of the plane.*

Proof By [34], we can assume that G is a real Lie group. Let G_0 be the component of the identity. Then G_0 is open and G/G_0 also has at most quadratic growth. Since G/G_0 is finitely generated, G/G_0 contains a subgroup of finite index isomorphic to \mathbb{Z}^i for $i \leq 2$. So, we may assume that $G/G_0 \simeq \mathbb{Z}^i$ for $i \leq 2$.

Now, G_0 has at most quadratic growth implies that $G_0 \simeq K \times \mathbb{R}^j$ for $j \leq 2$ and K is a compact group. Then there exists a characteristic compact subgroup L of G_0 contained in K such that K/L is a subgroup of $SO(2, \mathbb{R})$. Replacing G by G/L , we may assume that K is a subgroup of $SO(2, \mathbb{R})$.

If $j = 1$, $G_0 = K \times \mathbb{R}$ and K is characteristic in G_0 , hence normal in G . Replacing G by G/K we may assume that $G_0 = \mathbb{R}$. Then since G is unimodular G_0 is central in a subgroup of finite index in G . Thus, we may assume that G_0 is central in G , hence G is nilpotent. Since G has at most quadratic growth, G is abelian, hence is a closed subgroup of \mathbb{R}^2 .

Let $j = 2$ and $a \in G$ but $a \notin G_0$. Then the closed subgroup A generated by a is isomorphic to \mathbb{Z} and $A \cap G_0 = \{e\}$ as $G/G_0 \simeq \mathbb{Z}^i$. Let H_a be the closed subgroup generated by G_0 and a . Then $H_a = A \rtimes G_0$ where the action of A on G_0 is given by the conjugacy. Since $A \simeq \mathbb{Z}$, an easy computation shows that H_a has growth at least three. This is a contradiction to G having quadratic growth. Thus, $G = G_0$ which is isomorphic to a closed subgroup of \mathbb{G}_2 .

Proof of Proposition 8 Let G be H -recurrent and L be any compactly generated open subgroup of G . Let $\mu \in M^1(G)$ be H -recurrent. Let $\mu_L \in M^1(L)$ be the induced measure on L . Then as observed in section 2 μ_L is also H -recurrent. Hence we can assume $\mu_L \in M^1(L)$ is as in the conclusion of Lemma 4. Then since L is unimodular, we are in the situation of Theorem

VII 1.1 of [48]. It follows that L and hence G has polynomial growth of degree at most two and the rest follows from Proposition 9.

5 Examples of recurrent or transient behaviors for random walks on G -spaces

Here $E = G/H$ will be a homogeneous space and $\mu \in M^1(G)$ defines a random walk on E with trajectories $X_n(\omega)x$ ($x \in E$). We recall that the associated Markov operator P satisfies:

$$\sum_0^\infty P^k \psi(x) = \int \sum_0^\infty \psi(X_n(\omega)x) d\mathbb{P}(\omega)$$

for all $\psi \in C_b^+(E)$. If $E = G$, there is a natural P -invariant measure, that is left Haar measure. Here in various cases the discussion involves a natural P -invariant measure. For a given $x \in E$, we define the following properties that may or may not be satisfied.

R_x : There exists a compact set $K_x \subset E$ such that \mathbb{P} -a.e, $X_n(\omega)x \in K_x$ infinitely often.

T_x : For any compact set $K \subset E$, \mathbb{P} -a.e, there exists $n(\omega) \in \mathbb{N}$ with $X_n(\omega)x \notin K$ for $n \geq n(\omega)$.

We observe that if property R (defined in 2.3) is valid, then since E is a countable union of compact sets, property R_x is valid a.e. We will also consider the following property R^a for (E, P) :

R^a : There exists a compact set $K \subset E$ such that for each $x \in E$, \mathbb{P} -a.e, $X_n(\omega)x \in K$ infinitely often.

It is easy to see that property R^a implies R_x for all $x \in E$ with $K_x = K$ independent of $x \in E$. Clearly if R_x is valid, then $\sum_0^\infty P^n 1_{K_x}(x) = +\infty$. Also, if $\sum_0^\infty P^n 1_K(x) < +\infty$ for any compact K , then T_x is valid. If G_μ is "large" one can expect for "most" $x \in E$, the trajectories $X_n(\omega)x$ to have similar asymptotic behaviors. More precisely one can expect the existence of a large set of points $x \in E$ such that T_x (or its complement) is valid; see [28] for a discussion of analogous properties when μ has density. In the examples studied below, there exists a natural P -stationary measure η on E and we also discuss recurrence and ergodicity properties with respect to η : see [17] for a discussion on this problem and [40] for other examples of probabilistic

significance. Property R is valid with respect to η in examples 1-5, R_x is valid for any x in examples 1,2,4, while in example 6, T_x is valid η -a.e.

(1) **Homogeneous spaces with finite stationary measure:** If $E = G/H$ is compact, Markov-Kakutani theorem implies the existence of a μ -stationary probability η , that is, $\mu * \eta = \eta$. If η is extremal, then ergodicity of $(\Omega \times E, \theta, \mathbb{P} \otimes \eta)$, hence of its natural extension $(\hat{\Omega} \times E, \hat{\theta}, \hat{\eta})$ is valid. This situation takes place also for some noncompact spaces. For example, if \mathbb{F} is a local field we can take $G = GL(d, \mathbb{F}) \ltimes \mathbb{F}^d$, the affine group of \mathbb{F}^d , and we denote $g \in G$ as $g = (a(g), b(g))$ with $a(g) \in GL(d, \mathbb{F})$ and $b(g) \in \mathbb{F}^d$. Then if $\mu \in M^1(G)$ and

$$\int (|\log(\|a(g)\|)| + |\log(\|b(g)\|)|) d\mu(g) < +\infty$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \int \log \|a(g)\| d\mu^n(g) < 0$$

there exists a unique μ -stationary probability η on $\mathbb{F}^d = E = G/GL(d, \mathbb{F})$ (cf. [9], [31]). If the support of μ has no fixed point on \mathbb{F}^d , then η is not a Dirac measure. In this case (E, η) is a μ -boundary of (G, μ) (see [16]) and remarkable homogeneity properties of η at infinity have been described in [31]. Using proximality of the G_μ -action, it is easy to show that property R^a is valid for (E, P) in this case for any compact subset K having non-empty interior with $\eta(K) > 0$.

Another kind of example takes place when G is a semisimple noncompact real Lie group and E is a non-compact homogeneous space of finite volume $E = G/\Gamma$ where Γ is a lattice in G . Then if $\mu \in M^1(G)$ is such that the support of μ is compact and generates a Zariski-dense subgroup, then by [13] there exists a compact $K \subset E$ and $C_K > 0$ such that for any $x \in E$,

$$\liminf_{n \rightarrow \infty} \mu^n * \delta_x(K) \geq C_K.$$

It follows that if $u \in C_C^+(E)$ satisfies $u \geq 1$ on K , then $\sum_0^\infty P^k u = +\infty$ on E , hence property R is valid with respect to any μ -stationary measure. If G is almost simple and for any g in G the group $gG_\mu g^{-1}$ has no finite index subgroup contained in Γ , then it follows from [5] that Haar measure on E is the unique μ -stationary measure. Then, by Breiman's law of large numbers for Markov chains, property R^a is valid for any compact set K with positive measure. Finally, Haar measure is μ -stationary but, in general, there exists finitely supported μ -stationary measures.

(2) **Affine space and affine groups:** If \mathbb{F} is a local field with absolute value $|\cdot|$ and E is the corresponding affine line, that is, G is the affine group $'ax + b'$ of \mathbb{F} , then the conditions

$$\int (|\log |a(g)|| + |\log |b(g)||)^{2+\delta} d\mu(g) < +\infty \quad (\delta > 0)$$

$$\int \log |a(g)| d\mu(g) = 0$$

and the support of μ has no fixed point on E imply the existence of $u \in C_c^+(\mathbb{F})$ such that $\sum_0^\infty P^k u = +\infty$ on \mathbb{F} (see Babillot and others [3]), hence by Proposition 1, the existence of λ with $P\lambda = \lambda$. As shown in [3], the Radon measure λ has infinite mass and is unique up to a coefficient. Hence the equidistribution property of Proposition 1 is valid in this case. Here we complete the example with the following.

Proposition 10 *With the above notations and hypothesis, there exists a unique μ -stationary Radon measure λ on E , the mass of λ is infinite, hence the system $(\hat{\Omega} \times E, \hat{\theta}, \hat{\lambda})$ is ergodic. Furthermore, property R^a is valid.*

Proof The proof is based on the method of [3]. Let τ be the first descending ladder index of the random walk $\log |a(X_n)| \in \mathbb{R}$, that is, $\tau(\omega) = \inf\{n \in \mathbb{N} \mid |a(X_n(\omega))| < 1\}$. Let μ_τ be the law of $X_\tau(\omega) \in G$ and observe that $m_\tau = \int \log |a(g)| d\mu_\tau(g) < 0$. Also, using the fact that for $t > 0$,

$$\mathbb{P}\{\tau > t\} \leq ct^{-\frac{1}{2}}$$

with $c > 0$ which follows from fluctuation theory of random walks on \mathbb{R} (cf. [9]), we get that

$$\mathbb{E}(|\log(|a(X_\tau)|)| + |\log(|b(X_\tau)|)|) < +\infty.$$

Then we know from the above examples that there exists a unique μ_τ -stationary probability ν_τ on E . We denote by $X_{\tau,n}(\omega)$ the random walk on G defined by μ_τ , by P^τ the corresponding Markov operator on E . Then τ is a stopping time and $X_{\tau,n}(\omega)$ is a subprocess of $X_n(\omega)$. Since property R^a is valid for (E, P^τ) , it is also valid for (E, P) .

Let U be a relatively compact open subset of E with $\nu_\tau(U) > 0$. Then, from the examples considered above, we know that, for any $x \in U$, \mathbb{P} -a.e

$X_{\tau,n}(\omega)x \in U$ for some $n = n(\omega, x) \in \mathbb{N}$. Hence, also $X_n(\omega)x \in U$ for some $n \in \mathbb{N}$. Using the remark following definition of property R , we get that property R is valid for (E, P, λ) . The uniqueness property of P -invariant Radon measures implies the extremality of λ . Hence ergodicity of $(\hat{\Omega} \times E, \hat{\theta}, \hat{\lambda})$ follows from Corollary 3.

(3) **Fibered random walks:** We consider below two different situations with $E = G/\Delta'$ and Δ' is a normal subgroup of a closed subgroup Δ of G such that $G/\Delta = \overline{E}$ is compact and Δ/Δ' is isomorphic to $Z = \mathbb{R}$ or \mathbb{Z} . Here we give a general condition of recurrence for these choices which will be applied below in two different situations. Since Δ' is normal in Δ , $Z = \Delta/\Delta'$ acts on the right on $E = G/\Delta'$, this action commutes with the G -action and the corresponding factor space of E is \overline{E} . The action of $z \in Z$ on $y \in G/\Delta'$ will be denote by $y \cdot z$. Also we denote by \overline{P} the convolution operator on \overline{E} defined by μ , and we consider $\rho \in M^1(\overline{E})$ with $\overline{P}\rho = \rho$ and ρ is extremal with respect to this property. Then we consider the space $\Omega \times \overline{E}$ and the map $\overline{\theta}$ with $\overline{\theta}(\omega, x) = (\theta\omega, Y_1(\omega)x)$ if $(\omega, x) \in \Omega \times \overline{E}$ and θ is the shift on Ω . We endow $\Omega \times \overline{E}$ with the measure $\mathbb{P} \otimes \rho$ and we observe that $\mathbb{P} \otimes \rho$ is $\overline{\theta}$ -invariant and ergodic. We fix a Borel fundamental domain D of Z -action on E which is relatively compact and E is Borel isomorphic to $D \times Z$. We denote by l the measure on Z which is Lebesgue if $Z = \mathbb{R}$ or counting if $Z = \mathbb{Z}$. Then we can identify $\rho \otimes l$ with a Radon measure λ on E which satisfies $P\lambda = \lambda$. We define a Borel function $z(y)$ on E by $y = \overline{y} \cdot z(y)$ where $\overline{y} \in D$, $z(y) \in Z$. Also we write $z(g, \overline{x}) = z(gx)$ if $g \in G$ and $x \in D$ corresponds to $\overline{x} \in \overline{E}$. Then we have the cocycle relation

$$z(gh, \overline{x}) = z(g, h\overline{x}) + z(h, \overline{x})$$

for $g, h \in G$ and $\overline{x} \in \overline{E}$. It follows that for $g_i \in G$ ($1 \leq i \leq n$) and $\overline{x} \in \overline{E}$,

$$z(g_n \cdots g_1, \overline{x}) = \sum_1^n z(g_k, g_{k-1} \cdots g_1 \overline{x}).$$

If we write $f(\omega, \overline{x}) = z(Y_1(\omega), \overline{x})$, $S_n(\omega, \overline{x}) = z(Y_n \cdots Y_1, \overline{x})$ with $\omega \in \Omega$ and $\overline{x} \in \overline{E}$, we get $S_n(\omega, \overline{x}) = \sum_0^{n-1} f(\theta^k(\omega, x))$. With the above identification, we write $x = (\overline{x}, t) \in \overline{E} \times Z$ and $X_n(\omega)x = (X_n(\omega)\overline{x}, t + S_n(\omega, \overline{x}))$ where $X_n(\omega)\overline{x}$ is the random walk on \overline{E} defined by μ . We will also consider the map $\tilde{\theta}$ on $\Omega \times E$ defined by $\tilde{\theta}(\omega, y) = (\theta\omega, Y_1(\omega)y)$ and $\Omega \times E$ will be endowed with the $\tilde{\theta}$ -invariant measure $\mathbb{P} \otimes \lambda$. This is a skew-product as in [24] and [44]

with the base $\Omega \times \overline{E}$, the fiber Z and the function $f(\omega, x)$. We refer to [24] for a discussion of ergodic properties of such systems in case ρ is G -invariant. Since D is relatively compact for $g \in G$, $c(g) = \sup_{x \in D} |z(gx)| < +\infty$. Also $c(gh) \leq c(g) + c(h)$. In particular, if G is a closed subgroup of a linear group, it follows that for some $c > 0$, $c(g) \leq c(\log \|g\| + 1)$. Also it is easy to show that if (D, c) is replaced by (D', c') for another fundamental domain D' , then there exist constants $d, d' \geq 0$ such that $c'(g) \leq dc(g) + d'$. Thus, it follows for $\mu \in M^1(G)$ that the condition $\int c(g)d\mu(g) < +\infty$ is independent of (D, c) . It is also evident that if $\int c(g)d\mu(g) < +\infty$, then we get the $\mathbb{P} \otimes \rho$ -integrability of $f(\omega, \overline{x})$, hence we can apply ergodic theorems to the Birkhoff sum $S_n(\omega, \overline{x})$. Also in this case we can describe the measure $\hat{\lambda}$ on $\hat{\Omega} \times E$ in terms of $\lim_{n \rightarrow \infty} Y_{-n}(\hat{\omega}) \cdots Y_0(\hat{\omega})\rho = \rho_{\hat{\omega}}$ which is the limit of the bounded martingale $Y_{-n}(\hat{\omega}) \cdots Y_0(\hat{\omega})\rho$.

Proposition 11 *With the above notations, assume that $\mu \in M^1(G)$ is such that $\int c(g)d\mu(g) < +\infty$. Then, if $\int z(gx)d\mu(g)d\rho(\overline{x}) = 0$, the random walk on G/Δ' defined by μ satisfies property R with respect to $\lambda = \rho \otimes l$. We have $\hat{\lambda} = \int \delta_{\hat{\omega}} \otimes l_{\hat{\omega}} d\hat{\mathbb{P}}(\hat{\omega})$, where $\rho_{\hat{\omega}} = \lim_{n \rightarrow \infty} Y_{-n}(\hat{\omega}) \cdots Y_0(\hat{\omega})\rho$ and $l_{\hat{\omega}} = \rho_{\hat{\omega}} \otimes l$. If ρ is G_μ -invariant and μ is symmetric, then the condition*

$$\int z(gx)d\mu(g)d\rho(x) = 0$$

is satisfied.

Proof Let $I \subset Z$ be an open symmetric interval. We write $x = (\overline{x}, t)$ with $\overline{x} \in D, t \in I, X_n(\omega)x = (X_n(\omega)\overline{x}, t + S_n(\omega, \overline{x}))$. Since $\int f(\omega, \overline{x})d\mathbb{P}(\omega)d\rho(\overline{x}) = 0$, the Birkhoff sum $S_n(\omega, x)$ belongs to I infinitely often $\mathbb{P} \otimes \rho$ -a.e (see [44]). It follows that for any $t \in I$ and $\mathbb{P} \otimes \rho$ -a.e $X_n(\omega)x \in D \times I$ infinitely often. Since the actions of G and Δ/Δ' commute, the same property is true for $s \in Z$ with x replaced by $x \cdot s$ and I by $I + s$. Then using Poincaré recurrence theorem, for $D \times 1_I$ and $\rho \otimes (1_I l)$, property R with respect to λ follows. We observe that the structure of skew product implies $g(\rho \otimes l) = (g\rho) \otimes l$ if $g \in G$, hence for any $\psi \in C_c(E)$, $Y_{-n}(\hat{\omega}) \cdots Y_0(\hat{\omega})\lambda(\psi) = (Y_{-n}(\hat{\omega}) \cdots Y_0(\hat{\omega})\rho) \otimes l(\psi)$ is a bounded martingale which converges to $\rho_{\hat{\omega}} \otimes l(\psi)$. Hence from remark 4 we get that $\hat{\lambda} = \int \delta_{\hat{\omega}} \otimes l_{\hat{\omega}} d\hat{\mathbb{P}}(\hat{\omega})$.

The cocycle property of $z(g, \overline{x})$ gives $z(g, \overline{x}) = -z(g^{-1}, g\overline{x})$ on $G \times \overline{E}$. Then $\int z(g, \overline{x})d\mu(g)d\rho(\overline{x}) = -\int z(g^{-1}, g\overline{x})d\mu(g)d\rho(\overline{x})$. The G_μ -invariance of ρ and the symmetry of μ gives $\int z(g, \overline{x})d\mu(g)d\rho(\overline{x}) = -\int z(g, \overline{x})d\mu(g)d\rho(\overline{x})$. Thus proving the last assertion.

(4) **Pointed vector spaces:** Let $G = GL(d, \mathbb{R})$ with $d \geq 2$. We assume that $\mu \in M^1(G)$ satisfies the following conditions (H-1)-(H-3).

(H-1) no finite union of proper subspaces of \mathbb{R}^d is invariant under the support of μ , that is, G_μ is strongly irreducible on \mathbb{R}^d .

(H-2) G_μ contains an element g that has unique dominant eigenvalue λ_g with $|\lambda_g| = \lim_{n \rightarrow \infty} \|g^n\|^{\frac{1}{n}}$.

(H-3) the support of μ is compact.

Here the space E will be the factor space $V = (\mathbb{R}^d)^*/\{\pm Id\}$ considered as a G -homogeneous space where $(\mathbb{R}^d)^* = \{v \in \mathbb{R}^d \mid v \neq 0\}$. We denote by e the point of V corresponding to the first basis vector, by Δ' the stabilizer of e and Δ the stabilizer of the line $\mathbb{R}_+^* \subset V$ containing e . Then Δ' is normal in Δ with $\Delta/\Delta' \simeq \mathbb{R}_+^*$ and $E = V = G/\Delta'$, $\bar{E} = G/\Delta \simeq \mathbb{P}^{d-1}$, the projective space. Also $v \in V$ can be written as $v = (x, r)$ with $r \in \mathbb{R}_+^*$ and $x \in \mathbb{P}^{d-1}$ so that $V = \mathbb{P}^{d-1} \times \mathbb{R}_+^*$ and \mathbb{P}^{d-1} can be considered as a fundamental domain of the action of $\mathbb{R}_+^* \simeq \Delta/\Delta'$ on V by dilations. We identify $\mathbb{P}^{d-1} \subset V$ with the unit sphere modulo symmetry. In particular, if $g \in G$ and $x \in V$, the quantity $\|gx\|$ is well defined and gives a cocycle on $G \times \mathbb{P}^{d-1}$. Also, if $x, x' \in \mathbb{P}^{d-1}$, then $\bar{\delta}(x, x') = |\sin(x, x')|$ defines a distance on \mathbb{P}^{d-1} . We denote by H_ϵ the space of ϵ -Holder functions on \mathbb{P}^{d-1} endowed with the norm

$$\|\phi\|_\epsilon = \sup_{x \in \mathbb{P}^{d-1}} |\phi(x)| + \sup_{x, x' \in \mathbb{P}^{d-1}} \frac{|\phi(x) - \phi(x')|}{\bar{\delta}^\epsilon(x, x')}.$$

Then under condition (H) for ϵ sufficiently small, the operator \bar{P} on \mathbb{P}^{d-1} associated with μ has nice spectral properties on H_ϵ (cf. [27]). In particular, \bar{P} has a unique stationary measure $\rho = \nu$ on \mathbb{P}^{d-1} , and \bar{P} has a spectral gap on H_ϵ , i.e \bar{P} is the sum of a one-dimensional projection π and another operator Q with spectral radius less than one which satisfies $Q\pi = \pi Q = 0$. We need to consider also the family of operators P_t ($t \in \mathbb{R}$) defined by $P_t\phi(x) = \int \|gx\|^{it} \phi(gx) d\mu(g)$; we refer to [27] for a recent exposition of the spectral properties of the operators P_t . In particular, if $t \neq 0$, P_t has a spectral radius less than 1, and for t small one has $P_t = k(t)\pi_t + R_t$ where $k(t) \in \mathbb{C}$, π_t is a projection of rank one, R_t has a spectral radius less than $|k(t)|$ and commutes with π_t . All quantities depend analytically of t , and if $\int \log \|gx\| d\mu(g) d\nu(x) = 0$, then $k'(0) = 0$ and $\sigma^2 = -k''(0) > 0$ (see [7]).

We denote by l the Lebesgue measure $\frac{dr}{r}$ on \mathbb{R}_+^* , we observe that the Radon measure $\lambda = \nu \otimes l$ on $E = V$ is P -invariant and we have the following.

Proposition 12 *Assume $\mu \in M^1(G)$ satisfies conditions (H-1)-(H-3) and $\int \log \|gx\| d\mu(g) d\nu(x) = 0$. Then there exists a $\sigma > 0$ such that for any $\psi \in C_c(V)$ and any $v \in V = (\mathbb{R}^d)^*/\{\pm Id\}$,*

$$\lim_{n \rightarrow \infty} \sigma \sqrt{2\pi n} P^n \psi(v) = (\nu \otimes l)(\psi).$$

In particular, for any $u \in C_c^+(V)$ with $(\nu \otimes l)(u) > 0$, $\sum_0^\infty P^k u = +\infty$ on V . Furthermore property R^a is valid, $\nu \otimes l$ is P -invariant extremal, hence $(\hat{\Omega} \times V, \hat{\theta}, \hat{\lambda})$ is ergodic.

Proof A general formula as in the proposition was proved in [32], under the assumption that P has spectral gap and the spectral radius of P_t ($t \neq 0$) is less than one, which is valid in our situation (cf. [27]). Hence, here we only sketch the proof. For fixed $v \in V$, we consider the sequence of Radon measures μ_n defined by $\mu_n(\psi) = \sigma \sqrt{2\pi n} P^n \psi(v)$. It suffices to prove the convergence of μ_n to $\nu \otimes l(\psi)$ for functions of the form $\psi(x, r) = \phi(x) f(r)$ where $\phi \in H_\epsilon$ and $f \in L^1(\mathbb{R}_+^*)$ is such that its Fourier transform \hat{f} has compact support. Then we can write, using Fourier inversion formula $f(r) = \frac{1}{2\pi} \int \hat{f}(t) r^{-it} dt$ where $\hat{f}(t) = \int_{\mathbb{R}_+^*} r^{it} f(r) \frac{dr}{r}$. Then $P^n \psi(x, r) = \frac{1}{2\pi} \int P_t^n \phi(x) r^{-it} \hat{f}(t) dt$. Replacing t by $\frac{t}{\sqrt{n}}$ we get that

$$\mu_n(\psi) = \frac{\sigma}{\sqrt{2\pi}} \int P_{\frac{t}{\sqrt{n}}}^n \phi(x) r^{-i\frac{t}{\sqrt{n}}} \hat{f}\left(\frac{t}{\sqrt{n}}\right) dt.$$

Using the spectral properties of P_t we can replace $P_{\frac{t}{\sqrt{n}}}^n \phi(x)$ by $k^n\left(\frac{t}{\sqrt{n}}\right) \pi_{\frac{t}{\sqrt{n}}} \phi$ which converges to $e^{\frac{-\sigma^2 t^2}{2}} \nu(\phi)$. Then (see [32]) we get that

$$\lim \mu_n(\psi) = \nu(\phi) \hat{f}(0) \frac{\sigma}{\sqrt{2\pi}} \int e^{\frac{-\sigma^2 t^2}{2}} dt = (\nu \otimes l)(\psi).$$

In order to show property R^a , we use a result of [8]. Under conditions (H), for any $v \in V$ we have \mathbb{P} -a.e

$$\limsup_{n \rightarrow \infty} \|X_n(\omega)v\| = \infty, \quad \liminf_{n \rightarrow \infty} \|X_n(\omega)v\| = 0.$$

Since the support of μ is compact, there exists $c > 0$ such that for any $(\omega, v) \in \Omega \times V$, $n \in \mathbb{N}$,

$$\frac{1}{c} \leq \frac{\|X_{n+1}(\omega)v\|}{\|X_n(\omega)v\|} < c.$$

Then, for any v , the relatively compact open set

$$U_c = \{u \in V \mid \frac{1}{c} < \|u\| < c\}$$

is visited infinitely often \mathbb{P} -a.e. Hence property R^a is valid. The validity of property R follows from Proposition 11.

The extremality of $\lambda = \nu \otimes l$ is proved in Proposition 3.5 of [27], hence the ergodicity of $(\hat{\Omega} \times V, \hat{\theta}, \hat{\lambda})$ follows from Corollary 3.

Remark 7 In general, the support of ν in \mathbb{P}^{d-1} is of Cantor type, hence the support of $\nu \otimes l$ is a proper subset of V . If $u \in C_c^+(V)$ with $(\nu \otimes l)(u) = 0$, we can show that \mathbb{P} -a.e, for any v , $\sum_0^\infty u(X_n(\omega)v) < +\infty$, in particular, if K is a compact set that does not intersect the support of $\nu \otimes l$, then \mathbb{P} -a.e the random walk $X_n(\omega)v$ escapes from K , if v is not in the support of $\nu \otimes l$.

(5) **Covering spaces:** Let G be a real simple linear group of rank one, Δ be a cocompact lattice and Δ' be a normal subgroup in Δ such that $\Delta/\Delta' \simeq \mathbb{Z}$. It may be noted that if G has rank more than one, then G has Kazhdan property T which implies that such a Δ' does not exist. Let $E = G/\Delta'$, $\bar{E} = G/\Delta$ and D be a fundamental domain of the \mathbb{Z} -action on E as in (3). Let m (resp. \bar{m}) denote the Haar measure on E (resp. \bar{E}), hence with the notations of (3) $\rho = \bar{m}$ and $m = \rho \otimes l$. Such a situation arises if $G = SL(2, \mathbb{R})$ and Δ is the fundamental group of a compact Riemann surface S and in this case G/Δ (resp. G/Δ') can be identified with the tangent unit bundle of S (resp. of an abelian cover of S).

Proposition 13 *Assume $\mu \in M^1(G)$ is symmetric with $\int \log \|g\| d\mu(g) < +\infty$ and G_μ is non-amenable. Then $(\hat{\Omega} \times E, \hat{\theta}, \hat{\mathbb{P}} \otimes m)$ is ergodic.*

The proof depends on the next two lemmas.

Lemma 5 *Suppose $\mu \in M^1(G)$ is such that G_μ is non-amenable. Let α be a non-trivial character of Δ and T_α be the unitary representation of G induced by α . Then the spectral radius of $T_\alpha(\mu)$ is less than one.*

Proof In view of Theorem C of [45], it suffices to verify that T_α does not weakly contain the trivial one-dimensional representation of G . It is clear that the one-dimensional representation of Δ defined by α does not weakly contain the trivial one-dimensional representation of Δ . Since Δ is a cocompact lattice in G , it follows from 1.10 and 1.11, Chapter III of [37] that the induced representation T_α does not weakly contain the trivial one-dimensional representation of G .

If $u \in L^2(\overline{E}, \overline{m})$, $f \in l^2(\mathbb{Z})$, we can identify $u \otimes f$ with an element of $L^2(E, m)$, again denoted by $u \otimes f$. If $f \in l^1(\mathbb{Z})$ with $\sum f(k) = 0$, we write $f \in l_0^1(\mathbb{Z})$. In Proposition 3.6 of [27], if X is a compact metric space, a Markov operator Q acting on $Y = X \times \mathbb{R}^d$ which commutes with the \mathbb{R}^d translations was considered and it was proved that $\lim_{n \rightarrow \infty} \|Q^n(u \otimes f)\|_1 = 0$ for Holder functions $u \in H_\epsilon(X)$ and $f \in L^1(\mathbb{R}^d)$ with $\int f(x)dx = 0$. Essential points of the proof are polynomial growth of \mathbb{R}^d and a spectral gap property for the Q -action on functions of the form $u \otimes \alpha$ where $u \in H_\epsilon(X)$ and α is a given character of \mathbb{R}^d (see above).

Here we observe that the adjoint P^* of P in $L^2(E, m)$ is associated with $\check{\mu}$, the symmetric of μ which has the same properties as μ . Also, in view of Lemma 5, the condition of spectral gap of P^* is valid for the P^* -action on functions of the form $u \otimes \alpha$ where $u \in L^2(\overline{E}, \overline{m})$ and α is a fixed character of \mathbb{Z} . The polynomial growth condition is also satisfied here. Hence the proof in [27] gives the following.

Lemma 6 *Assume μ satisfies the condition in Lemma 5. Then for any $u \in L^2(\overline{E}, \overline{m})$ and $f \in l_0^1(\mathbb{Z})$, we have $\lim_{n \rightarrow \infty} \|P^{*n}(u \otimes f)\|_1 = 0$.*

Proof of Proposition 13 We begin by showing that if $h \in L^\infty(E, m)$ satisfies $Ph = h$, then h is constant m -a.e. If $u \in L^2(\overline{E}, \overline{m})$ and $f \in l_0^1(\mathbb{Z})$, then $\langle (P^*)^n(u \otimes f), h \rangle = \langle u \otimes f, h \rangle$. By Lemma 6, $\langle u \otimes f, h \rangle = 0$. Since f is arbitrary in $l_0^1(\mathbb{Z})$ this implies the \mathbb{Z} -invariance of h . Hence h defines an element $\overline{h} \in L^\infty(\overline{E}, \overline{m})$ such that $\overline{P}(\overline{h}) = \overline{h}$. This equation can be written as

$$\overline{P}(\overline{h})(\overline{x}) = \int \overline{h}(g\overline{x})d\mu(g) = \overline{h}(\overline{x}).$$

Since \overline{m} is G -invariant and $\overline{h} \in L^2(\overline{E}, \overline{m})$, strict convexity in $L^2(\overline{E}, \overline{m})$ implies that $\overline{h}(\overline{x}) = \overline{h}(g\overline{x})$ $\mu \otimes \overline{m}$ -a.e. Since G_μ is non-amenable, it is unbounded, hence the Moore ergodicity theorem implies the ergodicity of G_μ on $\overline{E} = G/\Delta$

with respect to \bar{m} (cf. Theorem 4 of [38]), hence \bar{h} is constant \bar{m} -a.e. This proves that h is constant m -a.e.

The above argument shows also the extremality of $\rho = \bar{m}$ as a \bar{P} -stationary measure. The condition $\int \log \|g\| d\mu(g) < +\infty$ implies that the condition $\int c(g) d\mu(g) < +\infty$ of Proposition 11 is satisfied. Then Proposition 11 gives that property R is valid with respect to $m = \bar{m} \otimes l$. Then as above, we can use Proposition 5 to obtain the required ergodicity.

Remark 8 In the situation of Proposition 13, a counter example to property R^a is the following. Assume $G_\mu = \Delta'$. Since Δ is non-amenable, the same is true of Δ' and all the conditions on μ can be seen to be satisfied. Since Δ' is normal in Δ , we observe that if $e \in E$ corresponds to Δ' and $z \in \Delta$, then $e \cdot z$ is Δ' -invariant. It follows that $P(e \cdot z, \cdot) = \delta_{e \cdot z}$ and if $u(e \cdot z) = 0$, that is $x = e \cdot z$ is not in the support of u , then $X_n(\omega)x$ is not in the support of u for any n and a.e ω . We conjecture that property R_x is valid for any x and if μ is adapted, then property R^a is valid.

(6) **Transient behavior on pseudo-Riemannian symmetric spaces:** Let G be a semisimple real algebraic group with no compact factors and H be a closed subgroup of G that is not Zariski dense in G . Assume that $E = G/H$ has a G -invariant measure m and $\mu \in M^1(G)$ is such that G_μ is non-amenable. Then P acts on $L^2(E, m)$ as a contraction. As an extension of the Borel density theorem it is proved in [23] that, since H is not Zariski-dense, G/H do not carry an invariant mean, that is, G/H is not amenable in the sense of Eymard (see also [46]). In particular, the spectral radius of P on $L^2(E, m)$ is strictly less than one. Hence for $\phi \in C_c(E)$, one has $\sum_0^\infty P^k \phi \in L^2(E, m)$. In particular $\sum_0^\infty P^k \phi$ is finite m -a.e. This implies property T_x is valid m -a.e.

We consider the case where the homogeneous space G/H is a symmetric pseudo-Riemann space, that is H is the set of fixed points of an involution. Then H is reductive, hence H is unimodular. Thus, G/H has a G -invariant measure m . Also, H is algebraic, hence not Zariski dense. Then the above discussion gives the following.

Proposition 14 *Assume $E = G/H$ is a pseudo-Riemannian symmetric space, $\mu \in M^1(G)$ is such that G_μ is non-amenable. Then for $\phi \in C_c(E)$, $\sum_0^\infty P^k \phi$ is finite m -a.e. In particular, T_x is valid m -a.e.*

Among the pseudo-Riemannian symmetric spaces we have the spaces $E =$

$SL(n, \mathbb{R})/SO(p, q)$ ($p + q < n$) and $E = SO(k, l)/SO(p, q)$ ($p + q < k + l$). The space $SL(2, \mathbb{C})/SL(2, \mathbb{R})$ locally isomorphic to $SO(3, 1)/SO(2, 1)$ was considered in [23].

(7) **On homogeneous spaces with property R** : Considering the above examples, we formulate the following questions for a G -homogeneous space with G a connected Lie group.

a) Characterize the systems (E, P, λ) with property R .

More precisely, if R is valid is it true that λ has at most quadratic growth in the following sense: if $V \subset G$ is a compact neighborhood of e , and x in the support of λ does there exist $c > 0$ such that, for any $n \in \mathbb{N}$, $\lambda(V^n x) \leq cn^2$. This property is valid in examples 2,4,5 with linear growth of $\lambda(V^n x)$. It may be noted that when P is given by spread-out probability on G , question (a) has definitive answer if G is a compact extension of simply connected nilpotent real Lie group [18] or if G is a p -adic algebraic group [42].

In view of Corollary 3, if λ is extremal, property R is equivalent to ergodicity of $(\hat{\Omega} \times E, \hat{\theta}, \hat{\lambda})$.

b) Characterize systems (E, P, λ) with λ of finite mass.

Examples of this situation are given in (1). Except for trivial situations, in the general case a combination of these kind of examples.

6 Singularity of stationary measures on the projective line

As is well known the group $G = SL(2, \mathbb{R})$ acts by fractional linear transformations on the Poincaré upper half plane $\mathbb{H} = \{z \in \mathbb{C} \mid y = \text{Im}z > 0\}$, preserving the Poincaré metric $\frac{|dz|^2}{y^2}$. If $a = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$, the group $\Gamma = \langle a, b \rangle$ is a free subgroup of index 6 in $SL(2, \mathbb{Z}) \subset G$. Furthermore the Γ -action on \mathbb{H} is free and totally discontinuous. We denote by Γ' the commutator subgroup of Γ , by $\bar{\gamma}$ the projection of $\gamma \in \Gamma$ in $\bar{\Gamma} = \Gamma/\Gamma' = \langle \bar{a}, \bar{b} \rangle = \mathbb{Z}^2$, and $|\bar{\gamma}|$ the word length of $\bar{\gamma}$ in \bar{a}, \bar{b} . We observe that as a Riemann surface, $\Gamma \backslash \mathbb{H}$ can be identified with the complement of $\{0, 1\}$ in \mathbb{C} ; also $\Gamma' \backslash \mathbb{H}$ is an abelian cover for $\Gamma \backslash \mathbb{H}$ with covering group $\mathbb{Z}^2 = \langle \bar{a}, \bar{b} \rangle$. Then $\Gamma \backslash G$ (resp. $\Gamma' \backslash G$) can be identified with the unit tangent bundle of $\Gamma \backslash \mathbb{H}$ (resp. $\Gamma' \backslash \mathbb{H}$).

Let MAN be the triangular subgroup of matrices of the form $\begin{pmatrix} a & b \\ 0 & \frac{1}{a} \end{pmatrix}$

($a \in \mathbb{R}^*$ and $b \in \mathbb{R}$). Here $M = \{\pm Id\}$, $A = \left\{ \begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix} \mid a > 0 \right\}$ and $N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in \mathbb{R} \right\}$. Then $G/MAN \simeq \mathbb{P}^1$ can be identified with the boundary of \mathbb{H} in \mathbb{C} . If $\mu \in M^1(\Gamma)$ is adapted, there exists a unique μ -stationary measure ν on $\mathbb{P}^1 = G/MAN$: $\mu * \nu = \sum_{\gamma \in \Gamma} \mu(\gamma)\gamma\nu = \nu$. It is known (see for instance [16] Proposition 4.1, p.207) that Lebesgue measure m on \mathbb{P}^1 is such a measure for a certain μ as above. From a geometrical point of view, such measures appear in the study of foliated Brownian motion on $\Gamma \backslash G$ along AN -orbits and corresponding harmonic measures. Here, as an application of recurrence-transience properties of random walk on $\Gamma \backslash G$ we give a proof of the following (mentioned in [25]; see [6] and [11] for recent different proofs).

Proposition 15 *Assume that $\mu \in M^1(\Gamma)$ is symmetric adapted and satisfies $\sum_{\gamma \in \Gamma} \mu(\gamma)|\bar{\gamma}|^2 < +\infty$. Then the unique μ -stationary measure on \mathbb{P}^1 is singular with respect to Lebesgue measure.*

The proof depends on the next two results which are of independent interest.

Lemma 7 *The action of Γ' on $(\mathbb{P}^1 \times \mathbb{P}^1, m \otimes m)$ is not ergodic.*

Proof The Lemma is a direct consequence of the results of [47] and [35]. From [35] we know that Brownian motion on $\Gamma' \backslash \mathbb{H}$ is transient. Then it follows from Theorem 4 of [47] that the geodesic flow on the manifold $\Gamma' \backslash \mathbb{H}$ is not ergodic. In other words the action of A on $\Gamma' \backslash G$ endowed with the Haar measure is not ergodic. By duality this means that the action of Γ' on G/A endowed with Haar measure is not ergodic. Now the result follows since G/AM and $(\mathbb{P}^1 \times \mathbb{P}^1, m \otimes m)$ are isomorphic as measured G -spaces.

Lemma 8 *The action of Γ' on $(\mathbb{P}^1 \times \mathbb{P}^1, \nu \otimes \nu)$ is ergodic.*

Proof As in 2.3, we denote $\hat{\Omega} = \Omega^- \times \Omega$, $\hat{\mathbb{P}} = \mathbb{P}^- \otimes \mathbb{P}$, we write $\hat{\Omega} = (\omega^-, \omega)$ if $\hat{\omega} \in \hat{\Omega}$. Then $\theta\hat{\omega} = (\omega^- Y_1(\omega), \theta\omega)$. We consider the transformation $\hat{\theta}$ of $\hat{\Omega} \times \Gamma$ defined by $\hat{\theta}(\hat{\omega}, \gamma) = (\theta\hat{\omega}, \gamma Y_1(\omega))$. We denote also by $\hat{\theta}$ the transformation of $\hat{\Omega} \times \bar{\Gamma}$ defined by $\hat{\theta}(\hat{\omega}, \bar{\gamma}) = (\theta\hat{\omega}, \bar{\gamma} \overline{Y_1(\omega)})$. Since the random walk on \mathbb{Z}^2 associated with μ is recurrent and adapted the skew product $(\hat{\Omega} \times \bar{\Gamma}, \hat{\theta}, \hat{\mathbb{P}} \otimes \lambda_{\bar{\Gamma}})$ is ergodic (see Corollary 2). If we denote by $z(\omega)$ the random variable on

\mathbb{P}^1 uniquely defined \mathbb{P} -a.e by $Y_1(\omega)z(\theta\omega) = z(\omega)$ we have, \mathbb{P} -a.e: $\delta_{z(\omega)} = \lim_n Y_1 \cdots Y_n \nu$ and ν is the law of $z(\omega)$. Since μ is symmetric, ν is also the law under \mathbb{P}^- of $z'(\omega^-)$ where $\delta_{z'(\omega^-)} = \lim_n Y_0^{-1}(\omega) \cdots Y_n^{-1}(\omega) \nu$ and $\mu \otimes \mathbb{P}^-$ -a.e: $z'(\omega^- Y_1(\omega)) = Y_1^{-1}(\omega) z'(\omega^-)$. Let ϕ be a Borel function on $\mathbb{P}^1 \times \mathbb{P}^1$ and denote $f(\hat{\omega}, \gamma) = \phi(\gamma z(\omega), \gamma z'(\omega^-))$. Then the functional equations above implies that $f(\theta\hat{\omega}, \gamma Y_1(\omega)) = f(\hat{\omega}, \gamma)$. Also, if ϕ is Γ' -invariant $f(\hat{\omega}, \eta\gamma) = f(\omega, \gamma)$ for all $\eta \in \Gamma'$. Then we can write $f(\hat{\omega}, \gamma) = \bar{f}(\hat{\omega}, \bar{\gamma})$ where \bar{f} is a $\hat{\theta}$ -invariant function on $\hat{\Omega} \times \bar{\Gamma}$. From above we know that \bar{f} is constant $\mathbb{P}^- \otimes \mathbb{P}$ -a.e. Since the law of $(z'(\omega^-), z(\omega))$ under $\mathbb{P}^- \otimes \mathbb{P}$ is $\nu \otimes \nu$, this means that ϕ is constant $\nu \otimes \nu$ -a.e. In other words $(\mathbb{P}^1 \times \mathbb{P}^1, \nu \otimes \nu)$ is Γ' -ergodic.

Proof of Proposition 15 Since $\Gamma \backslash G$ has finite volume, the geodesic flow on $\Gamma \backslash G$ is ergodic with respect to the Haar measure and the action of A on $\Gamma \backslash G$ is ergodic. By duality this gives the ergodicity of Γ on $(\mathbb{P}^1, m) = (G/MAN, m)$. In particular, any Γ -quasiinvariant and absolutely continuous measure on \mathbb{P}^1 is equivalent to m . If ν is not singular with respect to m , we can write $\nu = \nu_a + \nu_s$ where $\nu_a \neq 0$ is absolutely continuous and ν_s is singular (with respect to m). The equation $\nu = \mu * \nu$ implies $\mu * \nu_a = \nu_a$ and $\mu * \nu_s = \nu_s$, hence by uniqueness of the μ -stationary measure, we have $\nu_a = \nu$ and $\nu_s = 0$. From above, the Γ -quasiinvariance of $\nu = \nu_a$ implies that ν is equivalent to m . Then the two lemmas give the required contradiction.

7 Appendix: Groups whose closed subgroups are unimodular

Groups whose closed subgroups are unimodular plays a crucial role in our proof of quadratic growth conjecture. This motivates us to prove the following characterization of such groups among algebraic groups and almost connected locally compact groups in terms of polynomial growth.

Proposition 16 *Let G be either the group of \mathbb{F} -rational points of an algebraic group defined over a non-archimedean local field \mathbb{F} of characteristic zero or an almost connected locally compact group. Then closed subgroups of G are unimodular if and only if G has polynomial growth. Furthermore, in the first case, any compactly generated closed subgroup H of G has a basis of compact open normal subgroups (K_n) such that H/K_n has a finitely generated abelian subgroup of finite index.*

Proof If G is a Lie group over a local field, then $\text{Ad}(g)$ denotes the adjoint automorphism of the Lie algebra \mathcal{G} of G defined by g .

We first consider the case of group of \mathbb{F} -rational points of an algebraic group defined over a non-archimedean local field \mathbb{F} of characteristic zero. Since the Zariski-connected component is a subgroup of finite index, we may assume that G is Zariski-connected. Assume that closed subgroups of G are unimodular. Let H be a open compactly generated subgroup of G . Then by Corollary 7, H contains a basis (K_n) of compact open normal subgroups. On the other hand, since $\text{Ad}(G)$ is also an algebraic group, $\text{Ad}(G)$ is closed in $GL(\mathcal{G})$ and is isomorphic to G/Z where Z is the center of G ; as characteristic of \mathbb{F} is zero, center of G is the kernel of the adjoint homomorphism of G (ref. 0.15 and 0.24 of [37]). Since H is open in G , $\text{Ad}(H)$ is open in $\text{Ad}(G)$ and hence closed in $GL(\mathcal{G})$. Since the orbits of $\text{Ad}(H)$ in \mathcal{G} are relatively compact, $\text{Ad}(H)$ is also relatively compact, hence $\text{Ad}(H)$ is compact. Since $\text{Ad}(G) \simeq G/Z$, $Z \cap H$ is co-compact in H . Thus, the center of H is co-compact in H . The same property is valid in the finitely generated subgroup H/K_n . In particular, H as well as G has polynomial growth. Now if H is any compactly generated closed subgroup, let C be a compact generating set of H . Let V be a compact neighborhood of C and N be the subgroup generated by V . Then N has a basis (K_n) of compact open normal subgroup such that N/K_n has a finitely generated abelian subgroup of finite index. Let $L_n = K_n \cap H$. Since H is a closed subgroup of N , L_n is a compact open normal subgroup of H and H/L_n is continuously embedded in N/K_n . Thus, H/L_n also has a finitely generated abelian subgroup of finite index.

We now consider a connected real Lie group. Then for any g in $\text{Ad}(G)$ (with the quotient topology from G), C_g is a simply connected nilpotent Lie subgroup of $GL(\mathcal{G})$ and $C_g \subset \{v \in GL(\mathcal{G}) \mid \lim_{n \rightarrow \infty} g^n v g^{-n} = \text{Id in } GL(\mathcal{G})\} = \tilde{C}_g$, say. But \tilde{C}_g is a unipotent algebraic group. Since C_g is a connected Lie subgroup of \tilde{C}_g which is unipotent, we get that C_g is closed in $\text{Ad}(G)$. By Lemma 2, C_g is trivial. Now, for any $h \in G$, $\text{Ad}(C_h) \subset C_{\text{Ad}(h)} = \{e\}$. This implies that C_h is contained in the kernel of the homomorphism Ad which is the center of G . Thus, C_h is trivial, hence G is a type R Lie group. It follows from [20] that G has polynomial growth.

We now consider any almost connected group. Let G be an almost connected locally compact group. Then there exists a compact normal subgroup K such that G/K is a real Lie group. Let M be the closed subgroup of G containing K such that M/K is the component of identity in G/K . Since G is almost connected, M is a subgroup of finite index. Now any closed sub-

group M is also unimodular and since K is compact, any closed subgroup of M/K is unimodular. Since M/K is a connected Lie group, M/K has polynomial growth. Since K is compact, M has polynomial growth. Since M has finite index in G , G also has polynomial growth.

Converse follows from the facts that closed subgroups of polynomial growth also have polynomial growth and groups of polynomial growth are unimodular (cf. [20]).

References

- [1] H. Abels, Distal automorphism groups of Lie groups, *J. Reine Angew. Math.* 329 (1981), 82–87.
- [2] P. Baldi, Caractérisation des groupes de Lie connexes récurrents, *Ann. Inst. H. Poincaré Sect. B (N.S.)* 17 (1981), 281–308.
- [3] M. Babilot, P. Bougerol and L. Elie, The random difference equation $X_n = A_n X_{n-1} + B_n$ in the critical case, *Ann. Probab.* 25 (1997), 478–493.
- [4] U. Baumgartner and Willis, Contraction groups and scales of automorphisms of totally disconnected locally compact groups, *Israel J. Math.* 142 (2004), 221–248.
- [5] Y. Benoist and J.-F. Quint, Mesures stationnaires et fermés invariants des espaces homogènes, *C. R. Acad. Sci. Paris, Ser. I* 347 (2009), 9–13.
- [6] S. Blachère and P. Mathieu, Harmonic measures versus quasiconformal measures for hyperbolic groups, preprint, Arxiv december 2008.
- [7] P. Bougerol and J. Lacroix, Products of random matrices with applications to Schrödinger operators. *Progress in Probability and Statistics*, 8. Birkhäuser Boston, Inc., Boston, MA, 1985.
- [8] P. Bougerol, Oscillation des produits de matrices aléatoires dont l'exposant de Lyapunov est nul, *Lyapunov exponents (Bremen, 1984)*, 27–36, *Lecture Notes in Math.*, 1186, Springer, Berlin, 1986.
- [9] D. I. Cartwright, V. A. Kaimanovich and W. Woess, Random walks on the affine group of local fields and of homogeneous trees. *Ann. Inst. Fourier (Grenoble)* 44 (1994), 1243–1288.
- [10] J.-P. Conze and Y. Guivarc'h, Remarques sur la distalité dans les espaces vectoriels, *C. R. Acad. Sci. Paris Sér. A* 278 (1974), 1083–1086.
- [11] B. Deroin, V. Kleptsyn and A. Navas, On the question of ergodicity for minimal group action on the circle, Preprint Univ. Rennes 2008.

- [12] R. M. Dudley, Random walks on abelian groups, *Proc. Amer. Math. Soc.* 13 (1962), 447-450.
- [13] A. Eskin and G. Margulis, Recurrence properties of random walks on finite volume homogeneous manifolds, *Random walks and geometry*, 431-444, Walter de Gruyter, Berlin, 2004.
- [14] S. R. Foguel, The ergodic theory of Markov processes. *Van Nostrand Mathematical Studies*, No. 21. Van Nostrand Reinhold Co., New York-Toronto, Ont.-London 1969.
- [15] A. Furman, Random walks on groups and random transformations. *Handbook of dynamical systems*, Vol. 1A, 931-1014, North-Holland, Amsterdam, 2002.
- [16] H. Furstenberg, Boundary theory and stochastic processes on homogeneous spaces. *Harmonic analysis on homogeneous spaces (Proc. Sympos. Pure Math., Vol. XXVI, Williams Coll., Williamstown, Mass., 1972)*, 193-229. Amer. Math. Soc., Providence, R.I., 1973.
- [17] H. Furstenberg, Noncommuting random products. *Trans. Amer. Math. Soc.* 108 (1963), 377-428.
- [18] L. Gallardo and R. Schott, Marches aléatoires sur les espaces homogènes de certains groupes de type rigide. *Conference on Random Walks (Kleebach, 1979)*, pp. 149-170, 4, *Astérisque*, 74, Soc. Math. France, Paris, 1980.
- [19] M. Gromov, Groups of polynomial growth and expanding maps, *Inst. Hautes études Sci. Publ. Math.* No. (1981), 53-73.
- [20] Y. Guivarc'h, Croissance polynomiale et périodes des fonctions harmoniques, *Bull. Soc. Math. France*, 101 (1973), 333-379.
- [21] Y. Guivarc'h and M. Keane, Marches aléatoires transitoires et structure des groupes de Lie. *Symposia Mathematica*, Vol. XXI (Convegno sulle Misure su Gruppi e su Spazi Vettoriali, Convegno sui Gruppi e Anelli Ordinati, INDAM, Rome, 1975), 197-217. Academic Press, London, 1977.
- [22] Y. Guivarc'h, M. Keane and B. Roynette, *Marches aléatoires sur les groupes de Lie*, *Lecture Notes in Mathematics*, 624, Springer-Verlag, Berlin-New York, 1977.
- [23] Y. Guivarc'h, Quelques propriétés asymptotiques des produits de matrices aléatoires. *Eighth Saint Flour Probability Summer School—1978 (Saint Flour, 1978)*, pp. 177-250, *Lecture Notes in Math.*, 774, Springer, Berlin, 1980.
- [24] Y. Guivarc'h, Propriétés ergodiques, en mesure infinie, de certains systèmes dynamiques fibrés, *Ergodic Theory Dynam. Systems* 9 (1989), 433-453.

- [25] Y. Guivarc'h and Y. LeJan, Sur l'enroulement du flot géodésique, *C. R. Acad. Sci. Paris Sr. I Math.* 311 (1990), 645–648.
- [26] Y. Guivarc'h, *Marches aléatoires sur les groupes*, Development of Mathematics 1950–2000, 577–608, Birkhauser, Basel, 2000.
- [27] Y. Guivarc'h and A. N. Starkov, Orbits of linear group actions, random walks on homogeneous spaces and toral automorphisms. *Ergodic Theory Dynam. Systems* 24 (2004), 767–802.
- [28] H. Hennion and B. Roynette, Un théorème de dichotomie pour une marche aléatoire sur un espace homogène. *Conference on Random Walks (Kleebach, 1979)* (French), 99–122, 4, *Astrisque*, 74, Soc. Math. France, Paris, 1980.
- [29] W. Jaworski and C. R. E. Raja, The Choquet-Deny theorem and distal properties of totally disconnected locally compact groups of polynomial growth, *New York J. Math.* 13 (2007), 159–174.
- [30] H. Kesten, The Martin boundary of recurrent random walks on countable groups. 1967 *Proc. Fifth Berkeley Sympos. Math. Statist. and Probability* (Berkeley, Calif., 1965/66) Vol. II: Contributions to Probability Theory, Part 2, pp. 51–74. Univ. California Press, Berkeley, Calif.
- [31] H. Kesten, Random difference equations and renewal theory for products of random matrices, *Acta Math.* 131 (1973), 207–248.
- [32] E. Le Page, Théorèmes limites pour les produits de matrices aléatoires, *Probability measures on groups (Oberwolfach, 1981)*, pp. 258–303, *Lecture Notes in Math.*, 928, Springer, Berlin-New York, 1982.
- [33] M. Lin, Conservative Markov processes on a topological space, *Israel J. Math.* 8 (1970), 165–186.
- [34] V. Losert, On the structure of groups with polynomial growth. *Math. Z.* 195 (1987), 109–117.
- [35] T. J. Lyons and H. P. McKean, Winding of the plane Brownian motion, *Adv. in Math.* 51 (1984), 212–225.
- [36] D. Montgomery and L. Zippin, *Topological transformation groups*, Interscience Publishers, New York-London, 1955.
- [37] G. Margulis, Discrete subgroups of semisimple Lie groups, *Ergebnisse der Mathematik und ihrer Grenzgebiete* (3) 17. Springer-Verlag, Berlin, 1991.
- [38] C. C. Moore, Ergodicity of flows on homogeneous spaces. *Amer. J. Math.* (1966), 154–178.

- [39] A. Parreau, Sous-groupes elliptiques de groupes linéaires sur un corps valué, *J. Lie Theory* 13 (2003), 271–278.
- [40] M. Peigné, Systèmes d’itérations de transformations aléatoires: propriétés de récurrence, Preprint, Université de Tours, 2008.
- [41] C. R. E. Raja, On growth, recurrence and the Choquet-Deny Theorem for p -adic Lie groups, *Math. Z.* 251 (2005), 827–847.
- [42] C. R. E. Raja and R. Schott, Recurrent random walks on homogeneous spaces of p -adic algebraic groups of polynomial growth, *Arch. Math. (Basel)* 91 (2008), 379–384.
- [43] D. Revuz, Markov chains. North-Holland Mathematical Library, Vol. 11. North-Holland Publishing Co., Amsterdam-Oxford; American Elsevier Publishing Co., Inc., New York, 1975.
- [44] K. Schmidt, Cocycles of ergodic transformation groups. Macmillan Lectures in Mathematics, Vol. 1. Macmillan Company of India, Ltd., Delhi, 1977.
- [45] Y. Shalom, Explicit Kazhdan constants for representations of semisimple and arithmetic groups, *Annales de l’institut Fourier*, 50 (2000), 833-863
- [46] G. Stuck, Growth of homogeneous spaces, density of discrete subgroups and Kazhdan’s property (T), *Invent. Math.* 109 (1992), no. 3, 505–517
- [47] D. Sullivan, The density at infinity of a discrete group of hyperbolic motions, *Inst. Hautes tudes Sci. Publ. Math. No. 50* (1979), 171–202.
- [48] N. Th. Varopoulos, L. Saloff-Coste and T. Coulhon, Analysis and geometry on groups, Cambridge Tracts in Mathematics, 100. Cambridge University Press, Cambridge, 1992.
- [49] S. P. Wang, The Mautner phenomenon for p -adic Lie groups, *Math. Z.* 185 (1984), 403–412.
- [50] W. Woess, Random walks on infinite graphs and groups, Cambridge Tracts in Mathematics, 138. Cambridge University Press, Cambridge, 2000.

Y. Guivarc’h
 IRMAR
 Campus de Beaulieu
 Université de Rennes I
 35042 Rennes France
 yves.guivarch@univ-rennes1.fr

C. R. E. Raja
 Stat-Math Unit
 Indian Statistical Institute
 8th Mile Mysore Road
 Bangalore 560059.
 creraja@isibang.ac.in