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Remark on a paper of Sababheh and Khalil

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REMARK ON A PAPER OF SABABHEH AND KHALIL

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Abstract. We show that any Banach space X that has a sequence of unit vectors weakly converging to 0, has a closed and bounded convex set that is not remotal. This extends and gives a correct proof of the main result of [5].

1. Introduction

Let X be a Banach space and let $E \subset X$ be a closed and bounded set. We follow the notation and terminology of [5]. Let $x \in X$, E is said to be remotal at x, if there exists a $e_0 \in E$ such that $D(x, E) = sup\{\|x - e\| : e \in E\} = \|x - e_0\|$. E is said to be remotal (densely remotal) if it is remotal at all (on a dense set) $x \in X$. A well known result of Lau ([3]) says that any weakly compact set is densely remotal. These notions when E is the unit ball of a subspace $Y \subset X$, were recently studied in [1]. Sababheh and Khalil claim in [5] that in an infinite dimensional reflexive space X there is always a closed and bounded convex set E that is not remotal. Their proof depended on Theorem 2.6 in [5], which says that in reflexive spaces if the closed convex hull $\overline{[E]}$ is remotal at x, then so is E. Their argument was based on an incorrect application of Milman's theorem, ([2]page 151) where in they conclude that an extreme point of $\overline{[E]}$ lies in E. Unfortunately Milman's theorem only shows extreme points of $\overline{[E]}$ lie in $\overline{E}$, where the closure is taken in the weak topology. To overcome this difficulty we use the stronger fact that any infinite dimensional reflexive space has sequence of unit vectors weakly converging to 0, i.e., X fails the Schur property. By following different arguments we also extend Lemma 2.4 of [5] by showing that for any weakly compact convex set K, if $F(x, K) \neq \emptyset$, then it has an extreme point of K. As already noted, Lau's result implies that $F(x, K) \neq \emptyset$ on a dense subset of X.

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2. Main result

We first prove an extended version of Lemma 2.4 in [5]. Let K be a weakly compact convex set in X. We recall that $F \subset K$ is an extreme set, if for $x, y \in K$ and $\lambda \in (0,1)$, $\lambda x + (1-\lambda)y \in F$ implies $x, y \in F$. A convex extreme set is called a face. It is easy to see that a set is extreme if and only if it is a union of faces. With this notation it is easy to see that for any $x \in X$, $F(x, K)$ is an extreme set.

We need the following Theorem (Theorem 5.8 from [4]), in the proof of our Lemma.

Theorem 1. *Let K be a compact convex set in a locally convex space Y and let $F \subset K$ be a non-empty set, such that it is a union of faces of K (i.e., an extreme set) and $K \setminus F$ is a countable union of compact convex sets. Then F has an extreme point of K.*

Lemma 2. *Let $K \subset X$ be a weakly compact convex set. Let $x \in X$. If $F(x, K) \neq \emptyset$, then it has an extreme point of K.*

Proof. We use the above Theorem for the locally convex space, $Y = (X, weak)$. Let $z \in F(x, K)$. Since $F(x, K)$ is an extreme set, we need to verify only the second conditions in the above theorem. We note that $K \setminus F(x, K) = \bigcup_n \{k \in K : \|x - k\| \leq (1 - \frac{1}{n})\|x - z\|\}$. Since $\|.\|$ is lower semi-continuous in the weak-topology, the sets in this union are weakly closed and hence weakly compact. Also they are convex. Thus it follows from the above Theorem that $F(x, K)$ has an extreme point of K.

□

Theorem 3. *Let X be a Banach space with a sequence $\{x_n\}_{n \geq 1}$ of unit vectors such that $x_n \to 0$ in the weak topology. Then X has a closed and bounded convex set that is not remotal. In particular in any infinite dimensional reflexive Banach space there is a closed and bounded convex set that is not remotal.*

Proof. Let $E = \{(1 - \frac{1}{n})x_n\}_{n \geq 1}$. Clearly $\overline{E} = E \cup \{0\}$ is weakly compact (closure taken in the weak topology). Also E is not remotal with respect to 0. Note that $\overline{[E]} = \overline{[E \cup \{0\}]}$ is a weakly compact set (we recall that by Mazur' theorem ([2], Theorem1 on page 11) weak and norm closures of the convex hull are the same). Suppose $F(0, \overline{[E]}) \neq \emptyset$. Since $D(0, \overline{[E]}) = 1$, the extreme point, say x, of $F(0, \overline{[E]})$, that one gets from the above Lemma, is a unit vector. Now by applying Milman's theorem to the weakly compact convex set $\overline{[E]}$, we get that $x \in E \cup \{0\}$. A contradiction. Thus $F(0, \overline{[E]}) = \emptyset$ and hence $\overline{[E]}$ is a closed and bounded convex set that is not remotal at 0.

Now if X is an infinite dimensional and reflexive by the Josefson-Nissenzweig theorem [2](page 219) there exists a sequence $\{x_n\}_{n \geq 1}$ of unit vectors such that $x_n \to 0$ in the weak topology. One can also use the Eberlian-Smulian theorem ([2] page 18) and the fact that 0 is a weak-accumulation point of the unit sphere, to get the same conclusion.

□

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