On the existence of ergodic automorphisms in ergodic $\mathbb{Z}^d$-actions on compact groups

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Abstract

Let K be a compact metrizable group and Γ be a finitely generated group of commuting automorphisms of K. We show that ergodicity of Γ implies Γ contains ergodic automorphisms if center of the action, $Z(\Gamma) = \{\alpha \in \text{Aut } (K) \mid \alpha$ commutes with elements of $\Gamma\}$ has DCC. To explain that the condition on the center of the action is not restrictive, we discuss certain abelian groups which in particular, retrieves Theorems of Berend [Be] and Schmidt [Sc1] proved in this context.

1 Introduction

We shall be considering actions on compact groups. All compact groups considered in this article are assumed to be metrizable and all automorphisms are assumed to be continuous. Let K be a compact group and Aut (K) be the group of all continuous automorphisms of K.

Definition 1 Let K be a compact group and ω_K be the normalized Haar measure on K. Let Γ be a group of automorphisms of K. We say that Γ is ergodic on K if any Γ-invariant Borel set A of K satisfies $\omega_K(A) = 0$ or $\omega_K(A) = 1$. We say that $\alpha \in \text{Aut } (K)$ is ergodic on K if the action of $\{\alpha^n \mid n \in \mathbb{Z}\}$ on K is ergodic.

Ergodic action has been characterized in many forms, we now recall two such characterizations due to Berend (cf. Theorem 2.1 of [Be]): Γ is ergodic on K if and only if there exists $x \in K$ such that the orbit $\Gamma(x)$ is dense in K if and only if for any non-trivial irreducible unitary representation π, the orbit $\{\alpha(\pi) \mid \alpha \in \Gamma\}$ has

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infinitely many inequivalent unitary representations where $\alpha(\pi)$ is given by $\alpha(\pi)(x) = \pi(\alpha^{-1}(x))$ for all $x \in K$. Also, Berend [Be] proved the following result.

Berend's Theorem: Ergodic action of a group Γ of commuting epimorphisms on a compact connected finite-dimensional abelian group contains ergodic automorphisms.

Recently Berend's Theorem was proved for certain hereditarily ergodic actions of solvable groups on compact connected finite-dimensional abelian groups (cf. [BG]) and for ergodic actions of nilpotent groups on compact connected finite-dimensional abelian groups (cf. [Ra]): this type of result was called a local to global correspondence in [Ra] as no α in a group Γ is ergodic implies the whole group Γ is not ergodic. No analogue of Berend's Theorem is known for actions on general compact groups. We will now look at the following examples in this context.

Example 1.1 Let M be a compact group and Γ be a countably infinite group. Then the (left) shift action of Γ on M^Γ is defined for $\alpha \in \Gamma$ and $f \in M^\Gamma$, by

$$\alpha f(\beta) = f(\alpha^{-1}\beta)$$

for all $\beta \in \Gamma$. Then it can be seen that the shift action of Γ on M^Γ is ergodic (cf. [Ra] when M is abelian). If Γ is a torsion group (for example, we may take Γ to be the group of all finite permutations), then no element of Γ is ergodic. Thus, there are ergodic actions without having any ergodic automorphisms.

If Γ is assumed to be a finitely generated solvable group, then Γ is torsion implies Γ is finite. Hence actions as in Example 1.1 does not exist if Γ is assumed to be a finitely generated solvable group. However the following example shows that even if we assume Γ to be finitely generated abelian, Berend's Theorem need not be true for any compact group.

Example 1.2 Let M be a compact group with an ergodic automorphisms τ. For any $(i,j) \in \mathbb{Z}^2 \setminus (0,0)$, define $K_{i,j} = M$ with $\mathbb{Z}^2$-action defined by $(n,m) \mapsto \tau^{mi-nj}$ and define $K_{0,0} = M^{\mathbb{Z}^2}$ with shift action of $\mathbb{Z}^2$. Let $K = \prod_{i,j} K_{i,j}$ and define coordinate-wise $\mathbb{Z}^2$-action on K. Then it can be verified that this action is faithful, that is no non-zero (n,m) acts trivially on K. We first claim that $\mathbb{Z}^2$-action on K is ergodic. It is sufficient to claim that $\mathbb{Z}^2$-action on each $K_{i,j}$ is ergodic. Let $(i,j) \in \mathbb{Z}^2 \setminus (0,0)$. Since τ is ergodic on M, if $j \neq 0$, then $(i+1,j)$ whose action on $K_{i,j}$ is by τ^{-j} is ergodic on $K_{i,j}$ and if $i \neq 0$, then $(i,j+1)$ whose action on $K_{i,j}$ is by τ^i is ergodic on $K_{i,j}$. We know that the shift action of $\mathbb{Z}^2$ on $K_{0,0} = M^{\mathbb{Z}^2}$ is ergodic. Thus, $\mathbb{Z}^2$-action on K is ergodic.

We next claim that no element (i,j) of $\mathbb{Z}^2$ is ergodic on K. It is clear that $(0,0)$ is not ergodic on K. Now take any $(i,j) \in \mathbb{Z}^2 \setminus (0,0)$. Now, (i,j)-action on $K_{i,j}$ is

given by τ^{ji-ij}, identity. That is, (i,j) is not ergodic on $K_{i,j}$. Since $K_{i,j}$ is a factor of K, we get that (i,j) is not ergodic on K. Thus, no $(n,m) \in \mathbb{Z}^2$ is ergodic on K.

In the above example the $\mathbb{Z}^2$-action on the compact group has infinitely many invariant normal subgroups resulting in infinite chain of closed normal invariant subgroups. Motivated by example 1.2 we are lead to consider the following type of actions.

Definition 2 Action of a group Γ on a compact group K is said to have descending chain condition (abbr. as DCC) if any decreasing sequence $(K_i)_{i \geq 1}$ of Γ-invariant closed subgroups is finite, that is, there exists a $n \geq 1$ such that $K_n = K_{n+i}$ for all i.

The condition DCC was studied in the context of ergodic actions and other type of group actions (cf. [KS]). Kitchens and Schmidt [KS] proved that action of countable abelian groups is a projective limit of actions having DCC and action of a finitely generated abelian group Γ on a compact group K has DCC if and only if K is isomorphic to a shift-invariant subgroup of M^Γ for some compact real Lie group M (cf. Theorems 3.2 and 3.16 of [KS]): see Example 1.1 for definition of the shift action of Γ on M^Γ. It can be easily seen that K and Γ in Example 1.2 does not satisfy DCC: take $K_n = \prod_{i+j \geq n} K_{i+j}$, (K_n) is a decreasing sequence of closed (normal) Γ-invariant subgroups but $K_n \neq K_m$ for all $n \neq m$.

Schmidt [Sc1] proved that if Γ is a finitely generated abelian group of automorphisms of a compact abelian group K such that Γ has DCC on K, then Γ is ergodic on K if and only if Γ contains ergodic automorphisms (cf. Theorem 3.3 and Corollary 3.4 of [Sc1]).

We now introduce center of actions. If a group Γ acts on a compact group K, then center of the action denoted by $Z(\Gamma)$ is defined to be $Z(\Gamma) = \{\alpha \in \mathrm{Aut}\ (K) \mid \alpha\beta = \beta\alpha \ \ \text{for all} \ \ \beta \in \Gamma\}$. If Γ is generated by $\alpha_1, \cdots, \alpha_n \in \Gamma$ we may write $Z(\alpha_1, \alpha_2, \cdots, \alpha_n)$ instead of $Z(\Gamma)$.

In this article we study ergodic actions on compact groups and prove that ergodic actions of finitely generated abelian groups on compact groups contain ergodic automorphisms, hence extending Berend's Theorem provided that center of the action has DCC. Since a finitely generated abelian group Γ has a subgroup Γ of finite index isomorphic to $\mathbb{Z}^d$ for some $d \geq 0$ and Γ is ergodic if and only if any subgroup of finite index is ergodic, our results are about ergodic $\mathbb{Z}^d$-actions.

We provide examples of compact groups where the center of certain type of automorphisms have DCC, as a consequence ergodic actions of finitely generated abelian groups of automorphisms of such type contain ergodic automorphisms. This in particular, retrieves Berend's Theorem quoted above and Theorem 3.3 of Schmidt [Sc1] which explains that the condition on center of the action is not restrictive.

We will now see certain type of action which is needed for our purpose.

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Definition 3 We say that the action of a group Γ on a compact group K is distal if for any $x \in K \setminus (e)$, e is not in the closure of the orbit $\Gamma(x) = \{\alpha(x) \mid \alpha \in \Gamma\}$. We say that $\alpha \in \text{Aut}(K)$ is distal on K if the action of $\{\alpha^n \mid n \in \mathbb{Z}\}$ on K is distal.

Distal actions were studied by many in different contexts (cf. [El], [Fu] and [Ra]). As in [Ra] distality plays a useful role in our proofs. It is a well-known interesting fact that if Γ is distal on K, then Γ is distal on K/L for any Γ-invariant normal subgroup L of K (cf. [El]) and we quote this result as we use this result often without quoting.

2 Ergodic actions

In this section we prove a few basic results on ergodic actions that are needed for our purpose and we provide a proof of these results as we could not locate these results in the literature.

Let K be a compact group. If π is a continuous unitary representation of K, then for any automorphism α of K, $\alpha\pi$ is defined by

$$\alpha\pi(x) = \pi(\alpha^{-1}(x))$$

for all $x \in K$. It can be easily seen that if π and π' are equivalent unitary representations, then $\alpha(\pi)$ and $\alpha(\pi')$ are equivalent. Let $[\pi]$ be denote the equivalent class of unitary representations containing π. Define $\alpha[\pi] = [\alpha(\pi)]$ for any automorphism α on K and any unitary representation π of K. Let $\hat{K}$ be denote the equivalent classes of continuous irreducible unitary representations of K. Then $(\alpha, [\pi]) \mapsto \alpha[\pi] = [\alpha(\pi)]$ defines an action of $\text{Aut}(K)$ on $\hat{K}$ and this action is known as the dual action.

We now extend Berend's criterion Theorem 2.1 of [Be] to finite-dimensional unitary representations.

Lemma 2.1 *Let K be a compact group and Γ be a group of automorphisms of K. Then the following are equivalent:*

(i) Γ is ergodic on K;

(ii) $\Gamma[\pi]$ is infinite for any non-trivial $[\pi] \in \hat{K}$;

(iii) $\Gamma[\pi]$ is infinite for any non-trivial finite-dimensional unitary representation π of K.

Proof Equivalence of (i) and (ii) is proved in Theorem 2.1 of [Be] and since all irreducible unitary representation of compact groups are finite-dimensional, (iii) implies (ii). So, it is sufficient to prove that (ii) implies (iii). Let π be a finite-dimensional

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unitary representation of K such that $\Gamma[\pi]$ is finite. Let Γ_1 be the subgroup of Γ consisting of all $\alpha \in \Gamma$ such that $\alpha(\pi)$ is equivalent to π. Then Γ_1 is a subgroup of finite index.

Let $\pi = \oplus_{1 \le j \le m} \pi_j^{r_j}$ be the decomposition of π into irreducible unitary representations π_j of K. Then for any automorphism α of K, $\alpha(\pi) = \oplus_{1 \le j \le m} \alpha(\pi_j)^{r_j}$ and each $\alpha(\pi_j)$ is also irreducible. By uniqueness of decomposition, each $\alpha \in \Gamma_1$, defines a permutation on the set of equivalent classes of π_j, $1 \le j \le m$ (cf. 27.30 of [HR2]). Thus, there is a subgroup Γ_2 of finite index in Γ_1 such that $\alpha(\pi_j)$ is equivalent to π_j for all $\alpha \in \Gamma_2$ and all j. Since Γ_1 has finite index in Γ and Γ_2 has finite index in Γ_1, we get that Γ_2 has finite index in Γ. Using (ii) this implies that each π_j is trivial and hence π is trivial. Thus, (ii) implies (iii) is proved.

The following shows stage ergodicity implying ergodicity.

Lemma 2.2 *Let K be a compact group and Γ be a group of automorphisms of K. Suppose there is a decreasing sequence $(K_i)_{i \ge 0}$ of closed Γ-invariant subgroups with $K_0 = K$ and $\cap K_i = (e)$ such that K_i is normal in K_{i-1} and Γ is ergodic on K_{i-1}/K_i. Then Γ is ergodic on K.*

Proof Let π be an irreducible unitary representation of K such that $\Gamma[\pi]$ is finite. Then π is finite-dimensional. Suppose π is trivial on K_i. Then let $\tilde{\pi}$ be the restriction of π to K_{i-1}. Then $\tilde{\pi}$ is a unitary representation of K_{i-1}/K_i of finite-dimension and $\Gamma[\tilde{\pi}]$ is finite. Since Γ is ergodic on K_{i-1}/K_i, it follows from Lemma 2.1 that $\tilde{\pi}$ is trivial. Thus, π is trivial on K_{i-1} if π is trivial on K_i and hence it is sufficient to claim that π is trivial on some K_i.

Since π is finite-dimensional, there exists $n \ge 1$ such that $\pi(K)$ is a compact and hence a closed subgroup of $GL_n(\mathbb{C})$. Then by Cartan's Theorem $\pi(K)$ is a real Lie group (cf. Part II, Chapter V, Section 9, [Se]). Since each K_i is compact, we get that $(\pi(K_i))$ is a decreasing sequence of closed subgroups of $\pi(K)$. Considering dimension and using the fact that compact real Lie groups have only finitely many connected components, we get that there exists i_0 such that $\pi(K_i) = K_{i_0}$ for all $i \ge i_0$. Since $\cap K_i = (e)$, $\pi(K_i)$ is trivial for all $i \ge i_0$. Thus, π is trivial on K_{i_0}.

Lemma 2.3 *Let K be a compact group and Γ be a group of automorphisms of K. Suppose there exists a collection $(K_i)_{i \in I}$ of closed Γ-invariant subgroups K such that $\overline{\cup_{i \in I} K_i} = K$ and Γ is ergodic on K_i for all $i \in I$. Then Γ is ergodic on K.*

Proof Let π be an irreducible unitary representation of K such that $\Gamma[\pi]$ is finite. For $i \in I$, let π_i be the restriction of π to K_i. Then π_i is a finite-dimensional unitary representation of K_i and $\Gamma[\pi_i]$ is finite for all $i \in I$. Since Γ is ergodic on K_i, by Lemma 2.1, we get that π_i is trivial and hence π is trivial on K_i for all $i \in I$. Since $\overline{\cup K_i} = K$, π is trivial on K.

Lemma 2.4 *Let K be a compact group and Γ be a group of automorphisms of K. Suppose there exist subgroups Γ_1 and Γ_2 of Γ such that Γ_2 is a compact normal subgroup of Γ and $\Gamma = \Gamma_1\Gamma_2$. Then we have the following:*

(i) $\Gamma_2[\pi]$ is finite for any irreducible unitary representation π of K;

(ii) Γ is ergodic on K if and only if Γ_1 is ergodic on K.

Proof Let π be an irreducible unitary representation of K. Let $\mathcal{H}$ be the Hilbert space on which π is defined. Let n be the dimension of $\mathcal{H}$. Let R_n be the space of all irreducible unitary representations of K of dimension n and A_n be the equivalent classes of irreducible unitary representations of K of dimension n. Equip R_n with the smallest topology for which the functions $\pi \mapsto < \pi(x)u, v >$ are continuous on R_n for all $x \in K$ and all $u, v \in \mathcal{H}$. Consider the quotient map $\pi \mapsto [\pi]$ of R_n onto A_n and equip A_n with the quotient topology from R_n. It is well-known that A_n is discrete (cf. 3.5.8 and 18.4.3 of [Di]). Let $\alpha_n \to \alpha$ in Aut (K). Then for $x \in K$ and $u, v \in \mathcal{H}$, we have

$$< \alpha_n\pi(x)u, v >=< \pi(\alpha_n^{-1}x)u, v >\to< \pi(\alpha^{-1}x)u, v >=< \alpha\pi(x)u, v >$$

as π is continuous and $\alpha_n^{-1}(x) \to \alpha^{-1}(x)$. Thus, $\alpha_n[\pi] \to \alpha[\pi]$ in A_n. Since Γ_2 is compact, we get that $\Gamma_2[\pi]$ is compact in A_n. Since A_n is discrete, $\Gamma_2[\pi]$ is finite. Thus, proving (i).

Suppose Γ is ergodic on K. If π is an irreducible unitary representation of K such that $\Gamma_1[\pi]$ is finite, then since Γ_2 is normal, (i) implies that $\Gamma[\pi] = \Gamma_2\Gamma_1[\pi]$ is finite. Since Γ is ergodic, π is trivial. Thus, Γ_1 is ergodic. This proves (ii).

We next prove the existence of largest ergodic subgroup for Γ-actions.

Proposition 2.1 *Let K be a compact group and Γ be a group of automorphisms of K. Then there is a $Z(\Gamma)$-invariant (largest) closed normal subgroup L of K such that Γ is ergodic on L and Γ is distal on K/L.*

Proof Let $\mathcal{A}$ be the collection of all closed normal Γ-invariant subgroups L such that Γ is ergodic on L and elements of $\mathcal{A}$ are ordered by inclusion. Since the trivial subgroup is in $\mathcal{A}$, $\mathcal{A}$ is non-empty. Let $\mathcal{B}$ be a chain in $\mathcal{A}$. Let $M = \overline{\cup_{L\in\mathcal{B}}L}$. Since $\mathcal{B}$ is a chain, M is a closed normal Γ-invariant subgroup of K. By Lemma 2.3, Γ is ergodic on M and hence $M \in \mathcal{A}$ and $L \subset M$ for all $L \in \mathcal{B}$. Thus, every chain in $\mathcal{A}$ has upper bound. By Zorn's Lemma $\mathcal{A}$ has a maximal element and let N be the maximal element of $\mathcal{A}$.

Let Inn (K) be the group of all inner automorphisms of K. Then Inn (K) is a compact normal subgroup of Aut (K). Let $\tilde{\Gamma}$ be the group generated by Γ and all

inner automorphisms of K. Then N is $\tilde{\Gamma}$-invariant and $\tilde{\Gamma} = \Gamma\mathrm{Inn}\,(K)$. Suppose $\tilde{N}$ is a closed $\tilde{\Gamma}$-invariant subgroup of K containing N such that $\tilde{\Gamma}$ is ergodic on $\tilde{N}/N$. Since Γ is ergodic on N and $\Gamma \subset \tilde{\Gamma}$, $\tilde{\Gamma}$ is ergodic on N. Thus, $\tilde{\Gamma}$ is ergodic on $\tilde{N}$ (cf. Lemma 2.2). Now Lemma 2.4 implies that Γ is ergodic on $\tilde{N}$. Since $\tilde{N}$ is $\tilde{\Gamma}$-invariant, $\tilde{N}$ is normal in K and hence $\tilde{N} \in \mathcal{A}$ and $N \subset \tilde{N}$. Since N is a maximal element in $\mathcal{A}$, $N = \tilde{N}$. By Proposition 2.1 of [Ra], we get that $\tilde{\Gamma}$ is distal on K/N. Since $\Gamma \subset \tilde{\Gamma}$, Γ is distal on K/N.

Let $\beta \in Z(\Gamma)$. Since Γ is ergodic on N, by Theorem 2.1 of [Be], there exists a $x \in N$ such that $\Gamma(x)$ is dense in N. So, there exists a sequence (α_n) in Γ such that $\alpha_n(x) \to e$. Now $\alpha_n(\beta(x)) = \beta\alpha_n(x) \to e$. Since Γ is distal on K/N, $\beta(x) \in N$. Since N is Γ-invariant, this implies that $\beta(\alpha(x)) = \alpha(\beta(x)) \in \alpha(N) = N$ for all $\alpha \in \Gamma$. Since $\Gamma(x)$ is dense in N, $\beta(N) \subset N$. Thus, N is $Z(\Gamma)$-invariant.

If H is a closed Γ-invariant subgroup of K such that Γ is ergodic on H, then since N is normal in K, HN is also a closed Γ-invariant subgroup and hence Γ is ergodic on the closed subgroup HN (cf. Lemma 2.2). This implies that Γ is ergodic on HN/N but Γ is distal on K/N and hence distal on HN/N. Thus, $N = HN \supset H$.

3 Distal actions

In this section we prove that distality of generating elements is sufficient for distality of finitely generated abelian group actions on compact groups which is needed for our purpose and which may be of independent interest.

We first consider zero-dimensional compact groups. It may be noted that a compact group is zero-dimensional if and only if it is totally disconnected (see 3.5 and 7.7 of [HR1]).

Lemma 3.1 *Let Γ be a finitely generated abelian group of automorphisms of a compact zero-dimensional group K. Suppose $\alpha_1, \alpha_2, \cdots, \alpha_m \in \Gamma$ generate Γ. Then the following are equivalent:*

 (i) Γ action on K is distal;

 (ii) α_i is distal on K for $i = 1, 2, \cdots, m$;

(iii) K has arbitrarily small Γ-invariant compact open subgroups.

Proof It is sufficient to prove that (ii) implies (iii). We prove (iii) by induction on the number of generators of Γ. If Γ has only one generator, the result follows from Proposition 2.1 of [JR] (see also [BW]). If $m > 1$, then by induction assumption, for any compact open subgroup W of K, there exists a compact open subgroup $U \subset W$

that is invariant under $\alpha_1, \alpha_2, \cdots, \alpha_{n-1}$. By Proposition 2.1 of [JR], there exists a k such that $\cap_{i=0}^{k}\alpha_n^i(U)$ is α_n-invariant. Let $V = \cap_{i=0}^{k}\alpha_n^i(U)$. Then V is invariant under $\alpha_1, \alpha_2, \cdots, \alpha_{n-1}$ as α_is commute and V is invariant under α_n also, hence V is invariant under the group generated by $\alpha_1, \alpha_2, \cdots, \alpha_{n-1}$ and α_n which is Γ. Thus, proving (iii).

We next consider abelian groups and we recall a few results on duality of locally compact abelian groups. Let G be a locally compact abelian group. Then any continuous homomorphism of G into the circle group $\{z \in \mathbb{C} \mid |z| = 1\}$ is known as character of G. Let $\hat{G}$ be the group of characters on G. Then $\hat{G}$ with compact-open topology is known as the dual group of G. It is well-known that $\hat{G}$ is discrete if and only if G is compact, $\widehat{(\hat{G})} \simeq G$, and G is metrizable if and only if $\hat{G}$ is σ-compact (cf. Theorems 12, 23 and 29 of [Mo]). Thus, K is a compact (metrizable) abelian group if and only if $\hat{K}$ is a countable discrete group.

For any closed subgroup M of G, consider the subgroup $M^\perp$ of $\hat{G}$ defined by $M^\perp = \{\chi \in \hat{G} \mid \chi \text{ is trivial on } M\}$. It follows from Pontryagin duality theory that $\hat{M} \simeq \hat{G}/M^\perp$ and $M^\perp \simeq \widehat{(G/M)}$ (cf. Theorem 27 of [Mo]).

For any bi-continuous automorphism α of G and $\chi \in \hat{G}$, we define $\alpha\chi$ by $\alpha\chi(g) = \chi(\alpha^{-1}(g))$ for all $g \in G$ and hence $\chi \mapsto \alpha\chi$ defines a bi-continuous automorphism of $\hat{G}$ which would be called dual automorphism of α. Also, $(\alpha, \chi) \mapsto \alpha\chi$ defines an action of Aut (G) on $\hat{G}$ and it is known as the dual action.

We now apply duality theory of compact abelian groups to obtain the following.

Lemma 3.2 *Let K be a compact abelian group and Γ be a group of automorphisms of K. Suppose there is compact normal subgroup Δ of Γ and $\alpha_1, \cdots, \alpha_n$ are in Γ such that Γ/Δ is abelian and Γ is generated by Δ and $\alpha_1, \cdots, \alpha_n$. Then the following are equivalent:*

1. *Γ is distal on K;*

2. *each α_i is distal on K.*

Proof Assume that each α_i is distal on K. Proof is based on induction on n. Suppose L is a non-trivial Γ-invariant subgroup of K. Let A be the dual of L. Let Γ_1 be the group generated by Δ and $\alpha_1, \cdots, \alpha_{n-1}$. Then by induction hypothesis, Γ_1 is distal on K and hence the subgroup $A_1 = \{\chi \in A \mid \Gamma_1(\chi) \text{ is finite }\}$ is non-trivial (cf. Proposition 2.1 of [Ra]). Since Γ/Δ is abelian and Γ_1 contains Δ, we get that α_n normalizes Γ_1 and hence A_1 is α_n-invariant. Since α_n is distal on K, there is a non-trivial $\chi \in A_1$ such that $\alpha_n^m(\chi) = \chi$ for some $m \neq 0$ (cf. Proposition 2.1 of [Ra]). This implies that $\Gamma_1\alpha_n^j(\chi) \subset \cup_{i=1}^{m}\Gamma_1(\alpha_n^i(\chi))$ for all j and hence since $\chi \in A_1$ and A_1

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is α_n-invariant, the orbit $\Gamma(\chi)$ is finite. Thus, Γ is not ergodic on L. It follows from Proposition 2.1 of [Ra] that Γ is distal on K.

We now consider connected groups.

Lemma 3.3 *Suppose $\alpha_1, \cdots, \alpha_n \in$ Aut (K) are commuting automorphisms on a compact connected group K. Then the following are equivalent:*

1. *each α_i is distal on K;*

2. *the group generated by $\alpha_1, \cdots, \alpha_n$ is distal on K.*

Proof Assume that each α_i is distal on K. Let Γ be the group generated by $\alpha_1, \alpha_2, \cdots, \alpha_n$. Let $x \in K$ be such that e is in the closure of $\Gamma(x)$. Let T be a maximal pro-torus in K containing x (cf. Theorem 9.32 of [HM]). Then Aut $(K) =$ Inn $(K)\Omega$ where Inn (K) is the group of inner automorphisms of K and $\Omega = \{\tau \in$ Aut $(K) \mid \tau(T) = T\}$ (cf. Corollary 9.87 of [HM]). Let $\beta_i \in \Omega$ be such that $\alpha_i \in$ Inn $(K)\beta_i$. If $y \in K$ is such that $\beta_i^{k_n}(y) \to e$, then for each $n \geq 1$, $\alpha_i^{k_n} = \sigma_{k_n}\beta_i^{k_n}$ for some $\sigma_{k_n} \in$ Inn (K) as Inn (K) is normal in Aut (K). Since K is compact, Inn (K) is compact, hence by passing to a subsequence, if necessary, we may assume that $\sigma_{k_n} \to \sigma \in$ Inn (K) and hence $\alpha_i^{k_n}(y) = \sigma_{k_n}\beta_i^{k_n}(y) \to \sigma(e) = e$. Since α_i is distal, $y = e$. Thus, β_i is distal on K and hence in particular, β_i is distal on T.

Let Ω' be the group generated $\beta_1, \cdots, \beta_n$. Since Inn (K) is normal, Inn $(K)\Omega'$ is a group containing α_i. Also $\Omega' \subset$ Inn $(K)\Gamma$, we get that Inn $(K)\Gamma =$ Inn $(K)\Omega'$. This implies that

$$\Omega'/\text{Inn }(K) \cap \Omega' \simeq \text{Inn }(K)\Omega'/\text{Inn }(K) \simeq \Gamma\text{Inn }(K)/\text{Inn }(K) \simeq \Gamma/\Gamma \cap \text{Inn }(K).$$

Since Γ is abelian, $\Omega'/\text{Inn }(K) \cap \Omega'$ is abelian. It follows since each β_i is distal on T and from Lemma 3.2 that Ω' is distal on T.

Since e is in the closure of $\Gamma(x)$, there exists $\eta_n \in \Gamma$ such that $\eta_n(x) \to e$. Since $\Gamma \subset$ Inn $(K)\Omega'$, there exist $\tau_n \in$ Inn (K) and $\gamma_n \in \Omega'$ such that $\eta_n = \tau_n\gamma_n$. Since Inn (K) is compact, we may assume by passing to a subsequence, if necessary that $\tau_n \to \tau \in$ Inn (K). Then $\gamma_n(x) = \tau_n^{-1}(\eta_n(x)) \to \tau^{-1}(e) = e$. Since $x \in T$, $\gamma_n \in \Omega'$ and Ω' is distal on T, we get that $x = e$. This shows that Γ is distal on K

We now consider the general case.

Lemma 3.4 *Suppose $\alpha_1, \cdots, \alpha_n \in$ Aut (K) are commuting automorphisms on a compact group K. Then the following are equivalent:*

1. *each α_i is distal on K;*

2. the group generated by $\alpha_1, \cdots, \alpha_n$ is distal on K.

Proof Suppose each α_i is distal on K. Let Γ be the group generated by $\alpha_1, \alpha_2, \cdots, \alpha_n$. Let $x \in K$ be such that e is in the closure of $\Gamma(x)$. Let K_0 be the connected component of e in K. Then K_0 is a closed normal Γ-invariant subgroup of K and K/K_0 is a zero-dimensional compact group. Each α_i is distal on K implies that each α_i is distal on K/K_0. It follows from Lemma 3.1 that Γ is distal on K/K_0. Since e is in the closure of $\Gamma(x)$, $x \in K_0$. Now it follows from Lemma 3.3 that $x = e$.

The following example shows that Lemma 3.4 is not true for any acting group.

Example 3.1 Let M be any compact group and Γ be a countably infinite group. Consider the shift action of Γ on M^Γ defined for $\alpha \in \Gamma$ and $f \in M^\Gamma$ by $\alpha f(\beta) = f(\alpha^{-1}\beta)$ for all $\beta \in \Gamma$. Suppose $f \in M^\Gamma$ is such that $f(\beta)$ is identity in M for all but finitely many $\beta \in \Gamma$. We claim that identity in M^Γ is a limit point of $\Gamma(f)$. Let $\beta_1, \cdots, \beta_n$ be such that $f(\beta)$ is identity in M if $\beta \neq \beta_i$. Let U be a neighborhood of identity in M. Then for any $\alpha_1, \cdots, \alpha_m$, since Γ is infinite, there is a $\alpha \in \Gamma$ such that $\alpha^{-1}\alpha_i \notin \{\beta_i\}$ and hence $\alpha f(\alpha_i) \in U$. This shows that the shift action of Γ on M^Γ is not distal. If we choose Γ to be finitely generated, infinite and torsion, we get that each $\alpha \in \Gamma$ is distal but Γ is not distal (cf. [Go] for existence of such Γ.)

4 Berend's Theorem

We now prove Berend's Theorem for ergodic $\mathbb{Z}^d$-actions on compact groups provided that center of the action has DCC.

Theorem 4.1 *Let K be a compact group and Γ be a finitely generated abelian group of automorphisms of K. Suppose Γ is ergodic on K and $Z(\Gamma)$ has DCC on K. Then there exists $\alpha \in \Gamma$ such that α is ergodic on K. Furthermore, if Γ is generated by $\alpha_1, \cdots, \alpha_n$, then there exist positive integers $i_1, \cdots, i_n$ such that $\alpha_1^{i_1} \cdots \alpha_n^{i_n}$ is ergodic on K.*

We need the following results.

Lemma 4.1 *Let K be a compact group and α, β be two commuting automorphisms of K. Suppose α is ergodic on K and β is distal on K. Then $\alpha^i \beta^j$ is ergodic on K for all integers i and j with $i \neq 0$.*

Proof Since α^i is ergodic on K for any $i \neq 0$ and β^j is distal on K for all j, it is sufficient to prove that $\alpha\beta$ is ergodic on K. Applying Proposition 2.1 to $\alpha\beta$ on K, we get that there exists a closed normal $Z(\alpha, \beta) \subset Z(\alpha\beta)$-invariant subgroup K_1

such that $\alpha\beta$ is ergodic on K_1 and $\alpha\beta$ is distal on K/K_1. Since α and β commute, $\alpha, \beta \in Z(\alpha, \beta)$, hence K_1 is invariant under both α and β. Since β is distal on K, β is distal on K/K_1. By Lemma 3.4, the group generated by both $\alpha\beta$ and β is distal on K/K_1. It is easy to see that α is in the group generated by $\alpha\beta$ and β, hence α is also distal on K/K_1 but since α is ergodic on K, α is ergodic on K/K_1 and hence $K = K_1$ (cf. Proposition 2.1 of [Ra]).

Lemma 4.2 *Let α, β be two commuting automorphisms of K such that $Z(\alpha, \beta)$ has DCC. If α is ergodic on K, then $\alpha^i\beta$ is ergodic for all but finitely many $i > 0$.*

Proof We first note that since α and β commute, $\alpha, \beta \in Z(\alpha, \beta)$. Let $K_0 = K$. We now inductively construct a decreasing sequence $(K_i)_{i \geq 1}$ of $Z(\alpha, \beta)$-invariant closed subgroups such that K_i is normal in K_{i-1}, $\alpha^j\beta$ is ergodic on K_{i-1}/K_i for all $j \neq i$ and $\alpha^i\beta$ is ergodic on K_i.

Applying Proposition 2.1 to $\alpha\beta$ on K, we get that there exists $Z(\alpha, \beta) \subset Z(\alpha\beta)$-invariant closed subgroup K_1 such that $\alpha\beta$ is ergodic on K_1 and $\alpha\beta$ is distal on K/K_1. Since α is ergodic on K, α is ergodic on K/K_1 and hence by Lemma 4.1, we get that $\alpha^i\beta$ is ergodic on K/K_1 for all $i \neq 1$.

If for $j \leq i$, $Z(\alpha, \beta)$-invariant closed subgroups K_j are chosen so that $\alpha^l\beta$ is ergodic on K_{j-1}/K_j for all $l \neq j$ and $\alpha^j\beta$ is ergodic on K_j. We now choose K_{i+1}. Applying Proposition 2.1 to $\alpha^{i+1}\beta$ on K_i, we get that there exists a $Z(\alpha, \beta) \subset Z(\alpha^{i+1}\beta)$-invariant normal subgroup K_{i+1} of K_i such that $\alpha^{i+1}\beta$ is ergodic on K_{i+1} and distal on K_i/K_{i+1}. Since $\alpha^i\beta$ is ergodic on K_i, $\alpha^i\beta$ is ergodic on K_i/K_{i+1} and hence $\beta^{-1}\alpha^{-i}$ is ergodic on K_i/K_{i+1}. Since $\alpha^{i+1}\beta$ is distal K_i/K_{i+1}, by Lemma 4.1, $\alpha = \beta^{-1}\alpha^{-i}\alpha^{i+1}\beta$ is ergodic on K_i/K_{i+1}. Again by Lemma 4.1, we get that $\alpha^j\beta$ is ergodic on K_i/K_{i+1} for all $j \neq i+1$. Thus, there exists a decreasing sequence $(K_i)_{i \geq 1}$ of $Z(\alpha, \beta)$-invariant closed subgroups such that K_i is normal in K_{i-1}, $\alpha^j\beta$ is ergodic on K_{i-1}/K_i for all $j \neq i$ and $\alpha^i\beta$ is ergodic on K_i.

Since $Z(\alpha, \beta)$ has DCC on K, there exists a i such that $K_i = K_j$ for all $j > i$. This implies that for $j > i$, $\alpha^j\beta$ is ergodic on K_{m-1}/K_m for $1 \leq m \leq i$ and $\alpha^j\beta$ is ergodic on $K_j = K_i$. Thus, by Lemma 2.2 we get that $\alpha^j\beta$ is ergodic on K for all $j > i$.

Proposition 4.1 *Let K be a compact group and Γ be a finitely generated group of commuting automorphisms of K generated by $\alpha_1, \cdots, \alpha_n$. Then there exists a series $(e) = K_0 \subset K_1 \subset \cdots \subset K_n \subset K$ of closed normal $Z(\Gamma)$-invariant subgroups such that*

1. *α_i is ergodic on K_i/K_{i-1} and distal on K/K_i,*

2. *Γ is ergodic on K_n and distal on K/K_n.*

Moreover, if Γ is ergodic on K, $K = K_n$.

Proof We first observe that since Γ is abelian, $\Gamma \subset Z(\Gamma)$ and hence any $Z(\Gamma)$-invariant subgroup is also Γ-invariant. It follows from Proposition 2.1 that there exists a closed normal $Z(\Gamma) \subset Z(\alpha_1)$-invariant subgroup K_1 such that α_1 is ergodic on K_1 and distal on K/K_1. For $i \geq 1$ and for all $1 \leq j \leq i$, if closed normal $Z(\Gamma)$-invariant subgroups K_j are chosen such that α_j is ergodic on K_j/K_{j-1} and distal on K/K_j. We now choose K_{i+1}. Applying Proposition 2.1 to α_{i+1} on K/K_i, there exists a closed normal $Z(\Gamma) \subset Z(\alpha_{i+1})$-invariant subgroup K_{i+1} containing K_i such that α_{i+1} is ergodic on K_{i+1}/K_i and α_{i+1} is distal on K/K_{i+1}. Thus, we have closed normal $Z(\Gamma)$-invariant subgroups $(e) = K_0 \subset K_1 \subset \cdots \subset K_n \subset K$ such that α_i is ergodic on K_i/K_{i-1} and distal on K/K_i.

Since $\alpha_i \in \Gamma$, Γ is ergodic on K_i/K_{i-1} for all $1 \leq i \leq n$, hence by Lemma 2.2, Γ is ergodic on K_n.

Since K/K_n is a quotient of K/K_i for all $1 \leq i \leq n$, each α_i is distal on K/K_n. Thus by Lemma 3.4, Γ is distal on K/K_n.

If Γ is ergodic on K, then Γ is ergodic on K/K_n but Γ is distal on K/K_n, hence $K = K_n$ (cf. Proposition 2.1 of [Ra]).

Proof of Theorem 4.1 Let $\alpha_1, \alpha_2, \cdots, \alpha_n \in \Gamma$ be a set of generators of Γ. The proof is by induction on n. Suppose Γ has more than one-generator. By Proposition 4.1, there exist closed normal Γ-invariant subgroups $(e) = K_0 \subset K_1 \subset \cdots \subset K_n = K$ such that α_i is ergodic on K_i/K_{i-1} and α_i is distal on K/K_i.

Since α_n is ergodic on K/K_{n-1} and α_{n-1} is distal on K/K_{n-1}, Lemma 4.1 implies that $\alpha_n^i \alpha_{n-1}^j$ is ergodic on K/K_{n-1} for all i and j with $i \neq 0$. In particular, $\alpha_n \alpha_{n-1}^j$ is ergodic on K/K_{n-1} for all j.

Since $Z(\Gamma) \subset Z(\alpha_n, \alpha_{n-1})$, $Z(\alpha_n, \alpha_{n-1})$ has DCC on K. Since α_{n-1} is ergodic on K_{n-1}/K_{n-2}, applying Lemma 4.2 to α_{n-1} and α_n on K_{n-1}/K_{n-2}, we get that $\alpha_n \alpha_{n-1}^j$ is ergodic on K_{n-1}/K_{n-2} for all but finitely many $j > 0$. Since $\alpha_n \alpha_{n-1}^j$ is ergodic on K/K_{n-1} for all j, we get that $\alpha_{n-1}^j \alpha_n$ is ergodic on K/K_{n-2} for some $j > 0$ (cf. Lemma 2.2).

Let Δ be the group generated by $\alpha_1, \cdots, \alpha_{n-2}, \alpha_{n-1}^j \alpha_n$. Then $Z(\Delta)$ contains $Z(\Gamma)$ and hence $Z(\Delta)$ has DCC on K. Since each α_i is ergodic on K_i/K_{i-1}, we get that Δ is ergodic on K_i/K_{i-1} for all $1 \leq i \leq n-2$, hence by Lemma 2.2, Δ is ergodic on K_{n-2}. Since $\alpha_{n-1}^j \alpha_n$ is ergodic on K/K_{n-2}, we get that Δ is ergodic on K (cf. Lemma 2.2). Since Δ is generated by $n-1$ elements, result follows from induction assumption.

The next result is a consequence of a classical result of Rokhlin [Ro] and Theorem 4.1.

Corollary 4.1 *Let K be a compact group and Γ be a finitely generated group of commuting automorphisms of K. Suppose $Z(\Gamma)$ has DCC on K. Then the following are equivalent:*

(1) Γ is ergodic on K;

(2) there exists $\alpha \in \Gamma$ such that α is mixing of all orders.

The following results and results in section 5 show that the additional condition on $Z(\Gamma)$ in Theorem 4.1 is not restrictive.

Corollary 4.2 *Let K be a compact group. If the center of Aut (K) has DCC on K, then any ergodic action of finitely generated abelian group contains ergodic automorphisms.*

Proof Since the center of Aut (K) is contained in $Z(\Gamma)$ for any subgroup Γ of automorphisms, $Z(\Gamma)$ has DCC on K if the center of Aut (K) has DCC on K. Now the result follows from Theorem 4.1.

Let M be a compact group and Γ be a countably infinite group. Consider the shift action of Γ on M^Γ defined for $\alpha \in \Gamma$ and $f \in M^\Gamma$ by $\alpha f(\beta) = f(\alpha^{-1}\beta)$ for all $\beta \in \Gamma$. In this situation we have the following:

Theorem 4.2 *Let Γ be a finitely generated abelian group and M be a compact Lie group. Let K be a shift-invariant subgroup M^Γ. Suppose the action of a finitely generated abelian group Δ on K commutes with the shift action of Γ on K. Then Δ is ergodic on K implies that Δ contains ergodic automorphisms.*

Proof By Theorem 3.2 of [KS] we get that the shift action of Γ has DCC on M^Γ and hence on K. Since the action of Δ on K commutes with the shift action on K, $Z(\Delta)$ has DCC. Now the result follows from Theorem 4.1.

5 Some compact abelian groups

5.1 Duals of countable vector spaces

Let $\mathbb{F}$ be a countable field with discrete topology. Let $\mathbb{F}^*$ denote the multiplicative group of non-zero elements in $\mathbb{F}$ and $\mathbb{F}^r$ be the vector space over $\mathbb{F}$ of dimension $r \geq 1$. Then $\mathbb{F}^r$ is countable discrete group. Let K_r be the dual of $\mathbb{F}^r$. Then K_r is a compact group. Let $\mathrm{Aut}_\mathbb{F}(\mathbb{F}^r)$ be the group of all $\mathbb{F}$-linear transformation on $\mathbb{F}^r$ and $\mathrm{Aut}_\mathbb{F}(K_r)$ be the corresponding group of dual automorphisms on K_r. In this situation we have the following:

Theorem 5.1 *Let $\mathbb{F}$ be a countable field with discrete topology. Let K_r be the dual of $\mathbb{F}^r$ for some integer $r > 0$. Then we have the following:*

(1) the center of $\mathrm{Aut}_{\mathbb{F}} (K_r)$ has DCC;

(2) if Γ is a finitely generated abelian subgroup of $\mathrm{Aut}_{\mathbb{F}} (K_r)$ such that Γ is ergodic, then Γ contains ergodic automorphisms.

Remark We will now show by an example that finite generation condition on Γ in Theorem 5.1 (2) can not be relaxed. Let p be a prime number. Then for any $n \geq 1$, we can find a unique finite field $\mathbb{F}_n$ with p^n number of elements and F_n contains F_m if and only if m divides n (cf. Corollary 2, Section 1, Chapter 1, [We]). Let $\mathbb{F} = \cup_{k=1}^{\infty} F_{n^k}$ for some fixed $n > 1$. Then F is a countably infinite field. Consider the $\mathbb{F}^*$-action on $\mathbb{F}$ by scalar multiplication. Then for any non-zero $a \in \mathbb{F}$, $F^* a$ is infinite. Thus, the dual action of $\mathbb{F}^*$ on the dual K_1 of $\mathbb{F}$ is ergodic but every element in $\mathbb{F}^*$ has finite order and hence no element of $\mathbb{F}^*$ is ergodic.

Proof Consider the $\mathbb{F}^*$-action on $\mathbb{F}^r$ given by $q(q_i) = (qq_i)$ for all $q \in \mathbb{F}^*$ and $(q_i) \in \mathbb{F}^r$. It can be easily seen that $\mathbb{F}^*$ is in the center of $\mathrm{Aut}_{\mathbb{F}} (\mathbb{F}^r)$. If (L_n) is a decreasing sequence of closed subgroups in K_r that are $\mathbb{F}^*$-invariant, then let $A_n = \{\chi \in \mathbb{F}^r \mid \chi$ is trivial on $L_n\}$. Then (A_n) is a increasing sequence of subgroups of $\mathbb{F}^r$ that are $\mathbb{F}^*$- invariant. Since any subgroup that is $\mathbb{F}^*$-invariant is a $\mathbb{F}$-subspace of $\mathbb{F}^r$, (A_n) is a increasing sequence of $\mathbb{F}$-subspaces of $\mathbb{F}^r$. Since $\mathbb{F}^r$ is finite dimensional, $A_n = A_m$ for all large n and m. Since $L_n = \widehat{(\mathbb{F}^r / A_n)}$, we get that $L_n = L_m$ for all large n and m. Thus, the center of $\mathrm{Aut}_{\mathbb{F}} (K_r)$ has DCC on K_r.

Let Γ be a finitely generated abelian subgroup of $\mathrm{Aut}_{\mathbb{F}} (K_r)$. Then $Z(\Gamma)$ contains the center of $\mathrm{Aut}_{\mathbb{F}} (K_r)$. Since the center of $\mathrm{Aut}_{\mathbb{F}} (K_r)$ has DCC, we get that $Z(\Gamma)$ has DCC. Now the result follows from Theorem 4.1.

If we take $K = \mathbb{Q}$, we may obtain Berend's Theorem of [Be] for actions of automorphisms as shown below.

Theorem 5.2 *Let K be a compact connected finite-dimensional abelian group and Γ be a group of commuting automorphisms of K. If Γ is ergodic on K, then there exists $\alpha \in \Gamma$ such that α is ergodic on K.*

Proof Let K be a compact connected abelian group of finite-dimension. Let r be the dimension of K. Let B_r be the dual of the discrete group $\mathbb{Q}^r$. Then K is a quotient of B_r and any automorphism of K lifts to an automorphism of B_r. A group of automorphisms is ergodic on K if and only if the corresponding group of lifts is ergodic on B_r (cf. [Be] or [Ra]). Thus, we may assume that $K = B_r$.

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Let Γ be a group of commuting automorphisms of B_r and Γ is ergodic on B_r. By Lemma 5.9 of [Ra], there exist a series $(e) = K_0 \subset K_1 \subset \cdots \subset K_m = B_r$ of closed connected Γ-invariant subgroups with each $K_i \simeq B_{r_i}$ and automorphisms $\alpha_1, \cdots, \alpha_m$ in Γ such that α_i is ergodic on K_i/K_{i-1}. This implies that the group generated by $\alpha_1, \cdots, \alpha_m$ is ergodic on $K_m = B_r$ (cf. Lemma 2.2). Thus, replacing Γ by its subgroup generated by $\alpha_1, \cdots, \alpha_m$, if necessary, we may assume that Γ is a finitely generated abelian group.

It is easy to see that any homomorphism of $\mathbb{Q}^r$ into $\mathbb{Q}^r$ is a $\mathbb{Q}$-linear transformation. This shows that $\mathrm{Aut}\,(B_r) \simeq \mathrm{Aut}_\mathbb{Q}\,(B_r)$. Now the result follows from Theorem 5.1.

Remark   Using the results proved here we can prove Theorem 5.2 for commuting semigroup of epimorphisms of compact connected finite-dimensional abelian groups (as lifts of epimorphisms are automorphisms) which would retrieve Berends result in its full generality but as this would cause a digression we would not go into the details.

5.2   Duals of modules

We now discuss compact abelian groups arising as duals of modules over rings: see [Sh] for any details on rings and modules. Let R be a commutative ring and for any $d \geq 1$, let $\mathbf{R} = \mathbf{R}_d = R[u_1^{\pm 1}, \cdots, u_d^{\pm 1}]$ be the ring of Laurent polynomials in the commuting variables $u_1, \cdots, u_d$ with coefficients in R. Let $\mathcal{M}$ be a $\mathbf{R}$-module. Let $X^\mathcal{M}$ be the dual of $\mathcal{M}$. Then $X^\mathcal{M}$ is a compact abelian group.

For $\mathbf{n} = (n_1, \cdots, n_d) \in \mathbb{Z}^d$ we define an automorphism $\alpha_\mathbf{n}$ of $\mathcal{M}$ by

$$\alpha_\mathbf{n} a = u_1^{n_1} \ldots u_d^{n_d} a$$

for all $a \in \mathcal{M}$. By considering the dual we obtain an automorphism $\alpha_\mathbf{n}$ of $X^\mathcal{M}$. It can be easily seen that $\mathbf{n} \mapsto \alpha_\mathbf{n}$ defines a $\mathbb{Z}^d$-action on $X^\mathcal{M}$.

Suppose the additive structure of R is cyclic. Then it can be easily seen that the afore-defined $\mathbb{Z}^d$-action has DCC on $X^\mathcal{M}$ if and only if $\mathcal{M}$ is a Noetherian $\mathbf{R}$-module: a module A over a ring Q is called a Noetherian module if any increasing sequence $A_1 \subset A_2 \subset \cdots \subset A_n \subset A_{n+1} \subset \cdots$ of submodules is finite, that is, $A_n = A_m$ for large n and m and a ring Q is called a Noetherian ring if Q is a Noetherian Q-module.

If $R = \mathbb{Z}$, Theorem 11.2 of [KS] proved that $\mathbb{Z}^d$-action defined as above has DCC on $X^\mathcal{M}$ if (and only if) $\mathcal{M}$ is a finitely generated $\mathbf{R}$-module. Theorem 3.3 of [Sc1] has shown that if $R = \mathbb{Z}$, $\mathcal{M}$ is a finitely generated $\mathbf{R}$-module and the $\mathbb{Z}^d$-action defined as above is ergodic on $X^\mathcal{M}$, then there is a $\mathbf{n}$ such that $\alpha_\mathbf{n}$ is ergodic on $X^\mathcal{M}$. In this respect we prove the following using Theorem 4.1 when the additive group structure of R is a cyclic group.

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Theorem 5.3 *Assume that the additive group structure of R is cyclic and $\mathcal{M}$ is a finitely generated $\mathbf{R}$-module. Then we have the following:*

(1) $\{\alpha_{\mathbf{n}} \mid \mathbf{n} \in \mathbb{Z}^d\}$ has DCC on $X^{\mathcal{M}}$;

(2) if Γ is a finitely generated abelian group of $\mathbb{R}$-module homomorphisms of $\mathcal{M}$ and the dual action of Γ is ergodic on $X^{\mathcal{M}}$, then there is a $\alpha \in \Gamma$ such that the dual action of α is ergodic on $X^{\mathcal{M}}$.

Remark If we take $R = \mathbb{Z}$ and $\Gamma = \{\alpha_{\mathbf{n}} \mid \mathbf{n} \in \mathbb{Z}^d\}$, we retrieve Theorem 3.3 of [Sc1].

Proof We first claim that $\mathcal{M}$ is Noetherian. Since the additive structure of R is cyclic, R is either finite or $R = \mathbb{Z}$, hence R is Noetherian. By Hilbert's Basis Theorem, the polynomial ring $R[X_1, \cdots, X_d, Y_1, \cdots, Y_d]$ is also Noetherian (cf. 8.7 and 8.8 of [Sh]). Let $f \colon R[X_1, \cdots, X_d, Y_1, \cdots, Y_d] \to \mathbf{R}$ be the map defined $f(X_i) = u_i$ and $f(Y_i) = u_i^{-1}$ for all $1 \leq i \leq d$. Then f is a surjective ring homomorphism and hence by Lemma 8.2 of [Sh], we get that $\mathbf{R}$ is a Noetherian ring. Since $\mathcal{M}$ is a finitely generated $\mathbf{R}$-module, we get from Corollary 7.22 (i) of [Sh] that $\mathcal{M}$ is Noetherian. Since $\mathbb{Z}^d$-action having DCC on $X^{\mathcal{M}}$ is equivalent to $\mathcal{M}$ being Noetherian, (1) is proved.

Let α be a $\mathbf{R}$-module homomorphism. Then for any $\mathbf{n} = (n_1, \cdots, n_d) \in \mathbb{Z}^d$,

$$\alpha\alpha_{\mathbf{n}}(a) = \alpha(u_1^{n_1} \cdots u_d^{n_d} a) = u_1^{n_1} \cdots u_d^{n_d}\alpha(a) = \alpha_{\mathbf{n}}\alpha(a)$$

for all $a \in \mathcal{M}$. Thus, $Z(\Gamma)$ contains all α_n and hence by (1), $Z(\Gamma)$ has DCC. Now (2) follows from Theorem 4.1.

Remark Let $\mathcal{M}$ be free $\mathbf{R}$-module of rank k (we may take $\mathcal{M}$ to be direct sum of k-copies of $\mathbf{R}$). Then the group of invertible $\mathbf{R}$-module homomorphisms of $\mathcal{M}$, can be realized as invertible matrices with coefficients from the ring $\mathbf{R}$. Thus, Γ in (2) of Theorem 5.3 may be taken as a finitely generated abelian group of invertible matrices with coefficients from R.

We now give one more variation of examples considered in [KS]. Let $\mathbb{F}$ be a countable field and for any $d \geq 1$, let $\mathbf{S} = \mathbb{F}[u_1^{\pm 1}, \cdots, u_d^{\pm 1}]$ be the ring of Laurent polynomials in the commuting variables $u_1, \cdots, u_d$ with coefficients in $\mathbb{F}$. Let $\mathcal{N}$ be a $\mathbf{S}$-module. Let $K^{\mathcal{N}}$ be the dual of $\mathcal{N}$. Then $K^{\mathcal{N}}$ is a compact abelian group.

Consider the $\mathbb{Z}^d$-action defined as above, that is, for any $\mathbf{n} = (n_1, \cdots, n_d) \in \mathbb{Z}^d$ we define an automorphism $\alpha_{\mathbf{n}}$ of $\mathcal{N}$ by

$$\alpha_{\mathbf{n}}a = u_1^{n_1} \ldots u_d^{n_d} a$$

for all $a \in \mathcal{N}$.

Let $\Delta = \mathbb{F}^* \times \mathbb{Z}^d$. For $(q, \mathbf{n}) \in \Delta = \mathbb{F}^* \times \mathbb{Z}^d$, we define $\alpha_{(q,\mathbf{n})}$ by

$$\alpha_{(q,\mathbf{n})}(a) = q\alpha_{\mathbf{n}}a$$

for all $a \in \mathcal{N}$. By considering the dual, we get an automorphism of $K^{\mathcal{N}}$. Thus, we get an action of Δ on $K^{\mathcal{N}}$.

It can be easily seen that Δ having DCC on $K^{\mathcal{N}}$ is equivalent to $\mathcal{N}$ being a Noetherian $\mathbf{S}$-module. In this situation we have the following.

Theorem 5.4 *Suppose $\mathcal{N}$ is finitely generated as a $\mathbf{S}$-module. Then we have the following:*

(1) *Δ has DCC on $K^{\mathcal{N}}$, the dual of $\mathcal{N}$.*

(2) *If Γ is a finitely generated abelian group of $\mathbf{S}$-module homomorphisms of $\mathcal{N}$ and the dual action of Γ is ergodic on $K^{\mathcal{N}}$, then there is a $\alpha \in \Gamma$ such that the dual action of α is ergodic on $K^{\mathcal{N}}$.*

(3) *If $\Gamma = \{\alpha_{\mathbf{n}} \mid \mathbf{n} \in \mathbb{Z}^d\}$, then the dual action of Γ is ergodic on $K^{\mathcal{N}}$ implies that there is a $\mathbf{n} \in \mathbb{Z}^d$ such that the dual action of $\alpha_{\mathbf{n}}$ is ergodic on $K^{\mathcal{N}}$.*

Proof Since Δ having DCC on $K^{\mathcal{N}}$ is equivalent to $\mathcal{N}$ being Noetherian, it is sufficient to show that $\mathcal{N}$ is a Noetherian module over the ring $\mathbf{S}$ if $\mathcal{N}$ is finitely generated.

Since $\mathbb{F}$ is a field, it is easy to see that $\mathbb{F}$ is a Noetherian ring as $\mathbb{F}$ is the only non-zero ideal. By Hilbert's Basis Theorem, the polynomial ring $F[X_1, \cdots, X_d, Y_1, \cdots, Y_d]$ is also Noetherian (cf. 8.7 and 8.8 of [Sh]). Now $X_i \mapsto u_i$ and $Y_i \mapsto u_i^{-1}$ induces a ring homomorphism of $F[X_1, \cdots, X_d, Y_1, \cdots, Y_d]$ onto $\mathbf{S} = \mathbb{F}[u_1^{\pm 1}, \cdots, u_d^{\pm 1}]$ and hence by Lemma 8.2 of [Sh], we get that $\mathbf{S}$ is a Noetherian ring. Since $\mathcal{N}$ is a finitely generated $\mathbf{S}$-module, Corollary 7.22 (i) of [Sh] implies that $\mathcal{N}$ is Noetherian. This proves (1).

It is easy to see that any $\mathbf{S}$-module homomorphism commutes with Δ-action and hence by (1) $Z(\Gamma)$ has DCC if Γ is a group of $\mathbf{S}$-module homomorphisms. Thus, (2) follows from Theorem 4.1.

Since each $\alpha_{\mathbf{n}}$ is a $\mathbf{S}$-module homomorphism, (3) follows from (2).

We would like to note that compact groups considered in Theorem 5.1 are finite-dimensional whereas compact groups considered in this subsection need not be of finite-dimension. In fact, if $R = \mathbb{Z}$ or $\mathbb{F} = \mathbb{Q}$ and modules are free-modules, then compact groups in Theorems 5.3 and 5.4 are infinite-dimensional.

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