isibang/ms/2008/8
May 5th, 2008
http://www.isibang.ac.in/~statmath/eprints

Maximal Commutative Subalgebras Invariant for CP-Maps: (Counter-)Examples

B.V. Rajarama Bhat, Franco Fagnola, and Michael Skeide

Indian Statistical Institute, Bangalore Centre
8th Mile Mysore Road, Bangalore, 560059 India

Maximal Commutative Subalgebras Invariant for CP-Maps: (Counter-)Examples*

B.V. Rajarama Bhat, Franco Fagnola, and Michael Skeide

April 2008

Abstract

We solve, mainly by counterexamples, many natural questions regarding maximal commutative subalgebras invariant under CP-maps or semigroups of CP-maps on a von Neumann algebra. In particular, we discuss the structure of the generators of norm continuous semigroups on $\mathcal{B}(G)$ leaving invariant a maximal commutative subalgebra and show that there exists Markov CP-semigroups on M_d without invariant maximal commutative subalgebras for any $d > 2$.

1 Introduction

Markov semigroups, that is, semigroups of normal unital completely positive (CP-)maps on a von Neumann algebra $\mathcal{B} \subset \mathcal{B}(G)$ (G a Hilbert space) are models for irreversible evolutions both of classical and of quantum systems.

The structure of the generator of a norm-continuous CP semigroups on a von Neumann algebra or C^*-algebra is well-known after Lindblad [Lin76] and Christensen and Evans [CE79]. In several classes of CP semigroups this structure is further specialized revealing an unexpectedly rich algebraic structure. Here we recall some of them without claiming that this is a full list.

Quasi-free semigroups are well-known from the '70 and were the key example in the study of the irreversible evolution of open quantum systems. Another remarkable class of semigroups, called *generic*, arising from the weak coupling limit of a generic system interacting with a Boson reservoir was recently studied in [AFH06]. Davies [Dav79] gave a detailed description of the generators of those strongly continuous quantum dynamical semigroups which possess a pure stationary state. Generators of quantum Markov semigroups that are covariant with respect to the action of a unitary group were studied by Holevo [Hol96] in connection with semigroups

*BVRB is supported by the Department of Science and Technology (India) under the Swarnajayanthi Fellowship Project. FF and MS are supported by the Italian MUR (PRIN 2005). MS is supported by research funds of the Dipartimento S.E.G.e S. of University of Molise.

of instruments in quantum continual measurements [BP96]. Albeverio and Goswami [AG02] showed that the generators of trace symmetric quantum dynamical semigroups on von Neumann algebras are sums of double commutators with self-adjoint operators. This was extended to norm-continuous semigroups that are symmetric with respect to the scalar product induced on the algebra by an invariant state by Fagnola and Umanità in [FU07] who characterized the generator of detailed balance quantum Markov semigroups.

This paper is concerned with the problem of finding invariant commutative subalgebras $\mathcal{C}$ of a CP-semigroup on $\mathcal{B} \subset \mathcal{B}(G)$ and of its generator. Indeed, if G is separable, then the commutative von Neumann algebra $\mathcal{C}$ is isomorphic to $L^\infty(\Omega, \mathcal{F}, P)$ for some probability space, and a Markov semigroup is, indeed, the semigroup induced on $L^\infty(\Omega, \mathcal{F}, P)$ by a classical Markov semigroup of transition probabilities. More generally, if a Markov semigroup $T = (T_t)_{t \in \mathbb{R}_+}$ on a not necessarily commutative von Neumann algebra $\mathcal{B}$ leaves a commutative subalgebra $\mathcal{C}$ ***invariant*** (that is, $T_t(\mathcal{C}) \subset \mathcal{C}$ for all $t \in \mathbb{R}_+$), then the restriction to $\mathcal{C}$ gives rise to a classical Markov semigroup. Finding invariant commutative subalgebras means, thus, recognizing classical subsystems as embedded into a quantum one.

A commutative subalgebra $\mathcal{C}$ with $\mathbf{1} = \mathbf{1}_\mathcal{B} \in \mathcal{C} \subset \mathcal{B} \subset \mathcal{B}(G)$ is a ***maximal commutative subalgebra*** of $\mathcal{B}$, if $\mathcal{C} \subset \mathcal{D} \subset \mathcal{B}$ for a commutative subalgebra $\mathcal{D}$ implies $\mathcal{D} = \mathcal{C}$. A maximal commutative subalgebra of $\mathcal{B} = \mathcal{B}(G)$ is a called a ***maximal abelian subalgebra*** or a ***masa***. If G is separable and if $\mathcal{C} \subset \mathcal{B}(G)$ is a masa isomorphic to $L^\infty(\Omega, \mathcal{F}, P)$, then $G \cong L^2(\Omega, \mathcal{F}, P)$. If $\mathcal{C}$ is a maximal comutative subalgebra of $\mathcal{B} \subset \mathcal{B}(G)$, then we obtain a description of the system by *classical* (or *macroscopic*) parameters that is not improvable by measuring a set of classical observables. If $\mathcal{C}$ is a masa, then this description is complete.

Rebolledo [Reb05a] (see also [Reb05b]) proved the following sufficient criterion in the case $\mathcal{B} = \mathcal{B}(G)$: Let T be a normal CP-map on $\mathcal{B}(G)$ given by some ***Kraus decomposition*** $T(b) = \sum_i L_i^* b L_i$ ($L_i \in \mathcal{B}(G)$). Suppose that $\mathcal{C} \subset \mathcal{B}(G)$ is a masa generated by a single self-adjoint element $c \in \mathcal{C}$, and suppose that there are self-adjoint elements $c_i \in \mathcal{C}$ such that

$$cL_i - L_i c \;=\; c_i L_i.$$

Then $T(\mathcal{C}) \subset \mathcal{C}$. If T is the ***CP-part*** of the ***Lindblad generator*** [Lin76]

$$L(b) \;=\; \sum_i L_i^* b L_i + b\beta + \beta^* b$$

($\beta \in \mathcal{B}(G)$) of a uniformly continuous CP-semigroup $T_t = e^{tL}$ on $\mathcal{B}(G)$, then invariance of the CP-part plus invariance of the ***effective Hamitonian*** $b \mapsto b\beta + \beta^* b$ implies that the whole CP-semigroup leaves $\mathcal{C}$ invariant. In the case of a Markov semigroup (where L has to be normalized to $L(\mathbf{1}) = 0$) we get

$$L(b) \;=\; \sum_i L_i^* b L_i - \tfrac{b\left(\sum_i L_i^* L_i\right) + \left(\sum_i L_i^* L_i\right) b}{2} + i[b, h],$$

for the self-adjoint $h = \mathsf{Im}\beta \in \mathcal{B}(G)$. As the CP-part T alone, by Rebolledo's criterion, leaves C invariant, we have, in particular, that $\sum_i L_i^* L_i = T(\mathbf{1}) \in C$. So, if (and only if; see [FS07, Lemma 4.4]) also $h \in C$ so that the ***Hamiltonian*** $b \mapsto i[b, h]$ leaves C invariant, then all T_t leave C invariant.

Fagnola and Skeide [FS07] proved the following generalization of Rebolledo, which now provides a sufficient and necessary criterion.

1.1 Theorem [FS07]. *Let T be a normal CP-map on $\mathcal{B}(G)$ with Kraus decomposition $T(b) = \sum_{i \in I} L_i^* b L_i$. Then T leaves a maximal abelian von Neumann algebra $C \subset \mathcal{B}(G)$ invariant, if and only if for every $c \in C$ there exist coefficients $c_{ij}(c) \in C$ $(i, j \in I)$ such that*

$$1.)\quad c_{ij}(c^*) \;=\; c_{ji}(c)^*, \qquad\qquad 2.)\quad cL_i - L_i c \;=\; \sum_{j \in I} c_{ij}(c) L_j,$$

for all $c \in C$.

Theorem 1.1 is a special case of [FS07, Theorem 3.1] for general von Neumann algebras. Fagnola and Skeide also provide the sufficient and necessary criterion [FS07, Theorem 4.2] for the generator of a uniformly continuous CP-semigroup on a general von Neumann algebra. We state here the result of the specialization to $\mathcal{B}(G)$. A proof is delegated to the appendix.

1.2 Theorem. *Let L be the generator of a uniformly continuous normal CP-semigroup on $\mathcal{B}(G)$ with Lindblad form $L(b) = \sum_{i \in I} L_i^* b L_i + b\beta + \beta^* b$. Then L, or equivalently, all $T_t = e^{tL}$, leave a maximal abelian von Neumann algebra $C \subset \mathcal{B}(G)$ invariant, if and only if there exist coefficients $\gamma = \gamma^*, c_i \in C$, and for every $c \in C$ there exist coefficients $c_{ij}(c) \in C$ $(i, j \in I)$ such that*

$$1.)\quad c_{ij}(c^*) \;=\; c_{ji}(c)^*, \qquad\qquad 2.)\quad cL_i - L_i c \;=\; \sum_{j \in I} c_{ij}(c)(L_j - c_j),$$

$$3.)\quad L(c) \;=\; \sum_{i \in I} (L_i - c_i)^* c (L_i - c_i) + \gamma c$$

for all $c \in C$.

1.3 Remark. We would like to mention that in both theorems (like in Theorems A.1 and A.1, from which the former are derived) maximal commutativity of C easily guarantees sufficiency. The stated conditions are necessary (in all four theorems) for invariance of the unital commutative subalgebra C, even if C is not maximal commutative.

Like [Par92, Theorem 30.16], the following theorem characterizes the possibilities to transform a generator in minimal Lindblad form into another. The proof illustrates the power of techniques from product systems of Hilbert modules. But as we do not need these techniques in the rest of these notes, we postpone also this proof to the appendix.

1.4 Theorem. *Let L be the generator of a uniformly continuous normal CP-semigroup on $\mathcal{B}(G)$ in minimal Lindblad form $L(b) = \sum_{i \in I} L_i^* b L_i + b\beta + \beta^* b$, and let $K(b) = \sum_{j \in J} K_j^* b K_j + b\alpha + \alpha^* b$ be another generator.*

Then $K = L$, if and only if there exists a matrix $\begin{pmatrix} \gamma & \eta^* \\ \eta' & M \end{pmatrix} \in M_{(1+\#J)\times(1+\#I)}$, with $\eta' \in \mathbb{C}^{\#J}$ arbitrary, $M = (a_{ji})_{ji} \in M_{\#J\times\#I}$ an isometry, $\eta = -M^*\eta' \in \mathbb{C}^{\#I}$, and $\gamma = ih - \frac{\langle \eta', \eta' \rangle}{2} \in \mathbb{C}$ ($h \in \mathbb{R}$ arbitrary), such that

$$\alpha = \beta + \gamma \mathbf{1} + \sum_{i \in I} \overline{\eta}_i L_i, \qquad\qquad K_j = \eta'_j \mathbf{1} + \sum_{i \in I} a_{ji} L_i.$$

This holds for arbitrary cardinalities $\#I$ and $\#J$, if infinite sums are understood as strongly convergent.

1.5 Corollary. *A similar result holds if the Lindblad form of L is not necessarily minimal. In that case M may be just a partial isometry and η' must be such that $MM^*\eta' = \eta'$.*

Proof. Observe that the minimal L_i in the theorem may be recoverd as $L_i = \eta_i \mathbf{1} + \sum_{j \in J} \overline{a}_{ji} K_j$. So, in order to compare two not necessarily minimal Lindblad forms we may simply "factor" through a minimal one. ■

There are several natural questions around about Theorems 1.1 and 1.2 and how they are related with Rebolledo's original criterion. Most of them are motivated by the examples with 2×2–matrices that have been studied in [FS07]. The goal of these notes is to give answers to these questions, and Theorem 1.4 will play a crucial role. As our results here show, the answers sometimes are typical only for M_2 and look different already for M_3. Therefore, in the following list of questions and throughout the answers later on in these notes we will have to distinguish between M_2 and higher dimensional settings.

We explain briefly why counterexamples for a single map furnish also counterexamples for the semigroup case.

1.6 Observation. The CP-semigroup $T_t = e^{tL}$ leaves a subalgebra invariant, if and only if its generator L leaves that subalgebra invariant. So, for all questions about invariance for CP-semigroups we are done if we answer the single mapping case. (If T is CP-map with a certain invariance property, then e^{tT} shares that property.) Similarly, if T is a **unital** CP-map leaving a certain subalgebra invariant or not, then $L := T - \mathsf{id}$ is the generator of a Markov semigroup sharing this property. (This is so, simply because id leaves every subalgebra invariant so that L and, therefore, T_t share the invariance properties of T.)

We now list our questions and the answers we obtain later on in the remainder of these notes.

1. Does every CP-semigroup on $\mathcal{B}(G)$ leave some masa invariant?

Answer: No, by Example 2.1 already for a single CP-map on M_2 and, therefore, also for a CP-semigroup on M_2 (and, therefore, for all $\mathcal{B}(G)$).

2. Does every Markov semigroup on $\mathcal{B}(G)$ leave some masa invariant?

   Answer: Yes, for M_2 by Theorem 2.4 both for Markov semigroups and for unital CP-maps.

   Answer: No, for M_3 by Example 3.3 and for $\mathcal{B}(G)$ (separable G for any dimension $d \geq 3$ including $d = \infty$) by Section 4, both for Markov semigroups and for unital CP-maps.

3. Is Rebolledo's criterion equivalent to the one in Theorem 1.1? More precisely, does every normal CP-map on $\mathcal{B}(G)$ that leaves a masa invariant, admit a Kraus decomposition fulfilling Rebolledo's criterion?

   Answer: No, already for M_2, by Example 2.2 for Markov semigroups and for unital CP-maps.

4. Suppose we have a generator leaving a masa invariant. Does every such generator decompose, like in Rebolledo's criterion, into a CP-part and a Hamiltonian part that leave the masa invariant, separately?

   Answer: No for the CP-part, already in the case of a Markov semigroup on M_2 by Example 2.8. This answer extends to all $\mathcal{B}(G)$.

   Answer: Yes for the Hamiltonian part, in the case of CP-semigroups on M_2 by Corollary 2.6. No, in the case of CP-semigroups on M_3 and higher dimension, by Example 3.2.

In Section 2 we study everything related to $\mathcal{B} = M_2$, while Section 3 is dedicated to $\mathcal{B} = M_3$. In the final Section 4 we deal also with an infinite-dimensional example.

We would like to mention that a further natural question asked in [FS07], namely, whether the necessary and sufficient criterion in [FS07] remains valid for unbounded generators, has a negative answer, too. There exist generators in terms of double commutators and the CCR that leave invariant a masa but that do not fufill the (unbounded analogue of the) criterion in [FS07]. We will study these generators elsewhere systematically. Here we restrict ourselves to the bounded case.

Conventions. For every $n \in \mathbb{N}$ we denote by $M_n = M_n(\mathbb{C}) = \mathcal{B}(\mathbb{C}^n)$ the von Neumann algebra of $n \times n$–matrices with complex entries. By M_∞ we mean the von Neumann algebra $\mathcal{B}(G)$ for a separable infinite-dimensional Hilbert space G. The elements of $\mathcal{B}(G)$ are considered as matrices with respect to a fixed orthonormal basis $(e_n)_{n\in\mathbb{N}_0}$ of G. By $\mathcal{D}_n$ ($n \in \mathbb{N}\cup\{\infty\}$) we denote the respective subalgebras of diagonal matrices.

2 Examples and results for M_2

We start with some counterexamples for things that do not even work for M_2.

2.1 Example. Consider the CP map $T\colon M_2(\mathbb{C}) \to M_2(\mathbb{C})$ defined by

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} a+b+c+d & b+d \\ c+d & d \end{pmatrix}.$$

If T leaves a masa $C \subset M_2$ invariant, then $\{\mathbf{1}, T(\mathbf{1}), T^2(\mathbf{1}), \ldots\} \subset C$ should all commute. But clearly this is not the case as $T(\mathbf{1}) = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ and $T^2(\mathbf{1}) = \left(\begin{smallmatrix} 5 & 2 \\ 2 & 1 \end{smallmatrix}\right)$ do not commute. So neither the CP-map T nor the CP-semigroup e^{tT} leave a masa of M_2 invariant.

Any CP-map T may be extended to a CP-map $\widehat{T}(X) = T(\mathbf{1}_2 X \mathbf{1}_2)$ on M_d for any $d \geq 3$ including ∞. Again $\widehat{T}(\mathbf{1})$ and $\widehat{T}^2(\mathbf{1})$ do not commute. So, $\widehat{T}$ has no invariant masa and the CP-semigroup $e^{t\widehat{T}}$ $(= \widehat{e^{tT}})$ shares this property.

2.2 Example. Define the CP-map $T\colon M_2 \to M_2$ by

$$T(X) = L_1^* X L_1 + L_2^* X L_2$$

where

$$L_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \qquad\qquad L_2 = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Then

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(a + \frac{b}{\sqrt{2}} + \frac{c}{\sqrt{2}} + d) & \frac{b-c}{2\sqrt{2}} \\ \frac{c-b}{2\sqrt{2}} & \frac{1}{2}(a + \frac{b}{\sqrt{2}} + \frac{c}{\sqrt{2}} + d) \end{pmatrix}.$$

We see that T is unital and that it leaves the diagonal subalgebra $\mathcal{D}_2$ of M_2 invariant.

Now suppose $T(X) = \sum_j K_j^* X K_j$ is another Kraus decomposition of T. Then each K_j, is a linear combination of L_1, L_2; see Observation A.3. Say $K_1 = aL_1 + bL_2$, $a, b \in \mathbb{C}$. Now suppose this decomposition satisfies Rebolledo's condition. Then for every diagonal matrix $D = \left(\begin{smallmatrix} d_1 & 0 \\ 0 & d_2 \end{smallmatrix}\right) \in \mathcal{D}_2$ there exists $D' = \left(\begin{smallmatrix} d_1' & 0 \\ 0 & d_2' \end{smallmatrix}\right) \in \mathcal{D}_2$ (depending upon D) such that

$$D' K_1 = K_1 D.$$

So

$$\begin{pmatrix} d_1' & 0 \\ 0 & d_2' \end{pmatrix}\begin{pmatrix} \frac{a}{\sqrt{2}} & \frac{b}{\sqrt{2}} \\ \frac{a-b}{2} & \frac{a+b}{2} \end{pmatrix} = \begin{pmatrix} \frac{a}{\sqrt{2}} & \frac{b}{\sqrt{2}} \\ \frac{a-b}{2} & \frac{a+b}{2} \end{pmatrix}\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$$

or

$$\begin{pmatrix} d_1' \frac{a}{\sqrt{2}} & d_1' \frac{b}{\sqrt{2}} \\ d_2' \left(\frac{a-b}{2}\right) & d_2' \left(\frac{a+b}{2}\right) \end{pmatrix} = \begin{pmatrix} d_1 \frac{a}{\sqrt{2}} & d_2 \frac{b}{\sqrt{2}} \\ d_1 \left(\frac{a-b}{2}\right) & d_2 \left(\frac{a+b}{2}\right) \end{pmatrix}$$

It is easily seen that no non-zero K_1 satisfies this condition. We conclude that Rebolledo's condition is not a necessary condition.

We now discuss several things that work only for M_2. The counterexamples in the general case for the statements we prove here for M_2, must wait until Section 3 on M_3.

2.3 Lemma. *Let α be a linear $*$–map on M_2 such that $\alpha(\mathbf{1}) \in \mathbb{C}\mathbf{1}$. Then α leaves a masa of M_2 invariant.*

Proof. The *Cayley-Hamilton theorem* asserts that for every matrix $Y \in M_n$ the characteristic polynomial P of Y gives $P(Y) = 0$. It follows that for every $Y \in M_2$ the subalgebra of M_2 generated by Y has the form $\mathcal{C}_Y := \mathbb{C}\mathbf{1}_2 + \mathbb{C}Y$. Therefore, if we find a self-adjoint $Y = Y^* \notin \mathbb{C}\mathbf{1}_2$ such that $\alpha(Y) \in \mathcal{C}_Y$, then $\mathcal{C}_Y$ is a masa of M_2 invariant for α.

Define the 4–dimensional real subspace $S = \{X \in M_2 \colon X = X^*\}$ of self-adjoint elements of M_2. By tr we denote the normalized trace on M_2. Then $\mathsf{id}_S - \mathsf{tr}\,\mathbf{1} \colon X \mapsto X - \mathsf{tr}(X)\mathbf{1}$ defines a projection onto the subspace $S_0 := S \cap \ker \mathsf{tr}$ of self-adjoint zero-trace operators. The linear map $(\mathsf{id}_S - \mathsf{tr}\,\mathbf{1}) \circ \alpha$ leaves the 3–dimensional real vector space S_0 invariant. Therefore, $\beta := (\mathsf{id}_S - \mathsf{tr}\,\mathbf{1}) \circ \alpha \upharpoonright S_0$ has an eigenvector Y to some real eigenvalue. Clearly, $\alpha(Y) \in \mathcal{C}_Y$ and $Y \notin \mathbb{C}\mathbf{1}_2$, so that $\mathcal{C}_Y$ is a masa invariant for α. (Of course, it is an easy exercise to check directly that $Y^2 \in \mathbb{C}\mathbf{1}$ for every self-adjoint zero-trace operator $Y \in M_2$, showing that $\mathcal{C}_Y$ is an algebra without reference to the Cayley-Hamilton theorem.) ■

The following theorem is a simple corollary of the lemma.

2.4 Theorem. *Every unital CP-map T on M_2 has an invariant masa. Every generator L of a Markov semigroup on M_2 has an invariant masa.*

Proof. T is a linear $*$–map that maps $\mathbf{1}$ to $1 \cdot \mathbf{1}$ and L is a linear $*$–map that maps $\mathbf{1}$ to $0 \cdot \mathbf{1}$. ■

Once assured existence of an invariant masa of M_2, by a basis transformation we may always assume that this invariant subalgebra is $\mathcal{D}_2$. We now investigate when a generator leaving $\mathcal{D}_2$ invariant can be split such that also its CP-part or at least its Hamiltonian part leaves $\mathcal{D}_2$ invariant. Note that by Corollary 2.6 and Example 2.8 these two properties need not coincide.

2.5 Theorem. *Suppose the minimal Lindblad generator $L(X) = \sum_{i=1}^d L_i^* X L_i + XB + B^* X$ of a CP-semigroup on M_n leaves $\mathcal{D}_n$ invariant. Then L admits a (minimal) Lindblad form whose CP-part leaves $\mathcal{D}_n$ invariant separately, if and only if there is a linear combination $K := \sum_{i=1}^d \overline{\eta}_i L_i$ such that $B + K \in \mathcal{D}_n$.*

Proof. Note that $T(X) = \sum_{i=1}^d L_i^* X L_i$ leaves $\mathcal{D}_n$ invariant, if and only if $\Delta \colon X \mapsto XB + B^* X = \{X, \mathsf{Re}\, B\} + i[X, \mathsf{Im}\, B]$ does. We show that this happens, if and only if $B \in \mathcal{D}_n$. "If" being clear, for "only if" suppose that Δ leaves $\mathcal{D}_n$ invariant. Then $\Delta(\mathbf{1}) = 2\,\mathsf{Re}\, B \in \mathcal{D}_n$ and, therefore

$\{X, \operatorname{Re} B\} \in \mathcal{D}_n$ for all $X \in \mathcal{D}_n$. Clearly, $X \mapsto [X, \operatorname{Im} B]$ leaves $\mathcal{D}_n$ invariant, if and only if $\operatorname{Im} B \in \mathcal{D}_n$.

Suppose A, K_j are the coefficients of another Lindblad form of L. So, in order that the CP-part $\sum_{j=1}^{d'} K_j^* X K_j$ leaves $\mathcal{D}_n$ invariant, it is necessary and sufficient that $A \in \mathcal{D}_n$. By Theorem 1.4 the only possibility to achieve this, is adding linear combinations of the L_i (and $\mathbf{1}$) to B. So, the condition $\exists K = \sum_{i=1}^{d} \overline{\eta}_i L_i \colon B + K \in \mathcal{D}_n$ is necessary. On the other hand, suppose that K exists. In view of Theorem 1.4 put $M = \mathbf{1}_d$, $\eta' = -\eta$, and $\gamma = -\frac{\langle \eta, \eta \rangle}{2}$. Then the Lindblad generator with coefficients $A = B + \gamma\mathbf{1} + K$ and $K_i = L_i - \eta_i \mathbf{1}$ coincides with L and $X \mapsto XA + A^* X$ leaves $\mathcal{D}_n$ invariant. ■

2.6 Corollary. *Every generator L of a CP-semigroup on M_2 leaving $\mathcal{D}_2$ invariant can be written in a Lindblad form where also the Hamiltonian part leaves $\mathcal{D}_2$ invariant.*

Proof. Either all L_i are in $\mathcal{D}_2$ so that also $B \in \mathcal{D}_2$, or there exists at least one L_k that is not diagonal. $H = \operatorname{Im} B$ is self-adjoint and $\operatorname{Im}(\overline{\eta}_k L_k)$ will eliminate the off-diagonal from H for suitable η_k. ■

2.7 Remark. Note that this is true for arbitrary generators (not necessarily leaving $\mathcal{D}_2$ invariant) as soon as the CP-part does not leave $\mathcal{D}_2$ invariant (assuring existence of a nondiagonal L_k).

2.8 Example. Let $L(X) = L_1^* X L_1 + L_2^* X L_2 + XB + B^* X$ with

$$B := -\frac{1}{2}\begin{pmatrix} 7 & 6 \\ 10 & 8 \end{pmatrix}, \qquad L_1 := \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad L_2 := \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}.$$

One easily verifies that L leaves $\mathcal{D}_2$ invariant and that $L(\mathbf{1}) = 0$. However, all linear combinations of L_1 and L_2 have equal off-diagonal elements, and B has not. Therefore, none of the linear combinations $B + \gamma\mathbf{1} + \overline{\eta}_1 L_1 + \overline{\eta}_2 L_2$ will be diagonal. In conclusion, it is not possible to find a Lindblad form with effective Hamiltonian and CP-part that leave $\mathcal{D}_2$ invariant separately.

This example extends easily to arbitrary higher dimension $\mathcal{B}(G)$, if we embed all coefficients it into the M_2–corner of $\mathcal{B}(G)$.

3 Examples for M_3

Apart from the counterexamples, the preceding section contained also some positive results which were, however, specific for M_2. In the present section we give counterexamples to the analogue statements in M_3.

3.1 Remark. This behaviour, a qualitative jump for what is possible when passing from dimension 2 to dimension 3, reminds us somehow of *Gleason's theorem*. That theorem gives a characterization of those *probability* functions defined on the lattice of projections of $\mathcal{B}(G)$ that extend as normal states to all of $\mathcal{B}(G)$, the so-called *frame functions*; see, for instance, the book Parthasarthy [Par92]. Gleason's theorem holds for all dimensions $n \geq 3$ of G, but not for $n = 2$.

We start with an example in M_3 that contradicts the statement of Corollary 2.6 for M_2.

3.2 Example. Let $L(X) = L_1^* X L_1 + L_2^* X L_2 + XB + B^* X$ with

$$B := -\frac{1}{2}\begin{pmatrix} 7 & 6 & 0 \\ 2 & 11 & 0 \\ 4 & 10 & 26 \end{pmatrix}, \qquad L_1 := \begin{pmatrix} 1 & 3 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 5 \end{pmatrix}, \qquad L_2 := \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}.$$

One calculates

$$L\begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix} = \begin{pmatrix} -6d_1 + 2d_2 + 4d_3 & 0 & 0 \\ 0 & 9d_1 - 10d_2 + d_3 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

so that L leaves $\mathcal{D}_3$ invariant and $L(\mathbf{1}) = 0$. One easily computes

$$2B' := c_1 L_1 + c_2 L_2 + 2B = \begin{pmatrix} c_1 - 7 & 3c_1 - 6 & 0 \\ c_1 + c_2 - 2 & c_2 - 11 & 0 \\ 2c_2 - 4 & c_1 - 10 & 5c_1 + c_2 - 26 \end{pmatrix}.$$

For that $B' - B'^*$ is diagonal we obtain the three equations $c_1 - 10 = 0$, $2c_2 - 4 = 0$, and $3c_1 - 6 - \overline{c}_1 - \overline{c}_2 + 2 = 0$. Inserting $c_1 = 10$ and $c_2 = 2$ into the third equation gives $30 - 6 - 10 - 2 + 2 = 14 \neq 0$. We conclude that no other Lindblad form of L has a Hamiltonian part leaving $\mathcal{D}_3$ invariant.

Also this example extends easily to arbitrary higher dimensional $\mathcal{B}(G)$, if we embed all coefficients into the M_3–corner of $\mathcal{B}(G)$.

We now construct an example in M_3 that contradicts the statement of Theorem 2.4 for M_2.

3.3 Example. We wish to construct a unital CP-map T on M_3 that does not leave any masa invariant. The technique of Example 2.1 to take an element X of the commutative subalgebra and to show that the family $X, T(X), T^2(X), \ldots$ is not commuting, is the same. What made Example 2.1 so simple, was that we could choose $X = \mathbf{1}$ and $\mathbf{1}$ is contained in every masa. Here $T(\mathbf{1}) = \mathbf{1}$. So, we must find another element X. But now, such an X will depend on the subalgebra in question. The fact, that we want to have the statement for all masas, is what makes the problem considerably more difficult. Moreover, this difficulty increases with dimension, because the "number" of possible masas increases rapidly. Here we discuss the case M_3, while the case of arbitrary dimension (> 2) is postponed to the following section.

Our strategy is to limit the "number" of candidates for an invariant masa, by making the range of T as small as possible, though noncommutative. So the range must contain at least two elements A_1 and A_2 that do not commute and $\mathbf{1}$. (That is, in particular, the triple $\mathbf{1}, A_1, A_2$ must be linearly independent.) And the mapping should be CP. To avoid problems with complete positivity we try a map T that factors through the conditional expectation $\mathbb{E}_3$

$$\mathbb{E}_3\begin{pmatrix} x_{00} & x_{01} & x_{02} \\ x_{10} & x_{11} & x_{22} \\ x_{20} & x_{21} & x_{22} \end{pmatrix} := \begin{pmatrix} x_{00} & 0 & 0 \\ 0 & x_{11} & 0 \\ 0 & 0 & x_{22} \end{pmatrix}$$

onto $\mathcal{D}_3$. The fact that every conditional expectation is CP, shows that for arbitrary positive A_1 and A_2 such that also $A_0 := \mathbf{1} - A_1 - A_2$ is positive, the map T on M_3 defined by setting

$$T\begin{pmatrix} x_{00} & x_{01} & x_{02} \\ x_{10} & x_{11} & x_{22} \\ x_{20} & x_{21} & x_{22} \end{pmatrix} := x_{00}A_0 + x_{11}A_1 + x_{22}A_2$$

is CP and unital.

Our choice for A_1 and A_2 is

$$A_1 := \frac{1}{4}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad\qquad A_2 := \frac{1}{4}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

It is easy to check that $A := A_1 + A_2 = \frac{1}{4}\left(\begin{smallmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{smallmatrix}\right)$ has norm $\|A\| \leq 1$ so that $A_0 := \mathbf{1} - A$ is positive. (See Section 4 for a more general estimate in higher dimension. Actually, $\|A\| = \frac{3}{4}$ but this estimate depends on the dimension.) The commutator is $[A_1, A_2] = \frac{1}{16}\left(\begin{smallmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{smallmatrix}\right) \neq 0$.

Let $X = \lambda_0\mathbf{1} + 4\lambda_1 A_1 + 4\lambda_2 A_2$ be an arbitrary element in the range $T(M_n)$. For checking whether X and $T(X)$ commute, the value of λ_0 is irrelevant ($\mathbf{1}$ commutes with everything and $T(\mathbf{1}) = \mathbf{1}$). Henceforth, we put $\lambda_0 = 0$. Then

$$T(X) = \lambda_1 A_0 + (\lambda_1 + \lambda_2)A_1 + \lambda_2 A_2 = \lambda_1\mathbf{1} + \lambda_2 A_1 + (\lambda_2 - \lambda_1)A_2.$$

Once more, for checking whether X and $T(X)$ commute, the summand $\lambda_1\mathbf{1}$ is irrelevant. In general, the commutator of $\lambda_1 A_1 + \lambda_2 A_2$ and $\mu_1 A_1 + \mu_2 A_2$ is

$$[\lambda_1 A_1 + \lambda_2 A_2, \mu_1 A_1 + \mu_2 A_2] = (\lambda_1\mu_2 - \lambda_2\mu_1)[A_1, A_2].$$

Since $[A_1, A_2] \neq 0$, for that X and $T(X)$ commute we find the necessary and sufficient condition

$$\lambda_1(\lambda_2 - \lambda_1) - \lambda_2^2 = 0. \tag{3.1}$$

If $\lambda_1 = 0$, then so is λ_2. But we are interested in more than just the trivial solution $X = \lambda_0\mathbf{1}$. If $\lambda_1 \neq 0$, then (by dividing X and $T(X)$ by λ_1) we may assume $\lambda_1 = 1$. Then (3.1) transforms into the second order equation

$$\lambda_2 - 1 - \lambda_2^2 = 0$$

for λ_2 with solutions $\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$. As these numbers both are not real but $\lambda_1 = 1 \in \mathbb{R}$, this means the problem of finding $X \in T(M_3)$ such that $[X, T(X)] = 0$ has no self-adjoint solution different from the multiples of the identity. Since every masa of M_3 is spanned by three orthogonal rank-one projections, this means there is no masa that is left invariant by T. The same is true for the Markov semigroup $e^{t(T-\mathrm{id})}$.

4 An example for $M_\infty = \mathcal{B}(G)$

In this section we generalize Example 3.3 to arbitrary (countable) dimension. We discuss first the infinite-dimensional case, while its modification to finite dimensions is the simple Observation 4.5.

Fix a Hilbert space G with an ONB $(e_n)_{n\in\mathbb{N}_0}$. Denote by $\mathbb{E}_\infty := \sum_{n=0}^{\infty} e_n e_n^* \bullet e_n e_n^*$ the conditional expectation onto the commutative subalgebra $\mathcal{D}_\infty := \{\sum_{n=1}^{\infty} \lambda_n e_n e_n^* \colon \sup_{n\in\mathbb{N}_0} |\lambda_n| < \infty\}$.

For every $n \in \mathbb{N}$ define the operator $A_n = \frac{1}{4}(e_{n-1}e_{n-1}^* + e_{n-1}e_n^* + e_n e_{n-1}^* + e_n e_n^*)$. In terms of the ONB, A_n has the matrix representation

$$A_n \;=\; \begin{array}{cc} \frac{1}{4}\begin{pmatrix} & & & & \\ & 1 & 1 & & \\ & 1 & 1 & & \\ & & & & \end{pmatrix} & \begin{array}{c} n-1 \\ n \end{array} \\ \begin{array}{cc} n-1 & n \end{array} & \end{array}$$

with 1's in the intersections of the $(n-1)$st and nth row, respectively, with the $(n-1)$st and nth column, respectively, and 0's elsewhere. We define $A = \sum_{n=0}^{\infty} A_n$. As for every $k, \ell \in \{0, 1\}$ the sum $\sum_{n=1}^{\infty} e_{n-k} e_{n-\ell}^*$ has norm one, the norm of A is not greater than $4 \cdot \frac{1}{4} = 1$. (Actually, it is not difficult to show that $\|A\| = 1$.) It follows that the map

$$\tau(e_0 e_0^*) \;:=\; \mathbf{1} - A \qquad\qquad \tau(e_n e_n^*) \;:=\; A_n \quad (n \in \mathbb{N})$$

extends to a unique normal and unital CP-map $\tau\colon \mathcal{D}_\infty \to \mathcal{B}(G)$.

It follows that $T := \tau \circ \mathbb{E}_\infty$ is a normal unital CP-map on $\mathcal{B}(G)$. We shall show that that no masa of $\mathcal{B}(G)$ is left invariant by T.

Let $R := \{\lambda_0(\mathbf{1} - A) + \sum_{n=1}^{\infty} \lambda_n A_n \colon \sup_{n\in\mathbb{N}_0} |\lambda_n| < \infty\}$ denote the range of T. Clearly,

$$\lambda_0(\mathbf{1} - A) + \sum_{n=1}^{\infty} \lambda_n A_n \;=\; \lambda_0 \mathbf{1} + \sum_{n=1}^{\infty} (\lambda_n - \lambda_0) A_n,$$

and $\sup_{n\in\mathbb{N}_0} |\lambda_n| < \infty$, if and only if $\sup_{n\in\mathbb{N}_0} |\lambda_n - \lambda_0| < \infty$. Thus,

$$R \;=\; \Big\{\lambda_0 \mathbf{1} + \sum_{n=1}^{\infty} \lambda_n A_n \colon \sup_{n\in\mathbb{N}_0} |\lambda_n| < \infty\Big\}.$$

Clearly, the coefficients λ_n $(n \in \mathbb{N})$ are unique. (Apply the normal functionals $\langle e_n, \bullet e_{n-1}\rangle$.) So, also λ_0 is unique. This shows that τ is injective.

4.1 Proposition. *Two elements $X = \lambda_0\mathbf{1} + \sum_{n=1}^{\infty} \lambda_n A_n$ and $Y = \mu_0\mathbf{1} + \sum_{n=1}^{\infty} \mu_n A_n$ of R commute, if and only if the coefficients fulfill*

$$\lambda_n \mu_{n+1} \;=\; \lambda_{n+1}\mu_n \tag{4.1}$$

for all $n \in \mathbb{N}$.

Proof. We denote $C_n = [A_n, A_{n+1}]$ and observe that $[A_n, A_m] = 0$ for $|m - n| \geq 2$. So,

$$[X, Y] \;=\; \sum_{i=1}^{\infty} (\lambda_n \mu_{n+1} - \lambda_{n+1}\mu_n) C_n.$$

Calculating explicitly the C_n and applying the normal linear functionals $\langle e_{n+1}, \bullet e_{n-1}\rangle$, the condition follows. ■

Fix an element $X = \lambda_0\mathbf{1} + \sum_{n=1}^{\infty} \lambda_n A_n \in R$. Define a sequence $i_1 < j_1 < i_2 < j_2 < \ldots$ of elements in $\mathbb{N} \cup \{\infty\}$ by setting

$$i_1 \;:=\; \min\{n \geq 1 \colon \lambda_n \neq 0\}, \quad j_n \;:=\; \min\{n > i_n \colon \lambda_n = 0\}, \quad i_{n+1} \;:=\; \min\{n > j_n \colon \lambda_n \neq 0\},$$

($n \in \mathbb{N}$), where we use the conventions $\inf \emptyset = \infty$ and, for formal reasons, $\infty < \infty$. (We may also say, the sequence terminates once we encounter ∞ for the first time.) That is, the family $(\lambda_n)_{n \in \mathbb{N}}$ (we are not really interested in λ_0) has the form

$$(0, \ldots, 0, 0, \lambda_{i_1}, \ldots, \lambda_{j_1 - 1}, 0, 0, \ldots, 0, 0, \lambda_{i_2}, \ldots, \lambda_{j_2 - 1}, 0, \ldots),$$

where all mentioned λ_n are different from 0, and where the numbers of zeroes in between these blocks is at least one.

4.2 Proposition. *$Y = \mu_0\mathbf{1} + \sum_{n=1}^{\infty} \mu_n A_n \in R$ commutes with X, if and only if the family $(\mu_n)_{n \in \mathbb{N}}$ has the form*

$$(\mu_1, \ldots, \mu_{i_1 - 2}, 0, \alpha_1\lambda_{i_1}, \ldots, \alpha_1\lambda_{j_1 - 1}, 0, \mu_{j_1 + 1}, \ldots, \mu_{i_2 - 2}, 0, \alpha_2\lambda_{i_2}, \ldots, \alpha_2\lambda_{j_2 - 1}, 0, \ldots),$$

where all mentioned μ_n and α_m are arbitrary. (For that in between the mth and the $(m+1)$st block of non-zero λ's there remains some free μ, the number $i_{m+1} - j_m$ of zeros must be at least 3.)

Proof. Simple inductive verification of the Equations (4.1), step by step. ■

Note that also $\mu_{j_m} = 0 = \alpha_m \cdot 0 = \alpha_m \lambda_{j_m}$. So we can say the blocks $(\mu_{i_m}, \ldots, \mu_{j_m})$ must be multiples of the blocks $(\lambda_{i_m}, \ldots, \lambda_{j_m})$. Where $i_m > 1$, this extends even to the blocks $(\mu_{i_m - 1}, \ldots, \mu_{j_m})$. Only those μ_n that are not contained in any of these blocks, can be chosen freely.

4.3 Lemma. *Fix a non-zero element $X = \lambda_0 \mathbf{1} + \sum_{n=1}^{\infty} 4\lambda_n A_n \in R$ and determine the sequence $i_1 < j_1 < i_s < j_2 < \ldots$ as above. Then X commutes with $T(X)$, if and only if one of the following conditions is fulfilled.*

1. *$i_1 = \infty$, so that $\lambda_n = 0$ for $n \in \mathbb{N}$ and $X = \lambda_0 \mathbf{1}$.*

2. *$i_1 = 1$ and $j_m \neq \infty \Rightarrow i_{m+1} = j_m + 1 \neq \infty$. For every block $(\lambda_{i_m}, \ldots, \lambda_{j_m})$ with $j_m \neq \infty$ we have*

$$\lambda_n = \lambda_1 \frac{1 - \beta_m^{n-i_m+1}}{1 - \beta_m} \qquad (n = i_m, \ldots, j_m)$$

for $\beta_m \in \mathbb{C}$ an $(i_{m+1} - i_m)$th root of 1 with the property that $\beta_m^n \neq 1$ for $n < i_{m+1} - i_m$. If there is a last $i_m < \infty$ so that $j_m = \infty$, then

$$\lambda_n = \lambda_1 \frac{1 - \beta_m^{n-i_m+1}}{1 - \beta_m} \qquad (n = i_m, i_m + 1, \ldots)$$

for $\beta_m \in \mathbb{C}$ with $|\beta_m| \leq 1$ and $\beta_m^k \neq 1$ for all $k \in \mathbb{N}$.

Proof. Excluding the trivial first case, we may assume that $\lambda_n \neq 0$ for at least one $n \in \mathbb{N}$. In this case, however, we may very well assume that $\lambda_0 = 0$. ($T(\mathbf{1}) = \mathbf{1}$ and $\mathbf{1}$ commutes with everything.)

Suppose that $i_1 > 1$. In this case $\langle e_0, X e_0 \rangle = 0$, so that

$$T(X) = \lambda_{i_1} A_{i_1 - 1} + \sum_{n=i_1}^{\infty} (\lambda_n + \lambda_{n+1}) A_n = \sum_{n=i_1-1}^{\infty} \mu_n A_n$$

with $\mu_{i_1-1} := \lambda_{i_1} \neq 0$ and $\mu_n := \lambda_n + \lambda_{n+1}$ for $n \geq i_1$. Then $\mu_{i_1-1}\lambda_{i_1} = \lambda_{i_1}^2 \neq 0 = \lambda_{i_1-1}\mu_i$. By Proposition 4.1, X and $T(X)$ do not commute.

Now assume $i_1 = 1$, so that $\lambda_1 \neq 0$. In this case, we have

$$T(X) = \lambda_1 \mathbf{1} + \sum_{n=1}^{\infty} (\lambda_n + \lambda_{n+1} - \lambda_1) A_n = \mu_0 \mathbf{1} + \sum_{n=1}^{\infty} \mu_n A_n$$

with $\mu_0 := \lambda_1$ and $\mu_n := \lambda_n + \lambda_{n+1} - \lambda_1$ per $n \in \mathbb{N}$.

By Proposition 4.1, X and $T(X)$ commute if and only if for every m with $i_m \neq \infty$, there exists an $\alpha_m \in \mathbb{C}$ such that

$$\alpha_m \lambda_n = \mu_n = \lambda_n + \lambda_{n+1} - \lambda_1 \tag{4.2}$$

for $n \in \mathbb{N}$ with $i_m \leq n \leq j_m$.

Suppose now that X and $T(X)$ commute. Suppose $j_m \neq \infty$. Then

$$0 = \lambda_{j_m} = \mu_{j_m} = \lambda_{j_m} + \lambda_{j_m+1} - \lambda_1 = \lambda_{j_m+1} - \lambda_1.$$

so $\lambda_{j_m+1} = \lambda_1 \neq 0$. This implies $i_{m+1} = j_m + 1$ and the statement $\lambda_{i_m} = \lambda_1$ for all m with $i_m \neq \infty$.

It is clear that for every α_m the eigenvalue equation in (4.2) defines a recursion for the λ_n in the block $n \in \mathbb{N}$ with $i_m \leq n \leq j_m$ with initial condition $\lambda_{i_m} = \lambda_1$, which is the unique solution, no matter whether j_m is finite or infinite. We leave it to the reader to verify that this solution is

$$\lambda_n \;=\; \lambda_1 \frac{1-\beta_m^{n-i_m+1}}{1-\beta_m}$$

with $\beta_m := \alpha_m - 1$. However, if $j_m \neq \infty$, then we know also that $\lambda_n \neq 0$ for all $i_m \leq n < j_m$ and that $\lambda_{j_m} = 0$. This leaves precisely those β_m that have the form as stated in the lemma, for only then the the first zero occurs precisely when $n = j_m$. If $j_m = \infty$, then only if $|\beta_m| \leq 1$ the sequence remains bounded. The condition $\beta_m^k \neq 1$ for all $k \in \mathbb{N}$ assures that there are no more zeroes for $n \geq i_m$.

Clearly, all the choices lead to a solution where X and $T(X)$ commute. ■

4.4 Theorem. *$\mathcal{B}(G)$ does not contain a masa left invariant by T.*

Proof. Any masa of $\mathcal{B}(G)$ contains a self-adjoint element $X \neq 0$ different from $\lambda\mathbf{1}$ ($\lambda \in \mathbb{R}$). Suppose $X = \sum_{n=1}^{\infty} 4\lambda_n A_n$ is such an element with λ_n according to Lemma 4.3, such that X and $T(X)$ commute. As $\lambda_1 \neq 0$ we may divide all λ_n by λ_1 so that the new λ_1 is now 1.

Suppose there are finite blocks. In order that all λ_n are real, a finite block must have length two, that is, after $\lambda_{i_m} = 1$ the next λ_{i_m+1} must already be 0. In particular, if there are finite blocks, then we have $\lambda_1 = 1$ and $\lambda_2 = 0$ and $\lambda_3 = 1$ again, that is, $X = A_1 + A_3 + \lambda_4 A_4 \ldots$ and

$$\begin{aligned} T(X) \;&=\; \mathbf{1} - A_1 - A_2 - A_3 - A_4 \ldots \\ &\qquad + A_1 + A_2 + (1+\lambda_4)A_3 + (\lambda_4+\lambda_5)A_4 \ldots \;=\; \mathbf{1} + \lambda_4 A_3 + (\lambda_4+\lambda_5-1)A_4 + \ldots \quad . \end{aligned}$$

As the coefficient of A_1 in $T(X)$ is 0, the only possibility that $T(X)$ fulfills again the requirements of Lemma 4.3 (so that $T^2(X)$ commutes with $T(X)$) is that all following coefficients of $T(X)$ are zero. This happens if and only $\lambda_{2n-1} = 1$ and $\lambda_{2n} = 0$ for all $n \in \mathbb{N}$. In that case $T(X) = \mathbf{1}$, so that all $X, T(X), T^2(X), \ldots$ commute. Since $X^2 = \frac{1}{2}X$ we see that $\mathbb{C}\mathbf{1} + \mathbb{C}X$ is even a commutative von Neumann algebra. Of course, it is not a masa, so there must be more linearly independent elements. By Proposition 4.1, $\{X\}' \cap R = \mathbb{C}\mathbf{1} + \mathbb{C}X$. So, a masa containing X must be contained in $\mathbb{C}\mathbf{1} + \mathbb{C}X + \ker T$. Recall that $\mathbb{E}_\infty$ is faithful onto $\mathcal{D}_\infty$ and the map τ is injective. So, $\ker T$ does not contain any new projections, and the only commutative algebra containing X and invariant for T is $\mathbb{C}\mathbf{1} + \mathbb{C}X$, which is not a masa of $\mathcal{B}(G)$.

Suppose now that all $\lambda_n \neq 0$. So, there is $\beta \in \mathbb{C}$ with $|\beta| \leq 1$ and $\beta^k \neq 1$ for all $k \in \mathbb{N}$ such that $\lambda_n = \frac{1-\beta^n}{1-\beta}$. For that all λ_n are real, β must be real, too. Here $T(X) = (1+\beta)X$. But, once more $\{X\}' \cap R = \mathbb{C}\mathbf{1} + \mathbb{C}X$, and faithfulness of T excludes that a bigger commutative subalgebra is mapped into that subspace of R. ■

4.5 Observation. This example also works for every finite dimensional M_n ($n \geq 3$). The Equations (4.1) terminate with $n-1$. But the following equation would still be fulfilled with $\lambda_{n+1} = 0 = \mu_{n+1}$. What happens in Lemma 4.3, is just that there remain precisely those cases that have $\lambda_{n+1} = 0$. Regarding the proof of Theorem 4.4, we are just in the finite case, and the result stands also here.

Appendix

For the proofs in this appendix we do not make any attempt to be self-contained. Instead, we assume that the reader is familiar with the notions as introduced in Fagnola and Skeide [FS07, Section 2] for the proofs of Theorems 1.1 and 1.2, plus the necessary notions from Barreto, Bhat, Liebscher and Skeide [BBLS04, LS01] about morphisms of time ordered product systems for the proof of Theorem 1.4. Theorems 1.1 and 1.2 are versions specialized to $\mathcal{B}(G)$ of the results [FS07, Theorem 3.1 and 4.2] for general von Neumann algebras $\mathcal{B} \subset \mathcal{B}(G)$. As the intuition of the proof of necessity in the latter results cannot be grasped without a good portion of experience with Hilbert modules, it appears useless to produce a proof for $\mathcal{B}(G)$, independent of [FS07], that would not even approximately reveal why it works and where it comes from.

The following result from [FS07] about invariance of a maximal commutative subalgebra under CP-maps for general von Neumann algebras is just [FS07, Theorem 3.1] supplemented by the statement in [FS07, Observation 3.3].

A.1 [FS07, Theorem 3.1]. *Let $\mathcal{B} \subset \mathcal{B}(G)$ be a von Neumann algebra on the Hilbert space G and let T be a normal CP-map T on $\mathcal{B}$. Suppose E is a von Neumann correspondence over $\mathcal{B}$ and $\xi \in E$ one of its elements such that $T(b) = \langle \xi, b\xi \rangle$. Furthermore, let $C \ni \mathrm{id}_G$ be a maximal commutative von Neumann subalgebra of $\mathcal{B}$.*

Then T leaves C invariant, if and only if there exists a $*$–map $\alpha\colon C \to \mathcal{B}^a(E)$ fulfilling the following properties:

1. *The range of α commutes with the left action of elements of C on E, that is, for all $c_1, c_2 \in C$ and $x \in E$ we have*
$$c_1\alpha(c_2)x \;=\; \alpha(c_2)c_1x.$$
2. *For all $c \in C$ we have*
$$\alpha(c)\xi \;=\; c\xi - \xi c.$$

A.2 Remark. Every normal CP-map on a von Neumann algebra can be obtained in that way. For people who like modules: Do the GNS-construction to obtain a correspondence E_0 over $\mathcal{B}$ with a cyclic vector $\xi \in E_0$ having the correct matrix elements; see [FS07, Section 2.1]. Then close E_0 suitable to obtain a von Neumann correspondence E following the procedure from Skeide [Ske00] as explained in [FS07, Section 2.3]. For people who like the classical approach: Do the Stinespring construction [Sti55] to obtain a Hilbert space H with a nondegenerate normal representation π of $\mathcal{B}$ and a map $\xi \in \mathcal{B}(G, H)$ such that $T(b) = \xi^*\pi(b)\xi$; see [FS07, Section 2.2]. The GNS-module is, then, the strong closure in $\mathcal{B}(B, H)$ of $\operatorname{span}\pi(\mathcal{B})\xi\mathcal{B}$; see [FS07, Section 2.3].

As we need the same argument in the proof of Theorem 1.2, we repeat from [FS07] the reduction of Theorem 1.1 to Theorem A.1.

Proof of Theorem 1.1. If $\mathcal{B} = \mathcal{B}(G)$, then $E = \mathcal{B}(G, G \otimes \mathfrak{H})$; see [FS07, Section 2.4]. Let $(e_i)_{i\in I}$ denote an ONB of $\mathfrak{H}$. The family $(\mathsf{id}_G \otimes e_i)_{i\in I}$ (where $\mathsf{id}_G \otimes e_i$ denotes the mapping $g \mapsto g \otimes e_i$) is, then, an ONB of E in the obvious sense. (See [Ske00] for quasi ONBs.) Denote by $L_i := \langle \mathsf{id}_G \otimes e_i, \xi\rangle$ the coefficients of ξ with respect to this ONB. Then

$$T(b) \;=\; \sum_{i\in I} L_i^* b L_i$$

is a Kraus decomposition of the CP-map T on $\mathcal{B}(G)$; see [FS07, Section 2.4]. Moreover, every Kraus decomposition can be obtained in that way. (Simply take $\mathfrak{H} := \mathbb{C}^{\#I}$ with the canonical ONB and define $\xi := \sum_{i\in I} L_i \otimes e_i$.) The correspondence between maps $\alpha\colon C \to \mathcal{B}^a(E)$ fulfilling the hypothesis of Theorem A.1 and coefficients $c_{ij}(c) \in C$ fulfilling the hypothesis of Theorem 1.1 is, then, given by

$$c_{ij}(c) \;:=\; \langle(\mathsf{id}_G \otimes e_i), \alpha(c)(\mathsf{id}_G \otimes e_j)\rangle.$$

(Note that the conditions of Theorem 1.1 are, clearly, sufficient. Therefore an α exists and it is easy to see that $\alpha(c)$ can be chosen to have the expansion coefficients $c_{ij}(c)$.) ■

A.3 Observation. Suppose $T(X) = \sum_{i=I} L_i^* X L_i = \sum_{j\in J} K_j^* X K_j$. Put $\mathfrak{H} := \mathbb{C}^{\#I}$ and $\mathfrak{G} := \mathbb{C}^{\#J}$ and denote by $(e_i)_{i\in I}$ and $(f_j)_{j\in J}$, respectively, their canonical ONBs. Then $T = \langle\xi, \bullet\xi\rangle = \langle\zeta, \bullet\zeta\rangle$ for the elements $\xi := \sum_{i\in I} L_i \otimes e_i$ and $\zeta := \sum_{j\in J} K_j \otimes f_j$ of the von Neumann $\mathcal{B}(G)$–correspondences $E := \mathcal{B}(G, G \otimes \mathfrak{H})$ and $F := \mathcal{B}(G, G \otimes \mathfrak{G})$, respectively. It follows that $v\xi = \zeta$ defines a unique partial isometry $v \in \mathcal{B}^{a,bil}(E, F)$ that vanishes on $(\mathcal{B}(G)\xi\mathcal{B}(G))^\perp$, whose adjoint sends ζ to $v^*\zeta = \xi$ and vanishes on $(\mathcal{B}(G)\zeta\mathcal{B}(G))^\perp$. The superscript ^{bil} refers to that the operators are bilinear, that is, they commute with the action of $\mathcal{B}(G)$. It follows that v must have the form $v = \mathsf{id}_G \otimes V \in \mathcal{B}(G \otimes \mathfrak{H}, G \otimes \mathfrak{G}) = \mathcal{B}^a(E, F)$ for some partial isometry $V \in \mathcal{B}(\mathfrak{H}, \mathfrak{G})$. If v_{ji} are the matrix elements of V with respect to the canonical ONBs, we find that $K_j = \sum_{i\in I} v_{ji} L_i$ and $L_i = \sum_{j\in J} v_{ji}^* K_j$. We see that the (strongly closed) linear hull is invariant under the choice of the Kraus decomposition. Moreover, v is injective, if and only if ξ generates E, and v is surjective, if and only if ζ generates F. If v is bijective, so that it is unitary, then the dimensions of $\mathfrak{H}$ and $\mathfrak{G}$ must coincide, and no Kraus decomposition can have fewer summands than that ***minimal*** dimension.

We now quote the criterion for the (bounded) generators of normal CP-semigroups.

A.4 [FS07, Theorem 4.2]. *Let $\mathcal{B} \subset \mathcal{B}(G)$ be a von Neumann algebra on the Hilbert space G and let L be a (bounded) normal CCP-map on $\mathcal{B}$. Suppose E is a von Neumann correspondence over $\mathcal{B}$ and $d\colon \mathcal{B} \to E$ a bounded derivation such that*

$$\langle d(b), d(b')\rangle \;=\; L(b^*b') - b^*L(b') - L(b^*)b' + b^*L(\mathbf{1})b'.$$

Furthermore, let $C \ni \mathsf{id}_G$ be a maximal commutative von Neumann subalgebra of $\mathcal{B}$.

Then L leaves C invariant, if and only if there exist an element $\zeta \in E$ that reproduces $d \upharpoonright C$ as

$$d(c) \;=\; c\zeta - \zeta c,$$

a $*$–map $\alpha\colon C \to \mathcal{B}^a(E)$ and a self-adjoint element $\gamma \in C$ such that the following conditions are satisfied:

1. *The range of α commutes with the left action of elements of C on E, that is, for all $c_1, c_2 \in C$ and $x \in E$ we have*

$$c_1\alpha(c_2)x \;=\; \alpha(c_2)c_1x.$$

2. *For all $c \in C$ we have*

$$\alpha(c)\zeta \;=\; c\zeta - \zeta c.$$

3. *For all $c \in C$ we have*

$$L(c) - \langle\zeta, c\zeta\rangle \;=\; \gamma c.$$

Proof of Theorem 1.2. Let L be given in the Lindblad form as stated in Theorem 1.2 and fix the Hilbert space $\mathfrak{H} := \mathbb{C}^{\#I}$ with its canonical basis $(e_i)_{i\in I}$. We observe that if L fulfills the three conditions in Theorem 1.2, then by Theorem 1.1 applied to the generator with coefficients $K_i := L_i - c_i$ we see that L leaves C invariant.

Suppose now, conversely, that L leaves C invariant. By [FS07, Sections 2.6 –2.8] the vector $\xi := \sum_{i\in I} L_i \otimes e_i$ in the von Neumann $\mathcal{B}(G)$–correspondence $E := \mathcal{B}(G, G \otimes \mathfrak{H})$ generates a derivation $d(b) := b\xi - \xi b$ that has the required inner products. By Theorem A.4, there exists a vector $\zeta = \sum_{i\in I} K_i \otimes e_i$ in E such that $d(c) = c\zeta - \zeta c$, that is,

$$c(\xi - \zeta) \;=\; (\xi - \zeta)c$$

for all $c \in C$. Therefore, the coefficients c_i of $\xi_i - \zeta = \sum_{i\in I}(L_i - K_i) \otimes e_i = \sum_{i\in I} c_i \otimes e_i$ must be elements of C. The rest follows by applying appropriately the other properties that, by Theorem 1.2, must be fulfilled. ■

Proof of Theorem 1.4. Recall that by [LS01] the (continuous) units of the time ordered product system over a von Neumann $\mathcal{B}$–correspondence E are parameterized as $\xi^{\odot}(\beta, \xi) = (\xi_t(\beta, \xi))_{t \in \mathbb{R}_+}$ where $\beta \in \mathcal{B}, \xi \in E$. The family of mappings $b \mapsto \langle \xi_t(\beta, \xi), b\xi_t(\beta, \xi) \rangle$ ($t \in \mathbb{R}_+$) form a uniformly continuous CP-semigroup with generator $L(b) = \langle \xi, b\xi \rangle + b\beta + \beta^* b$. In the case of a Lindblad generator on $\mathcal{B}(G)$ we have, as in the preceding proofs, $E = \mathcal{B}(G, G \otimes \mathfrak{H})$ and $\xi = \sum_{i \in I} L_i \otimes e_i$. The Lindblad form is minimal, if and only if the single unit $\xi^{\odot}(\beta, \xi)$ generates the whole time ordered product system. Suppose $F = \mathcal{B}(G, G \otimes \mathfrak{G})$ ($\mathfrak{G} = \mathbb{C}^{\# J}$) is another von Neumann $\mathcal{B}(G)$–correspondence with elements $\alpha \in B, \zeta \in F$ such that $L(b) = \langle \zeta, b\zeta \rangle + b\alpha + \alpha^* b$. Sending $\xi_t(\beta, \xi)$ to $\xi_t(\alpha, \zeta)$ defines, then, an isometric morphism from the time ordered product system over E into that over F. By [BBLS04, Theorem 5.2.1] morphisms are parameterized by matrices $\begin{pmatrix} \gamma & \eta^* \\ \eta' & a \end{pmatrix} \in \mathcal{B}^{a,bil}(\mathcal{B}(G) \oplus E, \mathcal{B}(G) \oplus F)$ such that the parameters of the units transform as

$$(\beta, \xi) \longmapsto (\beta + \gamma + \langle \eta, \xi \rangle \, , \, \eta' + a\xi).$$

By [BBLS04, Corollary 5.2.4] such a morphism is isometric, if and only if a is isometric, η' is arbitrary, $\eta = -a^*\eta'$, and $\gamma = ih - \frac{\langle \eta', \eta' \rangle}{2}$. Interpreting all this properly in terms of the concrete $\mathcal{B}(G)$–correspondences and their elements ξ, ζ, and taking also into account that $\mathcal{B}^{a,bil}(E, F) = \mathcal{B}^{a,bil}(\mathcal{B}(G, G \otimes \mathfrak{H}), \mathcal{B}(G, G \otimes \mathfrak{G})) = \mathcal{B}(\mathfrak{H}, \mathfrak{G})$ (because all elements must commute with $\mathcal{B}(G)$, the coefficients can just be scalar multiples of $\mathbf{1}$), gives the statement of Theorem 1.4. ■

References

[AFH06] L. Accardi, F. Fagnola, and S. Hachicha, *Generic q–Markov semigroups and speed of convergence of q–algorithms*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **9** (2006), 567–594.

[AG02] S. Albeverio and D. Goswami, *A remark on the structure of symmetric quantum dynamical semigroups on von Neumann algebras*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **5** (2002), 571–579.

[BBLS04] S.D. Barreto, B.V.R. Bhat, V. Liebscher, and M. Skeide, *Type I product systems of Hilbert modules*, J. Funct. Anal. **212** (2004), 121–181, (Preprint, Cottbus 2001).

[BP96] A. Barchielli and A.M. Paganoni, *A note on a formula of the Lévy-Khinchin type in quantum probability*, Nagoya Math. J. **141** (1996), 29–43.

[CE79] E. Christensen and D.E. Evans, *Cohomology of operator algebras and quantum dynamical semigroups*, J. London Math. Soc. **20** (1979), 358–368.

[Dav79] E.B. Davies, *Generators of dynamical semigroups*, J. Funct. Anal. **34** (1979), 421–432.

[FS07] F. Fagnola and M. Skeide, *Restrictions of CP-semigroups to maximal commutative subalgebras*, Noncommutative harmonic analysis with applications to probability

(M. Bozejko, A. Krystek, W. Mlotkowski, and J. Wysoczanski, eds.), Banach Center Publications, vol. 78, Polish Academy of Sciences — Institute of Mathematics, 2007, (arXiv: math.OA/0703001), pp. 121–132.

[FU07] F. Fagnola and V. Umanità, *Generators of detailed balance quantum Markov semigroups*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **10** (2007), 335–363.

[Hol96] A.S. Holevo, *Covariant quantum Markovian evolutions*, J. Math. Phys. **37** (1996), 1812–1832.

[Lin76] G. Lindblad, *On the generators of quantum dynamical semigroups*, Commun. Math. Phys. **48** (1976), 119–130.

[LS01] V. Liebscher and M. Skeide, *Units for the time ordered Fock module*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **4** (2001), 545–551, (Rome, Volterra-Preprint 2000/0411).

[Par92] K.R. Parthasarathy, *An introduction to quantum stochastic calculus*, Birkhäuser, 1992.

[Reb05a] R. Rebolledo, *A view on decoherence via master equations*, Information Dynamics & Open Systems **12** (2005), 37–54.

[Reb05b] ______, *Decoherence of quantum Markov semigroups*, Ann. I.H.P. Prob. Stat. **41** (2005), 349–373.

[Ske00] M. Skeide, *Generalized matrix C^*–algebras and representations of Hilbert modules*, Mathematical Proceedings of the Royal Irish Academy **100A** (2000), 11–38, (Cottbus, Reihe Mathematik 1997/M-13).

[Sti55] W.F. Stinespring, *Positive functions on C^*–algebras*, Proc. Amer. Math. Soc. **6** (1955), 211–216.

B.V. Rajarama Bhat: *Statistics and Mathematics Unit, Indian Statistical Institute Bangalore, R. V. College Post, Bangalore 560059, India*, E-mail: `bhat@isibang.ac.in`,
Homepage: `http://www.isibang.ac.in/Smubang/BHAT/`

Franco Fagnola: *Dipartimento di Matematica, Politecnico di Milano, Piazza Leonardo da Vinci 32, 20133 Milano, Italy*, E-mail: `franco.fagnola@polimi.it`,
Homepage: `http://www.mate.polimi.it/˜qp`

Michael Skeide: *Dipartimento S.E.G.e S., Università degli Studi del Molise, Via de Sanctis, 86100 Campobasso, Italy*, E-mail: `skeide@math.tu-cottbus.de`,
Homepage: `http://www.math.tu-cottbus.de/INSTITUT/lswas/_skeide.html`