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Limit Theorems for Functionals of Sums that Converge to Fractional Brownian and Stable Motions

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Abstract. Consider $S_k = \sum_{j=1}^{k} X_j$, where $X_k = \sum_{j=0}^{\infty} c_j \xi_{k-j}$, $k \geq 1$, with ξ_j, $-\infty < j < \infty$, iid belonging to the domain attraction of a strictly stable law with index $0 < \alpha \leq 2$. Under certain conditions on c_j, it is known that for $\gamma_n = n^H \tau_n$, $0 < H < 1$, with τ_n slowly varying, $\gamma_n^{-1} S_{[nt]}$ converges in distribution to a fractional stable motion. In addition, if $f(y)$ is such that $\int \left(|f(y)| + |f(y)|^2\right) dy < \infty$, then for β_n such that $\beta_n \to \infty$ and $\frac{\beta_n}{n} \to 0$ (in particular $\beta_n = \gamma_n$), $\frac{\beta_n}{n} \sum_{k=1}^{n} f\left(\frac{\beta_n}{\gamma_n} S_k\right)$ converges in distribution to $L_1^0 \int_{-\infty}^{\infty} f(y)\, dy$, where L_t^x is the local time of the fractional stable motion at x upto time t. In this paper we obtain further results, motivated by asymptotic inference for certain nonlinear models.

We obtain the convergence in distribution for $\frac{\beta_n}{n} \sum_{k=1}^{n} h\left(\frac{\beta_n}{\gamma_n} S_k, \frac{\beta_n}{\gamma_n} S_{k+r}\right)$, $r \geq 1$, as well as for $\frac{\beta_n}{n} \sum_{k=1}^{n} f\left(\frac{\beta_n}{\gamma_n} S_k\right) \sigma(\omega_k)$ and $\frac{\beta_n}{n} \sum_{k=1}^{n} h\left(\frac{\beta_n}{\gamma_n} S_k, \frac{\beta_n}{\gamma_n} S_{k+r}\right) \sigma(\omega_k)\, \sigma(\omega_{k+r})$, $r \geq 1$, for suitable $f(x)$ and $h(x, y)$ and for suitable $\sigma(\omega_k)$, where $\omega_k = \sum_{j=-\infty}^{k} d_{k-j} \eta_j$ such that (ξ_j, η_j), $-\infty < j < \infty$, are iid with $E[\eta_1^2] < \infty$ but possibly with $E[\eta_1] \neq 0$. For $h(x, y)$, the limits are different for the cases $\beta_n = \gamma_n$, $\frac{\beta_n}{\gamma_n} \to 0$ and $\frac{\beta_n}{\gamma_n} \to \infty$.

We further show that if in addition $\int_{-\infty}^{\infty} f(y)\, dy = 0$, then when $1/3 < H < 1$ (which probably cannot be relaxed), $\sqrt{\frac{\gamma_n}{n}} \sum_{k=1}^{n} f(S_k)$ converges in distribution. Similarly but when possibly $\int_{-\infty}^{\infty} f(y)\, dy \neq 0$, the same is true for $\sqrt{\frac{\gamma_n}{n}} \sum_{k=1}^{n} f(S_k) \omega_k$, where ω_k is as before but with $E[\eta_1] = 0$.

All the above convergencies are also shown to hold jointly.

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1 INTRODUCTION

Consider a sequence $\xi_j, -\infty < j < \infty$, of iid random variables belonging to the domain of attraction of a strictly stable law with index $0 < \alpha \le 2$. We recall that this is equivalent to the statement that for a suitable slowly varying function δ_n,

$$t \longmapsto \left(n^{1/\alpha}\delta_n\right)^{-1} \sum_{j=1}^{[nt]} \xi_j \overset{fdd}{\Longrightarrow} Z_\alpha(t)\,,\ t > 0, \tag{1}$$

where $\{Z_\alpha(t), t > 0\}$ is an α-stable Levy motion, that is, has stationary independent increments such that, for each $0 < t < \infty$,

$$E[e^{iuZ_\alpha(t)}] = \begin{cases} e^{-t|u|^\alpha\left(1-i\beta\,\mathrm{sign}(u)\,\tan\left(\frac{\pi\alpha}{2}\right)\right)} & \text{if } \alpha \neq 1 \\ e^{-t|u|} & \text{if } \alpha = 1 \end{cases}$$

with $|\beta| \le 1$. (Above and in the rest of the paper, the notation $\overset{fdd}{\Longrightarrow}$ signifies the convergence in distribution of random processes in the sense of convergence in distribution of all finite dimensional distributions.) For the details of the above statement, see for instance Ibragimov and Linnik (1965, Chapter 2, Section 6) or Bingham et al (1987, page 344). Note that this definition of strict α-stability for the case $\alpha = 1$ differs from the usual one in that we take the skewness parameter β to be 0. When $\alpha = 2$, $Z_2(t)$ becomes the Brownian Motion with variance 2.

Now consider the linear process

$$X_k = \sum_{j=0}^{\infty} c_j \xi_{k-j},\ k \ge 1,$$

where ξ_j, $-\infty < j < \infty$, are as earlier with index $0 < \alpha \le 2$, and c_j, $j \ge 0$, with $c_0 = 1$, are constants. Let

$$S_k = \sum_{j=1}^{k} X_j.$$

Under suitable conditions (specified in Section 2 below) on the constants c_j it is known that for a suitable H, $0 < H < 1$, and for normalizing constants of the form

$$\gamma_n = n^H \tau_n$$

with τ_n slowly varying, the process

$$\frac{1}{\gamma_n} S_{[nt]} \overset{fdd}{\Longrightarrow} \Lambda_{\alpha,H}(t),$$

where the limit $\{\Lambda_{\alpha,H}(t), t \geq 0\}$ is a *Linear Fractional Stable Motion* (LFSM). It is defined by

$$\Lambda_{\alpha,H}(t) = a \int_{-\infty}^{0} \left\{ (t-u)^{H-1/\alpha} - (-u)^{H-1/\alpha} \right\} Z_\alpha(du) + a \int_{0}^{t} (t-u)^{H-1/\alpha} Z_\alpha(du)$$

if $H \neq 1/\alpha$, and

$$\Lambda_{\alpha,H}(t) = Z_\alpha(t) \quad \text{ if } H = 1/\alpha$$

where a is a non-zero constant and $\{Z_\alpha(t), t \in R\}$ is an α-stable Levy motion, taken to be $Z_\alpha(t)$ as defined earlier for $0 < t < \infty$, and for $-\infty < t < 0$, it is taken to be $Z_\alpha(t) = Z^*_\alpha(-t)$ with $\{Z^*_\alpha(u), 0 < u < \infty\}$ an independent copy of $\{Z_\alpha(u), 0 < u < \infty\}$. See Samorodnitsky and Taqqu (1994) for the details of LFSM.

Note that when $H = 1/\alpha$, *the restriction* $0 < H < 1$ *reduces to* $1 < \alpha \leq 2$. When $\alpha = 2$, the LFSM reduces to the *Fractional Brownian Motion.*

Now let $f(y)$ be a function such that $\int \left(|f(y)| + |f(y)|^2 \right) dy < \infty$. Further let β_n be constants (here and throughout the rest of the paper) such that

$$\beta_n \to \infty \quad \text{ and } \quad \frac{\beta_n}{n} \to 0. \text{ (In particular one can take } \beta_n = \gamma_n.\text{)}$$

Then, under certain further restriction on the distribution of ξ_1, it is shown in Jeganathan (2004, Theorems 2 and 3) that

$$\frac{\beta_n}{n} \sum_{k=1}^{n} f\left(\frac{\beta_n}{\gamma_n} S_k \right) \Longrightarrow L_1^0 \int_{-\infty}^{\infty} f(y)\, dy,$$

where L_t^x is the *local time* of the LFSM $\Lambda_{\alpha,H}(t)$ at x upto the time t.

In this paper further results motivated by large sample inference in certain nonlinear time series models are obtained. The first main result directly related to the preceding convergence states that (Theorem 1 of Section 2) under suitable restrictions on the function $f(x,y)$, for any integer $r \geq 1$,

$$\frac{\beta_n}{n} \sum_{k=1}^{n} f\left(\frac{\beta_n}{\gamma_n} S_k, \frac{\beta_n}{\gamma_n} S_{k+r} \right) \Longrightarrow \begin{cases} L_1^0 \int_{-\infty}^{\infty} E\left[f(x, x+S_r) \right] dx & \text{if } \beta_n = \gamma_n \\ L_1^0 \int_{-\infty}^{\infty} f(x,x)\, dx & \text{if } \frac{\beta_n}{\gamma_n} \to 0 \\ 0 & \text{if } \frac{\beta_n}{\gamma_n} \to \infty. \end{cases}$$

Next let $\nu_n \geq 1$ be integers, possibly $\nu_n \to \infty$ as $n \to \infty$ such that $\frac{\nu_n}{n} \to 0$. Define

$$\omega_{k,\nu_n} = \sum_{j=k-\nu_n+1}^{k} d_{k-j}\eta_j = \eta_k + d_1\eta_{k-1} + ... + d_{\nu_n-1}\eta_{k-\nu_n+1}, \tag{2}$$

where (ξ_j, η_j), $-\infty < j < \infty$, are iid (ξ_j are as before) and

$$\sum_{j=0}^{\infty} |d_j| \max(1, |g(j)|) < \infty \quad \text{where } g(j) = \sum_{i=0}^{j} c_i. \tag{3}$$

Then we show that the quantities

$$\frac{\beta_n}{n}\sum_{k=1}^{n} f\left(\frac{\beta_n}{\gamma_n}S_k\right)\sigma(\omega_{k,\nu_n}) \quad \text{and} \quad \frac{\beta_n}{n}\sum_{k=1}^{n} f\left(\frac{\beta_n}{\gamma_n}S_k, \frac{\beta_n}{\gamma_n}S_{k+r}\right)\sigma(\omega_{k,\nu_n})\,\sigma(\omega_{k+r,\nu_n}), \quad r \geq 1, \tag{4}$$

converge in distribution with suitable limits, if $\sigma(x)$ is continuous and, for some $q \geq 0$,

$$|\sigma(\omega_{k,\nu_n})| \leq C + C|\omega_{k,\nu_n}|^q \qquad \text{with } E\left[\eta_1^{2q}\right] < \infty. \tag{5}$$

We note that the preceding convergence holds also when ω_{k,ν_n} in (4) is replaced by $\sum_{j=-\infty}^{k} d_{k-j}\eta_j$ (Theorem 3) or more generally by a suitable multilinear sum, see the Remark 3 in Section 2 below.

The fourth main result (Theorem 4 in Section 2) includes in particular the result that if for a function $f(y)$ the restrictions

$$\int |f(y)|^i \, dy < \infty,\ i = 1, 2, 3, 4, \tag{6}$$

$$\int_{-\infty}^{\infty} |yf(y)|\, dy < \infty, \tag{7}$$

$$\int_{-\infty}^{\infty} f(y)\, dy = 0,$$

$$\frac{1}{3} < H < 1$$

hold, then

$$\sqrt{\frac{\gamma_n}{n}}\sum_{k=1}^{n} f(S_k) \Longrightarrow W\sqrt{bL_1^0} \tag{8}$$

where W has the standard normal distribution independent of the local time L_1^0, and b is a nonnegative constant having an explicit expression in terms of the distributions of S_k, $k \geq 1$. *We remark that the restriction $\frac{1}{3} < H < 1$ probably cannot be relaxed* because it cannot be relaxed in the continuous time version of (8), see Jeganathan (2006c). The convergence (8) may be viewed as an analogue of the central limit theorem if the convergence

$\frac{\gamma_n}{n}\sum_{k=1}^{n} f\left(S_k\right) \Longrightarrow L_1^0 \int_{-\infty}^{\infty} f\left(y\right)dy$ is viewed as an analogue of the law of large numbers. In obtaining (8), as well as the convergence (11) below, we shall further assume that

$$\text{When } \alpha = 2, \quad E\left[\xi_1\right] = 0 \text{ and } E\left[\xi_1^2\right] < \infty. \tag{9}$$

The convergence (8) is known for the random walk case $S_k = \sum_{j=1}^{k} \xi_j$, see Borodin and Ibragimov (1995, Theorem 3.3 of Chapter IV). For the symmetric Bernoulli random walk case, it was originally discovered by Dobrushin (1955). But note however that many of the structural simplifications available in the random walk case (for example the fact that $S_{l+k} - S_k$ is independent of S_k and has the same distribution as that of S_l) are not available for the present case.

Next, let ω_{k,ν_n} be as in (2) but with

$$E\left[\eta_1\right] = 0, \quad E\left[\eta_1^4\right] < \infty \quad \text{and} \quad E\left[\left|\eta_1 \xi_1\right|\right] < \infty. \tag{10}$$

Then the fifth main result (Theorem 5, Section 2) will include the convergence

$$\sqrt{\frac{\gamma_n}{n}} \sum_{k=1}^{n} f(S_k)\omega_{k,\nu_n} \Longrightarrow W\sqrt{b^* L_1^0} \tag{11}$$

(and the same with ω_{k,ν_n} replaced by$\sum_{j=-\infty}^{k} d_{k-j}\eta_j$), where $f(y)$ satisfies (6) and (7) but now $\int_{-\infty}^{\infty} f\left(y\right)dy = 0$ need not hold, that is, possibly

$$\int_{-\infty}^{\infty} f\left(y\right)dy \neq 0.$$

The constant b^* in (11) will have the form similar to that of b in (8).

As far as we can determine, the convergence (11) has not been known previously, even for the random walk situation $S_k = \sum_{j=1}^{k} \xi_j$ with $\omega_{k,\nu_n} = \eta_k$.

Note that the requirement $E\left[\left|\eta_1 \xi_1\right|\right] < \infty$ in (10) implicitly requires certain moment condition on ξ_1. It is satisfied when $\alpha = 2$ because then $E\left[\xi_1^2\right] < \infty$ (see (9); $E\left[\eta_1^2\right] < \infty$ already by assumption). It is also satisfied, using Cauchy-Schwarz inequality, when

$$E\left[\left|\eta_1\right|^{\frac{\gamma}{\gamma-1}}\right] < \infty \text{ for some } 1 < \gamma < \alpha \text{ when } 1 < \alpha < 2.$$

The convergence results in Theorems 1 - 3 and 5, together with the joint convergence with other quantities are needed in obtaining the asymptotic behavior of least squares or similar estimators in certain nonlinear time series models (Jeganathan and Phillips (2008)). The convergence results (8) and (11) are closely related in that the proof of (11) will use similar ideas involved in (8), though unfortunately (11) is not directly deducible from (8).

The plan of the paper is as follows. The required assumptions as well as the statements of the main results are stated in Section 2. The next two sections 3 and 4 give the proof of Theorems 1 - 3. In Section 5 it is noted that the convergencies (8) and (11) can be related to a form of a martingale CLT. (Such a relationship to a martingale CLT is implicit in Borodin and Ibragimov (1995) though the methods employed there are tied in many ways to the iid structure of the random walk case $S_k = \sum_{j=1}^{k} \xi_j$ treated there.) The proof of the Theorems 4 and 5 will then consists of the verification of the conditions of this martingale CLT, which verification is done in Section 6, based on the earlier Theorems 1 - 2 together with additional arguments. The Appendix (Section 7) contains the statement and the proof of a version of martingale CLT used in Section 5 that may be of independent interest.

It is convenient to mention some of the notations here that will be used throughout the paper. In addition to the $\stackrel{fdd}{\Longrightarrow}$ introduced earlier, the convergence in distribution of a sequence of random variables or random vectors will be signified as usual by $\Longrightarrow$. As above, L_t^x will stand for the local time of the LFSM $\Lambda_{\alpha,H}(t)$ at x upto the time t. Throughout below we let

$$\psi(\lambda) = E\left[e^{i\lambda\xi_1}\right].$$

For any Borel measurable function $f(y)$ with $\int |f(y)|\, dy < \infty$, $\widehat{f}(\lambda)$ stands for its Fourier transform, that is,

$$\widehat{f}(\lambda) = \int e^{i\lambda y} f(y) dy.$$

Corresponding to the coefficients c_j in $X_k = \sum_{j=0}^{\infty} c_j \xi_{k-j}$, we let

$$g(j) = \begin{cases} \sum_{i=0}^{j} c_i & \text{if } j \geq 0 \\ 0 & \text{if } j < 0. \end{cases}$$

The normalizing constant $b_n = n^{1/\alpha}\delta_n$ (where δ_n is as in (1)) will be used exclusively in the sense of (30) below. Similarly γ_n will be used in the sense of (16) or (31) below.

Throughout the paper the notation C stands for a generic constant that may take different values at different places of even the same expression in the same proof.

2 ASSUMPTIONS AND THE MAIN RESULTS

One of the following two mutually exclusive conditions will be imposed on the coefficients c_j of the process $X_k = \sum_{j=0}^{\infty} c_j \xi_{k-j}$, where recall that $c_0 = 1$.

(A1) (**The case** $H = 1/\alpha$, $0 < H < 1$).

$$\sum_{j=0}^{\infty} |c_j| < \infty \quad \text{and} \quad \sum_{j=0}^{\infty} c_j \neq 0.$$

In addition

$$\sup_{j\geq 1}|jc_j| < \infty. \tag{12}$$

(A2) (**The case** $H \neq 1/\alpha$, $0 < H < 1$). $c_j = j^{H-1-1/\alpha}u_j$, with $H \neq 1/\alpha$, $0 < H < 1$, where u_j is slowly varying at infinity, satisfying

$$\sum_{j=0}^{\infty} c_j = 0 \quad \text{ when } H - 1/\alpha < 0. \tag{13}$$

In addition, there is an integer $l_0 > 0$ and constants C_1 and C_2 such that

$$0 < C_1 \leq \left|\frac{u_{l+j_1}}{u_{l-j_2}}\right| \leq C_2 \quad \text{ for all } 0 \leq j_1, j_2 \leq [l/2] \text{ and } l \geq l_0. \tag{14}$$

We note that the restriction (14) is automatically satisfied if u_j is monotone in j, because of the assumption of u_j being slowly varying. For instance if u_j is nondecreasing, then $1 \leq \frac{u_{l+j_1}}{u_{l-j_2}} \leq \frac{u_{2l}}{u_{l/2}}$ when $0 \leq j_1, j_2 \leq [l/2]$, where $\frac{u_{2l}}{u_{l/2}} \to 4$ as $l \to \infty$. (We do not know if the monotonicity of u_j can be assumed without loss of generality, in which case the restriction (14) then holds automatically.)

Note that if (13) is violated, then the case $c_j = j^{H-1-1/\alpha}u_j$ with $H - 1/\alpha < 0$ comes under (A1). Also it is implicit that $u_j \neq 0$ for all sufficiently large j.

Remark 1. A motivation of the condition (A2) is what has been called a *Fractional ARIMA* model with stable innovations, a detailed discussion of which can be found for instance in Samorodnitsky and Taqqu (1994, Section 7.13, page 380). In a simplest case of this model, X_k takes the form

$$X_k = (1-B)^{-d}\xi_k = \sum_{j=0}^{\infty} c_j(-d)\, B^j \xi_k = \sum_{j=0}^{\infty} c_j(-d)\,\xi_{k-j} \tag{15}$$

where B is the back-shift operator $B\xi_i = \xi_{i-1}$. Here we have used the formal expansion $(1-B)^{-d} = \sum_{j=0}^{\infty} c_j(-d)\, B^j$, so that using Stirling's approximation,

$$c_j(-d) = \frac{\Gamma(j+d)}{\Gamma(d)\,\Gamma(j+1)} \sim \frac{1}{\Gamma(d)} j^{d-1} \text{ as } j \to \infty \quad \text{ if } d \neq 0, -1, ...$$

where $\Gamma(.)$ stands for the gamma function, and $c_j(-d) = 0$ for $j \geq d$ if $d = 0, -1, ...$

Hence if we take $H = d + \frac{1}{\alpha}$, the condition (A2) is satisfied, including (13) because $H - \frac{1}{\alpha} < 0$ is the same as $d < 0$ and hence

$$\sum_{j=0}^{\infty} c_j(-d) = \left.(1-x)^{-d}\right|_{x=1} = 0 \quad (d < 0).$$

In addition, when $0 < H < 1$, the series (15) converges with probability one (see Samorodnitsky and Taqqu (1994, Theorem 7.13.1, page 381)). ■

Now let

$$\gamma_n = \begin{cases} \left(\sum_{j=0}^{\infty} c_j\right) n^{1/\alpha}\delta_n & \text{if (A1) is satisfied} \\ n^H u_n \delta_n & \text{if (A2) is satisfied,} \end{cases} \tag{16}$$

where δ_n is as in (1) and u_n as in (A2). Then it is known that when (A2) is satisfied, the process $\gamma_n^{-1} S_{[nt]} \overset{fdd}{\Longrightarrow} \Lambda_{\alpha,H}(t)$, $H \neq 1/\alpha$, and similarly when $1 < \alpha \leq 2$ and (A1) is satisfied, $\gamma_n^{-1} S_{[nt]} \overset{fdd}{\Longrightarrow} Z_\alpha(t)$. (See for instance Kasahara and Maejima (1988, Theorems 5.1, 5.2 and 5.3)), Astrauskas (1983) and Avram and Taqqu (1986).) In view of our convention that $\Lambda_{\alpha,1/\alpha}(t) = Z_\alpha(t)$ when $1 < \alpha \leq 2$, the preceding statements will be combined in the form

$$\gamma_n^{-1} S_{[nt]} \overset{fdd}{\Longrightarrow} \Lambda_{\alpha,H}(t),$$

with the understanding that when (A1) is satisfied the limit is $Z_\alpha(t)$ *with* $1 < \alpha \leq 2$.

To proceed further, define the functions, corresponding to a given real valued function $h(y_1, ..., y_k)$ defined on $\mathbb{R}^k$,

$$\left.\begin{aligned} M_{h,\eta}(y_1, ..., y_k) &= \sup\{h(u_1, ..., u_k) : |u_j - y_j| \leq \eta,\ j = 1, ..., k\} \\ m_{h,\eta}(y_1, ..., y_k) &= \inf\{h(u_1, ..., u_k) : |u_j - y_j| \leq \eta,\ j = 1, ..., k\}. \end{aligned}\right\} \tag{17}$$

Throughout below, we shall employ the following classes of functions.

Class $\mathcal{G}_1$. This is the class consisting of all Borel measurable real valued functions $f(x)$ defined on $\mathbb{R}$ such that

$$\int \left(|f(x)| + |f(x)|^2\right) dx < \infty.$$

■

Class $\mathcal{G}_2$. This is the class consisting of all Borel measurable real valued functions $f(x)$ defined on $\mathbb{R}$ such that

$$\int \left(M_{|f|,\eta}(x) + \left(M_{|f|,\eta}(x)\right)^2\right) dx < \infty \qquad \text{for all } \eta > 0$$

and

$$\int \left(M_{f,\eta}(x) - m_{f,\eta}(x)\right) dx \to 0 \qquad \text{as } \eta \to 0.$$

■

Class $\mathcal{H}_1$. This is the class consisting of all Borel measurable real valued functions $f(x_1, ..., x_k)$ defined on $\mathbb{R}^k$ such that

$$\int |f(x_1, ..., x_k)|^i\, dx_0 ... dx_r < \infty,\ \ i = 1, 2, \quad \int \left(\int |f(x_1, ..., x_k)|^2\, dx_k\right)^{\frac{1}{2}} dx_1 ... dx_{k-1} < \infty.$$

■

Class $\mathcal{H}_2$. This is the class consisting of all Borel measurable real valued functions $f(x_1,...,x_k)$ defined on $\mathbb{R}^k$ such that

$$\int \left|M_{|f|,\eta}(x_1,...,x_k)\right|^i dx_0...dx_r < \infty,\ \ i=1,2,\quad \int \left(\int \left|M_{|f|,\eta}(x_1,...,x_k)\right|^2 dx_k\right)^{\frac{1}{2}} dx_1...dx_{k-1} < \infty$$

for all $\eta > 0$ and

$$\int (M_{f,\eta}(x_1,...,x_k) - m_{f,\eta}(x_1,...,x_k))\, dx \to 0 \qquad \text{as } \eta \to 0.$$

■

We are now in a position to state the results. Throughout below, and without further mentioning, the requirements (A1) and (A2) are assumed to hold. Also recall that the constants β_n are such that $\beta_n \to \infty$ and $\frac{\beta_n}{n} \to 0$.

Theorem 1. *(I). Assume that $0 < H < 1$. Further assume that*

$$\int \left|E\left[e^{i\lambda\xi_1}\right]\right|^2 d\lambda < \infty. \tag{18}$$

Let the function $f(x_0,...,x_r)$ be in the class $\mathcal{H}_1$, and for the case $\frac{\beta_n}{\gamma_n} \to 0$ assume in addition that $f(x_0,...,x_r)$ is of the product form $f(x_0,...,x_r) = f_0(x_0)...f_r(x_r)$ with each $f_j(x)$ in the class $\mathcal{G}_1$.Then for any $1 \le i_1 < ... < i_r$,

$$\frac{\beta_n}{n}\sum_{l=1}^{n} f\left(\frac{\beta_n}{\gamma_n}S_l, \frac{\beta_n}{\gamma_n}S_{l+i_1},..., \frac{\beta_n}{\gamma_n}S_{l+i_r}\right) \Longrightarrow \begin{cases} L_1^0 \int E\left[f(x, x+S_{i_1},...,x+S_{i_r})\right] dx & \text{if } \beta_n = \gamma_n \\ L_1^0 \int f(x,x,...,x)\, dx & \text{if } \frac{\beta_n}{\gamma_n} \to 0 \\ 0 & \text{if } \frac{\beta_n}{\gamma_n} \to \infty, \end{cases}$$

where the constants β_n and the local time L_1^0 are as before.

(II). Assume $0 < H < 1$. Suppose that the function $f(x_0,...,x_r)$ is in the class $\mathcal{H}_2$. Then the preceding convergence holds also when (18) is relaxed to the Cramér's condition

$$\limsup_{|\lambda|\to\infty} \left|E\left[e^{i\lambda\xi_1}\right]\right| < 1 \qquad (\limsup_{|\lambda|\to\infty} \left|E\left[e^{i\lambda\xi_1}\right]\right| = 0 \text{ in the case } \frac{\beta_n}{\gamma_n} \to \infty). \tag{19}$$

■

We note that for the case $\frac{\beta_n}{\gamma_n} \to 0$, the Statement (I) requires the product form $f(x_0,...,x_r) = f_0(x_0)...f_r(x_r)$, which is not the case in Statement (II). Also note that (19) is very much weaker than (18) but the Statement (II) assumes that $f(x_0,...,x_r)$ is in the class $\mathcal{H}_2$, which is restrictive than the class $\mathcal{H}_1$, but is still reasonable for statistical applications. In this sense Statement (II) is quite satisfactory.

Note that, using Plancherel's theorem,

$$\int_{-\infty}^{\infty} E\left[f\left(x, x+S_{i_1}, ..., x+S_{i_r}\right)\right] dx = \frac{1}{2\pi} \int \widehat{f}\left(-\mu, \mu, ..., \mu\right) E\left[e^{-i\mu\left(S_{i_1}+...+S_{i_r}\right)}\right] d\mu.$$

Remark 2. It can be seen from the proof of the Theorem 1 that it extends to the joint convergence in distribution of

$$\left(\frac{\beta_n}{n} \sum_{l=1}^{n} f_i\left(\frac{\beta_n}{\gamma_n} S_l, \frac{\beta_n}{\gamma_n} S_{l+i_1}, ..., \frac{\beta_n}{\gamma_n} S_{l+i_r}\right), \quad i = 1, ..., q\right)$$

when the functions $f_i\left(x_0, ..., x_r\right)$, $i = 1, ..., q$, satisfy the conditions of Theorem 1. The same remark applies to Theorems 2 and 3 below. ■

To state the next result, let

$$\omega_k = \sum_{j=-\infty}^{k} d_{k-j} \eta_j \qquad \text{(coefficients } d_j \text{ are as in (2)).} \tag{20}$$

Theorem 2. *(I). Assume that* $0 < H < 1$ *and that (18) holds. For the linear process* ω_{l,ν_n} *as in (2) satisfying (3), let* $\sigma\left(\omega_{l,\nu_n}\right)$ *be as in (5) with* $\sigma\left(x\right)$ *continuous. Further assume that the constants* ν_n *satisfy* $\frac{\nu_n}{n} \to 0$.

Then, for any $f_0\left(x\right)$ *in the class* $\mathcal{G}_1$,

$$\frac{\beta_n}{n} \sum_{l=1}^{n} f_0\left(\frac{\beta_n}{\gamma_n} S_l\right) \sigma\left(\omega_{l,\nu_n}\right) \Longrightarrow L_1^0 E\left[\sigma\left(\omega_0\right)\right] \int f_0\left(x\right) dx, \tag{21}$$

where ω_0 *is as in (20). Further, for any* $f\left(x_0, ..., x_r\right)$ *as in the Statement (I) of Theorem 1 and for any* $1 \leq i_1 < ... < i_r$,

$$\frac{\beta_n}{n} \sum_{l=1}^{n} f\left(\frac{\beta_n}{\gamma_n} S_l, \frac{\beta_n}{\gamma_n} S_{l+i_1}, ..., \frac{\beta_n}{\gamma_n} S_{l+i_r}\right) \sigma\left(\omega_{l,\nu_n}\right) \sigma\left(\omega_{l+i_1,\nu_n}\right) ... \sigma\left(\omega_{l+i_r,\nu_n}\right)$$

$$\Longrightarrow \begin{cases} L_1^0 \frac{1}{2\pi} \int \widehat{f}\left(-\mu, \mu, ..., \mu\right) E\left[\sigma\left(\omega_0\right) \sigma\left(\omega_{i_1}\right) ... \sigma\left(\omega_{i_r}\right) e^{-i\mu\left(S_{i_1}+...+S_{i_r}\right)}\right] d\mu & \text{if } \beta_n = \gamma_n \\ L_1^0 E\left[\sigma\left(\omega_0\right) \sigma\left(\omega_{i_1}\right) ... \sigma\left(\omega_{i_r}\right)\right] \frac{1}{2\pi} \int \widehat{f}\left(-\mu, \mu, ..., \mu\right) d\mu & \text{if } \frac{\beta_n}{\gamma_n} \to 0 \\ 0 & \text{if } \frac{\beta_n}{\gamma_n} \to \infty. \end{cases} \tag{22}$$

(II). Assume $0 < H < 1$. *Suppose that (18) is relaxed to (19). Then (21) holds for any* $f_0\left(x\right)$ *in the class* $\mathcal{G}_2$.

Further, for any $f\left(x_0, ..., x_r\right)$ *in the class* $\mathcal{H}_2$ *and for any* $1 \leq i_1 < ... < i_r$, *the convergence (22) holds.* ■

Remark 3. Without going into the details we mention that Theorem 2 extends also, as will become clear form its proof, to the case when $\sigma\left(\omega_{k,\nu_n}\right)$ is replaced by the multilinear sum $\Omega_{k,\nu_n} = \sum_{i_1=k-\nu_n+1}^{k} \cdots \sum_{i_q=k-\nu_n+1}^{k} d_{k-i_1,\ldots,k-i_q} \varphi\left(\eta_{1,i_1}, \ldots, \eta_{q,i_q}\right)$ for a suitable φ, where the vectors $(\eta_{1,j}, \ldots, \eta_{q,j})$, $-\infty < j < \infty$, are iid such that (assuming without loss of generality that $\varphi(x_1, \ldots, x_q)$ is symmetric) $E\left[\left|\varphi\left(\eta_{1,1}, \ldots, \eta_{i-1,1}, \eta_{i,i}, \ldots, \eta_{q,q}\right)\right|^2\right] < \infty$ for all $2 \le i \le q+1$, and

$$\sum_{i_1=k-\nu_n+1}^{k} \cdots \sum_{i_q=k-\nu_n+1}^{k} \left|d_{i_1,\ldots,i_q}\right| \max\left(1, \left|g\left(i_1\right)\right|\right) \ldots \max\left(1, \left|g\left(i_{1q}\right)\right|\right) < \infty.$$

Without mentioning the detailed conditions, we note that the Theorem 2 holds also when $\sigma\left(\omega_{k,\nu_n}\right)$ is replaced by $\sigma\left(\Omega_{k,\nu_n}\right)$. The same remarks apply for the next Theorem 3 but perhaps it would be better to leave the precise forms of the required conditions (as suggested from the proof) to the specific situations at hand. ■

The next result gives additional restrictions under which (21) and (22) hold even when ω_{l,ν_n} and ω_{l+r,ν_n} in the left hand sides are replaced by ω_l and ω_{l+r} defined in (20). As will become clear later, Theorem 3 will follow as direct a consequence of (21) and (22).

Theorem 3. *In addition to the assumptions in either the Statement (I) or the Statement (II) of Theorem 2 above, assume that $\sigma(x)$ is p times differentiable for some $p \ge 1$ such that $\left|\sigma^{(p)}(x)\right| \le C$, $E\left[\left|\sigma^{(j)}\left(\omega_{l,\nu_n}\right)\right|^2\right] \le C$ for $1 \le j \le p-1$ and $E\left[\left|\eta_1\right|^{2p}\right] < \infty$. Assume further that the constants ν_n satisfy the additional restriction*

$$\beta_n \sum_{j=\nu_n}^{\infty} d_j^2 + E\left[\eta_1\right] \sqrt{\beta_n} \sum_{j=\nu_n}^{\infty} \left|d_j\right| \to 0. \tag{23}$$

Then the sum $\frac{\beta_n}{n} \sum_{l=1}^{n} f_0\left(\frac{\beta_n}{\gamma_n} S_l\right) \sigma\left(\omega_l\right)$ (where ω_l is as in (20)) also converges in distribution to the same limit in (21).

The sum $\frac{\beta_n}{n} \sum_{l=1}^{n} f\left(\frac{\beta_n}{\gamma_n} S_l, \frac{\beta_n}{\gamma_n} S_{l+r}\right) \sigma\left(\omega_l\right) \sigma\left(\omega_{l+r}\right)$ also converges in distribution to the same limit in (22) under the stronger conditions $E\left[\left|\sigma^{(j)}\left(\omega_{l,\nu_n}\right)\right|^4\right] \le C$ for $1 \le j \le p-1$ and $E\left[\left|\eta_1\right|^{4\max(q,p)}\right] < \infty$ (recall $\left|\sigma(x)\right| \le C\left|x\right|^q$) and the other remaining conditions the same as in Theorem 2. ■

To state the next Theorems 4 and 5, we introduce

Class $\mathcal{G}_3$. This is the class consisting of all Borel measurable real valued functions $f(x)$ defined on $\mathbb{R}$ such that $\int \left(M_{|f|,\eta}(x)\right)^i dx < \infty$, $i = 1, 2, 3, 4$, for some $\eta > 0$ (where $M_{|f|,\eta}(x)$ is as in (17)) and

$$\int \left(M_{f,\eta}(x) - m_{f,\eta}(x)\right) dx \le C\left|\eta\right|^d \quad \text{for some } \eta > 0 \text{ and } 0 < d \le 1. \tag{24}$$

■

The requirement (9) is assumed to hold in addition, without further mentioning, in Theorems 4 and 5 below.

Theorem 4. *(I) Assume* $1/3 < H < 1$*. In addition to (18) assume further*

$$\int |\lambda|^3 \left| E\left[e^{i\lambda\xi_1}\right]\right|^p d\lambda < \infty \quad \text{for some } p > 0. \tag{25}$$

Let $f(x)$ *be Borel measurable such that (6), (7) and*

$$\int f(y)\,dy = 0 \tag{26}$$

hold. Further, let $h(y)$ *be in the class* $\mathcal{G}_1$*. Then*

$$\left(\frac{1}{\gamma_n} S_{[nt]}, \frac{\gamma_n}{n}\sum_{k=1}^{n} h(S_k), \sqrt{\frac{\gamma_n}{n}}\sum_{k=1}^{n} f(S_k)\right) \overset{fdd}{\Longrightarrow} \left(\Lambda_{\alpha,H}(t), L_1^0 \int h(y)\,dy, W\sqrt{bL_1^0}\right),$$

where L_1^0 *is the local time as before,* W *is standard normal independent of the process* $\Lambda_{\alpha,H}(t)$ *and*

$$0 \le b = \frac{1}{2\pi}\int \left|\widehat{f}(\mu)\right|^2 \left(1 + 2\sum_{r=1}^{\infty} E\left[e^{-i\mu S_r}\right]\right) d\mu < \infty.$$

(II) Assume $1/3 < H < 1$*. Further assume that (19) hold and that* $f(x)$ *as above but instead of (6) assume that it is in the class* $\mathcal{G}_3$*. Further, let* $h(y)$ *be in the class* $\mathcal{G}_2$*.*

Then also the convergence in distribution in the Statement (I) holds. ■

We note that the requirements on the functions $f(x)$ and $h(x)$ in the Statement (II) are stronger (though still mild) than those in the Statement (I) but the Statement (II) assumes only the Cramér's condition (19) (regarding the restrictions (18) and (25) of the statement (I), see the Remark 4 below). Also note that the marginal convergence of $\frac{\gamma_n}{n}\sum_{k=1}^{n} h(S_k)$ in the preceding statements are particular cases of those in Jeganathan (2004), see further Proposition 14 in Section 4 below, from where it follows that they hold for all $0 < H < 1$, that is, the restriction $1/3 < H < 1$ is not required. Further, in view of the next paragraph, the restriction $1/3 < H < 1$ in Theorem 4 cannot probably be relaxed.

See Jeganathan (2006c) for the continuous time analogues of Theorem 4, in the forms of generalizations of the appropriate results in for instance Papanicolaou, Strook and Varadhan (1977), Yor (1983) and Rosen (1991). Note that these generalizations do not follow directly from Theorem 4. The reason is that in the method employed in the present paper the central limit phenomenon is involved at two different levels, one at the familiar level of the partial sum S_k itself, but another at the level of the partial sum of $f(S_k)$ themselves.

For later purposes we note that because $\left|E\left[e^{i\lambda\xi_1}\right]\right| \leq 1$, (25) entails

$$\int \left|E\left[e^{i\lambda\xi_1}\right]\right|^p d\lambda < \infty. \tag{27}$$

(This is also implied by (18) for $p \geq 2$.)

Remark 4, on the restrictions (18) and (25). Though these restrictions are not involved in the Statement (II) of Theorem 4 (and Theorem 5 below), we now indicate that from the point of view of statistical applications indicated earlier, they are not very restrictive. The restriction (18) entails that the Lebesgue density of the distribution of ξ_1 exists (Kawata (1972, Theorem 11.6.1)). If we denote this density by $\varphi(x)$, then $\psi(\lambda) = \widehat{\varphi}(\lambda)$ (recall $\psi(\lambda) = E\left[e^{i\lambda\xi_1}\right]$) and, by Plancherel's theorem, $\int |\psi(\lambda)|^2 d\lambda = 2\pi \int |\varphi(x)|^2 dx$.

Now suppose that the preceding density $\varphi(x)$ has a *distributional derivative* $\varphi'(x)$ such that $\varphi'(x)$ induces a finite signed measure (which will in particular entail $\int |\varphi'(x)| dx < \infty$). Then it can be shown that $\widehat{\varphi}(\lambda) = i\widehat{\varphi'}(\lambda)\lambda^{-1}$ where $\widehat{\varphi'}(\lambda)$ is the Fourier transform of (the signed measure induced by) $\varphi'(x)$. (This follows from standard facts about Fourier transforms and distributional derivatives, see for instance Rudin (1991).) In this case, in addition to (18), (25) holds for $p = 5$ and hence for all $p \geq 5$.

This is the case for instance when $\varphi(x)$ is suitably piecewise differentiable. As a simple example suppose that $\varphi(x) = \frac{1}{2}\mathbb{I}_{[-1,1]}(x)$, the density function of the random variable uniformly distributed over the interval $[-1, 1]$. Then the corresponding distributional derivative $\varphi'(x) = -\frac{1}{2}(\delta_1(x) - \delta_{-1}(x))$, where δ_a is the Dirac delta function. ■

In addition to the condition (3), we need a further condition for the next Theorem 5:

$$\sum_{r=1}^{\infty} \sqrt{\frac{1}{\gamma_r} \sum_{j=r}^{\infty} |d_j|^2} < \infty \qquad \text{(coefficients } d_j \text{ are as in(2)).} \tag{28}$$

This is not very restrictive. For instance, if $|d_j| \leq Cj^{-\frac{3}{2}+H}$, then $\sum_{j=r}^{\infty} |d_j|^2 \leq Cr^{-2+2H}$, so that noting $\gamma_r = r^H \kappa_r$ for a slowly varying κ_r, (28) will be of the form $\sum_{r=1}^{\infty} \sqrt{\kappa_r^{-1} r^{-2+H}} < \infty$.

Note that in the next result $E[\eta_1] = 0$. Without going into the details we mention that the linear sum ω_{l,ν_n} in the next result can be replaced by a suitable multilinear sum mentioned in Remark 3 above when $E[\varphi(\eta_{1,1}, ..., \eta_{i-1,1}, \eta_{i,i}, ..., \eta_{q,q})] = 0$ for all $2 \leq i \leq q+1$.

Theorem 5. *In addition to the preceding requirement (28), suppose that all the assumptions in* either one *of the Statements (I) or (II) of Theorem 4 hold, except that now possibly*

$$\int f(y) dy \neq 0.$$

Let the sequence ω_{l,ν_n} be as in (2) with η_1 satisfying (10) (in particular $E[\eta_1] = 0$) and with the constants satisfying $\frac{\nu_n}{n} \to 0$. Then

$$\left(\frac{1}{\gamma_n} S_{[nt]}, \frac{\gamma_n}{n} \sum_{l=1}^{n} h(S_l), \sqrt{\frac{\gamma_n}{n}} \sum_{l=1}^{n} f(S_l)\, \omega_{l,\nu_n}\right) \xrightarrow{fdd} \left(\Lambda_{\alpha,H}(t), L_1^0 \int h(y)\, dy, W\sqrt{b^* L_1^0}\right),$$

where

$$0 \le b^* = \frac{1}{2\pi} \int \left|\widehat{f}(\mu)\right|^2 \left(E\left[\omega_0^2\right] + 2\sum_{r=1}^{\infty} E\left[\omega_0 \omega_r e^{-i\mu S_r}\right]\right) d\mu < \infty.$$

The preceding convergence holds also, under the above same conditions, when the sum $\sqrt{\frac{\gamma_n}{n}} \sum_{l=1}^{n} f(S_l)\, \omega_{l,\nu_n}$ involved in the left hand side is replaced by $\sqrt{\frac{\gamma_n}{n}} \sum_{l=1}^{n} f(S_l)\, \omega_l$, where ω_l is as in (20), provided ν_n satisfies the additional restriction

$$n \sum_{j=\nu_n}^{\infty} d_j^2 \to 0.$$

■

The final statement in Theorem 5 follows from the convergence in the first part for exactly the same reason that the Theorem 3 follows from the convergencies (21) and (22). Note however that the above restriction $n \sum_{j=\nu_n}^{\infty} d_j^2 \to 0$ is stronger than that in Theorem 3.

As noted earlier, Theorem 5 has not been known previously, even for the situation $S_k = \sum_{j=1}^{k} \xi_j$ with $\omega_{k,\nu_n} = \eta_k$. Its possible continuous time versions in some specific forms have also been unknown.

3 SOME PRELIMINARIES

To begin with recall the fact that ξ_1 belongs to the domain of attraction of a strictly stable law with index $0 < \alpha \le 2$, in the sense of Section 1 above, means in particular (see Ibragimov and Linnik (1965, Theorem 2.6.5, page 85)) that, *for all u in some neighborhood of 0,*

$$\psi(u) = E\left[e^{iu\xi_1}\right] = \begin{cases} e^{-|u|^\alpha G(|u|)\left(1 - i\beta\, \mathrm{sign}(u) \tan\left(\frac{\pi\alpha}{2}\right)\right)} & \text{if } \alpha \ne 1 \\ e^{-|u| G(|u|)} & \text{if } \alpha = 1 \end{cases}$$

with $|\beta| \le 1$, where $G(u)$ is slowly varying as $u \to 0$. In particular there are constants $\eta > 0$ and $d > 0$ such that

$$|\psi(u)| \le e^{-d|u|^\alpha G(|u|)} \quad \text{for all } |u| \le \eta. \tag{29}$$

In addition, if one lets

$$b_n^{-1} = \inf\left\{u > 0 : u^\alpha G(u) = n^{-1}\right\},$$

then $b_n^\alpha \backsim nG(b_n^{-1})$ as $n \to \infty$, and in (1) one can take $\delta_n \backsim G^{\frac{1}{\alpha}}(b_n^{-1})$, so that we henceforth assume for convenience that δ_n in (1) and the above b_n are such that

$$b_n = n^{\frac{1}{\alpha}} G^{\frac{1}{\alpha}}(b_n^{-1}) = n^{\frac{1}{\alpha}} \delta_n. \tag{30}$$

See for instance Bingham et al (1987, page 344) for the details of these facts. Then note that, (15) takes the form

$$\gamma_n = \begin{cases} \left(\sum_{j=0}^{\infty} c_j\right) b_n & \text{if the condition (A1) is satisfied} \\ n^{H-\frac{1}{\alpha}} u_n b_n & \text{if the condition (A2) is satisfied.} \end{cases} \tag{31}$$

The following result is essentially well-known, and we supply its proof for completeness.

Lemma 6. *Let η be as in (29) and b_n be as in (30). Let κ_j be integers such that for some integer $j_0 > 0$ and a constant $C > 0$,*

$$\kappa_j \geq Cj \quad \text{ for all } j \geq j_0. \tag{32}$$

Then for every $0 < c < \alpha$ there is a constant $a > 0$ such that

$$\left|\psi\left(\lambda b_j^{-1}\right)\right|^{\kappa_j} \leq Ce^{-a|\lambda|^c} \quad \text{ for all } |\lambda| \leq \eta b_j,\ j \geq 1.$$

Further, if the Cramér's condition $\limsup_{|\lambda|\to\infty} |\psi(\lambda)| < 1$ *holds, then for every $\delta > 0$ there is a $0 < \rho < 1$ such that*

$$\sup_{|\lambda| \geq \delta b_j} \left|\psi\left(\lambda b_j^{-1}\right)\right|^{\kappa_j} = \sup_{|\mu| \geq \delta} |\psi(\mu)|^{\kappa_j} \leq C\rho^j \quad \text{ for all } j \geq 1.$$

Proof. According to (29), $\left|\psi\left(\lambda b_j^{-1}\right)\right|^{\kappa_j} \leq e^{-d\kappa_j|\lambda|^\alpha b_j^{-\alpha} G\left(|\lambda| b_j^{-1}\right)}$ for all $|\lambda| \leq \eta b_j$. Therefore we first recall a bound for $b_j^{-\alpha} G\left(|\lambda| b_j^{-1}\right)$ for all sufficiently large j.

According to Potter's inequality (see Bingham et al (1987, Theorem 1.5.6, Statement (ii), page 25), for every $\delta > 0$ there is a $B > 0$ such that $|\frac{G(x)}{G(y)}| \leq B \max\{(x/y)^\delta, (x/y)^{-\delta}\}$ for all $x > 0, y > 0$. In particular $\left|\frac{G\left(b_j^{-1}\right)}{G\left(|\lambda| b_j^{-1}\right)}\right| \leq B \max\{|\lambda|^\delta, |\lambda|^{-\delta}\}$. Because $\max\{|\lambda|^\delta, |\lambda|^{-\delta}\} = |\lambda|^\delta$ if $|\lambda| \geq 1$, it then follows from (30) that there is a j_1 such that

$$b_j^{-\alpha} G\left(|\lambda| b_j^{-1}\right) \geq B^{-1} j^{-1} |\lambda|^{-\delta} \quad \text{ for all } j \geq j_1 \text{ and } |\lambda| \geq 1.$$

Therefore, by (29), for every $0 < c < \alpha$ there is a $a > 0$ such that

$$\left|\psi\left(\lambda b_j^{-1}\right)\right|^{\kappa_j} \leq e^{-d\kappa_j|\lambda|^\alpha b_j^{-\alpha} G\left(|\lambda| b_j^{-1}\right)} \leq e^{-a|\lambda|^c} \quad \text{for all } 1 \leq |\lambda| \leq \eta b_j,\ j \geq j_2$$

where $j_2 = \max(j_0, j_1)$ (j_0 as in (32)). On the other hand, if $j \le j_2$,

$$\left|\psi\left(\lambda b_j^{-1}\right)\right|^{\kappa_j} \le 1 = e^{a|\eta b_j|^c} e^{-a|\eta b_j|^c} \le \left(\max_{j \le j_2} e^{a|\eta b_j|^c}\right) e^{-a|\lambda|^c} \quad \text{for all } |\lambda| \le \eta b_j,\ j \le j_2.$$

Further,

$$\left|\psi\left(\lambda b_j^{-1}\right)\right|^{\kappa_j} \le 1 = ee^{-1} \le ee^{-|\lambda|^c} \quad \text{if } |\lambda| \le 1,\ j \ge 1.$$

Hence the proof of the first part follows from the preceding three inequalities.

Regarding the second part note that the Cramér's condition involved is equivalent to the statement that for every $\delta > 0$, there is a $0 < \tau = \tau(\delta) < 1$ such that $\sup_{|\lambda| \ge \delta} |\psi(\lambda)| \le \tau < 1$. Hence the second statement follows, completing the proof of the lemma. ■

The following consequences of Lemma 6 will be used below. First, for any $\kappa \ge 0$,

$$\int_{\{|\lambda| \le \eta b_l\}} |\lambda|^\kappa \left|\psi\left(\frac{\lambda}{b_l}\right)\right|^{[l/2]} d\lambda \le C \int |\lambda|^\kappa e^{-a|\lambda|^c} d\lambda \le C,, \quad l \ge 1, \tag{33}$$

using the first part of Lemma 6.

Next let l_0 be such that for some $0 < \gamma < 1$, $[l/2] - p \ge [l\gamma]$ for all $l \ge l_0$, where p is as in (25). Then, for any $\delta > 0$ and $0 \le \kappa \le 3$, using the second part of Lemma 6 and using (25) (when $\kappa = 0$, only (27) is required),

$$\begin{aligned} &\int_{\{|\lambda| > \delta b_l\}} |\lambda|^\kappa \left|\psi\left(\frac{\lambda}{b_l}\right)\right|^{[l/2]} d\lambda \\ \le\ & C\rho^l \int_{\{|\lambda| > \delta b_l\}} |\lambda|^\kappa \left|\psi\left(\frac{\lambda}{b_l}\right)\right|^p d\lambda = C\rho^l b_l^{1+\kappa} \int |\lambda|^\kappa |\psi(\lambda)|^p d\lambda \le C\rho_*^l, \quad l \ge l_0, \end{aligned} \tag{34}$$

for some constant $0 < \rho_* < 1$.

We shall also need to use the next inequality, a direct consequence of Hölder's inequality.

Lemma 7. *For any functions $\varphi_i(u) : R^k \to R, i = 1, ..., q$,*

$$\int \prod_{i=1}^q |\varphi_i(u)|\, du \le \prod_{i=1}^q \left(\int |\varphi_i(u)|^q du\right)^{\frac{1}{q}}, \ q \ge 1.$$

By replacing $|\varphi_i(u)|$ by $|\ell(u)|^{1/q} |\varphi_i(u)|$ in this inequality, we also have

$$\int |\ell(u)| \prod_{i=1}^q |\varphi_i(u)|\, du \le \prod_{i=1}^q \left(\int |\ell(u)|\, |\varphi_i(u)|^q du\right)^{1/q}, \quad q \ge 1. \tag{35}$$

We now state one consequence of this. First note that, when (A2) holds,

$$g(j) = \sum_{s=0}^j c_s \sim C j^{H-1/\alpha} u_j, \quad j \to \infty.$$

(Note that in the case $H-1/\alpha<0$, the requirement $\sum_{j=0}^{\infty} c_j = 0$ (see (13)) is invoked here.) Therefore the requirement (14) on u_j holds for $g(j)$ also, that is, there is an integer $l_0>0$ such that $g(l)\neq 0$ and constants C_1 and C_2 such that

$$0<C_1\leq\left|\frac{g(l+j_1)}{g(l-j_2)}\right|\leq C_2 \quad \text{for all } 0\leq j_1,j_2\leq[l/2]$$

for all $l\geq l_0$. This also entails that, recalling that $\gamma_l=l^{H-1/\alpha}u_l b_l$ so that $\frac{\gamma_l}{b_l|g(q)|}=\frac{l^{H-1/\alpha}u_l}{|g(q)|}\sim\left|\frac{g(l)}{g(q)}\right|$, there is an l_0 and positive constants D_1 and D_2 such that

$$0<D_1\leq\frac{\gamma_l}{b_l\,|g\,(q)|}\leq D_2 \quad \text{for } [l/2]\leq q<l, \quad l\geq l_0. \tag{36}$$

Then, for $\delta>0$ such that $D_1^{-1}\delta=\eta$ with η as in the first part of Lemma 6, we have for $l\geq l_0$ and $\kappa\geq 0$,

$$\begin{aligned}
\int_{\{|\lambda|\leq\delta b_l\}}|\lambda|^\kappa\prod_{j=0}^{l}\left|\psi\left(\lambda\frac{g\,(j)}{\gamma_l}\right)\right|d\lambda &\leq \int_{\{|\lambda|\leq\delta b_l\}}|\lambda|^\kappa\prod_{j=[l/2]+1}^{l}\left|\psi\left(\lambda\frac{g\,(j)}{\gamma_l}\right)\right|d\lambda\\
\leq \prod_{j=[l/2]+1}^{l}&\left(\int_{\{|\lambda|\leq\delta b_l\}}|\lambda|^\kappa\left|\psi\left(\lambda\frac{g\,(j)}{\gamma_l}\right)\right|^{l-[l/2]}d\lambda\right)^{\frac{1}{l-[l/2]}} \qquad \text{by (35)}\\
= \prod_{j=[l/2]+1}^{l}&\left(\left|\frac{\gamma_l}{g\,(j)\,b_l}\right|^{1+\kappa}\int_{\left\{\left|\frac{\gamma_l}{g(j)b_l}\lambda\right|\leq\delta b_l\right\}}|\lambda|^\kappa\left|\psi\left(\frac{\lambda}{b_l}\right)\right|^{l-[l/2]}d\lambda\right)^{\frac{1}{l-[l/2]}}\\
\leq D_2^{1+\kappa}&\int_{\left\{|\lambda|\leq D_1^{-1}\delta b_l\right\}}|\lambda|^\kappa\left|\psi\left(\frac{\lambda}{b_l}\right)\right|^{l-[l/2]}d\lambda\leq C, \quad l\geq l_0, \text{ by (33) and (36).}
\end{aligned} \tag{37}$$

In the same way, using (34) when (25) holds, there is an l_0 such that for every $\delta>0$, $0\leq\kappa\leq 3$,

$$\int_{\{|\lambda|>\delta b_l\}}|\lambda|^\kappa\prod_{j=0}^{l-1}\left|\psi\left(g\,(j)\,\frac{\lambda}{\gamma_l}\right)\right|d\lambda\leq C\int_{\left\{|\lambda|>D_2^{-1}\delta b_l\right\}}|\lambda|^\kappa\left|\psi\left(\frac{\lambda}{b_l}\right)\right|^{l-[l/2]}d\lambda\leq C\rho^l, \quad l\geq l_0, \tag{38}$$

where $0<\rho<1$.

In addition, noting that $g\,(0)=1$ and $|\psi\,(\lambda)|\leq 1$, for any constants v_l, w_l, h_l such that $\min_{0\leq l\leq l_0}|w_l|>0$ and $\min_{0\leq l\leq l_0}|v_l|>0$, we have for $0\leq l\leq l_0$

$$\begin{aligned}
\max_{l\leq l_0}\left|\int\prod_{q=0}^{l}\left|\psi\left(w_l\frac{\lambda}{g\,(q)}\right)\right|\left|\widehat{f}\,(v_l\lambda-h_l)\right|d\lambda\right|^2 &\leq \max_{l\leq l_0}\left|\int\left|\psi\,(w_l\lambda)\,\widehat{f}\,(v_l\lambda-h_l)\right|d\lambda\right|^2\\
\leq \max_{l\leq l_0}\frac{1}{|w_l v_l|}\int|\psi\,(\lambda)|^2\,d\lambda\int\left|\widehat{f}\,(\lambda)\right|^2 d\lambda\leq C,
\end{aligned} \tag{39}$$

where we have used (18) and the fact $\int \left|\widehat{f}(\lambda)\right|^2 d\lambda = 2\pi \int |f(x)|^2 dx < \infty$. ■

In the context of the Statements (II) of Theorems 1 - 5, under the Cramér's condition (19), we shall use a certain smoothing device. To state it, let η be a positive number and K_η be a probability measure on $\mathbb{R}$ satisfying

$$K_\eta(\{x : |x| \leq \eta\}) = 1.$$

Then $K_{\eta_1} \times ... \times K_{\eta_k}$ is a probability measure on $\mathbb{R}^k$. Let $h(x_1, ..., x_k)$ be real valued functions on $\mathbb{R}^k$ such that $M_{|h|,\eta}(x_1, ..., x_k)$ is integrable with respect to μ. (Here $M_{h,\eta}(x)$ as well as $m_{h,\eta}(x)$ used below are as defined in (17).) Then clearly, for any finite measure ω on $\mathbb{R}^k$. Let

$$\int h(x_1, ..., x_k)\, d\omega(x_1, ..., x_k) \begin{array}{l} \geq \int m_{h,\eta}(x_1, ..., x_k)\, d(\omega * (K_\eta \times ... \times K_\eta))(x_1, ..., x_k) \\ \leq \int M_{h,\eta}(x_1, ..., x_k)\, d(\omega * (K_\eta \times ... \times K_\eta))(x_1, ..., x_k), \end{array} \tag{40}$$

where $*$ stands for the convolution. The probability measure K_η here will be chosen such that its characteristic function $\widehat{K_\eta}(\lambda)$ satisfies

$$\left|\widehat{K_\eta}(\lambda)\right| \leq C \exp\{-(\eta|\lambda|)^{1/2}\} \tag{41}$$

for all real λ, where C is a constant (independent of η). This is possible in view of Bhattacharya and Ranga Rao (1976, Corollary 10.4, page 88), where K_η is used extensively as a smoothing device.

Now, similar to (38), we have for every $\kappa > 0$ and for $l \geq l_0$,

$$\begin{aligned} & \int_{\{|\lambda|>\delta b_l\}} |\lambda|^\kappa \prod_{j=0}^{l-1} \left|\psi\left(g(j)\frac{\lambda}{\gamma_l}\right)\right| \left|\widehat{K_\eta}\left(\frac{\lambda}{\gamma_l}\right)\right| d\lambda \\ \leq\ & C\rho^l \prod_{j=[l/2]+1}^{l} \left(\int |\lambda|^\kappa \left|\widehat{K_\eta}\left(\frac{\lambda}{g(j)\, b_l}\right)\right| d\lambda\right)^{\frac{1}{l-[l/2]}} \leq C\rho^l \left(\frac{\gamma_l}{\eta}\right)^{1+\kappa}, \qquad l \geq l_0, \end{aligned} \tag{42}$$

where we have used (41), together with $\left|\frac{\gamma_l}{g(j)b_l}\right| \geq D_1 > 0$, see (36). Note that (42) is true for all $l \geq 1$ because the left hand side is bounded by

$$\int_{\{|\lambda|>\delta b_l\}} |\lambda|^\kappa \left|\widehat{K_\eta}\left(\frac{\lambda}{\gamma_l}\right)\right| d\lambda \leq C\left(\frac{\gamma_l}{\eta}\right)^{1+\kappa} \leq C\rho^{-l_0}\rho^l \left(\frac{\gamma_l}{\eta}\right)^{1+\kappa} \qquad \text{for } 1 \leq l \leq l_0. \tag{43}$$

Similarly, because in addition $\left|\widehat{K_\eta}(\lambda)\right| \leq C$ and similar to (37) using (33), we also have, for every $\kappa > 0$,

$$\int_{\{|\lambda|\leq\delta b_l\}} |\lambda|^\kappa \prod_{j=0}^{l-1} \left|\psi\left(g(j)\frac{\lambda}{\gamma_l}\right)\right| \left|\widehat{K}_\eta\left(\frac{\lambda}{\gamma_l}\right)\right| d\lambda \leq C \qquad l \geq 1. \tag{44}$$

It is important to note that (42) - (44) do not invoke the restrictions (18) and (25).

We next obtain some inequalities that will be used later on. For this purpose we note that, using the condition (7), we have $\left|\widehat{f}(\lambda_1) - \widehat{f}(\lambda_2)\right| \leq C|\lambda_1 - \lambda_2|$. Now (26) entails that $\widehat{f}(0) = \int_{-\infty}^{\infty} f(y)\,dy = 0$. Hence $\left|\widehat{f}(\lambda)\right| \leq C|\lambda|$. We also have $\left|\widehat{f}(\lambda)\right| \leq C$ using $\int |f(y)|\,dy < \infty$. Thus,

$$\left|\widehat{f}(\lambda)\right| \leq C\min(|\lambda|, 1) \quad \text{under (6), (7) and (26).} \tag{45}$$

Further, corresponding to $M_{f,\eta}(x)$, though $\int_{-\infty}^{\infty} M_{f,\eta}(y)dy \neq 0$, we have

$$\left|\widehat{M_{f,\eta}}(\lambda) - \widehat{f}(\lambda)\right| \leq \int (M_{f,\eta}(x) - f(x))\,dx \leq C|\eta|^d \qquad \text{under (24).}$$

The preceding two inequalities imply

$$\left|\widehat{M_{f,\eta}}(\lambda)\right| \leq C\min(|\lambda|, 1) + C|\eta|^d \qquad \text{under (6), (7), (24) and (26).} \tag{46}$$

To obtain further preliminaries, we next introduce a decomposition for S_k which will be repeatedly used throughout below. Recall that

$$S_k = \sum_{j=-\infty}^{0} (g(k-j) - g(-j))\xi_j + \sum_{j=1}^{k} g(k-j)\xi_j,$$

where recall that $g(j) = \sum_{s=0}^{j} c_s$. The indicated decomposition is

$$S_k = S_{k,l} + S_{k,l}^*, \qquad 1 \leq l \leq k, \tag{47}$$

where

$$S_{k,l} = \sum_{j=-\infty}^{0} (g(k-j) - g(-j))\xi_j + \sum_{j=1}^{k-l} g(k-j)\xi_j$$

and

$$S_{k,l}^* = \sum_{j=k-l+1}^{k} g(k-j)\xi_j = \sum_{q=0}^{l-1} g(q)\,\xi_{k-q}.$$

Here it is important to note that

$$S_{k,l} \text{ and } S_{k,l}^* \text{ are independent.}$$

In addition note that the marginal distribution of $S_{k,l}^*$ is the same as that of $T_l = \sum_{i=1}^{l} g(l-i)\,\xi_i$.

The second part of the next Lemma 8 will be used only in Theorem 4, in which $\beta_n = \gamma_n$.

Lemma 8. *If either(18) and* $\left|\widehat{f}(\lambda)\right| \le C$ *hold or (19) and* $\max\left(\left|\widehat{M_{f,\eta}}(\lambda)\right|, \left|\widehat{m_{f,\eta}}(\lambda)\right|\right) \le C$ *for all* $\eta > 0$ *hold, then*

$$\left|E_k\left[\frac{\beta_n}{\gamma_n} f(S_{k+l})\right]\right| \le \frac{C}{\gamma_l}\frac{\gamma_n}{\beta_n} \quad \text{for all } l \ge 1.$$

Further if $\left|\widehat{f}(\lambda)\right| \le C\min(|\lambda|, 1)$ *hold and either (18) hold or (19) and (24) hold, then*

$$\left|E_k\left[\frac{\beta_n}{\gamma_n} f(S_{k+l})\right]\right| \le C\frac{1}{\gamma_l^2}\left(\frac{\gamma_n}{\beta_n}\right)^2 \quad \text{for all } l \ge 1.$$

Here recall that E_k*,* $k \ge 1$*, stands for the conditional expectation given* $\{\xi_j, j \le k\}$*.*

Proof. First assume that (18) holds. Then using (47) and noting $f(y) = \frac{1}{2\pi}\int e^{-i\lambda y}\widehat{f}(\lambda)\,d\lambda$, we have

$$f\left(\frac{\beta_n}{\gamma_n}S_{k+l}\right) = \frac{1}{2\pi}\int e^{-i\lambda\frac{\beta_n}{\gamma_n}\left(S_{k+l,l}+S^*_{k+l,l}\right)}\widehat{f}(\lambda)\,d\lambda.$$

Therefore, because $S_{k+l,l}$ and $S^*_{k+l,l}$ are independent with $S_{k+l,l}$ being a function of $\{\xi_j, j \le k\}$,

$$\begin{aligned} \left|E_k\left[\frac{\beta_n}{\gamma_n} f(S_{k+l})\right]\right| &\le \frac{C}{\gamma_l}\frac{\gamma_n}{\beta_n}\int \left|E\left[e^{-i\frac{\lambda}{\gamma_l}S^*_{k+l,l}}\right]\right|\left|\widehat{f}\left(\frac{\gamma_n}{\beta_n}\frac{\lambda}{\gamma_l}\right)\right|d\lambda \\ &= \frac{C}{\gamma_l}\frac{\gamma_n}{\beta_n}\int \left|\widehat{f}\left(\frac{\gamma_n}{\beta_n}\frac{\lambda}{\gamma_l}\right)\right|\prod_{q=0}^{l-1}\left|\psi\left(\frac{g(q)}{\gamma_l}\lambda\right)\right|d\lambda, \end{aligned} \tag{48}$$

where we have used $\left|E\left[e^{-i\frac{\lambda}{\gamma_l}S^*_{k+l,l}}\right]\right| = \left|\prod_{i=0}^{l-1}\psi\left(\frac{\lambda g(i)}{\gamma_l}\right)\right|$.

Now, in view of (37) - (39), we have

$$\int \left|\widehat{f}\left(\frac{\gamma_n}{\beta_n}\frac{\lambda}{\gamma_l}\right)\right|\prod_{q=0}^{l-1}\left|\psi\left(\frac{g(q)}{\gamma_l}\lambda\right)\right|d\lambda \le \begin{cases} C & \text{if } \left|\widehat{f}(\lambda)\right| \le C \\ \frac{C}{\gamma_l}\frac{\gamma_n}{\beta_n} + C\rho^l \le \frac{C}{\gamma_l}\frac{\gamma_n}{\beta_n} & \text{if } \left|\widehat{f}(\lambda)\right| \le C\min(|\lambda|, 1) \end{cases}$$

using $0 < \rho < 1$. This gives the lemma when (18) holds.

We next show that the preceding bound holds under (19) also. According to (40),

$$\left|E_k\left[\frac{\beta_n}{\gamma_n} f(S_{k+l})\right]\right| \le \max\left(\left|E_k\left[M_{f,\eta}\left(\frac{\beta_n}{\gamma_n}(S_{k+l}+V_\eta)\right)\right]\right|, \left|E_k\left[m_{f,\eta}\left(\frac{\beta_n}{\gamma_n}(S_{k+l}+V_\eta)\right)\right]\right|\right)$$

for all $\eta > 0$, where V_η has the distribution K_η (having the characteristic function (41)) and is independent of $\{\xi_j, -\infty < j < \infty\}$, in particular independent of S_{k+l}. The same arguments above then give

$$\left|E_k\left[M_{f,\eta}\left(\frac{\beta_n}{\gamma_n}(S_{k+l}+V_\eta)\right)\right]\right| \le \frac{C}{\gamma_l}\frac{\gamma_n}{\beta_n}\int \left|\widehat{M_{f,\eta}}\left(\frac{\gamma_n}{\beta_n}\frac{\lambda}{\gamma_l}\right)\right|\prod_{q=0}^{l-1}\left|\psi\left(\frac{g(q)}{\gamma_l}\lambda\right)\right|\left|\widehat{K_\eta}\left(\frac{\lambda}{\gamma_l}\right)\right|d\lambda.$$

In the case $\left|\widehat{M_{f,\eta}}(\lambda)\right| \leq C$, we have using (42) - (44),

$$\int \left|\widehat{M_{f,\eta}}\left(\frac{\gamma_n}{\beta_n}\frac{\lambda}{\gamma_l}\right)\right| \prod_{q=0}^{l-1}\left|\psi\left(\frac{g(q)}{\gamma_l}\lambda\right)\right|\left|\widehat{K_\eta}\left(\frac{\lambda}{\gamma_l}\right)\right| d\lambda \leq C.$$

In the same way when $\left|\widehat{f}(\lambda)\right| \leq C \min(|\lambda|, 1)$, using (46) we obtain the bound

$$\begin{aligned}\int \left|\widehat{M_{f,\eta}}\left(\frac{\gamma_n}{\beta_n}\frac{\lambda}{\gamma_l}\right)\right| \prod_{q=0}^{l-1}\left|\psi\left(\frac{g(q)}{\gamma_l}\lambda\right)\right|\left|\widehat{K_\eta}\left(\frac{\lambda}{\gamma_l}\right)\right| d\lambda &\leq \left(\frac{C}{\gamma_l}\frac{\gamma_n}{\beta_n} + C|\eta|^d\right)\left(1 + \rho^l\frac{\gamma_l}{\eta}\right) \\ &\leq \frac{C}{\gamma_l}\frac{\gamma_n}{\beta_n} \text{ by choosing } \eta = \left(\frac{C}{\gamma_l}\frac{\gamma_n}{\beta_n}\right)^{\frac{1}{d}}.\end{aligned}$$

The preceding bounds hold also when $\widehat{m_{f,\eta}}\left(\frac{\gamma_n}{\beta_n}\frac{\lambda}{\gamma_l}\right)$ is involved in place of $\widehat{M_{f,\eta}}\left(\frac{\gamma_n}{\beta_n}\frac{\lambda}{\gamma_l}\right)$. ■

The analogue of the second part of the preceding Lemma 8 for $|E_k[w(S_{k+l}, S_{k+l+r})]|$ (for the case $\beta_n = \gamma_n$) will be obtained later and used in the context of the proofs of Theorems 4 and 5.

Next, similar to the inequality in the first part of the preceding Lemma 8, we obtain an inequality for $\left|E_k\left[w\left(\frac{\beta_n}{\gamma_n}S_{k+l}, \frac{\beta_n}{\gamma_n}S_{k+l+r}\right)\right]\right|$. These inequalities will be used below, in particular in the proofs of Theorems 1 and 2.

Recall that $S_{k+l} = S_{k+l,l} + S^*_{k+l,l}$ and $S_{k+l+r} = S_{k+l+r,l+r} + S^*_{k+l+r,l+r}$, see (47). Here $(S_{k+l,l}, S_{k+l+r,l+r})$ is a function of $\{\xi_j, j \leq k\}$ and is independent of $\left(S^*_{k+l,l}, S^*_{k+l+r,l+r}\right)$. In addition, the distribution of $\left(S^*_{k+l,l}, S^*_{k+l+r,l+r}\right)$ is the same as that of (T_l, T_{l+r}). (Recall $T_l = \sum_{j=1}^{l} g(l-j)\xi_j$.) Therefore, as in the proof of Lemma 8, we have

$$\left|E_k\left[f\left(\frac{\beta_n}{\gamma_n}S_{k+l}, \frac{\beta_n}{\gamma_n}S_{j+l+r}\right)\right]\right| \leq \int \left|E\left[e^{-i\lambda_1\frac{\beta_n}{\gamma_n}T_l - i\lambda_2\frac{\beta_n}{\gamma_n}T_{l+r}}\right]\right|\left|\widehat{f}(\lambda_1, \lambda_2)\right| d\lambda_1 d\lambda_2.$$

Here

$$\lambda_1 T_l + \lambda_2 T_{l+r} = \sum_{j=1}^{l}(\lambda_1 g(l-j) + \lambda_2 g(l+r-j))\xi_j + \sum_{j=l+1}^{l+r}\lambda_2 g(l+r-j)\xi_j,$$

where the first sum on the right hand side is independent of the second sum. Therefore

$$E\left[e^{-i\lambda_1 T_l - i\lambda_2 T_{l+r}}\right] = \left(\prod_{j=0}^{l-1}\psi(-\lambda_1 g(j) - \lambda_2 g(r+j))\right)\left(\prod_{j_1=0}^{r-1}\psi(-g(j_1)\lambda_2)\right).$$

Substituting this above, using the notation

$$g(j,r) = g(j+r) - g(j)$$

and making the transformation $(\lambda_1+\lambda_2,\lambda_2)\longmapsto(\lambda_1,\lambda_2)$, we obtain

$$\begin{aligned}
&\left|E_k\left[f\left(\frac{\beta_n}{\gamma_n}S_{k+l},\frac{\beta_n}{\gamma_n}S_{j+l+r}\right)\right]\right| \\
\leq\ & \int\left(\prod_{j=0}^{l-1}\left|\psi\left(-\frac{\beta_n}{\gamma_n}\lambda_1 g(j)-\frac{\beta_n}{\gamma_n}\lambda_2 g(j,r)\right)\right|\right)\prod_{j_1=0}^{r-1}\left|\psi\left(-g(j_1)\frac{\beta_n}{\gamma_n}\lambda_2\right)\right|\left|\widehat{f}(\lambda_1-\lambda_2,\lambda_2)\right|d\lambda_2 d\lambda_1 \\
\leq\ & \frac{1}{\gamma_l\gamma_r}\left(\frac{\gamma_n}{\beta_n}\right)^2\int\left(\prod_{j=[l/2]}^{l-1}\left|\psi\left(-\frac{\lambda_1 g(j)}{\gamma_l}-\frac{\lambda_2}{\gamma_r}g(j,r)\right)\right|\right) \\
&\times\prod_{j_1=[r/2]}^{r-1}\left|\psi\left(-g(j_1)\frac{\lambda_2}{\gamma_r}\right)\right|\left|\widehat{f}\left(\frac{\gamma_n}{\beta_n}\left(\frac{\lambda_1}{\gamma_l}-\frac{\lambda_2}{\gamma_r}\right),\frac{\gamma_n}{\beta_n}\frac{\lambda_2}{\gamma_r}\right)\right|d\lambda_2 d\lambda_1,
\end{aligned}\tag{49}$$

where we have used the fact that $\prod_{j=0}^{l-1}|\psi(\lambda)|\leq\prod_{j=[l/2]}^{l-1}|\psi(\lambda)|$. Making the transformations

$$\lambda_2=\mu_2\quad\text{and}\quad\lambda_1-\lambda_2\frac{\gamma_l}{g(j)}\frac{1}{\gamma_r}g(j,r)=\mu_1,$$

the last bound takes the form

$$\begin{aligned}
&\frac{1}{\gamma_l\gamma_r}\left(\frac{\gamma_n}{\beta_n}\right)^2\int_{R^2}\left(\prod_{j=[l/2]}^{l-1}\left|\psi\left(\frac{\mu_1 g(j)}{\gamma_l}\right)\right|\right)\prod_{j_1=[r/2]}^{r-1}\left|\psi\left(-g(j_1)\frac{\mu_2}{\gamma_r}\right)\right| \\
&\times\left|\widehat{f}\left(\frac{\gamma_n}{\beta_n}\left(\frac{\mu_1}{\gamma_l}+\frac{g(j,r)}{\gamma_r g(j)}\mu_2-\frac{\mu_2}{\gamma_r}\right),\frac{\gamma_n}{\beta_n}\frac{\mu_2}{\gamma_r}\right)\right|d\mu_1 d\mu_2.
\end{aligned}\tag{50}$$

Here note that the right hand side is nonrandom.

We also have, in view of (4),

$$E_k\left[f\left(\frac{\beta_n}{\gamma_n}S_{k+l},\frac{\beta_n}{\gamma_n}S_{k+l+r}\right)\right]\left\{\begin{array}{l}\leq E_k\left[M_{f,\eta}\left(\frac{\beta_n}{\gamma_n}S_{k+l}+V_\eta^{(1)},\frac{\beta_n}{\gamma_n}S_{k+l+r}+V_\eta^{(2)}\right)\right]\\ \geq E_k\left[m_{f,\eta}\left(\frac{\beta_n}{\gamma_n}S_{k+l}+V_\eta^{(1)},\frac{\beta_n}{\gamma_n}S_{k+l+r}+V_\eta^{(2)}\right)\right],\end{array}\right.$$

where $V_\eta^{(1)}$ and $V_\eta^{(2)}$ are independent random variables with the same distributions K_η, independent of (T_l,T_{l+r}). Hence the same bound (50) holds also when $\widehat{f}(\lambda,\mu)$ involved there is replaced by

$$\left|\widehat{K_\eta}(\lambda)\right|\left|\widehat{K_\eta}(\mu)\right|\max\left(\left|\widehat{M_{f,\eta}}(\lambda,\mu)\right|,\left|\widehat{m_{f,\eta}}(\lambda,\mu)\right|\right).\tag{51}$$

From these bounds we now obtain the next Lemmas 9 and 10 for $f(x_0,x_1)$. Note that in the cases $\beta_n=\gamma_n$ and $\frac{\beta_n}{\gamma_n}\to\infty$, the second inequality follows from the first but not in the case $\frac{\beta_n}{\gamma_n}\to 0$. For the second inequality in the case $\frac{\beta_n}{\gamma_n}\to 0$ and under (18), we take $f(x_0,x_1)=f_0(x_0)f_1(x_1)$.

Lemma 9. *Let $f(x_0, x_1)$ be such that either* $\max\left(\left|\widehat{M_{f,\eta}}(\lambda,\mu)\right|, \left|\widehat{m_{f,\eta}}(\lambda,\mu)\right|\right) \le C$ *and (19) hold or* $\left|\widehat{f}(\lambda,\mu)\right| \le C$ *and (18) hold. Then*

$$\left|E_k\left[f\left(\frac{\beta_n}{\gamma_n}S_{k+l}, \frac{\beta_n}{\gamma_n}S_{k+l+r}\right)\right]\right| \le \frac{C}{\gamma_l\gamma_r}\left(\frac{\gamma_n}{\beta_n}\right)^2 \quad \text{for all } l,r \ge 1.$$

Further (taking $f(x_0,x_1) = f_0(x_0) f_1(x_1)$ for the case $\frac{\beta_n}{\gamma_n} \to 0$ under (18)), for $r \ge 1$,

$$\left|E_k\left[f\left(\frac{\beta_n}{\gamma_n}S_{k+l}, \frac{\beta_n}{\gamma_n}S_{k+l+r}\right)\right]\right| \le \frac{C}{\gamma_l}\frac{\gamma_n}{\beta_n} \quad \text{for all } l \ge 1.$$

Proof. Consider the first inequality. For the case where (18) holds, it follows from (49) and (50) using (37) - (39).

For the other case, as noted above the same bound (50) but with $\widehat{f}(\lambda,\mu)$ replaced by (51) holds. Because $\max\left(\left|\widehat{M_{f,\eta}}(\lambda,\mu)\right|, \left|\widehat{m_{f,\eta}}(\lambda,\mu)\right|\right) \le C$, the resulting bound is bounded by

$$\begin{aligned} &\frac{1}{\gamma_l\gamma_r}\left(\frac{\gamma_n}{\beta_n}\right)^2 \int \left(\prod_{j=[l/2]}^{l-1}\left|\psi\left(\frac{\mu_1 g(j)}{\gamma_l}\right)\right|\right) \prod_{j_1=[r/2]}^{r-1}\left|\psi\left(-g(j_1)\frac{\mu_2}{\gamma_r}\right)\right| \\ &\times \left|\widehat{K_\eta}\left(\frac{\gamma_n}{\beta_n}\left(\frac{\mu_1}{\gamma_l} + \frac{g(j,r)}{\gamma_r g(j)}\mu_2 - \frac{\mu_2}{\gamma_r}\right)\right)\right| \left|\widehat{K_\eta}\left(\frac{\gamma_n}{\beta_n}\frac{\mu_2}{\gamma_r}\right)\right| d\mu_1 d\mu_2, \end{aligned} \tag{52}$$

where, following (42) - (44), we have when $l \ge 1$,

$$\begin{aligned} &\int \left|\widehat{K_\eta}\left(\frac{\gamma_n}{\beta_n}\left(\frac{\mu_1}{\gamma_l} + \frac{g(j,r)}{\gamma_r g(j)}\mu_2 - \frac{\mu_2}{\gamma_r}\right)\right)\right| \prod_{j=[l/2]}^{l-1}\left|\psi\left(\frac{\mu_1 g(j)}{\gamma_l}\right)\right| d\mu_1 \\ \le\ & C + C\rho^l \prod_{j=[l/2]+1}^{l}\left(\int\left|\widehat{K_\eta}\left(\frac{\gamma_n}{\beta_n}\left(\frac{\mu_1}{\gamma_l} + \frac{g(j,r)}{\gamma_r g(j)}\mu_2 - \frac{\mu_2}{\gamma_r}\right)\right)\right| d\mu_1\right)^{\frac{1}{l-[l/2]}} \\ \le\ & C + C\rho^l \frac{\gamma_l}{\eta}\frac{\beta_n}{\gamma_n} \le C \quad (\text{by choosing } \eta = \frac{\beta_n}{\gamma_n}) \end{aligned} \tag{53}$$

because (using (41)) $\int\left|\widehat{K_\eta}\left(\frac{\mu_1}{g(j)b_l} + \frac{g(j,r)}{\gamma_r g(j)}\mu_2 - \frac{\mu_2}{\gamma_r}\right)\right| d\mu_1 = \int\left|\widehat{K_\eta}\left(\frac{\gamma_n}{\beta_n}\frac{\mu_1}{g(j)b_l}\right)\right| d\mu_1 \le C\frac{\gamma_l}{\eta}\frac{\beta_n}{\gamma_n}$. Hence (52) is bounded by

$$\frac{C}{\gamma_l\gamma_r}\left(\frac{\gamma_n}{\beta_n}\right)^2 \int_R \left|\widehat{K_\eta}\left(\frac{\mu_2}{\gamma_r}\right)\right| \prod_{j_1=[r/2]}^{r-1}\left|\psi\left(-g(j_1)\frac{\mu_2}{\gamma_r}\right)\right| d\mu_2 \le \frac{C}{\gamma_l\gamma_r}\left(\frac{\gamma_n}{\beta_n}\right)^2,$$

using arguments similar to the above. This completes the proof of the first part of the lemma.

Regarding the second part, as noted earlier it follows from the first for the cases $\beta_n = \gamma_n$ and $\frac{\beta_n}{\gamma_n} \to \infty$. For the remaining case, first consider it under (18), so that by assumption $f(x_0, x_1) = f_0(x_0) f_1(x_1)$, and hence $\widehat{f}(\lambda, \mu) = \widehat{f_0}(\lambda) \widehat{f_1}(\mu)$. Using $\prod_{j_1=[r/2]}^{r-1} |\psi(\lambda)| \leq 1$ further, the bound (50) is bounded by

$$\frac{1}{\gamma_l \gamma_r} \left(\frac{\gamma_n}{\beta_n}\right)^2 \int \prod_{j=[l/2]}^{l-1} \left|\psi\left(\frac{\mu_1 g(j)}{\gamma_l}\right)\right| \left|\widehat{f_0}\left(\frac{\gamma_n}{\beta_n}\left(\frac{\mu_1}{\gamma_l} + \frac{g(j,r)}{\gamma_r g(j)}\mu_2 - \frac{\mu_2}{\gamma_r}\right)\right) \widehat{f_1}\left(\frac{\gamma_n}{\beta_n}\frac{\mu_2}{\gamma_r}\right)\right| d\mu_1 d\mu_2. \tag{54}$$

Here we have

$$\begin{aligned} &\int \left|\widehat{f_0}\left(\frac{\gamma_n}{\beta_n}\left(\frac{\mu_1}{\gamma_l} + \frac{g(j,r)}{\gamma_r g(j)}\mu_2 - \frac{\mu_2}{\gamma_r}\right)\right) \widehat{f_1}\left(\frac{\gamma_n}{\beta_n}\frac{\mu_2}{\gamma_r}\right)\right| d\mu_2 \\ = \; & \frac{\beta_n \gamma_r}{\gamma_n} \int \left|\widehat{f_0}\left(\frac{\gamma_n}{\beta_n}\frac{\mu_1}{\gamma_l} + \frac{g(j,r)}{g(j)}\mu_2 - \mu_2\right) \widehat{f_1}(\mu_2)\right| d\mu_2 \\ \leq \; & \frac{\beta_n \gamma_r}{\gamma_n} \sqrt{\left|\widehat{f_1}(\mu_2)\right|^2 d\mu_2 \left|\int \widehat{f_0}\left(\frac{\gamma_n}{\beta_n}\frac{\mu_1}{\gamma_l} + \frac{g(j,r)}{g(j)}\mu_2 - \mu_2\right)\right|^2 d\mu_2} \leq C\frac{\beta_n \gamma_r}{\gamma_n}, \end{aligned} \tag{55}$$

where we use $\left|\int \widehat{f_0}\left(\frac{\gamma_n}{\beta_n}\frac{\mu_1}{\gamma_l} + \frac{g(j,r)}{g(j)}\mu_2 - \mu_2\right)\right|^2 d\mu_2 = \left|\int \widehat{f_0}(\mu_2)\right|^2 d\mu_2$. Substituting this in (54), we see that the bound in (54) is bounded by $\frac{1}{\gamma_l}\left(\frac{\gamma_n}{\beta_n}\right) \int \left(\prod_{j=[l/2]}^{l-1}\left|\psi\left(\frac{\mu_1 g(j)}{\gamma_l}\right)\right|\right) d\mu_1 \leq \frac{C}{\gamma_l}\frac{\gamma_n}{\beta_n}$. This gives the second part under (18).

For the second part under (19), we use the same method above but with the role of $\widehat{f_0}(\lambda) \widehat{f_1}(\mu)$ now played by $\widehat{K_\eta}(\lambda) \widehat{K_\eta}(\mu)$. This completes the proof. ■

We next consider further generalizations of (49). Similar to (49), and in exactly the same way as in (50), one obtains (recall $g(j,r) = g(j+r) - g(j)$)

$$\begin{aligned} &(2\pi)^3 \left|E\left[f\left(\frac{\beta_n}{\gamma_n}S_l, \frac{\beta_n}{\gamma_n}S_{l+r}, \frac{\beta_n}{\gamma_n}S_{l+r+q}\right)\right]\right| \leq \\ &\frac{C}{\gamma_l \gamma_r \gamma_q}\left(\frac{\gamma_n}{\beta_n}\right)^3 \int \left(\prod_{j_1=[l/2]}^{l-1}\left|\psi\left(\frac{\mu_1 g(j_1)}{\gamma_l}\right)\right|\right)\left(\prod_{j_2=[r/2]}^{r-1}\left|\psi\left(\frac{\mu_2 g(j_2)}{\gamma_r}\right)\right|\right) \prod_{j_3=[q/2]}^{q-1}\left|\psi\left(\frac{\mu_3 g(j_3)}{\gamma_q}\right)\right| \\ &\times \left|\widehat{f}\left(\frac{\gamma_n}{\beta_n}\left(\frac{\lambda_1}{\gamma_l} - \frac{\lambda_2}{\gamma_r}\right), \frac{\gamma_n}{\beta_n}\left(\frac{\lambda_2}{\gamma_r} - \frac{\lambda_3}{\gamma_q}\right), \frac{\gamma_n}{\beta_n}\frac{\lambda_3}{\gamma_q}\right)\right| d\mu_1 d\mu_2 d\mu_3, \end{aligned} \tag{56}$$

where

$$\lambda_1 + \lambda_2 \frac{\gamma_l g(j_1, r)}{\gamma_r g(j_1)} + \lambda_3 \frac{\gamma_l g(j_1 + r, q)}{\gamma_q g(j_1)} = \mu_1, \; \lambda_2 + \lambda_3 \frac{\gamma_r g(j_2, q)}{\gamma_q g(j_2)} = \mu_2 \text{ and } \lambda_3 = \mu_3.$$

In the same way we have

$$
\begin{aligned}
&(2\pi)^4\left|E\left[f\left(\frac{\beta_n}{\gamma_n}S_l,\frac{\beta_n}{\gamma_n}S_{l+r},\frac{\beta_n}{\gamma_n}S_{l+r+q},\frac{\beta_n}{\gamma_n}S_{l+r+q+s}\right)\right]\right| \\
\leq\; & \frac{C}{\gamma_l\gamma_r\gamma_q\gamma_s}\left(\frac{\gamma_n}{\beta_n}\right)^4\int\prod_{j_1=[l/2]}^{l-1}\left|\psi\left(\frac{\mu_1 g(j_1)}{\gamma_l}\right)\right|\prod_{j_2=[r/2]}^{r-1}\left|\psi\left(\frac{\mu_2 g(j_2)}{\gamma_r}\right)\right| \\
&\times\prod_{j_3=[q/2]}^{q-1}\left|\psi\left(\frac{\mu_3 g(j_3)}{\gamma_q}\right)\right|\prod_{j_4=[s/2]}^{s-1}\left|\psi\left(\frac{\mu_4 g(j_4)}{\gamma_s}\right)\right| \\
&\times\left|\widehat{f}\left(\frac{\gamma_n}{\beta_n}\left(\frac{\lambda_1}{\gamma_l}-\frac{\lambda_2}{\gamma_r}\right),\frac{\gamma_n}{\beta_n}\left(\frac{\lambda_2}{\gamma_r}-\frac{\lambda_3}{\gamma_q}\right),\frac{\gamma_n}{\beta_n}\left(\frac{\lambda_3}{\gamma_q}-\frac{\lambda_4}{\gamma_s}\right),\frac{\gamma_n}{\beta_n}\frac{\lambda_4}{\gamma_s}\right)\right|d\mu_1d\mu_2d\mu_3d\mu_4 \qquad (57)
\end{aligned}
$$

where

$$
\left.
\begin{array}{r}
\lambda_1+\lambda_2\frac{\gamma_l g(j_1,r)}{g(j_1)\gamma_r}+\lambda_3\frac{\gamma_l g(j_1+r,q)}{g(j_1)\gamma_q}+\lambda_4\frac{\gamma_l g(j_1+r+q,s)}{g(j_1)\gamma_s}=\mu_1 \\
\lambda_2+\lambda_3\frac{\gamma_r g(j_2,q)}{g(j_2)\gamma_q}+\lambda_4\frac{\gamma_r g(j_2+q,s)}{g(j_2)\gamma_s}=\mu_2 \\
\lambda_3+\lambda_4\frac{\gamma_q g(j_3,s)}{g(j_3)\gamma_s}=\mu_3,\qquad \lambda_4=\mu_4.
\end{array}
\right\} \qquad (58)
$$

As a consequence of (57), we obtain the next lemma, analogous to Lemma 9 for the case $f(x_0,x_1,x_2,x_3)$. As in Lemma 9 the second part follows from the first for the cases $\beta_n=\gamma_n$ and $\frac{\beta_n}{\gamma_n}\to\infty$. Also, for the second part in the case $\frac{\beta_n}{\gamma_n}\to 0$ and under (18), we take $f(x_0,x_1,x_2,x_3)=f_0(x_0)f_1(x_1)f_0(x_2)f_1(x_3)$.

Lemma 10. *Let $f(x_0,x_1,x_2,x_3)$ be such that either $\left|\widehat{f}(\upsilon_0,\upsilon_1,\upsilon_2,\upsilon_3)\right|\leq C$ and (18) hold or $\max\left(\left|\widehat{M_{f,\eta}}(\upsilon_0,\upsilon_1,\upsilon_2,\upsilon_3)\right|,\left|\widehat{m_{w,\eta}}(\upsilon_0,\upsilon_1,\upsilon_2,\upsilon_3)\right|\right)\leq C$ and (19) hold. Then*

$$
\left|E\left[f\left(\frac{\beta_n}{\gamma_n}S_l,\frac{\beta_n}{\gamma_n}S_{l+r},\frac{\beta_n}{\gamma_n}S_{l+r+q},\frac{\beta_n}{\gamma_n}S_{l+r+q+s}\right)\right]\right|\leq\frac{C}{\gamma_l\gamma_r\gamma_q\gamma_s}\left(\frac{\gamma_n}{\beta_n}\right)^4\quad\text{for all } l,r,q,s\geq 1.
$$

Further (taking $f(x_0,x_1,x_2,x_3)=f_0(x_0)f_1(x_1)f_0(x_2)f_1(x_3)$ for the case $\frac{\beta_n}{\gamma_n}\to 0$ under (18)), for each fixed $r\geq 1$ and $s\geq 1$,

$$
\left|E\left[f\left(\frac{\beta_n}{\gamma_n}S_l,\frac{\beta_n}{\gamma_n}S_{l+r},\frac{\beta_n}{\gamma_n}S_{l+r+q},\frac{\beta_n}{\gamma_n}S_{l+r+q+s}\right)\right]\right|\leq\frac{C}{\gamma_l\gamma_q}\left(\frac{\gamma_n}{\beta_n}\right)^2\quad\text{for all } l,q\geq 1.
$$

Proof. The proof of the first part is obtained from (57) in exactly the same way the first part of Lemma 9 was obtained from (50).

The proof of the second part is also similar to that of the second part of Lemma 9. Under (18), we use (57) with $\widehat{f}(\upsilon_0,\upsilon_1,\upsilon_2,\upsilon_3)=\widehat{f}_0(\upsilon_0)\widehat{f}_1(\upsilon_1)\widehat{f}_2(\upsilon_2)\widehat{f}_3(\upsilon_3)$. Similar to (55), we have $\int\left|\widehat{f}_1\left(\frac{\gamma_n}{\beta_n}\frac{\lambda_4}{\gamma_s}\right)\right|\left|\widehat{f}_0\left(\frac{\gamma_n}{\beta_n}\left(\frac{\lambda_3}{\gamma_q}-\frac{\lambda_4}{\gamma_s}\right)\right)\right|d\mu_4\leq C\frac{\beta_n\gamma_s}{\gamma_n}$. Substituting this in (57), and using $\prod_{j_2=[r/2]}^{r-1}|\psi(\lambda)|\le 1$ and $\prod_{j_4=[s/2]}^{s-1}|\psi(\lambda)|\le 1$, (57) then gives the bound

$$\begin{aligned}
&\left|E\left[f_0\left(\frac{\beta_n}{\gamma_n}S_l\right)f_1\left(\frac{\beta_n}{\gamma_n}S_{l+r}\right)f_0\left(\frac{\beta_n}{\gamma_n}S_{l+r+q}\right)f_1\left(\frac{\beta_n}{\gamma_n}S_{l+r+q+s}\right)\right]\right|\\
\le\ & \frac{C}{\gamma_l\gamma_q\gamma_r}\left(\frac{\gamma_n}{\beta_n}\right)^3\int\prod_{j_1=[l/2]}^{l-1}\left|\psi\left(\frac{\mu_1 g(j_1)}{\gamma_l}\right)\right|\prod_{j_3=[q/2]}^{q-1}\left|\psi\left(\frac{\mu_3 g(j_3)}{\gamma_q}\right)\right|\\
&\times\left|\widehat{f}_0\left(\frac{\gamma_n}{\beta_n}\left(\frac{\lambda_1}{\gamma_l}-\frac{\lambda_2}{\gamma_r}\right)\right)\widehat{f}_1\left(\frac{\gamma_n}{\beta_n}\left(\frac{\lambda_2}{\gamma_r}-\frac{\lambda_3}{\gamma_q}\right)\right)\right|d\mu_1 d\mu_2 d\mu_3.
\end{aligned}$$

Similar to (55), we further have $\int\left|\widehat{f}_0\left(\frac{\gamma_n}{\beta_n}\left(\frac{\lambda_1}{\gamma_l}-\frac{\lambda_2}{\gamma_r}\right)\right)\widehat{f}_1\left(\frac{\gamma_n}{\beta_n}\left(\frac{\lambda_2}{\gamma_r}-\frac{\lambda_3}{\gamma_q}\right)\right)\right|d\mu_2\le C\frac{\beta_n\gamma_r}{\gamma_n}$. Substituting this in the right hand side of the above inequality we see that its left hand side is bounded by

$$\frac{C}{\gamma_l\gamma_q}\left(\frac{\gamma_n}{\beta_n}\right)^2\int\prod_{j_1=[l/2]}^{l-1}\left|\psi\left(\frac{\mu_1 g(j_1)}{\gamma_l}\right)\right|\prod_{j_3=[q/2]}^{q-1}\left|\psi\left(\frac{\mu_3 g(j_3)}{\gamma_q}\right)\right|d\mu_1 d\mu_3\le\frac{C}{\gamma_l\gamma_q}\left(\frac{\gamma_n}{\beta_n}\right)^2.$$

This proves the second part under (18).

Under (19), in the same way as in the proof of Lemma 9, the same method above is used, with the role of $\widehat{f}_0(\upsilon_0)\,\widehat{f}_1(\upsilon_1)\,\widehat{f}_2(\upsilon_2)\,\widehat{f}_3(\upsilon_3)$ being played by $\widehat{K_\eta}(\upsilon_0)\,\widehat{K_\eta}(\upsilon_1)\,\widehat{K_\eta}(\upsilon_2)\,\widehat{K_\eta}(\upsilon_3)$, completing the proof of the lemma. ■

The inequalities in the preceding Lemmas 9 and 10 will help to deal with the sums of the form $\sum f(S_k)$ involved in Theorems 1 and 4. We next obtain similar inequalities that will help to deal with the sums of the forms $\sum f(S_k)\,\sigma(\omega_{k,\nu})$ and $\sum f(S_k)\,\omega_{k,\nu}$ involved respectively in Theorems 2 and 5. The main idea consists of reducing the situations to essentially to those of Lemmas 8 - 10. We start with the analogue of Lemma 8.

Lemma 11. *Let the linear process $\omega_{l,\nu}$ be as in (2) and (3). If the assumptions of the first part of Lemma 8 hold with f replaced by $|f|$, and if $\sigma(\omega_{k,\nu})$ is as in (5), then*

$$E\left[\left|f\left(\frac{\beta_n}{\gamma_n}S_l\right)\sigma(\omega_{k,\nu})\right|\right]\le\frac{C}{\gamma_l}\frac{\gamma_n}{\beta_n}\qquad\text{for all } l\ge 1.$$

Further, if the assumptions of the second part of Lemma 8 hold, together with the condition $E[\eta_1]=0$ and $E[|\eta_1\xi_1|]<\infty$, then

$$\left|E_k\left[\frac{\beta_n}{\gamma_n}f(S_{k+l})\,\omega_{k+l,\nu}\right]\right|\le\frac{C}{\gamma_l^2}\frac{\gamma_n}{\beta_n}\qquad\text{for all } l\ge\nu \text{ and for all } \delta>0.$$

Proof. Let us consider the first part for the case $\sigma(\omega_{k,\nu})=\omega_{k,\nu}$ and then indicate that essentially the same proof holds for the general case also. We have $E\left[\left|f\left(\frac{\beta_n}{\gamma_n}S_l\right)\omega_{l,\nu}\right|\right]\le$

$\sum_{j=l-\nu+1}^{l} |d_{l-j}|\, E\left[|\eta_j| \left|f\left(\frac{\beta_n}{\gamma_n}S_l\right)\right|\right]$ because $\omega_{l,\nu} = \sum_{j=l-\nu+1}^{l} d_{l-j}\eta_j$. Consider $E\left[|\eta_j| \left|f\left(\frac{\beta_n}{\gamma_n}S_l\right)\right|\right]$ where recall that $\left|f\left(\frac{\beta_n}{\gamma_n}S_{k+l}\right)\right| = \frac{1}{2\pi}\int e^{-i\lambda\frac{\beta_n}{\gamma_n}S_l}\widehat{|f|}(\lambda)\, d\lambda$. Suppose that $j \leq 0$. Then (recall $S_l = S_{l,l} + S_{l,l}^*$)

$$\left|E\left[|\eta_j|\, e^{-i\lambda\frac{\beta_n}{\gamma_n}S_l}\right]\right| = \left|E\left[|\eta_j|\, e^{-i\lambda\frac{\beta_n}{\gamma_n}S_{l,l}}\right]\right| \left|E\left[e^{-i\lambda\frac{\beta_n}{\gamma_n}S_{l,l}^*}\right]\right|,$$

where $\left|E\left[|\eta_j|\, e^{-i\lambda\frac{\beta_n}{\gamma_n}S_{l,l}}\right]\right| \leq C$ if $E[|\eta_j|] \leq C$. Thus

$$\left|E\left[|\eta_j|\, e^{-i\lambda\frac{\beta_n}{\gamma_n}S_l}\right]\right| \leq C\left|E\left[e^{-i\lambda\frac{\beta_n}{\gamma_n}S_{l,l}^*}\right]\right| \quad \text{when } j \leq 0.$$

In the same way (recall $S_{l,l}^* = \sum_{i=1}^{l} g(l-i)\xi_i$)

$$\left|E\left[|\eta_j|\, e^{-i\lambda\frac{\beta_n}{\gamma_n}S_l}\right]\right| \leq C\left|E\left[e^{-i\lambda\frac{\beta_n}{\gamma_n}\sum_{i=1,i\neq j}^{l} g(l-i)\xi_i}\right]\right| \quad \text{when } j > 0.$$

Therefore, using the same arguments of the first part of Lemma 8, we have $E\left[\left|\eta_j f\left(\frac{\beta_n}{\gamma_n}S_l\right)\right|\right] \leq \frac{C}{\gamma_l}\frac{\gamma_n}{\beta_n}$. Thus

$$E\left[\left|f\left(\frac{\beta_n}{\gamma_n}S_l\right)\omega_{l,\nu}\right|\right] \leq \sum_{j=l-\nu+1}^{l} |d_{l-j}| \left|E\left[\left|\eta_j f\left(\frac{\beta_n}{\gamma_n}S_l\right)\right|\right]\right| \leq \frac{C}{\gamma_l}\frac{\gamma_n}{\beta_n}\sum_{j=0}^{\infty}|d_j| \leq \frac{C}{\gamma_l}\frac{\gamma_n}{\beta_n},$$

establishing the first part of the lemma for the case $\sigma(\omega_{k,\nu}) = \omega_{k,\nu}$.

For the general case when $|\sigma(\omega_{k,\nu})| \leq C|\omega_{k,\nu}|^q$, $q \geq 1$, we have

$$E\left[\left|f\left(\frac{\beta_n}{\gamma_n}S_l\right)\sigma(\omega_{k,\nu})\right|\right] \leq \sum_{i_1=k-\nu_n+1}^{k} \cdots \sum_{i_q=k-\nu_n+1}^{k} |d_{k-i_1}| \ldots |d_{k-i_q}|\, E\left[(|\eta_{i_1}| \ldots |\eta_{i_q}|)\left|f\left(\frac{\beta_n}{\gamma_n}S_l\right)\right|\right],$$

and as can be seen easily that the bound obtained above for $E\left[\left|\eta_j f\left(\frac{\beta_n}{\gamma_n}S_l\right)\right|\right]$ holds true for $E\left[(|\eta_{i_1}| \ldots |\eta_{i_q}|)\left|f\left(\frac{\beta_n}{\gamma_n}S_l\right)\right|\right]$ also, so that the first part of lemma is proved.

For the second part, we have $E_k\left[\frac{\beta_n}{\gamma_n}f(S_{k+l})\,\omega_{k+l,\nu}\right] = \sum_{j=k+l-\nu+1}^{k+l} d_{k+l-j}E_k\left[\eta_j f\left(\frac{\beta_n}{\gamma_n}S_{k+l}\right)\right]$. Because $l \geq \nu$, we have $k+l-\nu+1 > k$ and hence $j > k$. Hence, $S_{k+l,l}$ and $\left(\eta_j, S_{k+l,l}^*\right)$ are independent in the identity $S_{k+l} = S_{k+l,l} + S_{k+l,l}^*$, so that, as in Lemma 8,

$$\left|E_k\left[\eta_j f\left(\frac{\beta_n}{\gamma_n}S_{k+l}\right)\right]\right| = \frac{1}{2\pi}\int\left|E\left[\eta_j e^{-i\lambda\frac{\beta_n}{\gamma_n}S_{k+l,l}^*}\right]\right|\left|\widehat{f}(\lambda)\right| d\lambda.$$

Here, noting that $S_{k+l,l}^* = \sum_{j=k+1}^{k+l} g(k+l-j)\xi_j$,

$$\left|E\left[\eta_j e^{-i\lambda\frac{\beta_n}{\gamma_n}S_{k+l,l}^*}\right]\right| \leq \left|E\left[\eta_j e^{-i\lambda\frac{\beta_n}{\gamma_n}g(k+l-j)\xi_j}\right]\right|\left|E\left[\eta_j e^{-i\lambda\frac{\beta_n}{\gamma_n}\sum_{i=k+1,i\neq j}^{k+l} g(k+l-i)\xi_i}\right]\right|,$$

with

$$\left|E\left[\eta_j e^{-i\lambda\frac{\beta_n}{\gamma_n}g(k+l-j)\xi_j}\right]\right| \leq |\lambda|\,\frac{\beta_n}{\gamma_n}\,|g(k+l-j)| \quad \text{using } E[\eta_j]=0,\ E[|\eta_j\xi_j|]<\infty.$$

Thus, using the same arguments of the second part of Lemma 8, but with the role of $\left|\widehat{f}(\lambda)\right| \leq C\min(|\lambda|,1)$ now played by the preceding inequality and $\left|\widehat{f}(\lambda)\right| \leq C$, we have $\left|E\left[\eta_j f\left(\frac{\beta_n}{\gamma_n}S_{k+l}\right)\right]\right| \leq C\frac{1}{\gamma_l^2}\frac{\gamma_n}{\beta_n}|g(k+l-j)|$. Therefore

$$\begin{aligned}\left|E\left[\eta_j f\left(\frac{\beta_n}{\gamma_n}S_{k+l}\right)\omega_{k+l,\nu}\right]\right| &\leq \sum_{j=k+l-\nu+1}^{k+l}|d_{k+l-j}|\left|E\left[\eta_j f\left(\frac{\beta_n}{\gamma_n}S_{k+l}\right)\right]\right| \\ &\leq C\frac{1}{\gamma_l^2}\frac{\gamma_n}{\beta_n}\sum_{j=l-\nu+1}^{l}|d_{l-j}|\,|g(l-j)| \leq C\frac{1}{\gamma_l^2}\frac{\gamma_n}{\beta_n}, \text{ using (3).}\end{aligned}$$

This proves the second part, completing the proof of the lemma. ■

Next, using the arguments of the proof of the first part of the preceding Lemma 11, it is clear that the following statement holds, where note that the conditional expectation $E_k[.]$ of Lemma 9 is replaced by $E[.]$.

Lemma 12. *Let $\sigma(\omega_{k,\nu})$ is as in (5). Then under the same conditions in Lemmas 9 and 10, the bounds in Lemmas 9 hold true also when $E_k\left[f\left(\frac{\beta_n}{\gamma_n}S_{k+l}, \frac{\beta_n}{\gamma_n}S_{k+l+r}\right)\right]$ in the left hand side is replaced by $E\left[f\left(\frac{\beta_n}{\gamma_n}S_l, \frac{\beta_n}{\gamma_n}S_{l+r}\right)\sigma(\omega_{l,\nu})\,\sigma(\omega_{l+r,\nu})\right]$, and similarly the bounds in Lemma 10 hold true when $E\left[f\left(\frac{\beta_n}{\gamma_n}S_l, \frac{\beta_n}{\gamma_n}S_{l+r}, \frac{\beta_n}{\gamma_n}S_{l+r+q}, \frac{\beta_n}{\gamma_n}S_{l+r+q+s}\right)\right]$ there is replaced by $E\left[f\left(\frac{\beta_n}{\gamma_n}S_l, \frac{\beta_n}{\gamma_n}S_{l+r}, \frac{\beta_n}{\gamma_n}S_{l+r+q}, \frac{\beta_n}{\gamma_n}S_{l+r+q+s}\right)\sigma(\omega_{l,\nu})\,\sigma(\omega_{l+r,\nu})\,\sigma(\omega_{l+r+q,\nu})\,\sigma(\omega_{l+r+q+s,\nu})\right]$*

4 PROOF OF THEOREMS 1 - 3

For simplicity we shall restrict to $\frac{\beta_n}{n}\sum_{l=1}^n f\left(\frac{\beta_n}{\gamma_n}S_l, \frac{\beta_n}{\gamma_n}S_{l+r}\right)$ in Theorem 1. Then, letting throughout below

$$n_{mk} = \left[n\frac{k}{m}\right] - \left[n\frac{k-1}{m}\right]$$

and noting $\sum_{l=1}^n f\left(\frac{\beta_n}{\gamma_n}S_l, \frac{\beta_n}{\gamma_n}S_{l+r}\right) = \sum_{k=1}^m\sum_{l=1}^{n_{mk}} f\left(\frac{\beta_n}{\gamma_n}S_{\left[n\frac{k-1}{m}\right]+l}, \frac{\beta_n}{\gamma_n}S_{\left[n\frac{k-1}{m}\right]+l+r}\right)$, it is clear that Theorem 1 follows from the next Propostion13.

Proposition 13. *Let the function $f(x_0,x_1)$ be as in Theorem 1. Assume that the assumptions of Theorem 1 are satisfied. Then, as $n\to\infty$ first and then $m\to\infty$,*

$$\frac{\beta_n}{n}\sum_{k=1}^m\sum_{l=1}^{n_{mk}} E_{\left[n\frac{k-1}{m}\right]}\left[f\left(\frac{\beta_n}{\gamma_n}S_{\left[n\frac{k-1}{m}\right]+l}, \frac{\beta_n}{\gamma_n}S_{\left[n\frac{k-1}{m}\right]+l+r}\right)\right]$$

$$\Longrightarrow \begin{cases} L_1^0\int_{-\infty}^{\infty}E\left[f_0(x)\,f_1(x+S_r)\right]dx & \text{if } \beta_n=\gamma_n \\ L_1^0\int_{-\infty}^{\infty}f_0(x)\,f_1(x)\,dx & \text{if } \frac{\beta_n}{\gamma_n}\to 0 \\ 0 & \text{if } \frac{\beta_n}{\gamma_n}\to\infty. \end{cases}$$

In addition, as $n \to \infty$ first and then $m \to \infty$,

$$E\left[\left\{\frac{\beta_n}{n}\sum_{k=1}^{m}\sum_{l=1}^{n_{mk}} f\left(\frac{\beta_n}{\gamma_n}S_{\left[n\frac{k-1}{m}\right]+l}, \frac{\beta_n}{\gamma_n}S_{\left[n\frac{k-1}{m}\right]+l+r}\right)\right.\right.$$
$$\left.\left.-\frac{\beta_n}{n}\sum_{k=1}^{m}\sum_{l=1}^{n_{mk}} E_{\left[n\frac{k-1}{m}\right]}\left[f\left(\frac{\beta_n}{\gamma_n}S_{\left[n\frac{k-1}{m}\right]+l}, \frac{\beta_n}{\gamma_n}S_{\left[n\frac{k-1}{m}\right]+l+r}\right)\right]\right\}^2\right] \to 0. \tag{59}$$

The proof of the preceding result will contain the proof of the following result

Proposition 14. *(I). Suppose that (18) holds and that the function $h(x)$ is in the class $\mathcal{G}_1$. Then, as $n \to \infty$ first and then $m \to \infty$,*

$$\frac{\beta_n}{n}\sum_{k=1}^{m}\sum_{l=l_0}^{n_{mk}} E_{\left[n\frac{k-1}{m}\right]}\left[h\left(\frac{\beta_n}{\gamma_n}S_{\left[n\frac{k-1}{m}\right]+l}\right)\right] \Longrightarrow \widehat{h}(0)\, L_1^0$$

In addition, as $n \to \infty$ first and then $m \to \infty$,

$$E\left[\left\{\frac{\beta_n}{n}\sum_{k=1}^{m}\sum_{l=1}^{n_{mk}}\left(h\left(\frac{\beta_n}{\gamma_n}S_{\left[n\frac{k-1}{m}\right]+l}\right) - E_{\left[n\frac{k-1}{m}\right]}\left[h\left(\frac{\beta_n}{\gamma_n}S_{\left[n\frac{k-1}{m}\right]+l}\right)\right]\right)\right\}^2\right] \to 0,$$

and hence

$$\frac{\beta_n}{n}\sum_{l=1}^{n} h\left(\frac{\beta_n}{\gamma_n}S_l\right) \Longrightarrow \widehat{h}(0)\, L_1^0 \quad \text{as } n \to \infty.$$

(II). Suppose that (19) holds and assume that $h(x)$ is in the class $\mathcal{G}_2$. Then also all the conclusions in the preceding Statement (I) hold.

Note that $\widehat{h}(0) = \int h(x)\,dx$. The conclusion $\frac{\beta_n}{n}\sum_{l=1}^{n} h\left(\frac{\beta_n}{\gamma_n}S_l\right) \Longrightarrow \widehat{h}(0)\, L_1^0$ in Proposition 14 is already known, see Jeganathan (2004) but we shall need below the other conclusions for the case $h(x) = f^2(x)$. Note that for this case, by Plancherel's theorem, $\widehat{h}(0) = \widehat{f^2}(0) = \int f^2(x)\,dx = \frac{1}{2\pi}\int \left|\widehat{f}(\mu)\right|^2 d\mu$.

Similar to Theorem 1, the next Proposition 15 will give Theorem 3.

Proposition 15. *Let the functions $f_0(x_0)$ and $f(x_0, x_1)$ be as in Theorem 2. Assume that the assumptions in the Statement (I) or those in the Statement (II) of Theorem 2 are satisfied. Then the conclusions of Proposition 14 hold also when $f_0\left(S_{\left[n\frac{k-1}{m}\right]+l}\right)\omega_{\left[n\frac{k-1}{m}\right]+l,\nu_n}$ is involved in place of $h\left(S_{\left[n\frac{k-1}{m}\right]+l}\right)$. Similarly, the conclusions of Propositions 13 hold when $f\left(S_{\left[n\frac{k-1}{m}\right]+l}, S_{\left[n\frac{k-1}{m}\right]+l+r}\right)\omega_{\left[n\frac{k-1}{m}\right]+l,\nu_n}\omega_{\left[n\frac{k-1}{m}\right]+l+r,\nu_n}$ is involved in place of $f\left(S_{\left[n\frac{k-1}{m}\right]+l}, S_{\left[n\frac{k-1}{m}\right]+l+r}\right)$.*

We now proceed with the proofs of Proposition 13. The required modifications needed for the proof of Proposition 15 will be described later on in this section. First we need to introduce some preliminaries. First recall from (47) that

$$S_{\left[n\frac{k-1}{m}\right]+q}=S_{\left[n\frac{k-1}{m}\right]+q,q}+\sum_{j=1}^{q}g\left(q-j\right)\xi_{\left[n\frac{k-1}{m}\right]+j}$$

where note that $S_{\left[n\frac{k-1}{m}\right]+q,q}$ and $\sum_{j=1}^{q}g\left(q-j\right)\xi_{\left[n\frac{k-1}{m}\right]+j}$ are functions of the respective collections $\left\{\xi_j:-\infty<j\leq\left[n\frac{k-1}{m}\right]\right\}$ and $\left\{\xi_j:j>\left[n\frac{k-1}{m}\right]\right\}$, which collections are independent of each other and do not depend on q. Further $\left\{\sum_{j=1}^{l}g\left(l-j\right)\xi_{\left[n\frac{k-1}{m}\right]+j};1<l\leq n_{mk}\right\}$ has the same distribution as that of $\{T_l;1<l\leq n_{mk}\}$, where

$$T_l=\sum_{j=1}^{l}g\left(l-j\right)\xi_j.$$

Hence one can write

$$\begin{aligned}
&E_{\left[n\frac{k-1}{m}\right]}\left[f\left(\frac{\beta_n}{\gamma_n}S_{\left[n\frac{k-1}{m}\right]+l},\frac{\beta_n}{\gamma_n}S_{\left[n\frac{k-1}{m}\right]+l+r}\right)\right]\\
=\;&E\left[f\left(y_1+\frac{\beta_n}{\gamma_n}T_l,y_2+\frac{\beta_n}{\gamma_n}T_{l+r}\right)\right]_{(y_1,y_2)=\left(\frac{\beta_n}{\gamma_n}S_{\left[n\frac{k-1}{m}\right]+l,l},\frac{\beta_n}{\gamma_n}S_{\left[n\frac{k-1}{m}\right]+l+r,l+r}\right)}. \qquad (60)
\end{aligned}$$

Letting, for any $0\leq\kappa_n<l$ (κ_n will also be allowed to tend to ∞ appropriately),

$$T^*_{nl}=\sum_{j=1}^{l-\kappa_n}g(l-j)\xi_j,\qquad T^*_{nl,r}=\sum_{j=1}^{l-\kappa_n}g(l+r-j)\xi_j,$$

we have

$$T_l=T^*_{nl}+\sum_{j=l-\kappa_n+1}^{l}g(l-j)\xi_j,\qquad T_{l+r}=T^*_{nl,r}+\sum_{j=l-\kappa_n+1}^{l+r}g(l+r-j)\xi_j.$$

(Note that T^*_{nl} and $T^*_{nl,r}$ depend on κ_n.) Hence, we have for any $0\leq\kappa_n<l$,

$$\begin{aligned}
&(2\pi)^2\,E\left[f\left(y_1+\frac{\beta_n}{\gamma_n}T_l,y_2+\frac{\beta_n}{\gamma_n}T_{l+r}\right)\right]\\
=\;&\int e^{-i\lambda y_1-i\mu y_2}E\left[e^{-i\lambda\frac{\beta_n}{\gamma_n}T_l-i\mu\frac{\beta_n}{\gamma_n}T_{l+r}}\right]\widehat{w}\left(\lambda,\mu\right)d\lambda d\mu\\
=\;&\int e^{-i\lambda y_1-i\mu y_2}E\left[e^{-i(\lambda+\mu)\frac{\beta_n}{\gamma_n}T^*_{nl}-i\mu\frac{\beta_n}{\gamma_n}\left(T^*_{nl,r}-T^*_{nl}\right)}\right]E\left[e^{-i\lambda\frac{\beta_n}{\gamma_n}\left(T_l-T^*_{nl}\right)-i\mu\frac{\beta_n}{\gamma_n}\left(T_{l+r}-T^*_{nl,r}\right)}\right]\widehat{f}\left(\lambda,\mu\right)d\lambda d\mu\\
=\;&\frac{1}{\beta_n}\int e^{-i\frac{\lambda}{\beta_n}y_1-i\mu(y_2-y_1)}E\left[e^{-i\lambda\frac{T^*_{nl}}{\gamma_n}-i\mu\frac{\beta_n}{\gamma_n}\left(T^*_{nl}-T^*_{nl,r}\right)}\right]E\left[e^{-i\frac{\lambda}{\gamma_n}\left(T_l-T^*_{nl}\right)-i\mu\frac{\beta_n}{\gamma_n}\left(T_{l+r}-T^*_{nl,r}-T_l+T^*_{nl}\right)}\right]\\
&\times\widehat{f}\left(\frac{\lambda}{\beta_n}-\mu,\mu\right)d\lambda d\mu. \qquad (61)
\end{aligned}$$

Now (recall $g(j)=0$ if $j<0$)

$$\begin{aligned}
& E\left[e^{-i\frac{\lambda}{\gamma_n}\left(T_l-T^*_{nl}\right)-i\mu\left(T_{l+r}-T^*_{nl,r}-T_l+T^*_{nl}\right)}\right] \\
= \; & E\left[e^{-i\frac{\lambda}{\gamma_n}\left(T_l-T^*_{nl}\right)-i\mu\frac{\beta_n}{\gamma_n}\sum_{j=l-\kappa_n+1}^{l+r}(g(l+r-j)-g(l-j))\xi_j}\right] \\
= \; & E\left[e^{-i\frac{\lambda}{\gamma_n}\left(T_l-T^*_{nl}\right)-i\mu\frac{\beta_n}{\gamma_n}\sum_{j=l-\kappa_n+1}^{l}\left(c_{l+1-j}+\ldots+c_{l+r-j}\right)\xi_j}\right]\prod_{j=0}^{r-1}\psi\left(-\frac{\beta_n}{\gamma_n}g\left(j\right)\mu\right),
\end{aligned}$$

where and throughout below we let

$$c_j=0 \text{ for } j<0.$$

Similarly

$$E\left[e^{-i\lambda\frac{T^*_{nl}}{\gamma_n}-i\mu\frac{\beta_n}{\gamma_n}\left(T^*_{nl,r}-T^*_{nl}\right)}\right]=\prod_{j=\kappa_n}^{l-1}\psi\left(\frac{\lambda g(j)}{\gamma_n}-\mu\frac{\beta_n}{\gamma_n}\left(c_{j+1}+\ldots+c_{j+r}\right)\right).$$

Hence

$$\begin{aligned}
& \left|E\left[e^{-i\lambda\frac{T^*_{nl}}{\gamma_n}-i\mu\frac{\beta_n}{\gamma_n}\left(T^*_{nl}-T^*_{nl,r}\right)}\right]E\left[e^{-i\frac{\lambda}{\gamma_n}\left(T_l-T^*_{nl}\right)-i\mu\frac{\beta_n}{\gamma_n}\left(T_{l+r}-T^*_{nl,r}-T_l+T^*_{nl}\right)}\right]\right| \\
\leq \; & \prod_{j=\kappa_n}^{l-1}\left|\psi\left(\frac{\lambda g(j)}{\gamma_n}-\mu\frac{\beta_n}{\gamma_n}\left(c_{j+1}+\ldots+c_{j+r}\right)\right)\right|\prod_{j_1=0}^{r-1}\left|\psi\left(-\frac{\beta_n}{\gamma_n}g\left(j_1\right)\mu\right)\right|. \qquad (62)
\end{aligned}$$

With these preliminaries, we now consider the proof of Propositions 13 through a series of steps. In order to state and prove the first step, we need the following result, which describes the intent of the condition involved in the class $\mathcal{H}_1$.

Lemma 16. *Let $f(x_0,\ldots,x_r)$, $r\geq 1$, be such that $\int\left(\int\left|f(x_0,\ldots,x_r)\right|^2dx_r\right)^{\frac{1}{2}}dx_0\ldots dx_{r-1}<\infty$.* Then

$$\sup_{\lambda_0,\ldots\lambda_{r-1},c}\int\left|\widehat{f}(\lambda_0,\ldots\lambda_{r-2},\lambda_{r-1}+c\mu,\mu)\right|^2d\mu\leq\int\left(\int\left|f(x_0,\ldots,x_r)\right|^2dx_r\right)^{\frac{1}{2}}dx_0\ldots dx_{r-1}.$$

In particular for $f(x_0,x_1)$ as in the Statement (I) of Theorem 1,

$$\sup_{c,\lambda}\int\left|\widehat{f}\left(\lambda+c\mu,\mu\right)\right|^2d\mu\leq\int\left(\int\left|f\left(x,y\right)\right|^2dy\right)^{\frac{1}{2}}dx\leq C.$$

Proof. We have by definition

$$\begin{aligned}
& \widehat{f}(\lambda_0,\ldots\lambda_{r-2},\lambda_{r-1}+c\mu,\mu) \\
= \; & \int e^{i\lambda_0x_0+\ldots+i\lambda_{r-2}x_{r-2}+i(\lambda_{r-1}+c\mu)x_{r-1}+i\mu x_r}f(x_0,\ldots,x_r)dx_0\ldots dx_r \\
= \; & \int e^{i\mu x_r}\left\{\int e^{i\lambda_0x_0+\ldots+i\lambda_{r-1}x_{r-1}}f(x_0,\ldots,x_{r-1},x_r-cx_{r-1})dx_0\ldots dx_{r-1}\right\}dx_r.
\end{aligned}$$

Then by Plancherel's theorem, for each $\lambda_0, ...\lambda_{r-1}, c$,

$$\begin{aligned}
& \int \left| \widehat{f}(\lambda_0, ...\lambda_{r-2}, \lambda_{r-1} + c\mu, \mu) \right|^2 d\mu \\
= & \int \left| \int e^{i\lambda_0 x_0 + ... + i\lambda_{r-1}x_{r-1}} f(x_0, ..., x_{r-1}, x_r - cx_{r-1}) dx_0 ... dx_{r-1} \right|^2 dx_r \\
\leq & \int \left| \int |f(x_0, ..., x_{r-1}, x_r - cx_{r-1})|^2 dx_r \right|^{1/2} dx_0 ... dx_{r-1} \\
= & \int \left(\int |f(x_0, ..., x_{r-1}, x_r)|^2 dx_r \right)^{1/2} dx_0 ... dx_{r-1},
\end{aligned}$$

where in obtaining the inequality we have used the generalized Minkowski inequality (see for instance Folland (1984, page 186)). This proves the result. ■

Below in Lemmas 17 and 18, recall that in the Statement (I) of Theorem 1 for the case $\frac{\beta_n}{\gamma_n} \to 0$ it is assumed that the function $f(x_0, x_1)$ is of the product form $f(x_0, x_1) = f_0(x_0) f_1(x_1)$.

Lemma 17. *Let $f(x_0, x_1)$ be as in Proposition 13, and assume that $T^*_{nl,r}$ and T^*_{nl} (see (61), defined previously, correspond to $2\kappa_n < [n\delta]$, $0 < \delta < 1$. Then the next two statements hold (recall $T_l = \sum_{j=1}^{l} g(l-j)\xi_j$)*

(I). Suppose that (18) holds. Let $R_n(y_1, y_2, a, \delta)$ be the difference between

$$(2\pi)^2 \frac{\beta_n}{n} \sum_{l=[n\delta]+1}^{n} E\left[f\left(y_1 + \frac{\beta_n}{\gamma_n} T_l, y_2 + \frac{\beta_n}{\gamma_n} T_{l+r} \right) \right] \tag{63}$$

and

$$\begin{aligned}
& \frac{1}{n} \sum_{l=[n\delta]+1}^{n} \int_{\{|\mu| \leq a, |\lambda| \leq a\}} U_n(\lambda, \mu, y_1, y_2) E\left[e^{-i\lambda \frac{T^*_{nl}}{\gamma_n} - i\mu \frac{\beta_n}{\gamma_n}\left(T^*_{nl,r} - T^*_{nl}\right)} \right] \\
& \times E\left[e^{-i\frac{\lambda}{\gamma_n}\left(T_l - T^*_{nl}\right) - i\mu\frac{\beta_n}{\gamma_n}\left(T_{l+1} - T^*_{nl,r} - T_l + T^*_{nl}\right)} \right] \widehat{f}\left(\frac{\lambda}{\beta_n} - \mu, \mu \right) d\lambda d\mu
\end{aligned} \tag{64}$$

where

$$U_n(\lambda, \mu, y_1, y_2) = e^{-i\frac{\lambda}{\beta_n} y_1 - i\mu(y_2 - y_1)}.$$

Then

$$\lim_{a\to\infty} \limsup_{n\to\infty} \left(\sup_{y_1, y_2} |R_n(y_1, y_2, a, \delta)| \right) = 0 \quad \text{ for each } \delta > 0.$$

(II). Suppose that (19) holds (instead of (18)). Let $V_\eta^{(1)}$ and $V_\eta^{(2)}$ be independent random variables with the same distributions K_η, independent of (T_l, T_{l+r}). Consider (63) with $E\left[M_{f,\eta}\left(y_1 + \frac{\beta_n}{\gamma_n} T_l + V_\eta^{(1)}, y_2 + \frac{\beta_n}{\gamma_n} T_{l+r} + V_\eta^{(2)} \right) \right]$ in place of $E\left[f\left(y_1 + \frac{\beta_n}{\gamma_n} T_l, y_2 + \frac{\beta_n}{\gamma_n} T_{l+r} \right) \right]$

and (64) with $\widehat{M_{f,\eta}}\left(\frac{\lambda}{\beta_n}-\mu,\mu\right)\widehat{K_\eta}\left(\frac{\lambda}{\beta_n}-\mu\right)\widehat{K_\eta}(\mu)$ *in place of* $\widehat{f}\left(\frac{\lambda}{\beta_n}-\mu,\mu\right)$. *Let* $R_n(y_1,y_2,a,\delta,\eta)$ *be the difference between these two. Then*

$$\lim_{a\to\infty}\limsup_{n\to\infty}\left(\sup_{y_1,y_2}|R_n(y_1,y_2,a,\delta,\eta)|\right)=0 \quad \text{for each } \delta,\ \eta>0.$$

The same holds when $m_{f,\eta}$ *is involved in place of* $M_{f,\eta}$.

Proof. First consider the Statement (I) under (18). Note that (63) involves the left hand side of the identity (61). Further when in (64) the $\int_{\{|\mu|\le a,|\lambda|\le a\}}$ is replaced by $\int_{R^2}$, it reduces to that involving the right hand side of (61). Therefore the difference $R_n(y_1,y_2,a,\delta)$ in the Statement (I) of the lemma is simply the same as (64) but with the integral $\int_{\{|\mu|\le a,|\lambda|\le a\}}$ replaced by the $\int_{\{|\mu|\le a,|\lambda|\le a\}^c}$, where $\{|\mu|\le a,|\lambda|\le a\}^c$ stands for the complement of $\{|\mu|\le a,|\lambda|\le a\}$.

For notational simplification, we treat the case $r=1$. Then, using (62) and noting that $|\psi(\lambda)|\le 1$, $|U_n(\lambda,\mu,y_1,y_2)|\le C$, and $\{|\mu|\le a,|\lambda|\le a\}^c\subset\{|\mu|>a,|\lambda|<\infty\}\cup\{|\mu|\le a,|\lambda|>a\}$, we have

$$\begin{aligned}&|R_n(y_1,y_2,a,\delta)|\\ \le\ &\frac{1}{n}\sum_{l=[n\delta]+1}^{n}\int_{\{|\mu|>a,|\lambda|<\infty\}\cup\{|\mu|\le a,|\lambda|>a\}}\left|\psi\left(-\frac{\beta_n}{\gamma_n}\mu\right)\widehat{f}\left(\frac{\lambda}{\beta_n}-\mu,\mu\right)\right|\prod_{j=\kappa_n}^{l-1}\left|\psi\left(\frac{\lambda g(j)}{\gamma_n}-\mu\frac{\beta_n}{\gamma_n}c_{j+1}\right)\right|d\lambda d\mu.\end{aligned}$$

Note that $\prod_{j=\kappa_n}^{l-1}\left|\psi\left(\frac{\lambda g(j)}{\gamma_n}-\mu\frac{\beta_n}{\gamma_n}c_{j+1}\right)\right|\le\prod_{j=[l/2]}^{l-1}\left|\psi\left(\frac{\lambda g(j)}{\gamma_n}-\mu\frac{\beta_n}{\gamma_n}c_{j+1}\right)\right|$ because $\kappa_n<[n\delta]/2\le[l/2]$.

We first deal with the integral over $\{|\mu|>a,|\lambda|<\infty\}$. Using (35),

$$\begin{aligned}&\int_{\{|\mu|>a,|\lambda|<\infty\}}\left|\psi\left(-\frac{\beta_n}{\gamma_n}\mu\right)\widehat{f}\left(\frac{\lambda}{\beta_n}-\mu,\mu\right)\prod_{j=[l/2]}^{l-1}\psi\left(\frac{\lambda g(j)}{\gamma_n}-\mu\frac{\beta_n}{\gamma_n}c_{j+1}\right)\right|d\lambda d\mu\\ \le\ &\prod_{j=[l/2]}^{l-1}N_{1j}^{\frac{1}{l-[l/2]}}\le\max_{[l/2]\le j\le l-1}N_{1j},\end{aligned}$$

where

$$\begin{aligned}N_{1j}&=\int_{\{|\mu|>a,|\lambda|<\infty\}}\left|\psi\left(-\frac{\beta_n}{\gamma_n}\mu\right)\widehat{f}\left(\frac{\lambda}{\beta_n}-\mu,\mu\right)\right|\left|\psi\left(\frac{\lambda g(j)}{\gamma_n}-\mu\frac{\beta_n}{\gamma_n}c_{j+1}\right)\right|^{l-[l/2]}d\lambda d\mu\\ &=\frac{\gamma_n}{|g(j)|\,b_l}\int_{\{|\mu|>a,|\lambda|<\infty\}}\left|\psi\left(-\frac{\beta_n}{\gamma_n}\mu\right)\widehat{f}\left(\frac{\gamma_n}{\beta_n}\frac{\lambda}{g(j)b_l}-\mu+\mu\frac{c_{j+1}}{g(j)},\mu\right)\right|\left|\psi\left(\frac{\lambda}{b_l}\right)\right|^{l-[l/2]}d\lambda d\mu,\end{aligned}$$

making the change of variable $\frac{\lambda g(j)}{\gamma_n}-\mu\frac{\beta_n}{\gamma_n}c_{j+1}\longmapsto\frac{\lambda}{b_l}$. Here note that $\int\left|\psi\left(\frac{\lambda}{b_l}\right)\right|^{l-[l/2]}d\lambda\le C$ (see (33) and (34)) and $\frac{\gamma_n}{|g(j)|b_l}\le C\frac{\gamma_n}{\gamma_l}$ using $\max_{[l/2]\le j\le l}\frac{\gamma_l}{|g(j)|b_l}\le C$ (see (36)). Therefore,

$$N_{1j}\le CQ_n(a)\frac{\gamma_n}{\gamma_l},$$

where

$$Q_n(a) = \max_{[n\delta]\le j\le n} \sup_{v} \int_{\{|\mu|>a\}} \left|\psi\left(-\frac{\beta_n}{\gamma_n}\mu\right)\widehat{f}\left(v-\mu+\mu\frac{c_{j+1}}{g(j)},\mu\right)\right| d\mu.$$

Now note that

$$\begin{aligned}
&\int_{\{|\mu|>a\}} \left|\psi\left(-\frac{\beta_n}{\gamma_n}\mu\right)\widehat{f}\left(\nu-\mu+\mu\frac{c_{j+1}}{g(j)},\mu\right)\right| d\mu \\
\le\ & \sqrt{\int \left|\widehat{f}\left(\nu-\mu+\mu\frac{c_{j+1}}{g(j)},\mu\right)\right|^2 d\mu \frac{\gamma_n}{\beta_n}\int_{\left\{|\mu|>\frac{\beta_n}{\gamma_n}a\right\}} |\psi(\mu)|^2 d\mu} \\
\le\ & C\left(\frac{\gamma_n}{\beta_n}\int_{\left\{|\mu|>\frac{\beta_n}{\gamma_n}a\right\}} |\psi(\mu)|^2 d\mu\right)^{1/2}, \qquad (65)
\end{aligned}$$

where in obtaining the last inequality we have used Lemma 16. In the case $\beta_n = \gamma_n$, the factor $\frac{\gamma_n}{\beta_n}\int_{\left\{|\mu|>\frac{\beta_n}{\gamma_n}a\right\}} |\psi(\mu)|^2 d\mu$ in the preceding bound reduces to $\int_{\{|\mu|>a\}} |\psi(\mu)|^2 d\mu \to 0$ as $a\to\infty$, and in the case $\frac{\beta_n}{\gamma_n}\to\infty$, it is bounded by $\frac{\gamma_n}{\beta_n}\int |\psi(\mu)|^2 d\mu \le C\frac{\gamma_n}{\beta_n} \to 0$ as $n\to\infty$.

In the case $\frac{\beta_n}{\gamma_n}\to 0$ recall that under (18) we have $f(x_0,x_1) = f_0(x_0)f_1(x_1)$, so that $\widehat{f}(\lambda,\mu) = \widehat{f}_0(\lambda)\widehat{f}_1(\mu)$. Hence, using $\left|\psi\left(-\frac{\beta_n}{\gamma_n}\mu\right)\right| \le 1$ and the Cauchy-Schwarz inequality,

$$\begin{aligned}
&\int_{\{|\mu|>a\}} \left|\psi\left(-\frac{\beta_n}{\gamma_n}\mu\right)\widehat{f}\left(\nu-\mu+\mu\frac{c_{j+1}}{g(j)},\mu\right)\right| d\mu \\
\le\ & \sqrt{\int \left|\widehat{f}_0\left(\nu-\mu+\mu\frac{c_{j+1}}{g(j)}\right)\right|^2 d\mu \int_{\{|\mu|>a\}} \left|\widehat{f}_1(\mu)\right|^2 d\mu} \\
\le\ & C\left(\int_{\{|\mu|>a\}} \left|\widehat{f}_1(\mu)\right|^2 d\mu\right)^{1/2} \to 0 \quad \text{as } a\to\infty, \qquad (66)
\end{aligned}$$

where we have used $\int \left|\widehat{f}_0\left(\nu-\mu+\mu\frac{c_{j+1}}{g(j)}\right)\right|^2 d\mu = \frac{g(j)}{g(j)-c_{j+1}}\int \left|\widehat{f}_0(\mu)\right|^2 \le C$.

Thus

$$\lim_{a\to\infty}\ \limsup_{n\to\infty} Q_n(a) = 0 \qquad (67)$$

Now consider the integral over $\{|\mu|\le a, |\lambda|>a\}$. We have $\prod_{j=\kappa_n}^{l-1}\left|\psi\left(\frac{\lambda g(j)}{\gamma_n}-\mu c_{j+1}\right)\right| \le \prod_{j=[[n\delta]/2]}^{[n\delta]}\left|\psi\left(\frac{\lambda g(j)}{\gamma_n}-\mu c_{j+1}\right)\right|$ because $\kappa_n < [n\delta]/2 < l/2$. Hence in the same way as earlier the integral over $\{|\mu|\le a, |\lambda|>a\}$ is bounded by $\max_{[[n\delta]/2]\le j\le[n\delta]} N_{2j}$, where

$$\begin{aligned}
N_{2j} &= \int_{\{|\mu|\le a,|\lambda|>a\}} \left|\psi\left(-\frac{\beta_n}{\gamma_n}\mu\right)\widehat{f}\left(\frac{\lambda}{\beta_n}-\mu,\mu\right)\right|\left|\psi\left(\frac{\lambda g(j)}{\gamma_n}-\mu\frac{\beta_n}{\gamma_n}c_{j+1}\right)\right|^{l-[l/2]} d\lambda d\mu \\
&\le C\frac{\gamma_n}{\gamma_l}Q_n^*\int_{\left\{|\lambda|>d_n a-\frac{\beta_n}{\gamma_n}e_n\right\}} \left|\psi\left(\frac{\lambda}{b_{[n\delta]}}\right)\right|^{[n\delta]-[[n\delta]/2]} d\lambda
\end{aligned}$$

with

$$e_n = ab_{[n\delta]} \max_{[[n\delta]/2]\le j\le [n\delta]} |c_{j+1}| \text{ and } d_n = \min_{[[n\delta]/2]\le j\le [n\delta]} \frac{\gamma_n}{|g(j)|\, b_{[n\delta]}}$$

and

$$Q_n^* = \max_{[[n\delta]/2]\le j\le [n\delta]} \sup_{\nu} \int \left| \psi\left(-\frac{\beta_n}{\gamma_n}\mu\right) \widehat{f}\left(\nu - \mu + \mu\frac{c_{j+1}}{g(j)}, \mu\right)\right| d\mu.$$

Similar to (65) and (66) (note that $Q_n^* = Q_n(0)$ except that Q_n^* involves $\max_{[[n\delta]/2]\le j\le [n\delta]}$ whereas $Q_n(a)$ involves $\max_{[n\delta]\le j\le n}$), we have

$$\sup_n Q_n^* \le \begin{cases} C & \text{if } \beta_n = \gamma_n \text{ or } \frac{\beta_n}{\gamma_n} \to 0 \\ C\frac{\gamma_n}{\beta_n} & \text{if } \frac{\beta_n}{\gamma_n} \to \infty. \end{cases} .$$

Further note that $d_n \ge d > 0$ for some $d > 0$ (see (36)). In addition $e_n \to 0$. To see this, assume for simplicity that $b_n \sim n^{\frac{1}{\alpha}}$, and $c_j \sim j^{H-1-\frac{1}{\alpha}}$ in the case of assumption (A2). Noting that $H - 1 - \frac{1}{\alpha} < 0$, we then have $e_n \sim Cn^{H-1}$. In the case of Assumption (A1), we have $|e_n| \le Cn^{\frac{1}{\alpha}-1}$ where $\frac{1}{\alpha} - 1 < 0$ because $1 < \alpha \le 2$.

Then, in the cases $\beta_n = \gamma_n$ or $\frac{\beta_n}{\gamma_n} \to 0$, there is an n_0 such that $\left\{|\lambda| > d_n a - \frac{\beta_n}{\gamma_n} e_n\right\} \subset \{|\lambda| > d_n a - e_n\} \subset \left\{|\lambda| > \frac{d}{2}a\right\}$ for all $n \ge n_0$. Further, using (33) and (34)

$$\int_{\{|\lambda|>d_n a - e_n\}} \left|\psi\left(\frac{\lambda}{b_{[n\delta]}}\right)\right|^{[n\delta]-[[n\delta]/2]} d\lambda \le C \int_{\left\{|\lambda|>\frac{d}{2}a\right\}} e^{-a|\lambda|^c} d\lambda + C\rho^{[n\delta]},$$

where $0 < \rho < 1$.

Thus $|R_n(y_1, y_2, a, \delta)|$ is bounded by

$$C\left(Q_n(a) + \int_{\left\{|\lambda|>\frac{d}{2}a\right\}} e^{-a|\lambda|^c} d\lambda + C\rho^{[n\delta]}\right) \frac{\gamma_n}{n}\sum_{l=1}^{n}\frac{1}{\gamma_l} \text{ in the cases } \beta_n = \gamma_n \text{ or } \frac{\beta_n}{\gamma_n} \to 0,$$

and by

$$C\left(Q_n(a) + C\frac{\gamma_n}{\beta_n}\right)\frac{\gamma_n}{n}\sum_{l=1}^{n}\frac{1}{\gamma_l} \text{ in the case } \frac{\beta_n}{\gamma_n} \to \infty.$$

for all $n \ge n_0$, which in view of (67) and the fact $\frac{\gamma_n}{n}\sum_{l=1}^{n}\frac{1}{\gamma_l} \le C$, completes the proof of the first statement.

For the Statement (II), the N_{1j} for the present situation will be the same as earlier but with $\widehat{f}(\lambda, \mu)$ replaced by $\widehat{M_{f,\eta}}(\lambda, \mu)\widehat{K_\eta}(\lambda)\widehat{K_\eta}(\mu)$. We also have $\left|\psi\left(-\frac{\beta_n}{\gamma_n}\mu\right)\right| \le 1$. Hence

$$\begin{aligned} N_{1j} \quad \le \quad & C\frac{\gamma_n}{\gamma_l}\int_{\{|\mu|>a, |\lambda|<\infty\}} \left|\widehat{M_{f,\eta}}\left(\frac{\gamma_n}{\beta_n}\frac{\lambda}{g(j)b_l} - \mu + \mu\frac{c_{j+1}}{g(j)}, \mu\right)\right| \\ & \times \left|\widehat{K_\eta}\left(\frac{\gamma_n}{\beta_n}\frac{\lambda}{g(j)b_l} - \mu + \mu\frac{c_{j+1}}{g(j)}\right)\widehat{K_\eta}(\mu)\right| \left|\psi\left(\frac{\lambda}{b_l}\right)\right|^{l-[l/2]} d\lambda d\mu. \end{aligned}$$

In the right hand side, the integral over $\{|\mu| > a, |\lambda| \le b_l\tau\}$, $\tau > 0$, is as in (33) bounded by, using $\left|\widehat{K_\eta}(\lambda)\right| \le C$ and using Lemma 16,

$$\sup_v C \int_{\{|\mu|>a\}} \left|\widehat{M_{f,\eta}}\left(v - \mu + \mu\frac{c_{j+1}}{g(j)}, \mu\right)\right| \left|\widehat{K_\eta}(\mu)\right| d\mu \le C\left(\int_{\{|\mu|>a\}} \left|\widehat{K_\eta}(\mu)\right|^2 d\mu\right)^{1/2}$$

The integral over $\{|\mu| > a, |\lambda| > b_l\tau\}$ is as in (34) bounded by, for some $0 < \rho < 1$,

$$C\rho^l \int \left|\widehat{K_\eta}\left(\frac{\gamma_n}{\beta_n}\frac{\lambda}{g(j)b_l} - \mu + \mu\frac{c_{j+1}}{g(j)}\right)\widehat{K_\eta}(\mu)\right| d\lambda d\mu \le C\rho^l \frac{\beta_n g(j) b_l}{\gamma_n} \le C\rho^{[n\delta]}\frac{\beta_n \gamma_l}{\gamma_n}, \quad [n\delta] \le j \le n.$$

Thus, $M_j \le C\frac{\gamma_n}{\gamma_l}\left(\int_{\{|\mu|>a\}} \left|\widehat{K_\eta}(\mu)\right|^2 d\mu\right)^{1/2} + C\rho_*^{[n\delta]}$ for all $[n\delta] \le j \le n$, for some $0 < \rho_* < 1$. In the same way, the N_{2j} for the present case is bounded by the same bound obtained earlier. This proves the lemma. ■

In the next lemma let

$$\pi(\mu) = \begin{cases} E\left[e^{i\mu S_r}\right] & \text{in the case } \beta_n = \gamma_n \\ 1 & \text{in the case } \frac{\beta_n}{\gamma_n} \to 0 \\ 0 & \text{in the case } \frac{\beta_n}{\gamma_n} \to \infty. \end{cases}$$

Lemma 18. *Let $R_n(y_1, y_2, a, \delta)$, $\delta > 0$, be the difference between*

$$(2\pi)^2 \frac{\beta_n}{n}\sum_{l=1}^n E\left[f\left(y_1 + \frac{\beta_n}{\gamma_n}T_l, y_2 + \frac{\beta_n}{\gamma_n}T_{l+r}\right)\right] \tag{68}$$

and

$$\frac{1}{n}\sum_{l=[n\delta]}^n \int_{\{|\mu|\le a, |\lambda|\le a\}} U_n(\lambda, \mu, y_1, y_2) E\left[e^{-i\frac{\lambda}{\gamma_n}T_l}\right]\pi(-\mu)\widehat{f}(-\mu, \mu)\, d\lambda d\mu \tag{69}$$

where $U_n(\lambda, \mu, y_1, y_2) = e^{-i\frac{\lambda}{\beta_n}y_1 - i\mu(y_2 - y_1)}$ as in Lemma 17. Then

$$\lim_{\delta\to 0}\lim_{a\to\infty}\limsup_{n\to\infty}\left(\sup_{y_1,y_2}|R_n(y_1, y_2, a, \delta)|\right) = 0.$$

Similarly $(2\pi)^2\frac{\beta_n}{n}\sum_{l=1}^n E\left[M_{f,\eta}\left(y_1 + \frac{\beta_n}{\gamma_n}T_l + V_\eta^{(1)}, y_2 + \frac{\beta_n}{\gamma_n}T_{l+r} + V_\eta^{(2)}\right)\right]$ corresponding to the Statement (II) of Lemma 17, has the same approximation given by (69) but with $\widehat{M_{f,\eta}}(-\mu, \mu)\widehat{K_\eta}(-\mu)\widehat{K_\eta}(\mu)$ involved in place of $\widehat{f}(-\mu, \mu)$.

The same holds for $(2\pi)^2\frac{\beta_n}{n}\sum_{l=1}^n E\left[m_{f,\eta}\left(y_1 + \frac{\beta_n}{\gamma_n}T_l + V_\eta^{(1)}, y_2 + \frac{\beta_n}{\gamma_n}T_{l+r} + V_\eta^{(2)}\right)\right]$.

Proof. First consider the first statement under (18). Note that the same right hand side bound in (49) holds for $\left|E\left[f\left(y_1+\frac{\beta_n}{\gamma_n}T_l, y_2+\frac{\beta_n}{\gamma_n}T_{l+r}\right)\right]\right|$ also, and hence according to Lemma 9 we have $\left|E\left[f\left(y_1+\frac{\beta_n}{\gamma_n}T_l, y_2+\frac{\beta_n}{\gamma_n}T_{l+r}\right)\right]\right| \le \frac{C}{\gamma_l}\frac{\gamma_n}{\beta_n}$. Hence

$$\sup_{y_1,y_2}\left|\frac{\beta_n}{n}\sum_{l=1}^{[n\delta]}\left|E\left[f\left(y_1+\frac{\beta_n}{\gamma_n}T_l, y_2+\frac{\beta_n}{\gamma_n}T_{l+r}\right)\right]\right|\right| \le C\frac{\gamma_n}{n}\sum_{l=1}^{[n\delta]}\frac{1}{\gamma_l}.$$

Clearly this converges to 0 as $n\to\infty$ first and then $\delta\to 0$.

Hence, in view of Lemma 17, letting $R_n^*\left(y_1,y_2,a,\delta\right)$ for the difference between (68) and (64),

$$\lim_{\delta\to 0}\lim_{a\to\infty}\limsup_{n\to\infty}\left(\sup_{y_1,y_2}\left|R_n^*\left(y_1,y_2,a,\delta\right)\right|\right)=0.$$

Therefore, letting $R_n^{**}\left(y_1,y_2,a,\delta\right)$ for the difference between (64) and (69), it is enough to show that

$$\limsup_{n\to\infty}\left(\sup_{y_1,y_2}\left|R_n^{**}\left(y_1,y_2,a,\delta\right)\right|\right)=0 \qquad \text{for each } a,\delta. \tag{70}$$

Note that without loss of generality, we can assume that κ_n, upon which $T_{nl,r}^*$ and T_{nl}^* of Lemma 17 depend, is such that $\kappa_n\to\infty$ and $\frac{\kappa_n}{n}\to 0$. Then, because $T_l-T_{nl}^*$ and $\sum_{s=0}^{\kappa_n-1}g\left(s\right)\xi_s$ have the same distribution,

$$\sup_{[n\delta]\le l\le n}P\left(\gamma_n^{-1}\left|T_l-T_{nl}^*\right|>\epsilon\right)=P\left(\left|\gamma_n^{-1}\sum_{s=0}^{\kappa_n-1}g\left(s\right)\xi_s\right|>\epsilon\right)\to 0,$$

where we have used the fact that $\gamma_{\kappa_n}^{-1}\sum_{s=0}^{\kappa_n-1}g\left(s\right)\xi_s$ converges in distribution and $\gamma_n^{-1}\gamma_{\kappa_n}\to 0$. Hence

$$\sup_{|\lambda|\le a,|\mu|\le a,[n\delta]\le l\le n}\left|E\left[e^{-i\frac{\lambda}{\gamma_n}\left(T_l-T_{nl}^*\right)-i\mu\frac{\beta_n}{\gamma_n}\left(T_{l+1}-T_{nl,r}^*-T_l+T_{nl}^*\right)}\right]-E\left[e^{-i\mu\frac{\beta_n}{\gamma_n}\left(T_{l+r}-T_{nl,r}^*-T_l+T_{nl}^*\right)}\right]\right|\to 0.$$

Now note that

$$E\left[e^{-i\mu\frac{\beta_n}{\gamma_n}\left(T_{l+r}-T_{nl,r}^*-T_l+T_{nl}^*\right)}\right]=\prod_{j=0}^{\kappa_n+r-1}\psi\left(-\left(c_j+...+c_{j-(r-1)}\right)\frac{\beta_n}{\gamma_n}\mu\right) \tag{71}$$

In the case $\beta_n=\gamma_n$, we then have, in view of

$$\prod_{j=0}^{\infty}\psi\left(-\left(c_j+...+c_{j-(r-1)}\right)\mu\right)=\psi_{S_r}\left(-\mu\right)$$

and (with r being fixed) because $\kappa_n\to\infty$,

$$\sup_{|\mu|\le a}\left|E\left[e^{-i\mu\left(T_{l+r}-T_{nl,r}^*-T_l+T_{nl}^*\right)}\right]-\psi_{S_r}\left(-\mu\right)\right|\to 0.$$

This also gives, for the case $\frac{\beta_n}{\gamma_n} \to 0$,

$$\sup_{|\mu|\le a}\left|E\left[e^{-i\mu\frac{\beta_n}{\gamma_n}\left(T_{l+r}-T^*_{nl,r}-T_l+T^*_{nl}\right)}\right]-1\right|\to 0 \quad \text{if } \frac{\beta_n}{\gamma_n}\to 0.$$

In the case $\frac{\beta_n}{\gamma_n} \to \infty$, let $0 \le j_0 < \kappa_n + r - 1$ be such that $c_{j_0} + ... + c_{j_0-(r-1)} \ne 0$. Then, noting that (18) entails $\psi(\mu) \to 0$ as $\mu \to \infty$,

$$\sup_{|\mu|\le a}\left|E\left[e^{-i\mu\frac{\beta_n}{\gamma_n}\left(T_{l+r}-T^*_{nl,r}-T_l+T^*_{nl}\right)}\right]\right| \le \sup_{|\mu|\le a}\left|\psi\left(-\left(c_{j_0}+...+c_{j_0-(r-1)}\right)\frac{\beta_n}{\gamma_n}\mu\right)\right| \to 0. \tag{72}$$

Now$\left|T^*_{nl,r} - T^*_{nl}\right| = \left|\sum_{j=\kappa_n}^{l-1}(c_{j+1} + ... + c_{j+r})\xi_j\right|$. Let $0 < \tau < \alpha$ be suitably close to α such that $\sum_{j=\kappa_n}^{\infty}|c_j|^\tau \to 0$. Then

$$\begin{aligned}\sup_{[n\delta]\le l<\infty} P\left(\left|T^*_{nl,r}-T^*_{nl}\right|>\varepsilon\right) &= \sup_{[n\delta]\le l<\infty} P\left(\left|\sum_{j=\kappa_n}^{l-1}(c_{j+1}+...+c_{j+r})\xi_j\right|>\varepsilon\right)\\ &\le Cr\sum_{j=\kappa_n}^{\infty}|c_j|^\tau \to 0, \end{aligned}\tag{73}$$

where the inequality is obtained using for instance Avram and Taqqu (1986, Lemma 1, Section 3, page 408)). Hence

$$\sup_{|\lambda|\le b,|\mu|\le a,[n\delta]\le l\le n}\left|E\left[e^{-i\lambda\frac{T^*_{nl}}{\gamma_n}-i\mu\frac{\beta_n}{\gamma_n}\left(T^*_{nl,r}-T^*_{nl}\right)}\right]-E\left[e^{-i\lambda\frac{T_l}{\gamma_n}}\right]\right|\to 0 \text{ if either } \beta_n=\gamma_n \text{ or } \frac{\beta_n}{\gamma_n}\to 0.$$

Hence (70) follows (in the case $\frac{\beta_n}{\gamma_n} \to \infty$, note that $\pi(-\mu) = 0$ in (69) so that (72) is sufficient to imply (70)).

To obtain the second statement, in which (19) is assumed, we apply the Statement (II) of Lemma 17. It is clear that the only place in the above proof that needs to be explained is (72) for the case $\frac{\beta_n}{\gamma_n} \to \infty$ where the condition $\limsup_{|\lambda|\to\infty}\left|E\left[e^{i\lambda\xi_1}\right]\right| = 0$, obtained as a consequence of (18), is used. But this restriction is assumed as part of (19) when $\frac{\beta_n}{\gamma_n} \to \infty$, see (19). This completes the proof of the Lemma. ■

The preceding Lemma 17 leads to the next statement where we define

$$\begin{aligned} S\left(\frac{k-1}{m},\frac{t}{m}\right) &= c\int_{-\infty}^{0}\left\{\left(\frac{t}{m}+\frac{k-1}{m}-u\right)^{H-1/\alpha}-(-u)^{H-1/\alpha}\right\}Z_\alpha(du)\\ &\quad +c\int_0^{\frac{k-1}{m}}\left(\frac{t}{m}+\frac{k-1}{m}-u\right)^{H-1/\alpha}Z_\alpha(du) \end{aligned}\tag{74}$$

and

$$T(t) = \int_0^t (t-u)^{H-1/\alpha} Z_\alpha(du). \tag{75}$$

Note that

$$S\left(\frac{k-1}{m},0\right)=\Lambda_{\alpha,H}\left(\frac{k-1}{m}\right).$$

Lemma 19. *For each integer $m\geq 1$,*

$$\frac{\beta_n}{n}\sum_{k=1}^{m}\sum_{l=1}^{n_{mk}}E_{\left[n\frac{k-1}{m}\right]}\left[f\left(\frac{\beta_n}{\gamma_n}S_{\left[n\frac{k-1}{m}\right]+l},\frac{\beta_n}{\gamma_n}S_{\left[n\frac{k-1}{m}\right]+l+r}\right)\right]$$

converges in distribution to

$$\left(\frac{1}{2\pi}\int\pi\left(-\mu\right)\widehat{f}\left(-\mu,\mu\right)d\mu\right)\frac{1}{m^{1-H}}\sum_{k=1}^{m}\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{0}^{1}e^{-i\lambda m^{H}S\left(\frac{k-1}{m},\frac{t}{m}\right)}E\left[e^{-i\lambda T(t)}\right]dtd\lambda$$

where $\pi\left(-\mu\right)$ is as in Lemma 18 and $S\left(\frac{k-1}{m},\frac{t}{m}\right)$ and $T\left(t\right)$ are as above in (74) and (75).

The same holds for $\frac{\beta_n}{n}\sum_{k=1}^{m}\sum_{l=1}^{n_{mk}}E_{\left[n\frac{k-1}{m}\right]}\left[M_{f,\eta}\left(\frac{\beta_n}{\gamma_n}S_{\left[n\frac{k-1}{m}\right]+l}+V_\eta^{(1)},\frac{\beta_n}{\gamma_n}S_{\left[n\frac{k-1}{m}\right]+l+r}+V_\eta^{(2)}\right)\right]$ but the limit will involve $\frac{1}{2\pi}\int\pi\left(-\mu\right)\widehat{M_{f,\eta}}\left(-\mu,\mu\right)\widehat{K}_\eta\left(-\mu\right)\widehat{K}_\eta\left(\mu\right)d\mu$ in place of $\frac{1}{2\pi}\int\pi\left(-\mu\right)\widehat{f}\left(-\mu,\mu\right)d\mu$. The same holds for $m_{f,\eta}$ also. Here $V_\eta^{(1)}$ and $V_\eta^{(2)}$ are as in Lemma 18.

Proof. We consider only the first statement because the proofs for the remaining statements are the same. Also note that in the right hand side of (69) is equal to 0 for the case $\frac{\beta_n}{\gamma_n}\to\infty$ because $\pi\left(-\mu\right)=0$. Therefore we only need to consider the cases $\beta_n=\gamma_n$ and $\frac{\beta_n}{\gamma_n}\to 0$.

Because $\frac{\gamma_{n_{mk}}}{n_{mk}}\frac{n}{\gamma_n}\sim m^{1-H}$, it is enough to show that, for each m and k,

$$\frac{\beta_n}{n}\sum_{l=1}^{n_{mk}}E_{\left[n\frac{k-1}{m}\right]}\left[f\left(\frac{\beta_n}{\gamma_n}S_{\left[n\frac{k-1}{m}\right]+l},\frac{\beta_n}{\gamma_n}S_{\left[n\frac{k-1}{m}\right]+l+r}\right)\right] \tag{76}$$

converges in distribution to

$$\left(\frac{1}{2\pi}\int\pi\left(-\mu\right)\widehat{f}\left(-\mu,\mu\right)d\mu\right)\frac{1}{m}\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{0}^{1}e^{-i\lambda S\left(\frac{k-1}{m},\frac{t}{m}\right)}E\left[e^{-i\lambda m^{-H}T(t)}\right]dtd\lambda, \tag{77}$$

where in obtaining the form of the limit we have used the transformation $\lambda m^{H}\longmapsto\lambda$.

Let $\left(y_1,y_2\right)$ be as in (60), that is

$$\left(y_1,y_2\right)=\left(\frac{\beta_n}{\gamma_n}S_{\left[n\frac{k-1}{m}\right]+l,l},\frac{\beta_n}{\gamma_n}S_{\left[n\frac{k-1}{m}\right]+l+r,l+r}\right). \tag{78}$$

With this $\left(y_1,y_2\right)$, let $R_n\left(a,\delta\right)$ be the difference between (76) and

$$\frac{1}{n\left(2\pi\right)^2}\sum_{l=[n\delta]}^{n_{mk}}\int_{\{|\lambda|\leq a,|\mu|\leq a\}}U_n\left(\lambda,\mu,y_1,y_2\right)E\left[e^{-i\lambda\gamma_n^{-1}T_l}\right]\pi\left(-\mu\right)\widehat{f}\left(-\mu,\mu\right)d\lambda d\mu, \tag{79}$$

where recall that

$$U_n(\lambda,\mu,y_1,y_2)=e^{-i\frac{\lambda}{\beta_n}y_1-i\mu(y_2-y_1)}.$$

It follows from Lemma 18 that, for each $\epsilon>0$,

$$\lim_{\delta\to0}\lim_{a\to\infty}\limsup_{n\to\infty}P\left(|R_n(a,\delta)|>\epsilon\right)=0.$$

Therefore it is enough to show that (79) converges in distribution to (77) by taking the limit as $n\to\infty$ first, then $a\to\infty$ and then $\delta\to0$.

To obtain the limit as $n\to\infty$, note that $U_n(\lambda,\mu,y_1,y_2)$ above involves

$$\frac{y_1}{\beta_n}=\frac{1}{\gamma_n}S_{\left[n\frac{k-1}{m}\right]+l,l}\quad\text{ and }\quad y_2-y_1=\frac{\beta_n}{\gamma_n}\left(S_{\left[n\frac{k-1}{m}\right]+l+r,l+r}-S_{\left[n\frac{k-1}{m}\right]+l,l}\right).$$

We have

$$S_{\left[n\frac{k-1}{m}\right]+l+r,l+r}-S_{\left[n\frac{k-1}{m}\right]+l,l}=\sum_{j=-\infty}^{\left[n\frac{k-1}{m}\right]}(c_{l+\left[n\frac{k-1}{m}\right]+1-j}+..+c_{l+\left[n\frac{k-1}{m}\right]+r-j})\xi_j,$$

and hence, similar to (73), under either of the cases $\beta_n=\gamma_n$ or $\frac{\beta_n}{\gamma_n}\to0$,

$$\begin{aligned}&\sup_{a_n<l<\infty}P\left(\frac{\beta_n}{\gamma_n}\left|S_{\left[n\frac{k-1}{m}\right]+l+r,l+r}-S_{\left[n\frac{k-1}{m}\right]+l,l}\right|>\varepsilon\right)\\&=\sup_{a_n<l<\infty}P\left(\frac{\beta_n}{\gamma_n}\left|\sum_{i=l}^{\infty}(c_{i+1}+..+c_{i+r})\xi_i\right|>\varepsilon\right)\to0\quad\text{ for any } a_n\uparrow\infty.\end{aligned}$$

Further, with $S_{mk}\left(\frac{t}{m}\right)$ and $T(t)$ as defined in (74) and (75),

$$\left(\gamma_n^{-1}S_{\left[n\frac{k-1}{m}\right]+[n_{mk}t],[n_{mk}t]},\gamma_n^{-1}T_{[n_{mk}t]}\right)\overset{fdd}{\Longrightarrow}\left(S\left(\frac{k-1}{m},\frac{t}{m}\right),m^{-H}T(t)\right).$$

It then follows (though the preceding convergence is only $\overset{fdd}{\Longrightarrow}$), in the same way as in Jeganathan (2004, Lemma 8), that (79) with (y_1,y_2) as in (78) converges in distribution to

$$\frac{1}{(2\pi)^2m}\int_{\{|\lambda|\le a,|\mu|\le a\}}\left\{\int_\delta^1 e^{-i\lambda S\left(\frac{k-1}{m},\frac{t}{m}\right)}E\left[e^{-i\lambda m^{-H}T(t)}\right]dt\right\}\pi(-\mu)\widehat{f}(-\mu,\mu)\,d\lambda d\mu$$

for each a and $\delta>0$.

Let $\Delta(a)$ be the difference between the preceding quantity and

$$\left(\frac{1}{2m\pi}\int\pi(-\mu)\widehat{f}(-\mu,\mu)\,d\mu\right)\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_\delta^1 e^{-i\lambda S\left(\frac{k-1}{m},\frac{t}{m}\right)}E\left[e^{-i\lambda m^{-H}T(t)}\right]dtd\lambda.$$

(Here m, k and δ are fixed.) We next show that $\Delta(a)\to0$ as $a\to\infty$.

Noting that $\left|e^{-i\lambda m^H S_{mk}\left(\frac{t}{m}\right)}\right| \leq 1$, we have

$$\begin{aligned} m\left(2\pi\right)^2 \Delta\left(a\right) &\leq \left(\int_{-\infty}^{\infty}\left|\widehat{f}\left(-\mu,\mu\right)\right|\left|\pi\left(\mu\right)\right|d\mu\right)\int_{\delta}^{1}\int_{\{|\lambda|>a\}}\left|E\left[e^{-i\lambda m^{-H}T(t)}\right]\right|d\lambda dt \\ &\quad +\left(\int_{\{|\mu|>a\}}\left|\widehat{f}\left(-\mu,\mu\right)\right|\left|\pi\left(\mu\right)\right|d\mu\right)\int_{\delta}^{1}\int_{-\infty}^{\infty}\left|E\left[e^{-i\lambda m^{-H}T(t)}\right]\right|d\lambda dt. \end{aligned}$$

Now note that

$$\begin{aligned} \int_{\{|\lambda|>a\}}\left|E\left[e^{-i\lambda m^{-H}T(t)}\right]\right|d\lambda &\leq C\int_{\{|\lambda|>a\}}e^{-c\left|\lambda t^H\right|^{\alpha}}d\lambda = Ct^{-H}\int_{\{|\lambda|>at^H\}}e^{-c|\lambda|^{\alpha}}d\lambda \\ &\leq C\delta^{-H}\int_{\{|\lambda|>a\delta^H\}}e^{-c|\lambda|^{\alpha}}d\lambda = R\left(a\right), \text{ say, if } \delta \leq t \leq 1. \end{aligned}$$

Hence

$$m\left(2\pi\right)^2 \Delta\left(a\right)\left(2\pi\right)^2 \Delta\left(a\right) \leq R\left(a\right)\int_{-\infty}^{\infty}\left|\widehat{f}\left(-\mu,\mu\right)\right|\left|\pi\left(\mu\right)\right|d\mu + R\left(0\right)\int_{\{|\mu|>a\}}\left|\widehat{f}\left(-\mu,\mu\right)\right|\left|\pi\left(\mu\right)\right|d\mu,$$

where note that

$$R\left(a\right) \to 0 \ \text{ as } \ a \to \infty \quad \text{and} \quad R\left(0\right) < \infty.$$

In the case $\pi\left(\mu\right) = \psi_{S_r}\left(\mu\right)$ (the case $\beta_n = \gamma_n$), we have, noting $\left|\psi_{S_r}\left(\mu\right)\right| \leq \left|\psi\left(\mu\right)\right|$,

$$\begin{aligned} \int_{\{|\mu|>a\}}\left|\widehat{f}\left(-\mu,\mu\right)\right|\left|\psi\left(\mu\right)\right|d\mu &\leq \sqrt{\int\left|\widehat{f}\left(-\mu,\mu\right)\right|^2 d\mu\int_{\{|\mu|>a\}}\left|\psi\left(\mu\right)\right|^2 d\mu} \\ &\leq C\sqrt{\int_{\{|\mu|>a\}}\left|\psi\left(\mu\right)\right|^2 d\mu} \to 0 \text{ as } a \to \infty, \end{aligned}$$

where we have used $\int\left|\widehat{f}\left(-\mu,\mu\right)\right|^2 d\mu \leq C$, see Lemma 16. In the case $\pi\left(\mu\right) = 1$ (the case (18) holds and $\frac{\beta_n}{\gamma_n} \to 0$), we have $\left|\widehat{f}\left(-\mu,\mu\right)\right| = \left|\widehat{f}_0\left(\mu\right)\right|\left|\widehat{f}_1\left(\mu\right)\right|$ so that

$$\begin{aligned} \int_{\{|\mu|>a\}}\left|\widehat{f}\left(-\mu,\mu\right)\right|\left|\psi\left(\mu\right)\right|d\mu &\leq \sqrt{\int\left|\widehat{f}_0\left(\mu\right)\right|^2 d\mu\int_{\{|\mu|>a\}}\left|\widehat{f}_1\left(\mu\right)\right|^2 d\mu} \\ &\leq C\sqrt{\int_{\{|\mu|>a\}}\left|\widehat{f}_1\left(\mu\right)\right|^2 d\mu} \to 0 \text{ as } a \to \infty. \end{aligned}$$

Thus $\Delta\left(a\right) \to 0$ as $a \to \infty$.

Next note that

$$\begin{aligned} \left|\int_{-\infty}^{\infty}\int_{0}^{\delta}e^{-i\lambda S_{mk}\left(\frac{t}{m}\right)}E\left[e^{-i\lambda m^{-H}T(t)}\right]dtd\lambda\right| &\leq C\left(\int_{0}^{\delta}t^{-H}dt\right)\left(\int_{-\infty}^{\infty}e^{-c|\lambda|^{\alpha}}d\lambda\right) \\ &\leq \frac{C\delta^{1-H}}{1-H} \to 0 \text{ as } \delta \to 0. \end{aligned}$$

This completes the proof of the first part of the lemma.

The proof of the second part is identical by allowing $\left|\widehat{K}_\eta(-\mu)\right|\left|\widehat{K}_\eta(\mu)\right|$ to play the role of $\left|\widehat{f}(-\mu,\mu)\right||\psi(\mu)|$ above. ■

To complete the proof of the first part of Propositions 13, we thus require

Lemma 20.

$$\frac{1}{m^{1-H}}\sum_{k=1}^{m}\int_0^1\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\lambda m^H S\left(\frac{k-1}{m},\frac{t}{m}\right)}E\left[e^{-i\lambda T(t)}\right]d\lambda\right]dt \Longrightarrow L_1^0 \qquad \text{as } m\to\infty.$$

Proof. We first show that

$$\lim_{\delta\to 0}\ \limsup_{m\to\infty}\int_0^\delta\frac{1}{m^{1-H}}E\left[\left|\sum_{k=1}^{m}\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\lambda m^H S\left(\frac{k-1}{m},\frac{t}{m}\right)}E\left[e^{-i\lambda T(t)}\right]d\lambda\right|\right]dt=0. \tag{80}$$

To see this note that, in view of (74), $S\left(\frac{k-1}{m},\frac{t}{m}\right)$ is α-stable with scale parameter σ_{tmk} such that

$$\sigma_{tmk}\geq C\left|\frac{t}{m}+\frac{k-1}{m}\right|^H.$$

(See Samorodnitsky and Taqqu (1994, page 345)). Hence

$$\begin{aligned}E\left[\left|\frac{1}{m^{1-H}}\sum_{k=2}^{m}\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\lambda m^H S\left(\frac{k-1}{m},\frac{t}{m}\right)}d\lambda\right|\right] &\leq \frac{1}{m}\sum_{k=2}^{m}\int\left|E\left[e^{i\lambda S\left(\frac{k-1}{m},\frac{t}{m}\right)}\right]\right|d\lambda\\ &\leq \frac{1}{m}\sum_{k=2}^{m}\frac{1}{\sigma_{mk}}\int e^{-c|\lambda|^\alpha}d\lambda\\ &\leq \frac{C}{m}\sum_{k=2}^{m}\left(\frac{m}{k-1}\right)^H\int e^{-c|\lambda|^\alpha}d\lambda\leq C\end{aligned}$$

because $\frac{1}{m}\sum_{k=2}^m\left(\frac{m}{k-1}\right)^H\leq C$. Here note that in the sum $\sum_{k=2}^m$ the leading term corresponding to $k=1$ is left out, but for this we have, in the same way as above, noting $\sigma_{tm1}\geq C\left|\frac{t}{m}\right|^H$,

$$E\left[\left|\frac{1}{m^{1-H}}\int_{-\infty}^{\infty}e^{-i\lambda m^H S\left(0,\frac{t}{m}\right)}d\lambda\right|\right]\leq C\frac{t^{-H}}{m^{1-H}}.$$

Hence, noting further that $\left|E\left[e^{-i\lambda T(t)}\right]\right|\leq 1$, the left hand side of (80) is bounded by

$$C\int_0^\delta\left(1+\frac{t^{-H}}{m^{1-H}}\right)dt=C\left(\frac{\delta^{1-H}}{m^{1-H}}+\delta\right),\quad \text{for all } m\geq 1, \text{ obtaining (80).}$$

Now consider

$$\begin{aligned}&\int_\delta^1\frac{1}{m^{1-H}}\sum_{k=1}^{m}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\lambda m^H S\left(\frac{k-1}{m},\frac{t}{m}\right)}E\left[e^{-i\lambda T(t)}\right]d\lambda\right]dt\\ =&\int_\delta^1\frac{1}{m^{1-H}}\sum_{k=1}^{m}h_t\left(-m^H S\left(\frac{k-1}{m},\frac{t}{m}\right)\right)dt\end{aligned} \tag{81}$$

where $h_t(y) \geq 0$ is the density function of $T(t)$, i.e.,

$$h_t(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda y}\widehat{h}_t(\lambda)\,d\lambda \text{ where } \widehat{h}_t(\lambda) = E\left[e^{-i\lambda T(t)}\right].$$

Note that for each fixed t, $\left\{S\left(\frac{k-1}{m},\frac{t}{m}\right), 0 \leq k \leq m\right\}$ has the same structure as that of $\left\{\Lambda_{\alpha,H}\left(\frac{k}{m}\right), 0 \leq k \leq m\right\}$. Hence Jeganathan (2004, Proposition 6) contains the fact that the difference between the integrand $\frac{1}{m^{1-H}}\sum_{k=1}^{m} h_t\left(-m^H S\left(\frac{k-1}{m},\frac{t}{m}\right)\right)$ in (81) and

$$\frac{1}{m^{1-H}}\sum_{k=1}^{m}\frac{1}{\sqrt{2\pi}}\int h_t\left(-m^H\left(S\left(\frac{k-1}{m},\frac{t}{m}\right)+\varepsilon z\right)\right)e^{-z^2/2}dz \tag{82}$$

converges to 0 in mean-square, as $m \to \infty$ first and then $\varepsilon \to 0$. In addition it is easy to see that the arguments in Jeganathan (2004) also give that this mean-square convergence uniformly over $\delta \leq t \leq 1$. (Note that this is a very specific case so that the steps in Jeganathan (2004) will take a rather simple and direct form.)

Now, note that $\frac{1}{m^{1-H}}\sum_{k=1}^{m}\int h_t\left(-m^H(y+\varepsilon z)\right)e^{-z^2/2}dz$ is sufficiently smooth in y (see Jeganathan (2004, Lemma 7)). Hence, for each $\varepsilon > 0$, it can be seen that (82) can be approximated, as $m \to \infty$, by

$$\frac{1}{m^{1-H}}\sum_{k=1}^{m}\frac{1}{\sqrt{2\pi}}\int h_t\left(-m^H\left(S\left(\frac{k-1}{m},0\right)+\varepsilon z\right)\right)e^{-z^2/2}dz$$

uniformly over $\delta \leq t \leq 1$, which in turn is approximated by $\frac{1}{m^{1-H}}\sum_{k=1}^{m}\frac{1}{\sqrt{2\pi}}h_t\left(-m^H S\left(\frac{k-1}{m},0\right)\right)$ as before as $m \to \infty$ first and then $\varepsilon \to 0$.

Noting that $S\left(\frac{k-1}{m},0\right) = \Lambda_{\alpha,H}\left(\frac{k-1}{m}\right)$, we thus have approximated (81) by

$$\int_{\delta}^{1}\frac{1}{m^{1-H}}\sum_{k=1}^{m} h_t\left(-m^H\Lambda_{\alpha,H}\left(\frac{k-1}{m}\right)\right)dt,$$

which in turn is approximated as before, as $m \to \infty$ first and then $\delta \to 0$, by

$$\begin{aligned}
&\int_0^1\frac{1}{m^{1-H}}\sum_{k=1}^{m} h_t\left(-m^H\Lambda_{\alpha,H}\left(\frac{k-1}{m}\right)\right)dt \\
=\ &\frac{1}{m^{1-H}}\sum_{k=1}^{m} g\left(-m^H\Lambda_{\alpha,H}\left(\frac{k-1}{m}\right)\right) \Longrightarrow \left(\int g(y)\,dy\right)L_1^0 = L_1^0
\end{aligned}$$

where $g(y) = \int_0^1 h_t(y)\,dt$. Note that $\int g(y)\,dy = \int_0^1\int h_t(y)\,dydt = 1$ because $\int h_t(y)\,dy = 1$ for each t. In obtaining the preceding convergence we have used Jeganathan (2004, Theorem 4). Note that $\int g^2(y)\,dy \leq \int_0^1\int h_t^2(y)\,dydt \leq C\int_0^1 t^{-H}dt \leq C$. This completes the proof of the lemma. ∎

Proof of Propositions 13. When (18) holds, the proof of the first part (weak convergence part) follows directly from Lemma 20 and the first part of the lemma 19.

Regarding the proof under (19), we have (with $V_\eta^{(1)}$, $V_\eta^{(2)}$ as in Lemma 17)

$$E_{[n\frac{k-1}{m}]}\left[f\left(\frac{\beta_n}{\gamma_n}S_{[n\frac{k-1}{m}]+l},\frac{\beta_n}{\gamma_n}S_{[n\frac{k-1}{m}]+l+r}\right)\right]$$
$$\begin{cases} \leq E_{[n\frac{k-1}{m}]}\left[M_{f,\eta}\left(\frac{\beta_n}{\gamma_n}S_{[n\frac{k-1}{m}]+l}+V_\eta^{(1)},\frac{\beta_n}{\gamma_n}S_{[n\frac{k-1}{m}]+l+r}+V_\eta^{(2)}\right)\right] \\ \geq E_{[n\frac{k-1}{m}]}\left[m_{f,\eta}\left(\frac{\beta_n}{\gamma_n}S_{[n\frac{k-1}{m}]+l}+V_\eta^{(1)},\frac{\beta_n}{\gamma_n}S_{[n\frac{k-1}{m}]+l+r}+V_\eta^{(2)}\right)\right]\end{cases}.$$

Therefore, in view of the second part of the Lemma 18, it only remains to show that $\int\pi(-\mu)\widehat{M_{f,\eta}}(-\mu,\mu)\widehat{K_\eta}(-\mu)\widehat{K_\eta}(\mu)\,d\mu\to\int\pi(-\mu)\widehat{f}(-\mu,\mu)\,d\mu$ as $\eta\to0$ and the same for $m_{f,\eta}$. To see this note that the left hand side is, for instance when $\pi(\mu)=\psi_{S_r}(\mu)$, equal to $2\pi\int E\left[M_{f,\eta}\left(x+V_\eta^{(1)},x+S_r+V_\eta^{(2)}\right)\right]dx$, with

$$\begin{aligned}\int M_{f,\eta}\left(x+V_\eta^{(1)},x+S_r+V_\eta^{(2)}\right)dx &\leq \int M_{f,2\eta}(x,x+S_r)\,dx\\ &\leq \int E\left[M_{f,2\eta}\left(x+V_a^{(1)},x+S_r+V_a^{(2)}\right)\right]dx,\end{aligned}$$

where $a>0$ is fixed. Here the left most side converges to $\int f(x,x+S_r)dx$ as $\eta\to0$ in view of the fact that $f(x,y)$ is the class $\mathcal{G}_2$, and the right most side to $\int E\left[f\left(x+V_a^{(1)},x+S_r+V_a^{(2)}\right)\right]dx$ using in addition the fact that $V_a^{(1)}$, S_r and $V_a^{(2)}$ are independent with $V_a^{(1)}$ and $V_a^{(2)}$ (and hence $S_r+V_a^{(2)}$) having bounded Lebesgue densities (a is fixed). The same holds for $\int m_{f,\eta}\left(x+V_\eta^{(1)},x+S_r+V_\eta^{(2)}\right)dx$ but reversing the inequalities. Hence, using a form of a Lebesgue dominated convergence theorem, it follows that

$$\int E\left[M_{f,\eta}\left(x+V_\eta^{(1)},x+S_r+V_\eta^{(2)}\right)\right]dx\to\int E\left[f(x,x+S_r)\right]dx \text{ as } \eta\to0$$

and the same for $\int E\left[m_{f,\eta}\left(x+V_\eta^{(1)},x+S_r+V_\eta^{(2)}\right)\right]dx$. This gives the first part.

Regarding the proof of the second part, that is, (59), let

$$h_n(x_0,x_1)=f\left(\frac{\beta_n}{\gamma_n}x_0,\frac{\beta_n}{\gamma_n}x_1\right).$$

Note that $\sum_{l=1}^{n_{mk}}h_n\left(S_{[n\frac{k-1}{m}]+l},S_{[n\frac{k-1}{m}]+l+r}\right)-\sum_{l=1}^{n_{mk}}E_{[n\frac{k-1}{m}]}\left[h_n\left(S_{[n\frac{k-1}{m}]+l},S_{[n\frac{k-1}{m}]+l+r}\right)\right],1\leq k\leq m$, form an array of martingale differences, and hence the expected value in (59) is bounded by

$$\begin{aligned}&\left(\frac{\beta_n}{n}\right)^2\sum_{k=1}^m E\left[\left(\sum_{l=1}^{n_{mk}}h_n\left(S_{[n\frac{k-1}{m}]+l},S_{[n\frac{k-1}{m}]+l+r}\right)\right)^2\right]\\ \leq\;&\left(\frac{\beta_n}{n}\right)^2\left\{\sum_{l=1}^n E\left[h_n^2(S_l,S_{l+r})\right]+2\sum_{l=1}^n\sum_{i=1}^{n_{mk}}\left|E\left[h_n(S_l,S_{l+r})\,h_n(S_{l+i},S_{l+r+i})\right]\right|\right\}.\end{aligned}\quad(83)$$

According to Lemma 9, we have $E\left[h_n^2\left(\frac{\beta_n}{\gamma_n}S_l,\frac{\beta_n}{\gamma_n}S_{l+r}\right)\right]=E\left[f_0^2\left(\frac{\beta_n}{\gamma_n}S_l\right)f_1^2\left(\frac{\beta_n}{\gamma_n}S_{l+r}\right)\right]\leq \frac{C}{\gamma_l}\left(\frac{\gamma_n}{\beta_n}\right)$ and hence using $\sum_{l=1}^{n}\frac{1}{\gamma_l}\sim C\frac{n}{\gamma_n}$, we have

$$\left(\frac{\beta_n}{n}\right)^2\sum_{l=1}^{n}E\left[h_n^2\left(S_l,S_{l+r}\right)\right]\leq C\left(\frac{\beta_n}{n}\right)^2\left(\frac{\gamma_n}{\beta_n}\right)\frac{n}{\gamma_n}=C\frac{\beta_n}{n}\to 0.$$

Similarly $\left|E\left[f\left(\frac{\beta_n}{\gamma_n}S_l,\frac{\beta_n}{\gamma_n}S_{l+r}\right)f\left(\frac{\beta_n}{\gamma_n}S_{l+i},\frac{\beta_n}{\gamma_n}S_{l+r+i}\right)\right]\right|\leq\frac{C}{\gamma_l\gamma_i}\left(\frac{\gamma_n}{\beta_n}\right)^2$, according to Lemma 10. Using $n_{mk}\sim\frac{n}{m}$ and $\gamma_{n_{mk}}\sim\gamma_n m^{-H}$, $\max_{1\leq k\leq m}\sum_{i=1}^{n_{mk}}\frac{1}{\gamma_i}\sim C\frac{n}{\gamma_n}\left(\frac{1}{m}\right)^{1-H}$, so that $\sum_{l=1}^{n}\sum_{i=1}^{n_{mk}}\frac{1}{\gamma_l\gamma_i}\leq C\left(\frac{1}{m}\right)^{1-H}\left(\frac{n}{\gamma_n}\right)^2$. Hence

$$\begin{aligned}&\left(\frac{\beta_n}{n}\right)^2\sum_{l=1}^{n}\sum_{i=1}^{n_{mk}}\left|E\left[h_n\left(S_l,S_{l+r}\right)h_n\left(S_{l+i},S_{l+r+i}\right)\right]\right|\\ \leq\ & C\left(\frac{1}{m}\right)^{1-H}\left(\frac{\beta_n}{n}\right)^2\left(\frac{\gamma_n}{\beta_n}\right)^2\left(\frac{n}{\gamma_n}\right)^2=C\left(\frac{1}{m}\right)^{1-H}.\end{aligned}$$

This completes the proof. ■

Proof of Proposition 14. Under (18), it is implicit in the proofs of Lemmas 17 and 18 that, for each $m\geq 1$, the difference between $\frac{\beta_n}{n}\sum_{k=1}^{m}\sum_{l=1}^{n_{mk}}E_{\left[n\frac{k-1}{m}\right]}\left[h\left(\frac{\beta_n}{\gamma_n}S_{\left[n\frac{k-1}{m}\right]+l}\right)\right]$ and

$$\widehat{h}(0)\frac{1}{m^{1-H}}\sum_{k=1}^{m}\frac{1}{n_{mk}2\pi}\sum_{l=[n\delta]}^{n_{mk}}\int_{\{|\lambda|\leq a\}}e^{-i\lambda\gamma_{n_{mk}}^{-1}S_{\left[n\frac{k-1}{m}\right]+l,l}}E\left[e^{-i\lambda n_{mk}^{-H}T_l}\right]d\lambda$$

converges to 0 in probability as $n\to\infty$ first, then $a\to\infty$ and then $\delta\to 0$, which in turn converges in distribution to $\widehat{h}(0)\frac{1}{m^{1-H}}\sum_{k=1}^{m}\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_0^1 e^{-i\lambda m^H S\left(\frac{k-1}{m},\frac{t}{m}\right)}E\left[e^{-i\lambda T(t)}\right]dtd\lambda$, see Lemma 19. Hence, similar to the preceding proof of Propositions 13 the proof of the first part under (18) follows by Lemma 20 and that of the second part follows in view of the inequalities of the first parts of Lemmas 8 and 9. Similarly to the preceding proof of Proposition 13, the proof under Cramer's condition in (19) also follows. ■

We next present the proof of Proposition 15. The proof will consist of reducing the situations to the framework of Propositions 13 and 14. We first obtain

Lemma 21. *Assume that the integers ν_n are such that $\frac{\nu_n}{n}\to 0$, in addition to the assumptions of Theorem 2. Then for each $k=1,...,m$,*

$$\frac{\beta_n}{n}\sum_{l=1}^{\nu_n}E\left[\left|f_0\left(\frac{\beta_n}{\gamma_n}S_{\left[n\frac{k-1}{m}\right]+l}\right)\sigma\left(\omega_{\left[n\frac{k-1}{m}\right]+l,\nu_n}\right)\right|\right]\to 0\qquad\text{as } n\to\infty,$$

and

$$\frac{\beta_n}{n}\sum_{l=1}^{\nu_n}E\left[\left|f\left(\frac{\beta_n}{\gamma_n}S_{\left[n\frac{k-1}{m}\right]+l},\frac{\beta_n}{\gamma_n}S_{\left[n\frac{k-1}{m}\right]+l+r}\right)\sigma\left(\omega_{\left[n\frac{k-1}{m}\right]+l,\nu_n}\right)\sigma\left(\omega_{\left[n\frac{k-1}{m}\right]+l+r,\nu_n}\right)\right|\right]\to 0.$$

Proof. According to Lemma 12 (and taking into account the second part in Lemma 9), the left hand side of the second part of the lemma is bounded by $C\frac{\beta_n}{n}\sum_{l=1}^{\nu_n}\frac{\gamma_n}{\beta_n}\frac{1}{\gamma_l} = C\frac{\gamma_n}{n}\sum_{l=1}^{\nu_n}\frac{1}{\gamma_l} \sim C\frac{\gamma_n}{n}\frac{\nu_n}{\gamma_{\nu_n}}$ because $\sum_{l=1}^{\nu_n}\frac{1}{\gamma_l} \sim \frac{\nu_n}{\gamma_{\nu_n}}$. We have $C\frac{\gamma_n}{n}\frac{\nu_n}{\gamma_{\nu_n}} \to 0$ because $\frac{\nu_n}{n} \to 0$. This proves the second part, and note that the preceding arguments hold for the first part of the lemma also, using the first part of Lemma 11. ■

Proof of Proposition 15. We shall present the proof for the case $\sigma(\omega_{k,\nu_n}) = \omega_{k,\nu_n}$ because all the arguments below hold when ω_{k,ν_n} is replaced by $\sigma(\omega_{k,\nu_n})$.

First consider the *counterpart of Proposition 14.* In view of the first part of the preceding Lemma 21, it is enough to consider $\frac{\beta_n}{n}\sum_{k=1}^{m}\sum_{l=\nu_n+1}^{n_{mk}} f_0\left(\frac{\beta_n}{\gamma_n}S_{\left[n\frac{k-1}{m}\right]+l}\right)\omega_{\left[n\frac{k-1}{m}\right]+l,\nu_n}$, as $n \to \infty$ first and then $m \to \infty$.

Recall that $\omega_{l,\nu_n} = \sum_{j=l-\nu_n+1}^{l} d_{l-j}\eta_j$. Also recall that $S_{j+l} = S_{j+l,\nu_n} + S^*_{j+l,\nu_n}$ where S_{j+l,ν_n} and S^*_{j+l,ν_n} are independent. Hence, if $j > \nu_n$, then for $q < j-\nu_n$,

$$E_q\left[f_0\left(\frac{\beta_n}{\gamma_n}S_{j+l}\right)\omega_{j+l,\nu_n}\right] = E_q\left[E_{j-\nu_n}\left[f_0\left(\frac{\beta_n}{\gamma_n}S_{j+l}\right)\omega_{j+l,\nu_n}\right]\right] = E_q\left[h_{\nu_n}\left(\frac{\beta_n}{\gamma_n}S_{j+l,\nu_n}\right)\right],$$

where

$$h_{\nu_n}(x) = E\left[f_0\left(x+\frac{\beta_n}{\gamma_n}S^*_{j+l,\nu_n}\right)\omega_{j+l,\nu_n}\right] = E\left[\omega_{\nu_n,\nu_n}f_0\left(x+\frac{\beta_n}{\gamma_n}S^*_{\nu_n,\nu_n}\right)\right].$$

Thus

$$E_{\left[n\frac{k-1}{m}\right]}\left[f_0\left(\frac{\beta_n}{\gamma_n}S_{\left[n\frac{k-1}{m}\right]+l}\right)\omega_{\left[n\frac{k-1}{m}\right]+l,\nu_n}\right] = E_{\left[n\frac{k-1}{m}\right]}\left[h_{\nu_n}\left(\frac{\beta_n}{\gamma_n}S_{\left[n\frac{k-1}{m}\right]+l,\nu_n}\right)\right] \quad \text{for } l > \nu_n. \tag{84}$$

We have

$$\widehat{h_{\nu_n}}(\lambda) = \widehat{f_0}(\lambda)\,E\left[\omega_{\nu_n,\nu_n}e^{-i\lambda\frac{\beta_n}{\gamma_n}S^*_{\nu_n,\nu_n}}\right].$$

In the case $\left|\widehat{f_0}(\lambda)\right| \le C$, it is clear that $\left|\widehat{h_{\nu_n}}(\lambda)\right| \le C$, and in addition $\sup_{|\lambda|\le M}\left|\widehat{h_{\nu_n}}\left(\frac{\lambda}{\beta_n}\right) - \widehat{h_{\nu_n}}(0)\right| \to 0$ for all $M > 0$ using the fact $\frac{1}{\gamma_n}S^*_{\nu_n,\nu_n} \xrightarrow{p} 0$ because $\frac{\nu_n}{n} \to 0$. Further note that $\int h_{\nu_n}(x)\,dx = E[\omega_{0,\nu_n}]\int f_0(x)\,dx = \widehat{h_{\nu_n}}(0)$, where $E[\omega_{0,\nu_n}] \to E[\omega_0]$. Furthermore $S_{\left[n\frac{k-1}{m}\right]+l,\nu_n}$ and $S_{\left[n\frac{k-1}{m}\right]+l}$ have has the same structure in addition to having the same limiting distributions. Hence in exactly the same way as in the first part of Proposition 13, we have

$$\frac{\beta_n}{n}\sum_{k=1}^{m}\sum_{l=\nu_n+1}^{n_{mk}} E_{\left[n\frac{k-1}{m}\right]}\left[f_0\left(\frac{\beta_n}{\gamma_n}S_{\left[n\frac{k-1}{m}\right]+l}\right)\omega_{\left[n\frac{k-1}{m}\right]+l,\nu_n}\right] \Longrightarrow L_1^0 E[\omega_0]\frac{1}{2\pi}\int\left|\widehat{f_0}(\mu)\right|^2 d\mu.$$

Recall that the proof of the second part of Proposition 14 relied on the bounds of the first parts of Lemmas 8 and 9. The same purpose is now served in the present context by the first part of Lemma 11 and the appropriate one in Lemma 12. Thus the counterpart of Proposition 14 is proved. ∎

Now consider the *counter part of Proposition 13*. It is enough to consider, similar to the preceding proof but now using the second part of Lemma 21,

$$\frac{\beta_n}{n}\sum_{k=1}^{m}\sum_{l=\nu_n+1}^{n_{mk}} E_{\left[n\frac{k-1}{m}\right]}\left[f\left(\frac{\beta_n}{\gamma_n}S_{\left[n\frac{k-1}{m}\right]+l},\frac{\beta_n}{\gamma_n}S_{\left[n\frac{k-1}{m}\right]+l+r}\right)\omega_{\left[n\frac{k-1}{m}\right]+l,\nu_n}\omega_{\left[n\frac{k-1}{m}\right]+l+r,\nu_n}\right],\qquad r\geq 1.$$

Let us first obtain a representation similar to (84). Because r is fixed, one can without loss of generality assume that $r\leq\nu_n$. Then for $j>\nu$ we have

$$\begin{aligned}&E_{j-\nu}\left[f\left(\frac{\beta_n}{\gamma_n}S_j,\frac{\beta_n}{\gamma_n}S_{j+r}\right)\omega_{j,\nu}\omega_{j+r,\nu}\right]\\ =\;& E_{j-\nu}\left[f\left(\frac{\beta_n}{\gamma_n}\left(S_{j,\nu}+S^*_{j,\nu}\right),\frac{\beta_n}{\gamma_n}\left(S_{\nu+r,r+\nu}+S^*_{\nu+r,r+\nu}\right)\right)\omega_{j,\nu}\omega_{j+r,\nu}\right]=f_{r,\nu}\left(\frac{\beta_n}{\gamma_n}S_{j,\nu},\frac{\beta_n}{\gamma_n}S_{\nu+r,r+\nu}\right)\end{aligned}$$

where

$$f_{r,\nu}\left(x,y\right)=E\left[f\left(x+\frac{\beta_n}{\gamma_n}S^*_{\nu,\nu},y+\frac{\beta_n}{\gamma_n}S^*_{\nu+r,r+\nu}\right)\omega_{\nu,\nu}\omega_{\nu+r,\nu}\right].$$

In particular, because $\left[n\frac{k-1}{m}\right]<\left[n\frac{k-1}{m}\right]+l-\nu_n$,

$$\begin{aligned}&E_{\left[n\frac{k-1}{m}\right]}\left[f\left(\frac{\beta_n}{\gamma_n}S_{\left[n\frac{k-1}{m}\right]+l},\frac{\beta_n}{\gamma_n}S_{\left[n\frac{k-1}{m}\right]+l+r}\right)\omega_{\left[n\frac{k-1}{m}\right]+l,\nu_n}\omega_{\left[n\frac{k-1}{m}\right]+l+r,\nu_n}\right]\\ &E_{\left[n\frac{k-1}{m}\right]}\left[f_{r,\nu_n}\left(\frac{\beta_n}{\gamma_n}S_{\left[n\frac{k-1}{m}\right]+l,\nu_n},\frac{\beta_n}{\gamma_n}S_{\left[n\frac{k-1}{m}\right]+l+r,\nu_n+r}\right)\right].\end{aligned}\tag{85}$$

Here we have

$$\begin{aligned}\widehat{f_{r,\nu_n}}\left(\lambda,\mu\right)&=\widehat{f}\left(\lambda,\mu\right)E\left[\omega_{\nu_n,\nu_n}\omega_{\nu_n+r,\nu_n}e^{-i\lambda\frac{\beta_n}{\gamma_n}S^*_{\nu_n,\nu_n}-i\mu\frac{\beta_n}{\gamma_n}S^*_{\nu_n+r,\nu_n+r}}\right]\\ &=\widehat{f}\left(\lambda,\mu\right)E\left[\omega_{\nu_n,\nu_n}\omega_{\nu_n+r,\nu_n}e^{-i\lambda\frac{\beta_n}{\gamma_n}\sum_{j=1}^{\nu_n}g(\nu_n-j)\xi_j-i\mu\frac{\beta_n}{\gamma_n}\sum_{j=1}^{\nu_n+r}g(\nu_n+r-j)\xi_j}\right]\\ &=\widehat{f}\left(\lambda,\mu\right)E\left[\omega_{0,\nu_n}\omega_{r,\nu_n}e^{-i\lambda\frac{\beta_n}{\gamma_n}\sum_{j=1-\nu_n}^{0}g(-j)\xi_j-i\mu\frac{\beta_n}{\gamma_n}\sum_{j=1-\nu_n}^{r}g(r-j)\xi_j}\right].\end{aligned}\tag{86}$$

First consider the analogue of the first part of Proposition 13 for the sum of (85) (recall that Proposition 13 involves the sum of $E_{\left[n\frac{k-1}{m}\right]}\left[h\left(S_{\left[n\frac{k-1}{m}\right]+l},S_{\left[n\frac{k-1}{m}\right]+l+r}\right)\right]$). Note that in the right hand side of (85), $S_{\left[n\frac{k-1}{m}\right]+l,\nu_n}$ has the same structure as that of $S_{\left[n\frac{k-1}{m}\right]+l}$, and similarly $S_{\left[n\frac{k-1}{m}\right]+l+r,\nu_n+r}$ has the same structure as that of $S_{\left[n\frac{k-1}{m}\right]+l+r}$.

To be more specific, in the identities (60) and (61), the roles of T_l and T_{l+r} are now respectively played by

$$\sum_{j=1}^{l-\nu_n}g\left(l-j\right)\xi_j=T^*_{nl}+\sum_{j=l-\kappa_n+1}^{l-\nu_n}g(l-j)\xi_j\quad\text{and}\quad\sum_{j=1}^{l-\nu_n}g(l+r-j)\xi_j=T^*_{nl,r}+\sum_{j=l-\kappa_n+1}^{l-\nu_n}g(l+r-j)\xi_j,$$

where note that the definition of T_{nl}^{*} and $T_{nl,r}^{*}$ remain the same as involved in (61).

Further note that the sum that contributed the factor $\prod_{j=0}^{r-1}\psi\left(-\frac{\beta_n}{\gamma_n}g(j)\mu\right)$ in (62), which in turn contributed the factor $\left|\psi\left(-\frac{\beta_n}{\gamma_n}\mu\right)\right|$ in the first line of (65) (which was required only for the Statement (I)), is now absorbed in $\widehat{f_{r,\nu_n}}(\lambda,\mu)$, which has the factor, see (86),

$$
\begin{aligned}
& \left|E\left[\omega_{0,\nu_n}\omega_{r,\nu_n}e^{-i\lambda\frac{\beta_n}{\gamma_n}\sum_{j=1-\nu_n}^{0}g(-j)\xi_j-i\mu\frac{\beta_n}{\gamma_n}\sum_{j=1-\nu_n}^{r}g(r-j)\xi_j}\right]\right| \\
\leq\ & \sum_{q=-\nu_n+1}^{0}\sum_{p=r-\nu_n+1}^{r}|d_{-q}|\,|d_{r-p}|\left|E\left[\eta_q\eta_p e^{-i\lambda\frac{\beta_n}{\gamma_n}\sum_{j=1-\nu_n}^{0}g(-j)\xi_j-i\mu\frac{\beta_n}{\gamma_n}\sum_{j=1-\nu_n}^{r}g(r-j)\xi_j}\right]\right| \\
\leq\ & \left|E\left[e^{j-i\mu\frac{\beta_n}{\gamma_n}\xi_r}\right]\right|\sum_{q=-\nu_n+1}^{0}\sum_{p=r-\nu_n+1}^{r-1}|d_{-q}|\,|d_{r-p}| \\
& +\left|E\left[e^{j-i\mu\frac{\beta_n}{\gamma_n}g(1)\xi_{r-1}}\right]\right|\sum_{q=-\nu_n+1}^{-1}|d_{-q}|+\left|E\left[e^{-i\lambda\frac{\beta_n}{\gamma_n}g(1)\xi_{-1}-i\mu\frac{\beta_n}{\gamma_n}g(r+1)\xi_{-1}}\right]\right| \\
\leq\ & C\left(\left|\psi\left(-\frac{\beta_n}{\gamma_n}\mu\right)\right|+\left|\psi\left(-\frac{\beta_n}{\gamma_n}g(1)\mu\right)\right|+\left|\psi\left(-\frac{\beta_n}{\gamma_n}g(1)\lambda-\frac{\beta_n}{\gamma_n}g(r+1)\mu\right)\right|\right).
\end{aligned}
$$

It is clear that when the factor $\left|\psi\left(-\frac{\beta_n}{\gamma_n}\mu\right)\right|$ in the first line of (65) is replaced by the preceding quantity, the conclusion in (65) still holds. Taking the preceding observations into account, the Lemma 17 holds true.

Now, to see that Lemma 18 also holds, note that the role of (71) is now taken by $\prod_{j=\nu_n}^{\kappa_n-1-r}\psi\left(-(g(j+r)-g(j))\frac{\beta_n}{\gamma_n}\mu\right)=E\left[e^{-i\mu\frac{\beta_n}{\gamma_n}\left(\sum_{j=-\kappa_n+1+r}^{-\nu_n}(g(r-j)-g(-j))\xi_j\right)}\right]$. In view of (86), we then have

$$
\begin{aligned}
& \widehat{f_{r,\nu_n}}(-\mu,\mu)\prod_{j=\nu_n}^{\kappa_n-1-r}\psi\left(-(g(j+r)-g(j))\frac{\beta_n}{\gamma_n}\mu\right) \\
=\ & \widehat{f}(-\mu,\mu)\,E\left[\omega_{0,\nu_n}\omega_{r,\nu_n}e^{-i\mu\frac{\beta_n}{\gamma_n}\left(\sum_{j=-\kappa_n+1+r}^{r}(g(r-j)-g(-j))\xi_j\right)}\right].
\end{aligned}
$$

Clearly, in the case $\frac{\beta_n}{\gamma_n}\to 0$, this converges to $\widehat{f}(-\mu,\mu)\,E[\omega_0\omega_r]$, and in the case $\beta_n=\gamma_n$, it converges to $\widehat{f}(-\mu,\mu)\,E\left[\omega_0\omega_r e^{-i\mu S_r}\right]$. In the case $\frac{\beta_n}{\gamma_n}\to\infty$ also, it converges to 0.

To see that this last claim is true, note that

$$
\begin{aligned}
& \left|E\left[\omega_{0,\nu_n}\omega_{r,\nu_n}e^{-i\mu\frac{\beta_n}{\gamma_n}\left(\sum_{j=-\kappa_n+1+r}^{r}(g(r-j)-g(-j))\xi_j\right)}\right]\right| \\
\leq\ & \left|E\left[\omega_{0,m}\omega_{r,m}e^{-i\mu\frac{\beta_n}{\gamma_n}\left(\sum_{j=-\kappa_n+1+r}^{r}(g(r-j)-g(-j))\xi_j\right)}\right]\right|+\left|E\left[\left|\omega_{0,\nu_n}\omega_{r,\nu_n}-\omega_{0,m}\omega_{r,m}\right|\right]\right|
\end{aligned}
$$

where, noting that $(\omega_{0,m},\omega_{r,m})$ is a function of $(\xi_{-m+1},...,\xi_r)$,

$$
\begin{aligned}
& \left|E\left[\omega_{0,m}\omega_{r,m}e^{-i\mu\frac{\beta_n}{\gamma_n}\left(\sum_{j=-\kappa_n+1+r}^{r}(g(r-j)-g(-j))\xi_j\right)}\right]\right| \\
\leq\ & \left|E\left[\left|\omega_{0,m}\omega_{r,m}\right|\right]\right|\left|\psi\left(-(g(m+r)-g(m))\frac{\beta_n}{\gamma_n}\mu\right)\right|\to 0\quad\text{as } n\to\infty \text{ for each } m,
\end{aligned}
$$

where we have used $|\psi(\mu)| \to 0$ as $\mu \to \infty$, which restriction follows from (18) (see (72)) and is also part of (19). Further $\lim_{m\to\infty} \lim_{n\to\infty} |E[|\omega_{0,\nu_n}\omega_{r,\nu_n} - \omega_{0,m}\omega_{r,m}|]| = 0$. Hence the claim holds. This completes the proof of Proposition 15. ■

Proof of the Theorem 3. This is obtained from Theorem 2. First, regarding the first part, note that

$$\left|\frac{\beta_n}{n}\sum_{l=1}^{n} f_0\left(\frac{\beta_n}{\gamma_n}S_l\right)(\sigma(\omega_l) - \sigma(\omega_{l,\nu_n}))\right|^2 \le \left(\frac{\beta_n}{n}\sum_{l=1}^{n} f_0^2\left(\frac{\beta_n}{\gamma_n}S_l\right)\right)\frac{\beta_n}{n}\sum_{l=1}^{n}(\sigma(\omega_l) - \sigma(\omega_{l,\nu_n}))^2.$$

Now according to Whittle's (1960) inequality, if $E[|\eta_1|^p] < \infty$ for $p \ge 2$, then for a constant C_p depending only on p,

$$E[|\omega_l - \omega_{l,\nu_n}|^p] = E\left[\left|\sum_{j=-\infty}^{k-\nu_n} d_{k-j}\eta_j\right|^p\right] \le C_p\left(\sum_{j=\nu_n}^{\infty} d_j^2\right)^{p/2} + C_p\left(E[\eta_1]\sum_{j=\nu_n}^{\infty}|d_j|\right)^p.$$

Then, using the Taylor expansion of $\sigma(\omega_l)$ around ω_{l,ν_n}, noting that $\omega_l - \omega_{l,\nu_n}$ and ω_{l,ν_n} are independent and using the given conditions $\left|\sigma^{(p)}(x)\right| \le C$, $E\left[\left|\sigma^{(j)}(\omega_{l,\nu_n})\right|^2\right] \le C$ for $1 \le j \le q-1$, we then have

$$E\left[(\sigma(\omega_l) - \sigma(\omega_{l,\nu_n}))^2\right] \le C\sum_{j=1}^{p}\sum_{k=1}^{p}\frac{1}{j!k!}E\left[|\omega_l - \omega_{l,\nu_n}|^{j+k}\right] \le C\sum_{j=\nu_n}^{\infty} d_j^2 + C\left(E[\eta_1]\sum_{j=\nu_n}^{\infty}|d_j|\right)^2,$$

so that $\frac{\beta_n}{n}E\left[\sum_{l=1}^{n}(\sigma(\omega_l) - \sigma(\omega_{l,\nu_n}))^2\right] \to 0$ if (23) holds. Further it follows from the first part of Lemma 8 that $E\left[\frac{\beta_n}{n}\sum_{l=1}^{n} f_0^2\left(\frac{\beta_n}{\gamma_n}S_l\right)\right] \le C\frac{\gamma_n}{n}\sum_{l=1}^{n}\frac{1}{\gamma_l} \le C$. Thus it follows that $\frac{\beta_n}{n}\sum_{l=1}^{n} f_0\left(\frac{\beta_n}{\gamma_n}S_l\right)(\sigma(\omega_l) - \sigma(\omega_{l,\nu_n})) \xrightarrow{p} 0$, and hence the first part of Theorem 3 follows from (21).

The second part also follows in the same way because, in view of the second part of Lemma 9, $E\left[\frac{\beta_n}{n}\sum_{l=1}^{n} f^2\left(\frac{\beta_n}{\gamma_n}S_l, \frac{\beta_n}{\gamma_n}S_{l+r}\right)\right] \le C$ and because

$$E\left[(\sigma(\omega_l)\sigma(\omega_{l+r}) - \sigma(\omega_{l,\nu_n})\sigma(\omega_{l+r,\nu_n}))^2\right] \le C\sum_{j=\nu_n}^{\infty} d_j^2 + C\left|E[\eta_1]\sum_{j=\nu_n}^{\infty}|d_j|\right|^2.$$

To obtain this last inequality, note that using $(a+b)^2 \le 2a^2 + 2b^2$ and the Cauchy - Schwarz inequality,

$$\begin{aligned}
&\left(E\left[(\sigma(\omega_l)\sigma(\omega_{l+r}) - \sigma(\omega_{l,\nu_n})\sigma(\omega_{l+r,\nu_n}))^2\right]\right)^2 \\
\le\ & 2E\left[|\sigma(\omega_{l+r})|^4\right]E\left[|\sigma(\omega_l) - \sigma(\omega_{l,\nu_n})|^4\right] + 2E\left[|\sigma(\omega_{l,\nu_n})|^4\right]E\left[|\sigma(\omega_{l+r}) - \sigma(\omega_{l+r,\nu_n})|^4\right].
\end{aligned}$$

Here in the right hand side we have $E\left[|\sigma(\omega_{l+r})|^4\right] \le C$ and $E\left[|\sigma(\omega_{l,\nu_n})|^4\right] \le C$ in view of $|\sigma(\omega_{l+r})| \le C|\omega_{l+r}|^q$ and the restriction $E\left[|\eta_1|^{4\max(q,p)}\right] < \infty$. Further, in the same way as above using Whittle's inequality, $E\left[\left|\sigma\left(\omega_l\right)-\sigma\left(\omega_{l,\nu_n}\right)\right|^4\right] \le C\sum_{j=\nu_n}^{\infty} d_j^2 + C\left(E\left[\eta_1\right]\sum_{j=\nu_n}^{\infty}\left|d_j\right|\right)^2$ and the same bound holds for $E\left[\left|\sigma\left(\omega_{l+r}\right)-\sigma\left(\omega_{l+r,\nu_n}\right)\right|^4\right]$. This completes the proof. ■

5 REDUCTION OF THEOREMS 4 AND 5 TO A MARTINGALE CLT

In this section we relate Theorems 4 and 5 to a martingale CLT. For this purpose, corresponding to Theorem 4 define, for each positive integer m,

$$\zeta_{nmk} = \sqrt{\frac{\gamma_n}{n}} \sum_{l=\left[n\frac{k-1}{m}\right]+1}^{\left[n\frac{k}{m}\right]} f\left(S_l\right), \quad k \ge 1, \tag{87}$$

Similarly, corresponding to Theorem 5 define (with ω_{l,ν_n} as in (2))

$$\zeta^*_{nmk} = \sqrt{\frac{\gamma_n}{n}} \sum_{l=\left[n\frac{k-1}{m}\right]+\nu_n+1}^{\left[n\frac{k}{m}\right]} f\left(S_l\right)\omega_{l,\nu_n}, \qquad R^*_{nmk} = \sqrt{\frac{\gamma_n}{n}} \sum_{l=\left[n\frac{k-1}{m}\right]+1}^{\left[n\frac{k-1}{m}\right]+\nu_n} f\left(S_l\right)\omega_{l,\nu_n}, \quad k \ge 1. \tag{88}$$

In these definitions we follow the usual convention that a sum is to be interpreted as 0 if it is with respect to an empty index set. Note that

$$\sqrt{\frac{\gamma_n}{n}}\sum_{l=1}^{n} f\left(S_l\right) = \sum_{k=1}^{m}\zeta_{nmk}, \qquad \sqrt{\frac{\gamma_n}{n}}\sum_{l=1}^{n} f\left(S_l\right)\omega_{l,\nu_n} = \sum_{k=1}^{m}\left(\zeta^*_{nmk}+R^*_{nmk}\right). \tag{89}$$

We shall show in the next Section 6 that

$$\lim_{m\to\infty}\limsup_{n\to\infty} P\left[\left|\sum_{k=1}^{m} R^*_{nmk}\right| > \epsilon\right] = 0 \quad \text{for all } \epsilon > 0, \tag{90}$$

and therefore, the respective limiting behaviors of the sums in (89) will be the same as those of $\sum_{k=1}^{m}\zeta_{nmk}$ and $\sum_{k=1}^{m}\zeta^*_{nmk}$.

In Sections 6 below we establish that the following facts hold (recall that E_l stands for the conditional expectation given $\sigma\left(\xi_j;\ j \le l\right)$).

(R1) There is a nonrandom $\Delta\left(n,m\right)$ such that

$$\sum_{k=1}^{m}\left|E_{\left[n\frac{k-1}{m}\right]}\left[\zeta_{nmk}\right]\right| \le \Delta\left(n,m\right) \to 0 \quad \text{as } n\to\infty\text{, for each } m.$$

(R2)

$$\sum_{k=1}^{m} E_{\left[n\frac{k-1}{m}\right]}\left[\zeta^2_{nmk}\right] \Longrightarrow bL^0_1$$

as $n\to\infty$ first and then $m\to\infty$, where the constant b is as specified in Theorem 4.

Recall that the convergence in distribution of a sequence of distribution functions is metrizable, for example by the Lévy distance (see for instance Loève (1963, page 215)). Then the preceding convergence means that the distribution of $\sum_{k=1}^{m} E_{\left[n\frac{k-1}{m}\right]}\left[\zeta_{nmk}^{2}\right]$ converges in such a metric to that of bL_1^0 as $n \to \infty$ first and then $m \to \infty$.

(R3)

$$\lim_{m\to\infty} \limsup_{n\to\infty} \sum_{k=1}^{m} E\left[\zeta_{nmk}^{4}\right] = 0.$$

The next condition (R4) pertains only to the case $\alpha = 2$. To state it define

$$\chi_{nmk} = \frac{1}{\sqrt{n}} \sum_{l=\left[n\frac{k-1}{m}\right]+1}^{\left[n\frac{k}{m}\right]} \xi_l. \tag{91}$$

(R4) *When* $\alpha = 2$ (in which case we have $E[\xi_1] = 0$ and $E[\xi_1^2] < \infty$, see (9))

$$\limsup_{n\to\infty} P\left[\sum_{k=1}^{m}\left|E_{\left[n\frac{k-1}{m}\right]}\left[\zeta_{nmk}\chi_{nmk}\right]\right| > \varepsilon\right] = 0 \text{ for each } m \text{ and } \varepsilon > 0.$$

- **(R*1) - (R*4)**: *In the case of Theorem 5, we shall verify the preceding conditions with ζ_{nmk}^{*} in place of ζ_{nmk}, in which case the corresponding conditions will be referred to as (R*1), (R*2), (R*3) and (R*4).*

Proposition 22. *Suppose that the conditions (R1) - (R4) are verified. Then the convergence in distribution conclusion of Theorem 4 holds.*

*Similarly, if the conditions (R*1) - (R*4), together with (90), are verified, then the convergence in distribution conclusion of Theorem 5 holds.*

The next Section 6 is devoted to the verification of the conditions of this proposition. The remaining part of this section is devoted to the proof of this proposition.

Note that the preceding conditions involve iterated limits in the sense that the limits are taken as $n \to \infty$ first and then $m \to \infty$. To proceed further it is convenient to note that they can be restated in an alternative form involving only the index n that goes to ∞. For this purpose recall that if $h(n,m)$ is a nonrandom function of n and m such that

$$\lim_{m\to\infty} \limsup_{n\to\infty} |h(n,m)| = 0$$

then one can find a sequence $m_n \uparrow \infty$ such that

$$h(n, m_n) \to 0.$$

If $G(n,m)$ is random, then note that

$$\lim_{m\to\infty}\limsup_{n\to\infty} P\left[|G(n,m)|\geq\eta\right]=0 \quad \text{for all } \eta>0,$$

is equivalent to $\lim_{m\to\infty}\limsup_{n\to\infty} E\left[\min\left(|G(n,m)|,1\right)\right]=0$, and therefore, taking $h(n,m)=E\left[\min\left(|G(n,m)|,1\right)\right]$, there is a sequence $m_n\uparrow\infty$ such that $E\left[\min\left(|G(n,m_n)|,1\right)\right]\to 0$, which is equivalent to

$$G(n,m_n)\xrightarrow{p}0.$$

Thus (noting that the convergence in (R2) can be restated in terms of a suitable metric), (R1) - (R4) entail that there is a sequence $m_n\uparrow\infty$ such that

$$\sum_{k=1}^{m_n}\left|E_{\left[n\frac{k-1}{m_n}\right]}\left[\zeta_{nm_nk}\right]\right|+\sum_{k=1}^{m_n}E\left[\zeta_{nm_nk}^4\right]\xrightarrow{p}0, \tag{92}$$

$$\sum_{k=1}^{m_n}\left|E_{\left[n\frac{k-1}{m_n}\right]}\left[\zeta_{nm_nk}\chi_{nm_nk}\right]\right|\xrightarrow{p}0 \quad (\text{for } \alpha=2) \tag{93}$$

and

$$\sum_{k=1}^{m_n}E_{\left[n\frac{k-1}{m_n}\right]}\left[\zeta_{nm_nk}^2\right]\Longrightarrow bL_1^0 \tag{94}$$

In the same way, the conditions (R*1) - (R*4) imply that (92) - (94) hold with ζ_{nmk} replaced by ζ_{nmk}^*.

We are now in a position to present the proof of the first part of Proposition 22. First, for convenience, we let

$$\zeta_{nk}=\zeta_{nm_nk},\quad \chi_{nk}=\chi_{nm_nk},\quad k=1,...,m_n.$$

Next, for the purpose of the proof, we

- *extend the array* ζ_{nk}, $1\leq k\leq m_n$, *to all* $k\geq 1$, *by taking* $\{\zeta_{nk};k=m_n+1,...\}$ *to be an array of iid Gaussian* $\left(0,\frac{1}{m_n}\right)$ *random variables, independent of* $\{\xi_j;-\infty<j<\infty\}$.

Further, we use the **notation** E_{nl} for the conditional expectation given the σ-field

$$\mathcal{F}_{nl}=\begin{cases}\sigma\left(\xi_j,\ j\leq\left[n\frac{l}{m_n}\right]\right) & \text{if } -\infty<l\leq m_n\\ \sigma\left(\xi_j,\ j\leq n, \text{ and } \zeta_{nk}, m_n+1\leq k\leq l\right) & \text{if } l>m_n.\end{cases}$$

Explicitly,

$$E_{nl}\left[\ .\ \right]=E\left[\ .\ \middle|\mathcal{F}_{nl}\right].$$

With this extension, (92) and (93) take the strengthened forms, for any $0 < \gamma < 1$,

$$\sum_{k=1}^{\left[m_n^{1+\gamma}\right]} \left|E_{n,k-1}\left[\zeta_{nk}\right]\right| \xrightarrow{p} 0, \tag{95}$$

$$\sum_{k=1}^{\left[m_n^{1+\gamma}\right]} E\left[\zeta_{nk}^4\right] \to 0, \tag{96}$$

and

$$\sum_{k=1}^{\left[m_n^{1+\gamma}\right]} \left|E_{n,k-1}\left[\zeta_{nk}\chi_{nk}\right]\right| \xrightarrow{p} 0 \quad (\text{for } \alpha = 2). \tag{97}$$

Now, define the martingale differences

$$\zeta'_{nk} = \zeta_{nk} - E_{n,k-1}\left[\zeta_{nk}\right], \quad k = 1, 2, ...$$

with respect to the σ-fields $\mathcal{F}_{nk}, k = 1, 2, ...$It is easily seen, in view of (95), that

$$\text{(96) and (97) hold with } \zeta_{nk} \text{ replaced by } \zeta'_{nk}. \tag{98}$$

In addition, if we define

$$T_n(q) = \sum_{k=1}^{q} E_{n,k-1}\left[\left|\zeta'_{nk}\right|^2\right] = \sum_{k=1}^{q} \left\{E_{n,k-1}\left[\zeta_{nk}^2\right] - \left(E_{n,k-1}\left[\zeta_{nk}\right]\right)^2\right\},$$

then, in view of (94) and (95) and because ζ_{nk}, $k = m_n + 1, ...$ are iid Gaussian $\left(0, \frac{1}{m_n}\right)$, for any $s \geq 1$,

$$T_n(sm_n) \Longrightarrow bL_1^0 + s - 1, \quad s \geq 1. \tag{99}$$

Now for each fixed $t > 0$, define

$$\tau_n(t) = \inf\left\{q \geq 1 : T_n(q) \geq t\right\}.$$

Note that

$$\tau_n(t) = m_n \qquad \text{if } t = T_n(m_n). \tag{100}$$

We have

$$\left\{\tau_n(t) \leq l\right\} = \left\{T_n(l) \geq t\right\} \in \mathcal{F}_{n,l-1}, \qquad l = 1, 2, ...,$$

so that for each n and $t > 0$,

$$\tau_n(t) \text{ is a stopping time with respect to the } \sigma\text{-fields } \mathcal{F}_{n,l-1}, l = 1, 2, \ldots.$$

Note that for any positive integer J, we have using (99),

$$P\left[\frac{\tau_n(t)}{m_n} > J\right] \le P\left[T_n(Jm_n) \le t\right] \to P\left[bL_1^0 + J - 1 \le t\right] = 0 \quad \text{if } J > t+1. \tag{101}$$

We thus have shown, in view of (95) - (98), (101) and because $m_n \uparrow \infty$,

$$\sum_{k=1}^{\tau_n(t)} E_{n,k-1}\left[\zeta_{nk}\right] \xrightarrow{p} 0, \tag{102}$$

$$\sum_{k=1}^{\tau_n(t)} E_{n,k-1}\left[\left|\zeta'_{nk}\right|^4\right] \xrightarrow{p} 0 \tag{103}$$

and

$$\sum_{k=1}^{\tau_n(t)} E_{n,k-1}\left[\left|\zeta'_{nk}\chi_{nk}\right|\right] \xrightarrow{p} 0 \quad (\text{for } \alpha = 2) \tag{104}$$

Further, because of (96), (98), (99) and (101),

$$E_{n,\tau_n(t)-1}\left[\left|\zeta'_{n,\tau_n(t)}\right|^2\right] \xrightarrow{p} 0.$$

Hence, because

$$T_n(\tau_n(t)) \ge t \ge T_n(\tau_n(t) - 1) = T_n(\tau_n(t)) - E_{n,\tau_n(t)-1}\left[\left|\zeta'_{n,\tau_n(t)}\right|^2\right],$$

$$T_n(\tau_n(t)) \xrightarrow{p} t. \tag{105}$$

Now let

$$W_n(t) = \begin{cases} \sum_{k=1}^{\tau_n(t)} \zeta'_{nk} & \text{if } t > 0 \\ 0 & \text{if } t \le 0. \end{cases} \tag{106}$$

Similarly, let $W(t)$ be the Brownian motion for $0 \le t < \infty$ and $W(t) \equiv 0$ for $t < 0$. We then have

Lemma 23. *Let $W(t)$ be as above, and as before let $Z_\alpha(t)$ be the α-stable motion. Then, for $0 < \alpha \leq 2$ and for every integer $l > 0$,*

$$t \longmapsto \left(\frac{1}{n^{1/\alpha}\delta_n} \sum_{j=-nl}^{[nt]} \xi_j, W_n(t) \right) \stackrel{fdd}{\Longrightarrow} \left(Z_\alpha(t) - Z_\alpha(-l), W(t) \right), \quad t \in [-l, \infty),$$

where (δ_n as in (1)) and

$$\text{the processes } W(t) \text{ and } Z_\alpha(t) \text{ are independent.}$$

■

The proof of this Lemma is given below separately in the Appendix, Section 7.

We now come back to the proof of the first part of Proposition 22 (assuming that (R1) - (R4) holds). Because Lemma 23 is true for every $l > 0$, it entails (keeping in mind the conditions (A1) and (A2), see Kasahara and Maejima (1988))

$$\left(\gamma_n^{-1} S_{[nt]}, W_n(t)\right) \stackrel{fdd}{\Longrightarrow} \left(\Lambda_{\alpha,H}(t), W(t)\right)$$

where the processes $W(t)$ and $\Lambda_{\alpha,H}(t)$ are independent. (Here in general the convergence of $\gamma_n^{-1} S_{[nt]}$ in the Skorokhod space does not hold, see Astrauskas (1983)). Further, according to the arguments given in the next Section 6 for the verification of (R2) (see the Remark immediately before the statement of Lemma 24), it follows that

$$T_n = T_n\left(_{m_n}\right) = \sum_{k=1}^{m_n} E_{n,k-1}\left[\left|\zeta'_{nk}\right|^2\right]$$

is approximated by a functional of the process $\gamma_n^{-1} S_{[nt]}$ such that T_n converges in distribution if $\gamma_n^{-1} S_{[nt]} \stackrel{fdd}{\Longrightarrow} \Lambda_{\alpha,H}(t)$. We then have

$$\left(\gamma_n^{-1} S_{[nt]}, W_n(t), T_n\right) \stackrel{fdd}{\Longrightarrow} \left(\Lambda_{\alpha,H}(t), W(t), bL_1^0\right). \tag{107}$$

The next step is to obtain the convergence of $\left(\gamma_n^{-1} S_{[nt]}, W_n(T_n)\right)$ from (107) (taking into account further that the marginal process $t \longmapsto W_n(t)$ will be tight, that is, uniformly equicontinuous in probability, see below). To present the details, let, with q a positive integer and $J > 0$,

$$0 = \tau_{q0} < \tau_{q1} < ...\tau_{q,q-1} < \tau_{qq} = J$$

be such that

$$\sup_{1 \leq i \leq q} |\tau_{qi} - \tau_{q,i-1}| \to 0 \text{ as } q \to \infty.$$

Define

$$T_{n,q,J} = \begin{cases} \tau_{qi} & \text{if } \tau_{qi} \leq T_n < \tau_{q,i+1},\ i = 0, 1, ...q-1, \\ J & \text{if } T_n \geq J. \end{cases}$$

Letting

$$T = bL_1^0,$$

define $T_{q,J}$ analogously. Now, taking $\tau_{q,q+1} = \infty$,

$$\{W_n(T_{n,q,J}) \leq v\} = \cup_{i=0}^q \{W_n(\tau_{qi}) \leq v, \tau_{qi} \leq T_n < \tau_{q,i+1}\}$$

where $\{W_n(\tau_{qi}) \leq v, \tau_{qi} \leq T_n < \tau_{q,i+1}\}$ are disjoint, and hence, for $0 \leq u_1 \leq ... \leq u_k < \infty$ and for any reals $a_j, j = 1, ..., k$,

$$\begin{aligned} & P\left(W_n(T_{n,q,J}) \leq v, \gamma_n^{-1} S_{[nu_j]} \leq a_j, j = 1, ..., k\right) \\ = \ & P\left(\cup_{i=0}^q \left\{W_n(\tau_{qi}) \leq v, \tau_{qi} \leq T_n < \tau_{q,i+1}, \gamma_n^{-1} S_{[nu_j]} \leq a_j, j = 1, ..., k\right\}\right) \\ = \ & \sum_{i=0}^q P\left(W_n(\tau_{qi}) \leq v, \tau_{qi} \leq T_n < \tau_{q,i+1}, \gamma_n^{-1} S_{[nu_j]} \leq a_j, j = 1, ..., k\right). \end{aligned}$$

One can assume without loss of generality that $\tau_{q1}, ...\tau_{qq}$ are continuity points of T. Then (107) together with the preceding identity entail that

$$\begin{aligned} & P\left(W_n(T_{n,q,J}) \leq v, \gamma_n^{-1} S_{[nu_j]} \leq a_j, j = 1, ..., k\right) \\ \to \ & \sum_{i=0}^q P\left(W(\tau_{qi}) \leq v, \tau_{qi} \leq T < \tau_{q,i+1}, \Lambda_{\alpha,H}(u_j) \leq a_j, j = 1, ..., k\right) \\ = \ & P\left(W(T_{q,J}) \leq v, \Lambda_{\alpha,H}(u_j) \leq a_j, j = 1, ..., k\right). \end{aligned}$$

In other words, we have

$$\left(W_n(T_{n,q,J}), \gamma_n^{-1} S_{[nt]}\right) \stackrel{fdd}{\Longrightarrow} \left(W(T_{q,J}), \Lambda_{\alpha,H}(t)\right).$$

(Note that $T_{q,J}$ is a function of L_1^0, which, being a functional of $\Lambda_{\alpha,H}(t)$, is independent of $W(t)$ by Lemma 23.) In addition, in view of (103) and (105), it is well known that the marginal process $W_n(t)$ satisfies the 'tightness' property

$$\lim_{h\to 0} \limsup_{n\to\infty} P\left[\sup_{|t-s|\leq h, t,s\in[0,M]} |W_n(t) - W_n(s)| > \varepsilon\right] = 0$$

for all $\varepsilon > 0$ and all $M > 0$. (Actually $W_n(t) \Longrightarrow W(t)$ in the Skorkhod space $D[0, M]$ with $W(t) \in C[0, M]$ for every $M > 0$.) Hence

$$\lim_{J\to\infty} \lim_{q\to\infty} \limsup_{n\to\infty} P\left[|W_n(T_{n,q,J}) - W_n(T_n)| > \varepsilon\right] = 0.$$

Similarly

$$\lim_{J\to\infty}\lim_{q\to\infty} P\left[\left|W\left(T_{q,J}\right)-W\left(T\right)\right|>\varepsilon\right]=0.$$

It follows that

$$\left(W_n\left(T_n\right),\gamma_n^{-1}S_{[nt]}\right)\overset{fdd}{\Longrightarrow}\left(W(T),\Lambda_{\alpha,H}(t)\right).$$

Noting that $\tau_n\left(T_n\right)=m_n$ (see (100)) so that $W_n\left(T_n\right)=\sum_{k=1}^{m_n}\zeta_{nk}$, and in view of the independence of the processes $W(t)$ and $\Lambda_{\alpha,H}(t)$ so that the distribution of $(W(T),\Lambda_{\alpha,H}(t))$ is the same as that of $\left(W\sqrt{bL_1^0},\Lambda_{\alpha,H}(t)\right)$ where W is standard normal independent of the process $\Lambda_{\alpha,H}(t)$ (recall $T=bL_1^0$), the preceding convergence takes the form

$$\left(\sum_{k=1}^{m_n}\zeta_{nk},\gamma_n^{-1}S_{[nt]}\right)\overset{fdd}{\Longrightarrow}\left(W\sqrt{bL_1^0},\Lambda_{\alpha,H}(t)\right) \tag{108}$$

(Recall that $\sum_{k=1}^{m_n}\zeta_{nk}=\sqrt{\frac{\gamma_n}{n}}\sum_{k=1}^{n}f\left(S_k\right)$.) Further, it follows from the arguments of the proof of Proposition 14 that $\frac{\gamma_n}{n}\sum_{k=1}^{n}h\left(S_k\right)$ occurring the statement of Theorem 1 is approximated by a functional of the process $\gamma_n^{-1}S_{[nt]}$ such that the former converges in distribution to $L_1^0\int h\left(y\right)dy$ if $\gamma_n^{-1}S_{[nt]}\overset{fdd}{\Longrightarrow}\Lambda_{\alpha,H}(t)$. Thus the convergence (108) holds jointly with $\frac{\gamma_n}{n}\sum_{k=1}^{n}h\left(S_k\right)$. This completes the proof of the first part of Proposition 22. The proof of the second part is identical to that of the first part. ■

6 VERIFICATION OF (R1) - (R4) AND (R*1) - (R*4) OF SECTION 5

Verification of (R1) and (R*1): First consider (R1) corresponding to ζ_{nmk}, defined in (87). According to the second part of Lemma 8, $\left|E_{\left[n\frac{k-1}{m}\right]}\left[f\left(S_{\left[n\frac{k-1}{m}\right]+l}\right)\right]\right|\leq\frac{C}{\gamma_l^2}$ for all $l\geq1$, because (26) holds, see (45). Hence

$$\left|E_{\left[n\frac{k-1}{m}\right]}\left[\zeta_{nmk}\right]\right|\leq\sqrt{\frac{\gamma_n}{n}}\sum_{l=1}^{\left[n\frac{k}{m}\right]-\left[n\frac{k-1}{m}\right]}\left|E_{\left[n\frac{k-1}{m}\right]}\left[f\left(S_{\left[n\frac{k-1}{m}\right]+l}\right)\right]\right|\leq C\sqrt{\frac{\gamma_n}{n}}\sum_{l=1}^{n}\frac{1}{\gamma_l^2}.$$

Here recall that $\gamma_n=n^Hu_n$, where u_n is slowly varying.

Hence if $1/2\leq H<1$, it is clear that $\sqrt{\frac{\gamma_n}{n}}\sum_{l=1}^{n}\frac{1}{\gamma_l^2}\to0$.

In the case $0<H<1/2$, we have $\sum_{l=1}^{n}\frac{1}{\gamma_l^2}\sim C\frac{n}{\gamma_n^2}$, so that

$$\sqrt{\frac{\gamma_n}{n}}\sum_{l=1}^{n}\frac{1}{\gamma_l^2}\sim C\sqrt{n}\gamma_n^{-\frac{3}{2}}=Cn^{-\frac{3H-1}{2}}u_n^{-\frac{3}{2}}.$$

Because $1/3<H<1$, this converges to 0, and hence (R1) is verified.

In the same way (R*2), which involves ζ_{nmk}^{*} defined in (88) (note that the sum ζ_{nmk}^{*} involves $f\left(S_{\left[n\frac{k-1}{m}\right]+l}\right)\omega_{\left[n\frac{k-1}{m}\right]+l,\nu_n}$for $l>\nu_n$ only), is verified using the second part of Lemma 11. ■

Verification of (R2) and (R*2): We first consider (R2) and then we shall indicate the modifications required for (R*2). We have (recall that $n_{mk} = \left[n\frac{k}{m}\right] - \left[n\frac{k-1}{m}\right]$)

$$\begin{aligned} E_{\left[n\frac{k-1}{m}\right]}\left[\zeta_{nmk}^2\right] &= \frac{\gamma_n}{n}\sum_{l=1}^{n_{mk}} E_{\left[n\frac{k-1}{m}\right]}\left[f^2\left(S_{\left[n\frac{k-1}{m}\right]+l}\right)\right] \\ &\quad +2\frac{\gamma_n}{n}\sum_{l=1}^{n_{mk}}\sum_{r=1}^{n_{mk}-l} E_{\left[n\frac{k-1}{m}\right]}\left[f\left(S_{\left[n\frac{k-1}{m}\right]+l}\right)f\left(S_{\left[n\frac{k-1}{m}\right]+l+r}\right)\right]. \end{aligned}$$

Noting that (R2) involves $\sum_{k=1}^{m} E_{\left[n\frac{k-1}{m}\right]}\left[\zeta_{nmk}^2\right]$, first observe that as a consequence of Proposition 13 and the Remark 2 following Theorem 1, we have for each $q \geq 1$,

$$\begin{aligned} &\frac{\gamma_n}{n}\sum_{k=1}^{m}\sum_{l=1}^{n_{mk}} E_{\left[n\frac{k-1}{m}\right]}\left[f^2\left(S_{\left[n\frac{k-1}{m}\right]+l}\right)\right] \\ &+2\frac{\gamma_n}{n}\sum_{k=1}^{m}\sum_{l=1}^{n_{mk}}\sum_{r=1}^{q} E_{\left[n\frac{k-1}{m}\right]}\left[f\left(S_{\left[n\frac{k-1}{m}\right]+l}\right)f\left(S_{\left[n\frac{k-1}{m}\right]+l+r}\right)\right] \\ \Longrightarrow\quad & L_1^0\frac{1}{2\pi}\int\left|\widehat{f}(\mu)\right|^2\left(1+2\sum_{r=1}^{q}E\left[e^{-i\mu S_r}\right]\right)d\mu \end{aligned}$$

as $n \to \infty$ first and then $m \to \infty$.

The preceding conclusion holds also (except for the form of the limit), in view of Proposition 15, when $f\left(S_{\left[n\frac{k-1}{m}\right]+l}\right)\omega_{\left[n\frac{k-1}{m}\right]+l,\nu_n}$ and $f\left(S_{\left[n\frac{k-1}{m}\right]+l+r}\right)\omega_{\left[n\frac{k-1}{m}\right]+l+r,\nu_n}$ are involved in place of $f\left(S_{\left[n\frac{k-1}{m}\right]+l}\right)$ and $f\left(S_{\left[n\frac{k-1}{m}\right]+l+r}\right)$.

Then clearly, (R2) is a consequence of the first parts of the next two Lemmas 24 and 25, that (R*2) is a consequence of the corresponding second parts.

Remark. Recall from Lemmas 18 and 19 of Section 4 that the left hand side of the preceding convergence was approximated by a continuous functional of the process $\frac{1}{\gamma_n}S_{[nt]}$ and then this functional was shown to convergence to right hand side of the preceding convergence. Therefore in view of the next Lemma 24, the same facts hold for $\sum_{k=1}^{m} E_{\left[n\frac{k-1}{m}\right]}\left[\zeta_{nmk}^2\right]$ also, as $n \to \infty$ first and then $m \to \infty$. In addition according to the usual diagonal arguments (see Section 5) one can obtain a sequence $m_n \to \infty$ such that the same approximation and the convergence hold for $\sum_{k=1}^{m_n} E_{\left[n\frac{k-1}{m}\right]}\left[\zeta_{nmk}^2\right]$ as $n \to \infty$. This fact has been used in Section 5. ■

Lemma 24. *For each* $1 \leq k \leq m$,

$$\max_{1\leq k\leq m}\frac{\gamma_n}{n}\sum_{l=l_0}^{n_{mk}}\sum_{r=q}^{n_{mk}} E\left[\left|E_{\left[n\frac{k-1}{m}\right]}\left[f\left(S_{\left[n\frac{k-1}{m}\right]+l}\right)f\left(S_{\left[n\frac{k-1}{m}\right]+l+r}\right)\right]\right|\right] \to 0$$

as $n \to \infty$ *first and then* $q \to \infty$.

The same conclusion holds also when $f\left(S_{\left[n\frac{k-1}{m}\right]+l}\right)\omega_{\left[n\frac{k-1}{m}\right]+l,\nu_n} f\left(S_{\left[n\frac{k-1}{m}\right]+l+r}\right)\omega_{\left[n\frac{k-1}{m}\right]+l+r,\nu_n}$ *is involved in place of* $f\left(S_{\left[n\frac{k-1}{m}\right]+l}\right) f\left(S_{\left[n\frac{k-1}{m}\right]+l+r}\right)$.

The proof of Lemma 24 will be given later below because it, as well as the verification of (R3) and (R*3), require Lemma 26 below.

Lemma 25. *Under the conditions of Theorem 4,*

$$\sum_{r=1}^{\infty}\int\left|E\left[e^{-i\mu S_r}\right]\right|\left|\widehat{f}(\mu)\right|^2 d\mu<\infty.$$

In particular the quantity b defined in Theorem 4 is finite.

In the same way, under the conditions of Theorem 5,

$$\sum_{r=1}^{\infty}\int\left|E\left[\omega_0\omega_r e^{-i\mu S_r}\right]\right|\left|\widehat{f}(\mu)\right|^2 d\mu<\infty.$$

Proof. Consider the first part. We have $\left|E\left[e^{-i\mu S_r}\right]\right|\le\left|E\left[e^{-i\mu S^*_{r,r}}\right]\right|=\left|\prod_{j=0}^{l-1}\psi\left(g(j)\mu\right)\right|$, in view of (47). Also $\int\left|\widehat{f}(\mu)\right|^2 d\mu<\infty$. Hence it is enough to show that

$$\sum_{l=l_0}^{\infty}\int\left|\prod_{j=0}^{l}\psi\left(g(j)\mu\right)\right|\left|\widehat{f}(\mu)\right|^2 d\mu=\sum_{l=l_0}^{\infty}\frac{1}{\gamma_l}\int\left|\prod_{j=0}^{l}\psi\left(g(j)\frac{\mu}{\gamma_l}\right)\right|\left|\widehat{f}\left(\frac{\mu}{\gamma_l}\right)\right|^2 d\mu<\infty \tag{109}$$

a suitable l_0. Because $\left|\widehat{f}\left(\frac{\mu}{\gamma_l}\right)\right|\le C\left|\frac{\mu}{\gamma_l}\right|$ (see (45)), we have using (37) and (38) (with $\kappa=0$ and with the role of (27) now being played by $\int\left|\widehat{f}(\mu)\right|^2 d\mu<\infty$),

$$\int\left|\prod_{j=0}^{l-1}\psi\left(g(j)\frac{\mu}{\gamma_l}\right)\right|\left|\widehat{f}\left(\frac{\mu}{\gamma_l}\right)\right|^2 d\mu\le\frac{C}{\gamma_l^2},\qquad l\ge l_0,$$

for a suitable l_0. Hence, (109) is bounded by $C\sum_{l=l_0}^{\infty}\frac{1}{\gamma_l^3}$, where note that $\sum_{l=l_0}^{\infty}\frac{1}{\gamma_l^3}<\infty$ when the assumed restriction $3H>1$ holds. Hence the first part follows.

Regarding the second part, recall from (20) that $\omega_k=\sum_{j=-\infty}^{k}d_{k-j}\eta_j$, so that

$$E\left[\omega_0\omega_r e^{-i\mu S_r}\right]=\sum_{q=-\infty}^{0}\sum_{p=-\infty}^{r}d_{-q}d_{r-p}E\left[\eta_q\eta_p e^{-i\mu S_r}\right].$$

Suppose that $p\neq q$ and $p\le 0$ ($q\le 0$ already). Then, noting that $S_r=S_{r,r}+S^*_{r,r}$ with $(\eta_q,\eta_p,S_{r,r})$ independent of $S^*_{r,r}$, we have

$$\left|E\left[\eta_q\eta_p e^{-i\mu S_r}\right]\right|\le\left|E\left[\eta_q\eta_p e^{-i\mu S_{r,r}}\right]\right|\left|E\left[e^{-i\mu S^*_{r,r}}\right]\right|.$$

Here, recalling $S_{r,r}=\sum_{s=-\infty}^{0}(g(r-s)-g(-s))\xi_s$, and using $E[\eta_1]=0$ and $E[|\eta_1\xi_1|]<\infty$,

$$\begin{aligned}\left|E\left[\eta_q\eta_p e^{-i\mu S_{r,r}}\right]\right| &\leq \left|E\left[\eta_q e^{-i\mu(g(r-q)-g(-q))\xi_q}\right]\right|\left|E\left[\eta_p e^{-i\mu(g(r-p)-g(-p))\xi_p}\right]\right| \\ &\leq |\mu|^2\,|g(r-q)-g(-q)|\,|g(r-p)-g(-p)|\,.\end{aligned}$$

Thus, the preceding inequality playing the role of $\left|\widehat{f}(\mu)\right|\leq C|\mu|$, the same arguments of the first part above then give

$$\int\left|E\left[\eta_q\eta_p e^{-i\mu S_r}\right]\right|\left|\widehat{f}(\mu)\right|^2 d\mu \leq \frac{C}{\gamma_r^3}\,|g(r-q)-g(-q)|\,|g(r-p)-g(-p)|\,.$$

Similarly, for $0<p\leq r$ we have

$$\left|E\left[\eta_q\eta_p e^{-i\mu S_r}\right]\right| \leq \left|E\left[\eta_q e^{-i\mu S_{r,r}}\right]\right|\left|E\left[\eta_p e^{-i\mu g(r-p)\xi_p}\right]\right|\left|E\left[\eta_p e^{-i\mu\sum_{j=1,\neq p}^{r} g(r-j)\xi_j}\right]\right|,$$

and hence

$$\int\left|E\left[\eta_q\eta_p e^{-i\mu S_r}\right]\right|\left|\widehat{f}(\mu)\right|^2 d\mu \leq \frac{C}{\gamma_r^3}\,|g(r-q)-g(-q)|\,|g(r-p)|\,.$$

Thus

$$\begin{aligned}&\sum_{q=-\infty}^{0}\sum_{p=-\infty,p\neq q}^{r}|d_{-q}|\,|d_{r-p}|\int\left|E\left[\eta_q\eta_p e^{-i\mu S_r}\right]\right|\left|\widehat{f}(\mu)\right|^2 d\mu \\ \leq\ &\frac{C}{\gamma_r^3}\sum_{q=-\infty}^{0}\sum_{p=-\infty}^{r}|d_{-q}|\,|d_{r-p}|\left(|g(r-q)|+|g(-q)|\right)\left(|g(r-p)|+|g(-p)|\right).\end{aligned}$$

Here $\sum_{q=-\infty}^{0}|d_{-q}|\,|g(-q)|<\infty$ by (3), and $\sum_{q=-\infty}^{0}|d_{-q}|\,|g(r-q)|=\sum_{q=0}^{\infty}|d_q|\,|g(r+q)|$. In the case $H-\frac{1}{\alpha}\leq 0$, $\sum_{q=0}^{\infty}|d_q|\,|g(r+q)|\leq C\sum_{q=0}^{\infty}|d_q|<\infty$. In the case $H-\frac{1}{\alpha}>0$,

$$\begin{aligned}\sum_{q=0}^{\infty}|d_q|\,|g(r+q)| &= \sum_{q=0}^{r}|d_q|\,|g(r+q)|+\sum_{q=r+1}^{\infty}|d_q|\,|g(r+q)| \\ &\leq |g(2r)|\sum_{q=0}^{\infty}|d_q|+\sum_{q=r+1}^{\infty}|d_q|\,|g(2q)|\,,\end{aligned}$$

so that

$$\sum_{q=-\infty}^{0}\sum_{p=-\infty,p\neq q}^{r}|d_{-q}|\,|d_{r-q}|\int\left|E\left[\eta_q\eta_p e^{-i\mu S_r}\right]\right|\left|\widehat{f}(\mu)\right|^2 d\mu \leq \frac{C}{\gamma_r^3}\max\left(1,|g(r)|\right).$$

Next, we have

$$\begin{aligned}&\sum_{q=-\infty}^{0}|d_{-q}|\,|d_{r-q}|\int\left|E\left[\eta_q^2 e^{-i\mu S_r}\right]\right|\left|\widehat{f}(\mu)\right|^2 d\mu \\ \leq\ &\frac{C}{\gamma_r}\sum_{q=0}^{\infty}|d_q|\,|d_{r+q}| \leq C\sqrt{\left(\sum_{q=0}^{\infty}|d_q|^2\right)\frac{1}{\gamma_r}\sum_{q=0}^{\infty}|d_{r+q}|^2} \leq C\sqrt{\frac{1}{\gamma_r}\sum_{q=0}^{\infty}|d_{r+q}|^2}.\end{aligned}$$

Thus, in view of the condition (28) and fact $\sum_{r=0}^{\infty} \frac{1}{\gamma_r^3} < \infty$ when $H > \frac{1}{3}$, it only remains to see that $\sum_{r=0}^{\infty} \frac{|g(r)|}{\gamma_r^3} < \infty$ when $H - \frac{1}{\alpha} > 0$. Assume for convenience that $\gamma_r \sim r^H$ and $g(r) \sim Cr^{H-1/2}$. Then $\frac{|g(r)|}{\gamma_r^3} \sim \frac{1}{r^{2H+\frac{1}{\alpha}}}$, where $2H+\frac{1}{\alpha} > \frac{2}{3}+\frac{1}{2} > 1$, and hence $\sum_{r=0}^{\infty} \frac{|g(r)|}{\gamma_r^3} < \infty$. This completes the proof of the lemma. ■

We next consider the

Verification of (R4). Here recall that (R4) pertains only to the case $\alpha = 2$ and hence $E[\xi_1] = 0$ and $E[\xi_1^2] < \infty$, see (9).

For notational convenience, we take $\gamma_r = r^H$ and $g(r) \sim Cr^{H-1/2}$. Then (recall from (91) that $\chi_{nmk} = \frac{1}{\sqrt{n}} \sum_{l=\left[n\frac{k-1}{m}\right]+1}^{\left[n\frac{k}{m}\right]} \xi_l$)

$$\zeta_{nmk}\chi_{nmk} = n^{-\frac{1}{2}-\frac{1-H}{2}} \left(I_{1,nmk} + I_{2,nmk} + I_{3,nmk}\right)$$

where

$$I_{1,nmk} = \sum_{l=1}^{n_{mk}} \sum_{r=l+1}^{n_{mk}} f\left(S_{\left[n\frac{k-1}{m}\right]+l}\right) \xi_{\left[n\frac{k-1}{m}\right]+r},$$

$$I_{2,nmk} = \sum_{l=1}^{n_{mk}} \sum_{r=l+1}^{n_{mk}} \xi_{\left[n\frac{k-1}{m}\right]+l} f\left(S_{\left[n\frac{k-1}{m}\right]+r}\right), \qquad I_{3,nmk} = \sum_{l=1}^{n_{mk}} f\left(S_{\left[n\frac{k-1}{m}\right]+l}\right) \xi_{\left[n\frac{k-1}{m}\right]+l}.$$

According to the first part of Lemma 11, we have $E\left[\left|f\left(S_{\left[n\frac{k-1}{m}\right]+l}\right) \xi_{\left[n\frac{k-1}{m}\right]+l}\right|\right] \leq \frac{C}{l^H}$ for all $l \geq 1$, and hence

$$E\left[n^{-\frac{1}{2}-\frac{1-H}{2}} \left|I_{3,nmk}\right|\right] \leq Cn^{-\frac{1}{2}} n^{-\frac{1-H}{2}} \sum_{l=1}^{n} \frac{1}{l^H} \leq Cn^{-\frac{1}{2}} n^{-\frac{1-H}{2}} n^{1-H} = Cn^{-\frac{H}{2}},$$

and therefore

$$\left|E_{\left[n\frac{k-1}{m}\right]}\left[n^{-\frac{1}{2}-\frac{1-H}{2}} I_{3,nmk}\right]\right| \xrightarrow{p} 0 \text{ for each } m \geq 1. \tag{110}$$

Clearly

$$E_{\left[n\frac{k-1}{m}\right]}\left[I_{1,nmk}\right] = 0. \tag{111}$$

To deal with $I_{2,nmk}$ we have (see (47)) $S_{\left[n\frac{k-1}{m}\right]+r} = S_{\left[n\frac{k-1}{m}\right]+r,r} + S^*_{\left[n\frac{k-1}{m}\right]+r,r}$, where recall that $S^*_{\left[n\frac{k-1}{m}\right]+r,r} = \sum_{q=0}^{r-1} g(q)\, \xi_{\left[n\frac{k-1}{m}\right]+r-q}$ and is independent of $S_{\left[n\frac{k-1}{m}\right]+r,r}$. We also have $f(S_r) = \frac{1}{2\pi} \int e^{-i\lambda S_r} \widehat{f}(\lambda)\, d\lambda$. Hence

$$\begin{aligned}
&\left|E_{\left[n\frac{k-1}{m}\right]}\left[\xi_{\left[n\frac{k-1}{m}\right]+l} f\left(S_{\left[n\frac{k-1}{m}\right]+r}\right)\right]\right| \\
\leq\;& \frac{1}{2\pi} \int \left|E\left[\xi_l e^{-i\lambda \sum_{q=0}^{r-1} g(q)\xi_{r-q}}\right]\right| \left|\widehat{f}(\lambda)\right| d\lambda \\
\leq\;& \frac{1}{\gamma_r} \int \left|E\left[\xi_1 e^{-i\frac{\lambda}{\gamma_r} g(r-l)\xi_1}\right]\right| \prod_{q=0, q\neq r-l}^{r-1} \left|\psi\left(\frac{\lambda}{\gamma_r} g(q)\right)\right| \left|\widehat{f}\left(\frac{\lambda}{\gamma_r}\right)\right| d\lambda,
\end{aligned} \tag{112}$$

Now, because $E[\xi_1]=0$ and $E[\xi_1^2]<\infty$ ((R4) pertains only to the case $\alpha=2$),

$$\left|E\left[\xi_1 e^{-i\frac{\lambda}{\gamma_r}g(r-l)\xi_1}\right]\right| = \left|E\left[\xi_1\left(e^{-i\frac{\lambda}{\gamma_r}g(r-l)\xi_1}-1\right)\right]\right| \le C\frac{|\lambda|}{\gamma_r}\left|g(r-l)\right|.$$

Further $\left|\widehat{f}\left(\frac{\lambda}{\gamma_r}\right)\right| \le C\frac{|\lambda|}{\gamma_r}$, see (45). Also $\int\prod_{q=0,q\neq r-l}^{r-1}\left|\psi\left(\frac{\lambda}{\gamma_r}g(q)\right)\right|d\lambda \le C$ by (37) - (39). Hence

$$\left|E_{\left[n\frac{k-1}{m}\right]}\left[\xi_{\left[n\frac{k-1}{m}\right]+l}f\left(S_{\left[n\frac{k-1}{m}\right]+r}\right)\right]\right| \le \gamma_r^{-3}\left|g(r-l)\right| \tag{113}$$

Thus, noting that $\gamma_r = r^H$ and $\sum_{l=1}^{r-1}|g(r-l)| \sim Cr^{H+1-1/2}$ because $g(s)\sim Cs^{H-1/2}$,

$$\begin{aligned}
n^{-\frac{1}{2}-\frac{1-H}{2}}\left|E_{\left[n\frac{k-1}{m}\right]}\left[I_{2,nmk}\right]\right| &= n^{-\frac{1}{2}-\frac{1-H}{2}}\left|\sum_{l=1}^{n_{mk}}\sum_{r=l+1}^{n_{mk}}E_{\left[n\frac{k-1}{m}\right]}\left[\xi_{l+\left[n\frac{k-1}{m}\right]}f\left(S_{r+\left[n\frac{k-1}{m}\right]}\right)\right]\right| \\
&\le Cn^{-\frac{1}{2}-\frac{1-H}{2}}\sum_{r=1}^{n_{mk}}\sum_{l=1}^{r-1}\gamma_r^{-3}\left|g(r-l)\right| \\
&\le Cn^{-\frac{1}{2}-\frac{1-H}{2}}n^{-2H+\frac{3}{2}} = Cn^{-\frac{3H-1}{2}}.
\end{aligned} \tag{114}$$

Because $3H-1>0$, this together with (110) and (111) complete the verification of (R4) in the situation of the Statement (I) of Theorem 1.

In the case of the situation of the Statement (II) of Theorem 4 also the bound (112) holds except that the factor $\left|\widehat{f}\left(\frac{\lambda}{\gamma_r}\right)\right|$ in the right hand side needs to be replaced by $\left|\widehat{K}_\eta\left(\frac{\lambda}{\gamma_r}\right)\right|\max\left(\left|\widehat{M_{f,\eta}}\left(\frac{\lambda}{\gamma_r}\right)\right|,\left|\widehat{m_{f,\eta}}\left(\frac{\lambda}{\gamma_r}\right)\right|\right)$, see for instance the proof of the second part of Lemma 8. Hence, using (46) as in the proof of the second part of Lemma 8, it is seen that (R4) holds in the present situation also. This completes the verification of (R4). ■

We next consider the

Verification of (R*4). Here the analogue of $I_{3,nmk}$ above is

$$\begin{aligned}
I_{3,nmk}^* &= \sum_{l=1}^{n_{mk}}f\left(S_{\left[n\frac{k-1}{m}\right]+l}\right)w_{\left[n\frac{k-1}{m}\right]+l,\nu}\xi_{\left[n\frac{k-1}{m}\right]+l} \\
&= \sum_{j=\left[n\frac{k-1}{m}\right]+l-\nu+1}^{\left[n\frac{k-1}{m}\right]+l}d_{\left[n\frac{k-1}{m}\right]+l-j}\sum_{l=1}^{n_{mk}}f\left(S_{\left[n\frac{k-1}{m}\right]+l}\right)\eta_j\xi_{\left[n\frac{k-1}{m}\right]+l},
\end{aligned}$$

where we have used $\omega_{q,\nu} = \sum_{j=q-\nu+1}^{q}d_{q-j}\eta_j$. The same arguments of the first part of Lemma 11 gives $E\left[\left|f\left(S_{\left[n\frac{k-1}{m}\right]+l}\right)\eta_j\xi_{\left[n\frac{k-1}{m}\right]+l}\right|\right] \le \frac{C}{l^H}$ for all $l\ge 1$, and therefore

$$E\left[\left|I_{3,nmk}^*\right|\right] \le C\sum_{l=1}^{n}\frac{1}{l^H}\left(\sum_{j=0}^{\nu-1}|d_j|\right) \le Cn^{1-H},$$

which is the same as the bound for $E\left[\left|I_{3,nmk}\right|\right]$ obtained above.

Defining $I_{2,nmk}^{*}$ analogous to $I_{2,nmk}$ above, we have

$$I_{2,nmk}^{*}=\sum_{l=1}^{n_{mk}}\sum_{r=l+1}^{n_{mk}}\sum_{j=\left[n\frac{k-1}{m}\right]+r-\nu+1}^{\left[n\frac{k-1}{m}\right]+r}\xi_{\left[n\frac{k-1}{m}\right]+l}f\left(S_{\left[n\frac{k-1}{m}\right]+r}\right)\eta_{j}d_{\left[n\frac{k-1}{m}\right]+r-j}.$$

Similar to (113), we have

$$\left|E_{\left[n\frac{k-1}{m}\right]}\left[\xi_{\left[n\frac{k-1}{m}\right]+l}f\left(S_{\left[n\frac{k-1}{m}\right]+r}\right)\eta_{j}\right]\right|\leq\gamma_{r}^{-3}\left|g\left(r-l\right)\right|\left|g\left(\left[n\frac{k-1}{m}\right]+r-j\right)\right|\quad\text{if } j\neq\left[n\frac{k-1}{m}\right]+l.$$

Thus

$$\begin{aligned}
&\sum_{l=1}^{n_{mk}}\sum_{r=l+1}^{n_{mk}}\sum_{j=\left[n\frac{k-1}{m}\right]+r-\nu+1,j\neq\left[n\frac{k-1}{m}\right]+l}^{\left[n\frac{k-1}{m}\right]+r}E\left[\left|\xi_{\left[n\frac{k-1}{m}\right]+l}f\left(S_{\left[n\frac{k-1}{m}\right]+r}\right)\eta_{j}\right|\right]\left|d_{\left[n\frac{k-1}{m}\right]+r-j}\right|\\
\leq\ &\sum_{l=1}^{n_{mk}}\sum_{r=l+1}^{n_{mk}}\gamma_{r}^{-3}\left|g\left(r-l\right)\right|\sum_{j=\left[n\frac{k-1}{m}\right]+r-\nu+1}^{\left[n\frac{k-1}{m}\right]+r}\left|g\left(\left[n\frac{k-1}{m}\right]+r-j\right)\right|\left|d_{\left[n\frac{k-1}{m}\right]+r-j}\right|\\
=\ &\sum_{l=1}^{n_{mk}}\sum_{r=l+1}^{n_{mk}}\gamma_{r}^{-3}\left|g\left(r-l\right)\right|\left(\sum_{j=0}^{\nu-1}\left|g\left(j\right)\right|\left|d_{j}\right|\right)\leq\sum_{l=1}^{n_{mk}}\sum_{r=l+1}^{n_{mk}}\gamma_{r}^{-3}\left|g\left(r-l\right)\right|,
\end{aligned}$$

which bound is the same as the one involved in (114). Regarding the remaining factor in $I_{2,nmk}^{*}$, we have

$$\begin{aligned}
&\sum_{l=1}^{n_{mk}}\sum_{r=l+1}^{n_{mk}}E\left[\left|\xi_{\left[n\frac{k-1}{m}\right]+l}f\left(S_{\left[n\frac{k-1}{m}\right]+r}\right)\eta_{\left[n\frac{k-1}{m}\right]+l}\right|\right]\left|d_{r-l}\right|\\
\leq\ &C\sum_{l=1}^{n_{mk}}\sum_{r=l+1}^{n_{mk}}\frac{\left|d_{r-l}\right|}{r^{H}}\leq C\sum_{l=1}^{n}\sum_{r=1}^{n}\frac{\left|d_{r}\right|}{\left(r+l\right)^{H}}\leq Cn^{1-H},
\end{aligned}$$

which bound is the same as for $E\left[\left|I_{3,nmk}\right|\right]$ obtained above.

For $I_{1,nmk}^{*}$, the analogue of $I_{1,nmk}$, we have $E_{\left[n\frac{k-1}{m}\right]}\left[I_{1,nmk}^{*}\right]=0$, for the same reason $E_{\left[n\frac{k-1}{m}\right]}\left[I_{1,nmk}\right]=0$. This completes the proof of the verification of (R*4). ■

It remains to prove Lemma 24 and to verify (R3) and (R*3). For this purpose we need the next Lemma 26, where and in the rest of the paper we let

$$g(j,r)=g(j+r)-g(j)=c_{j+1}+...+c_{j+r}.$$

Lemma 26. *Let $g(j,r)$ be as above. Let $\vartheta>0$ be such that*

$$0<\vartheta<\begin{cases}\min\left(1-H,H,\left|\frac{1}{\alpha}-H\right|,\frac{1}{\alpha}\right) & \text{if } H\neq\frac{1}{\alpha}\\ \min\left(1-\frac{1}{\alpha},\frac{1}{\alpha}\right) & \text{if } H=\frac{1}{\alpha}.\end{cases}\tag{115}$$

Then

$$\sup_{[l/2]\le j\le l, q\ge 1, r\ge 1}\left|b_l\frac{g(j+q,r)}{\gamma_r}\right|\le Cl^{\vartheta}\quad\text{for all } 1\le l\le n. \tag{116}$$

Proof. First consider the case $H\neq\frac{1}{\alpha}$, in which the requirement (A2) of Section 2 holds. Let $\delta=\frac{\vartheta}{3}$ so that (115) becomes

$$0<3\delta<\min\left(1-H,H,\left|\frac{1}{\alpha}-H\right|,\frac{1}{\alpha}\right). \tag{117}$$

Recall the Potter's inequality, mentioned in Lemma 6 of Section 3 above, that if $G(x)$ is slowly varying at ∞, then there is a $B>0$ such that $\left|\frac{G(x)}{G(y)}\right|\le B\max\{(x/y)^{\delta},(x/y)^{-\delta}\}$ for all $x>0,y>0$. Therefore one can assume that

$$\left|\frac{c_i}{i^{H-1-\frac{1}{\alpha}}}\right|\le Bi^{\delta},\qquad\left|\frac{g(i)}{i^{H-\frac{1}{\alpha}}}\right|\le Bi^{\delta},\qquad\frac{b_r}{r^{\frac{1}{\alpha}}}\le Br^{\delta},\qquad\frac{r^H}{\gamma_r}\le Br^{\delta}.$$

We in particular have

$$\frac{b_l}{\gamma_r}\le Cl^{\frac{1}{\alpha}+\delta}r^{-H+\delta}. \tag{118}$$

Further, noting $H-1-\frac{1}{\alpha}+\delta<0$ (see (117)), we have when $j\ge[l/2]$,

$$\begin{aligned}|g(j+q,r)| &= |c_{j+q+1}+..+c_{j+q+r}|\\ &\le C\left|(j+q+1)^{H-1-\frac{1}{\alpha}+\delta}+...+(j+q+r)^{H-1-\frac{1}{\alpha}+\delta}\right|\\ &\le Cr(j+q)^{H-1-\frac{1}{\alpha}+\delta}\le Cr(\min(l,q))^{H-1-\frac{1}{\alpha}+\delta},\quad j\ge[l/2].\end{aligned} \tag{119}$$

Here, in obtaining the second inequality we have used $j\ge[l/2]$ and $H-1-\frac{1}{\alpha}+\delta<0$.

Further, when $H-\frac{1}{\alpha}<0$ (in which case $H-\frac{1}{\alpha}+\delta<0$, see (117)), we have

$$\begin{aligned}|g(j+q,r)| &\le |g(j+q)|+|g(j+q+r)|\\ &\le C(j+q)^{H-\frac{1}{\alpha}+\delta}\le C(\min(l,q))^{H-\frac{1}{\alpha}+\delta},\qquad j\ge[l/2],\end{aligned} \tag{120}$$

and similarly when $H-\frac{1}{\alpha}>0$,

$$\begin{aligned}|g(j+q,r)| &\le C(j+q+r)^{H-\frac{1}{\alpha}+\delta}\\ &\le \begin{cases}\begin{cases}Cl^{H-\frac{1}{\alpha}+\delta} & \text{if } j\le l,\ r\le l\\ Cr^{H-\frac{1}{\alpha}+\delta} & \text{if } j\le l,\ r>l\end{cases} & H-\frac{1}{\alpha}>0,\ q\le l\\ \begin{cases}Cq^{H-\frac{1}{\alpha}+\delta} & \text{if } j\le l,\ r\le q\\ Cr^{H-\frac{1}{\alpha}+\delta} & \text{if } j\le l,\ r>q\end{cases} & H-\frac{1}{\alpha}>0,\ q>l.\end{cases}\end{aligned} \tag{121}$$

First consider the situation

$$q \leq l.$$

Using (118) and (119) and noting $1 - H - 2\delta > 0$ (see (117)),

$$\left| b_l \frac{g(j+q,r)}{\gamma_r} \right| \leq C l^{\frac{1}{\alpha}+\delta} r^{-H+\delta} r l^{H-1-\frac{1}{\alpha}+\delta} = \left(\frac{r}{l}\right)^{1-H-2\delta} r^{3\delta} \leq C l^{3\delta}, \text{ if } r \leq l,\ j \geq [l/2].$$

In addition, using (120) and (121) and noting $H - \delta > 0$ and $\frac{1}{\alpha} - 2\delta > 0$ (see (117)), we have

$$\left| b_l \frac{g(j+q,r)}{\gamma_r} \right| \leq \begin{cases} C l^{\frac{1}{\alpha}+\delta} r^{-H+\delta} l^{H-\frac{1}{\alpha}+\delta} = C r^{-H+\delta} l^{H-\delta} l^{3\delta} \leq C l^{3\delta}, & H - \frac{1}{\alpha} < 0,\ r > l,\ j \leq l \\ C l^{\frac{1}{\alpha}+\delta} r^{-H+\delta} r^{H-\frac{1}{\alpha}+\delta} = C r^{2\delta-\frac{1}{\alpha}} l^{\frac{1}{\alpha}-2\delta} l^{3\delta} \leq C l^{3\delta}, & H - \frac{1}{\alpha} > 0,\ r > l,\ j \leq l. \end{cases}$$

Now consider

$$q > l.$$

From (119) we have,

$$\left| b_l \frac{g(j+q,r)}{\gamma_r} \right| \leq C l^{\frac{1}{\alpha}+\delta} r^{-H+\delta} r q^{H-1-\frac{1}{\alpha}+\delta} = \left(\frac{r}{q}\right)^{1-H+\delta} \left(\frac{l}{q}\right)^{\frac{1}{\alpha}-2\delta} l^{3\delta} \leq C l^{3\delta}, \text{ if } r \leq q,\ j \geq [l/2].$$

When $H - \frac{1}{\alpha} < 0$, $r > q$, we obtain from (120) that

$$\left| b_l \frac{g(j+q,r)}{\gamma_r} \right| \leq C l^{\frac{1}{\alpha}+\delta} r^{-H+\delta} q^{H-\frac{1}{\alpha}+\delta} = \left(\frac{q}{r}\right)^{H-\delta} \left(\frac{l}{q}\right)^{\frac{1}{\alpha}-2\delta} l^{3\delta} \leq C l^{3\delta}.$$

When $H - \frac{1}{\alpha} > 0$, $r > q$, we have from (121) that

$$\left| b_l \frac{g(j+q,r)}{\gamma_r} \right| \leq C l^{\frac{1}{\alpha}+\delta} r^{-H+\delta} r^{H-\frac{1}{\alpha}+\delta} = \left(\frac{l}{r}\right)^{\frac{1}{\alpha}-2\delta} l^{3\delta} \leq C l^{3\delta}$$

because $\frac{1}{\alpha} - 2\delta > 0$ (see (117)) and $l < q < r$. This completes the proof of the lemma when $H \neq \frac{1}{\alpha}$.

Now consider the case $H = \frac{1}{\alpha}$. In this case, by (11), we have $\sup_{i \geq 1} |i c_i| \leq C$. In addition $\sup_{i \geq 1} |g(i)| \leq C$ by (A1). Therefore, the inequalities (118) - (121) hold when $H = \frac{1}{\alpha}$, and hence the remaining arguments also hold with $H = \frac{1}{\alpha}$. This completes the proof of the lemma. ■

Below we assume ϑ of Lemma 26 satisfies (in addition to (115))

$$3H - 6\vartheta > 1. \tag{122}$$

This is possible in view of the restriction $3H > 1$.

We are now in a position to proceed with the proof of Lemma 24 and the verification of (R3) and (R*3).

Proof of the first part of Lemma 24. First note that the bound in (50) holds for $\left|E_{\left[n\frac{k-1}{m}\right]}\left[w\left(S_{\left[n\frac{k-1}{m}\right]+l}, S_{\left[n\frac{k-1}{m}\right]+l+r}\right)\right]\right|$ also, by taking k in the left hand side of (49) to be $\left[n\frac{k-1}{m}\right]$.

We first consider the proof under (18). We need to apply the bound (50) with $\widehat{f}(\lambda,\mu) = \widehat{f}(\lambda)\widehat{f}(\mu)$. Recall that in the proof of the first part of Lemma 9, we used the fact that $\left|\widehat{f}(\lambda)\right| \leq C$. Now the fact $\left|\widehat{f}(\lambda)\right| \leq C|\lambda|$, which we now have in view of the condition (26), see (45), will be crucially used. Here note that, for any ϑ satisfying (115),

$$\left|\frac{\gamma_l g(j,r)}{\gamma_r g(j)}\right| = \left|\frac{\gamma_l}{b_l g(j)}\right|\left|\frac{b_l g(j,r)}{\gamma_r}\right| \leq Cl^{\vartheta}, \quad [l/2] \leq j \leq l, \ \ r \geq 1 \tag{123}$$

by (36) and Lemma 26. Therefore, using $\left|\widehat{f}(\lambda)\right| \leq C|\lambda|$,

$$\left|\widehat{f}\left(\frac{\lambda}{\gamma_l} + \frac{g(j,r)}{\gamma_r g(j)}\mu - \frac{\mu}{\gamma_r}\right)\widehat{f}\left(\frac{\mu}{\gamma_r}\right)\right| \leq C\left(\frac{|\lambda|}{\gamma_l} + \frac{|\mu|\,l^{\vartheta}}{\gamma_l} + \frac{|\mu|}{\gamma_r}\right)\frac{|\mu|}{\gamma_r}.$$

Hence, in exactly the same manner as in the first part of Lemma 9 (when $\widehat{f}(\lambda,\mu) = \widehat{f}(\lambda)\widehat{f}(\mu)$), we have

$$\left|E_{\left[n\frac{k-1}{m}\right]}\left[f\left(S_{\left[n\frac{k-1}{m}\right]+l}\right)f\left(S_{\left[n\frac{k-1}{m}\right]+l+r}\right)\right]\right| \leq C\frac{1}{\gamma_l\gamma_r}\left(\frac{l^{\vartheta}}{\gamma_l} + \frac{1}{\gamma_r}\right)\frac{1}{\gamma_r} \qquad l,r \geq 1. \tag{124}$$

Thus we need to show, under the restriction $1 < 3H$, that

$$\frac{\gamma_n}{n}\sum_{l=1}^{n}\sum_{r=q}^{n}\frac{1}{\gamma_l\gamma_r}\left(\frac{l^{\vartheta}}{\gamma_l} + \frac{1}{\gamma_r}\right)\frac{1}{\gamma_r} \to 0 \tag{125}$$

as $n \to \infty$ first and then $q \to \infty$. To see that this is true, take for convenience that

$$\gamma_n = n^H \qquad \text{for all } n \geq 1.$$

First note that, using the restriction $1 < 3H$,

$$\frac{\gamma_n}{n}\sum_{l=1}^{n}\sum_{r=q}^{n}\frac{1}{\gamma_l\gamma_r^3} = \left(\frac{\gamma_n}{n}\sum_{l=1}^{n}\frac{1}{\gamma_l}\right)\sum_{r=q}^{n}\frac{1}{\gamma_r^3} \leq Cq^{1-3H} \to 0 \qquad \text{as } q \to \infty,$$

where we have used $\frac{\gamma_n}{n}\sum_{l=1}^{n}\frac{1}{\gamma_l} \leq C$ and $\sum_{r=q}^{n}\frac{1}{\gamma_r^3} = \sum_{r=q}^{n}\frac{1}{r^{3H}} \leq Cq^{1-3H}$. Next

$$\sum_{l=1}^{n}\frac{l^{\vartheta}}{\gamma_l^2} = \sum_{l=1}^{n}\frac{1}{l^{2H-\vartheta}} \leq \begin{cases} C\log n & \text{if } 2H-\vartheta \geq 1 \\ Cn^{1-2H+\vartheta} & \text{if } 2H-\vartheta < 1. \end{cases}$$

Also, $\frac{\gamma_n}{n} = n^{H-1}$. Hence

$$\frac{\gamma_n}{n}\sum_{l=1}^{n}\sum_{r=q}^{n}\frac{l^{\vartheta}}{\gamma_l^2\gamma_r^2} \le \frac{\gamma_n}{n}\left(\sum_{l=1}^{n}\frac{l^{\vartheta}}{\gamma_l^2}\right)^2 \le \begin{cases} Cn^{H-1}\left(\log n\right)^2 & \text{if } 2H-\vartheta \ge 1 \\ Cn^{H-1+2-4H+2\vartheta} = Cn^{1-3H+2\vartheta} & \text{if } 2H-\vartheta < 1 \end{cases} \tag{126}$$

where note that $1-3H+2\vartheta < 0$ in view of (122) Thus (125) holds and hence the proof of the first part of Lemma 24 is complete under the restriction (18).

Under the restriction (19), we use the same bound (50) but with $\widehat{f}(\lambda,\mu)$ replaced by

$$\left|\widehat{K_{\eta_1}}(\lambda)\right|\left|\widehat{K_{\eta_2}}(\mu)\right|\max\left(\left|\widehat{M_{f,\eta_1}}(\lambda)\right|\left|\widehat{M_{f,\eta_2}}(\mu)\right|,\left|\widehat{m_{f,\eta_1}}(\lambda)\right|\left|\widehat{m_{f,\eta_2}}(\mu)\right|\right),$$

In this case, using the arguments in (53), together with (46), we have

$$\begin{aligned}
&\int_R\left|\widehat{M_{f,\eta_1}}\left(\frac{\lambda}{\gamma_l}+\frac{g(j,r)}{\gamma_r g(j)}\mu-\frac{\mu}{\gamma_r}\right)\right|\left|\widehat{K_{\eta_1}}\left(\frac{\lambda}{\gamma_l}+\frac{g(j,r)}{\gamma_r g(j)}\mu-\frac{\mu}{\gamma_r}\right)\right|\prod_{j=[l/2]}^{l-1}\left|\psi\left(\frac{\lambda g(j)}{\gamma_l}\right)\right|d\lambda \\
\le\ & C\left(\frac{1}{\gamma_l}+\frac{|\mu|\,l^{\vartheta}}{\gamma_l}+\frac{|\mu|}{\gamma_r}\right)+C\left|\eta_1\right|^d+C\rho^l\frac{\gamma_l}{\eta}.
\end{aligned}$$

The same holds when $\widehat{M_{f,\eta_1}}$ is replaced by $\widehat{m_{f,\eta_1}}$. Then the same arguments used in (53) give

$$\begin{aligned}
&\left|E_{\left[n\frac{k-1}{m}\right]}\left[f\left(S_{\left[n\frac{k-1}{m}\right]+l}\right)f\left(S_{\left[n\frac{k-1}{m}\right]+l+r}\right)\right]\right| \\
\le\ & \frac{C}{\gamma_l\gamma_r}\int\left(\left(\frac{1}{\gamma_l}+\frac{|\mu|\,l^{\vartheta}}{\gamma_l}+\frac{|\mu|}{\gamma_r}\right)+C\left|\eta_1\right|^d+C\rho^l\frac{\gamma_l}{\eta}\right)\max\left(\left|\widehat{M_{f,\eta_2}}\left(\frac{\mu}{\gamma_r}\right)\right|,\left|\widehat{m_{f,\eta_2}}\left(\frac{\mu}{\gamma_r}\right)\right|\right) \\
&\times\left|\widehat{K_{\eta_2}}\left(\frac{\mu}{\gamma_r}\right)\right|\left(\prod_{j_1=[r/2]}^{r-1}\left|\psi\left(-g\left(j_1\right)\frac{\mu}{\gamma_r}\right)\right|\right)d\mu \\
\le\ & \frac{C}{\gamma_l\gamma_r}\left(\frac{l^{\vartheta}}{\gamma_l}+\frac{1}{\gamma_r}+\left|\eta_1\right|^d+\rho^l\frac{\gamma_l}{\eta_1}\right)\left(\frac{1}{\gamma_r}+\left|\eta_2\right|^d+\rho^r\left(\frac{\gamma_r}{\eta_2}\right)^2\right).
\end{aligned}$$

By choosing $\eta_1 = \gamma_l^{-\frac{1}{d}}$ and $\eta_2 = \gamma_r^{-\frac{1}{d}}$, and noting (recall $0<\rho<1$) that $\rho^r\left(\frac{\gamma_r}{\eta_2}\right)^2 = \rho^r\gamma_r^{2+\frac{2}{d}} \le C\gamma_r^{-1}$ and similarly $\rho^l\frac{\gamma_l}{\eta_1} \le C\gamma_l^{-1}$, we see that the preceding bound reduces to that in (124). This completes the proof of the first part of Lemma 24 (for the situation of Theorem 4).

Proof of the second part of Lemma 24. The second part involves

$$\begin{aligned}
&f\left(S_{\left[n\frac{k-1}{m}\right]+l}\right)w_{\left[n\frac{k-1}{m}\right]+l,\nu_n}f\left(S_{\left[n\frac{k-1}{m}\right]+l+r}\right)w_{\left[n\frac{k-1}{m}\right]+l+r,\nu_n} \\
=\ & \sum_{i=l-\nu+1}^{l}\sum_{j=r-\nu+1}^{r}d_{l-i}d_{r-j}f\left(S_{\left[n\frac{k-1}{m}\right]+l}\right)\eta_{\left[n\frac{k-1}{m}\right]+i}f\left(S_{\left[n\frac{k-1}{m}\right]+l+r}\right)\eta_{\left[n\frac{k-1}{m}\right]+l+j}.
\end{aligned}$$

Recall that $l \geq \nu_n$, which entails that $\eta_{\left[n\frac{k-1}{m}\right]+i}$ is independent of $\left\{\xi_j; j \leq \left[n\frac{k-1}{m}\right]\right\}$ for $i \geq l-\nu_n+1$. Suppose in addition that $r \geq \nu_n$. Then in the preceding identity $l+j>i$. Then, similar to (49), we have (recall $S_{k,l}^* = \sum_{j=1}^{l} g(l-j)\xi_{k+j}$)

$$\begin{aligned}
& \left|E_{\left[n\frac{k-1}{m}\right]}\left[f\left(S_{\left[n\frac{k-1}{m}\right]+l}\right)\eta_{\left[n\frac{k-1}{m}\right]+i} f\left(S_{\left[n\frac{k-1}{m}\right]+l+r}\right)\eta_{\left[n\frac{k-1}{m}\right]+l+j}\right]\right| \\
\leq & \int\left|E\left[e^{-i\lambda_1\sum_{q=1,q\neq i}^{l} g(l-q)\xi_q - i\lambda_2\sum_{p=1,p\neq l+j}^{l+r} g(l+r-p)\xi_p}\right]\right| \\
& \times\left|E\left[\eta_i e^{-i\lambda_1 g(l-i)\xi_i}\right]\right|\left|\eta_{l+j}E\left[e^{-i\lambda_2 g(r-j)\xi_{l+j}}\right]\right|\left|\widehat{f}\left(\lambda_1\right)\right|\left|\widehat{f}\left(\lambda_2\right)\right| d\lambda_1 d\lambda_2.
\end{aligned}$$

Here, using $E\left[\eta_1\right]=0$ and $E\left[\left|\eta_1\xi_1\right|\right]<\infty$,

$$\left|E\left[\eta_i e^{-i\lambda_1\frac{\beta_n}{\gamma_n}g(l-i)\xi_i}\right]\right| \leq \left|\lambda_1\right|\left|g\left(l-i\right)\right|, \qquad \left|\eta_{l+j}E\left[e^{-i\lambda_2\frac{\beta_n}{\gamma_n}g(r-j)\xi_{l+j}}\right]\right| \leq \left|\lambda_2\right|\left|g\left(r-j\right)\right|.$$

Thus the role of $\left|\widehat{f}\left(\lambda\right)\right| \leq C\left|\lambda\right|$ in the proof of the first part above is now played by the preceding inequalities. Therefore in place of (124) we have

$$\begin{aligned}
& \left|E_k\left[f\left(S_{\left[n\frac{k-1}{m}\right]+l}\right)\eta_{\left[n\frac{k-1}{m}\right]+i} f\left(S_{\left[n\frac{k-1}{m}\right]+l+r}\right)\eta_{\left[n\frac{k-1}{m}\right]+l+j}\right]\right| \\
\leq & \ C\frac{1}{\gamma_l\gamma_r}\left(\frac{l^\vartheta}{\gamma_l}+\frac{1}{\gamma_r}\right)\frac{1}{\gamma_r}\left|g\left(l-i\right)\right|\left|g\left(r-j\right)\right| \quad \text{if } l \geq \nu_n,\ r \geq \nu_n.
\end{aligned}$$

This bound holds also when $r<\nu_n$ but $l+j \neq i$. Thus

$$\begin{aligned}
& \sum_{i=l-\nu_n+1}^{l}\ \sum_{j=r-\nu_n+1,l+j\neq i}^{r}\left|d_{l-i}\right|\left|d_{r-j}\right|\left|E_k\left[f\left(S_{\left[n\frac{k-1}{m}\right]+l}\right)\eta_{\left[n\frac{k-1}{m}\right]+i} f\left(S_{\left[n\frac{k-1}{m}\right]+l+r}\right)\eta_{\left[n\frac{k-1}{m}\right]+l+j}\right]\right| \\
\leq & \ C\frac{1}{\gamma_l\gamma_r}\left(\frac{l^\vartheta}{\gamma_l}+\frac{1}{\gamma_r}\right)\frac{1}{\gamma_r}\left(\sum_{j=0}^{\infty}\left|d_j\right|\left|g\left(j\right)\right|\right)^2 \leq C\frac{1}{\gamma_l\gamma_r}\left(\frac{l^\vartheta}{\gamma_l}+\frac{1}{\gamma_r}\right)\frac{1}{\gamma_r},
\end{aligned}$$

which bound is the same as that in (124).

Next consider the sum $\sum_{i=l-\nu_n+1}^{l}\sum_{j=r-\nu_n+1}^{r}$ for $r<\nu_n$ but $l+j=i$, that is,

$$\begin{aligned}
& \sum_{j=r-\nu_n+1}^{0}\left|d_{-j}\right|\left|d_{r-j}\right|\left|E_k\left[f\left(S_{\left[n\frac{k-1}{m}\right]+l}\right)\eta_{\left[n\frac{k-1}{m}\right]+l+j} f\left(S_{\left[n\frac{k-1}{m}\right]+l+r}\right)\eta_{\left[n\frac{k-1}{m}\right]+l+j}\right]\right| \\
\leq & \ C\frac{1}{\gamma_l\gamma_r}\sum_{j=r-\nu_n+1}^{0}\left|d_{-j}\right|\left|d_{r-j}\right| \leq C\frac{1}{\gamma_l\gamma_r}\sqrt{\left(\sum_{j=0}^{\infty}\left|d_j\right|^2\right)\sum_{j=0}^{\infty}\left|d_{r+j}\right|^2} \leq C\frac{1}{\gamma_l\gamma_r}\sqrt{\sum_{j=0}^{\infty}\left|d_{r+j}\right|^2}.
\end{aligned}$$

Thus, combining the preceding two inequalities,

$$\begin{aligned}
& \left|E_{\left[n\frac{k-1}{m}\right]}\left[f\left(S_{\left[n\frac{k-1}{m}\right]+l}\right)w_{\left[n\frac{k-1}{m}\right]+l,\nu_n} f\left(S_{\left[n\frac{k-1}{m}\right]+l+r}\right)w_{\left[n\frac{k-1}{m}\right]+l+r,\nu_n}\right]\right| \\
\leq & \ C\frac{1}{\gamma_l\gamma_r}\left(\frac{l^\vartheta}{\gamma_l}+\frac{1}{\gamma_r}\right)\frac{1}{\gamma_r}+C\frac{1}{\gamma_l\gamma_r}\sqrt{\sum_{j=0}^{\infty}\left|d_{r+j}\right|^2} \ \text{ for all } l \geq \nu_n,\ r \geq 1. \qquad (127)
\end{aligned}$$

Hence, in view of (125), it only remains to show that

$$\frac{\gamma_n}{n}\sum_{l=1}^{n}\sum_{r=q}^{n}\frac{1}{\gamma_l\gamma_r}\sqrt{\sum_{j=0}^{\infty}|d_{r+j}|^2}\leq C\sum_{r=q}^{\infty}\frac{1}{\gamma_r}\sqrt{\sum_{j=0}^{\infty}|d_{r+j}|^2}\to 0 \text{ as } n\to\infty \text{ first and then } q\to\infty,$$

which is true by (28). This completes the proof of the second part of Lemma 24. ■

We next verify (90).

Verification of (90). This is essentially contained in the preceding proof of the second part of Lemma 24. Recall that there the restriction $l \geq \nu_n$ was needed in obtaining the inequality (127), because the left hand side involved the conditional expectation $E_{\left[n\frac{k-1}{m}\right]}[.]$ and it was required to ensure that $\eta_{\left[n\frac{k-1}{m}\right]+i}$ is independent of $\left\{\xi_j; j\leq\left[n\frac{k-1}{m}\right]\right\}$ for $i\geq l-\nu_n+1$, in order to obtain the right hand side bound in (127). When only the expectation $E[.]$ is involved in the left hand side, it is easily seen that the restriction $l\geq\nu_n$ is not required, see the arguments of the proof of the second part of Lemma 25. In other words, $\left|E\left[f\left(S_{\left[n\frac{k-1}{m}\right]+l}\right)w_{\left[n\frac{k-1}{m}\right]+l,\nu_n}f\left(S_{\left[n\frac{k-1}{m}\right]+l+r}\right)w_{\left[n\frac{k-1}{m}\right]+l+r,\nu_n}\right]\right|$ is bounded above by the same bound in the right hand side of (127), for all $l\geq 1$, $r\geq 1$. Thus, using this bound, it follows in the same way as in the proof of the second part of Lemma 24 that

$$\frac{\gamma_{\nu_n}}{\nu_n}\sum_{l=1}^{\nu_n}\sum_{r=1}^{\nu_n}\left|E\left[f\left(S_{\left[n\frac{k-1}{m}\right]+l}\right)w_{\left[n\frac{k-1}{m}\right]+l,\nu_n}f\left(S_{\left[n\frac{k-1}{m}\right]+l+r}\right)w_{\left[n\frac{k-1}{m}\right]+l+r,\nu_n}\right]\right|\leq C \text{ for all } n\geq 1,$$

and hence

$$\frac{\gamma_n}{n}\sum_{l=1}^{\nu_n}\sum_{r=1}^{\nu_n}\left|E\left[f\left(S_{\left[n\frac{k-1}{m}\right]+l}\right)w_{\left[n\frac{k-1}{m}\right]+l,\nu_n}f\left(S_{\left[n\frac{k-1}{m}\right]+l+r}\right)w_{\left[n\frac{k-1}{m}\right]+l+r,\nu_n}\right]\right|\to 0 \text{ because } \frac{\nu_n}{n}\to 0.$$

We also have $\frac{\gamma_n}{n}\sum_{l=1}^{\nu_n}E\left[\left|f\left(S_{\left[n\frac{k-1}{m}\right]+l}\right)w_{\left[n\frac{k-1}{m}\right]+l,\nu_n}\right|^2\right]\to 0$, using the first part of Lemma 11. Now note that $E\left[|R^*_{nmk}|^2\right]$ is the sum of the preceding two quantities. Hence (90) follows. ■

Verification of (R3) and (R*3) First consider (R3). We show that (recall $n_{mk}=\left[n\frac{k}{m}\right]-\left[n\frac{k-1}{m}\right]$)

$$E\left[\zeta_{nmk}^4\right]\leq\frac{C}{n^{\omega}}+C\left(\frac{\gamma_n}{n}\sum_{l=\left[n\frac{k-1}{m}\right]+1}^{\left[n\frac{k}{m}\right]}\frac{1}{\gamma_l}\right)\left(\frac{\gamma_n}{n}\sum_{j=1}^{n_{mk}}\frac{1}{\gamma_j}\right),\quad \text{for some } \omega>0. \tag{128}$$

This will verify (R3), because then

$$\sum_{k=1}^{m}E\left[\zeta_{nmk}^4\right]\leq\frac{Cm}{n^{\delta}}+C\left(\frac{\gamma_n}{n}\sum_{l=1}^{n}\frac{1}{\gamma_l}\right)\left(\max_{1\leq k\leq m}\frac{\gamma_n}{n}\sum_{j=1}^{n_{mk}}\frac{1}{\gamma_j}\right)$$

where $\frac{\gamma_n}{n}\sum_{l=1}^{n}\frac{1}{\gamma_l}\le C$ and $\max_{1\le k\le m}\frac{\gamma_n}{n}\sum_{j=1}^{n_{mk}}\frac{1}{\gamma_j}\sim\frac{1}{1-H}\left(\frac{1}{m}\right)^{1-H}$ as $n\to\infty$.

We shall show in detail that

$$\left(\frac{\gamma_n}{n}\right)^2\sum_{l=\left[n\frac{k-1}{m}\right]+1}^{\left[n\frac{k}{m}\right]}\sum_{r=1}^{n_{mk}}\sum_{q=1}^{n_{mk}}\left|E\left[f\left(S_l\right)f\left(S_{l+r}\right)f^2\left(S_{l+r+q}\right)\right]\right| \tag{129}$$

and

$$\left(\frac{\gamma_n}{n}\right)^2\sum_{l=\left[n\frac{k-1}{m}\right]+1}^{\left[n\frac{k}{m}\right]}\sum_{r=1}^{n_{mk}}\sum_{q=1}^{n_{mk}}\sum_{s=1}^{n_{mk}}\left|E\left[f\left(S_l\right)f\left(S_{l+r}\right)f\left(S_{l+r+q}\right)f\left(S_{l+r+q+s}\right)\right]\right| \tag{130}$$

are bounded by the right hand side of (128). The same can be similarly shown to be true for the remaining analogues in the expansion of $E\left[\zeta_{nmk}^4\right]=\left(\frac{\gamma_n}{n}\right)^2E\left[\left(\sum_{l=\left[n\frac{k-1}{m}\right]+1}^{\left[n\frac{k}{m}\right]}f\left(S_l\right)\right)^4\right]$. We shall use Lemma 26 in a manner similar to the proof of Lemmas 24 above. *In addition, we shall give the details of the verification only for the situation of the Statement (I) of Theorems 4 and 5. The corresponding situation of the Statement (II) can be similarly verified using the ideas in the earlier proof of Lemma 24.*

According to (56), we have (noting $\left|\widehat{f^2}\left(\lambda\right)\right|\le C$),

$$\begin{aligned}&(2\pi)^3\left|E\left[f\left(S_l\right)f\left(S_{l+r}\right)f^2\left(S_{l+r+q}\right)\right]\right|\\ \le\;&\frac{C}{\gamma_l\gamma_r\gamma_q}\int\left(\prod_{j_1=[l/2]}^{l-1}\left|\psi\left(\frac{\mu_1 g(j_1)}{\gamma_l}\right)\right|\right)\left(\prod_{j_2=[r/2]}^{r-1}\left|\psi\left(\frac{\mu_2 g\left(j_2\right)}{\gamma_r}\right)\right|\right)\\ &\times\left(\prod_{j_3=[q/2]}^{q-1}\left|\psi\left(\frac{\mu_3 g\left(j_3\right)}{\gamma_q}\right)\right|\right)\left|\widehat{f}\left(\frac{\lambda_1}{\gamma_l}-\frac{\lambda_2}{\gamma_r}\right)\right|\left|\widehat{f}\left(\frac{\lambda_2}{\gamma_r}-\frac{\lambda_3}{\gamma_q}\right)\right|d\mu_1 d\mu_2 d\mu_3.\end{aligned} \tag{131}$$

where recall that (with $g(j,r)=g(j+r)-g(j)$), $\lambda_1+\lambda_2\frac{\gamma_l g(j_1,r)}{\gamma_r g(j_1)}+\lambda_3\frac{\gamma_l g(j_1+r,q)}{\gamma_q g(j_1)}=\mu_1$, $\lambda_2+\lambda_3\frac{\gamma_r g(j_2,q)}{\gamma_q g(j_2)}=\mu_2$ and $\lambda_3=\mu_3$, see (56). Here note that, in the same way as in (123) using (36) and Lemma 26, we have

$$\left|\frac{\gamma_r g(j_2,q)}{\gamma_q g\left(j_2\right)}\right|\le Cr^{\vartheta},\qquad\left|\frac{\gamma_l g(j_1,r)}{\gamma_r g(j_1)}\right|\le Cl^{\vartheta},\qquad\left|\frac{\gamma_l g(j_1+r,q)}{\gamma_q g(j_1)}\right|\le Cl^{\vartheta}$$

uniformly in the variables involved. (For instance, using (36) and Lemma 26, $\left|\frac{\gamma_l g(j_1+r,q)}{\gamma_q g(j_1)}\right|=\left|\frac{\gamma_l}{b_l g(j_1)}\right|\left|\frac{b_l g(j_1+r,q)}{\gamma_q}\right|\le Cl^{\vartheta}$, $[l/2]\le j_1\le l,\ \ r,q\ge 1$.) Therefore,

$$\begin{aligned}&\left|\widehat{f}\left(\frac{\lambda_1}{\gamma_l}-\frac{\lambda_2}{\gamma_r}\right)\right|\left|\widehat{f}\left(\frac{\lambda_2}{\gamma_r}-\frac{\lambda_3}{\gamma_q}\right)\right|\\ \le\;&\left(\frac{1}{\gamma_l}\left(|\mu_1|+|\mu_2|\,l^{\vartheta}+|\mu_3|\,l^{\vartheta}r^{\vartheta}+|\mu_3|\,l^{\vartheta}\right)+\frac{1}{\gamma_r}\left(|\mu_2|+|\mu_3|\,r^{\vartheta}\right)\right)\left(\frac{1}{\gamma_r}\left(|\mu_2|+|\mu_3|\,r^{\vartheta}\right)+\frac{|\mu_3|}{\gamma_q}\right).\end{aligned}$$

Substituting this in (131), the right hand side in (131) is bounded by (in the same way as in (124))

$$(2\pi)^3 \left|E\left[f(S_l) f(S_{l+r}) f^2(S_{l+r+q})\right]\right| \leq \frac{C}{\gamma_l\gamma_r\gamma_q}\left(\frac{l^\vartheta + l^\vartheta r^\vartheta}{\gamma_l} + \frac{r^\vartheta}{\gamma_r}\right)\left(\frac{r^\vartheta}{\gamma_r} + \frac{1}{\gamma_q}\right) \tag{132}$$

Thus we need to consider

$$\left(\frac{\gamma_n}{n}\right)^2 \sum_{l=\left[n\frac{k-1}{m}\right]+1}^{\left[n\frac{k}{m}\right]} \sum_{r=1}^{n_{mk}}\sum_{q=1}^{n_{mk}} \frac{1}{\gamma_l\gamma_r\gamma_q}\left(\frac{l^\vartheta + l^\vartheta r^\vartheta}{\gamma_l} + \frac{r^\vartheta}{\gamma_r}\right)\left(\frac{r^\vartheta}{\gamma_r} + \frac{1}{\gamma_q}\right).$$

We have, similar to (126),

$$\left(\frac{\gamma_n}{n}\right)^2 \sum_{l=\left[n\frac{k-1}{m}\right]+1}^{\left[n\frac{k}{m}\right]} \sum_{r=1}^{n_{mk}}\sum_{q=1}^{n_{mk}} \frac{l^\vartheta r^\vartheta}{\gamma_l^2\gamma_r^2\gamma_q} \leq \begin{cases} Cn^{H-1}(\log n)^2 & \text{if } 2H-\vartheta \geq 1 \\ Cn^{1-3H+2\vartheta} & \text{if } 2H-\vartheta < 1. \end{cases}$$

Essentially the same holds for all other terms in (132) except for

$$\left(\frac{\gamma_n}{n}\right)^2 \sum_{l=\left[n\frac{k-1}{m}\right]+1}^{\left[n\frac{k}{m}\right]} \sum_{r=1}^{n_{mk}}\sum_{q=1}^{n_{mk}} \frac{r^{2\vartheta}}{\gamma_l\gamma_r^3\gamma_q} = \left(\left(\frac{\gamma_n}{n}\right)^2 \sum_{l=\left[n\frac{k-1}{m}\right]+1}^{\left[n\frac{k}{m}\right]} \sum_{q=1}^{n_{mk}} \frac{1}{\gamma_l\gamma_q}\right)\left(\sum_{r=1}^{n_{mk}} \frac{r^{2\vartheta}}{\gamma_r^3}\right),$$

which is of the form of the bound in (128) because $\sum_{r=1}^{n_{mk}} \frac{r^{2\vartheta}}{\gamma_r^3} < \infty$ in view of $3H - 2\vartheta > 1$ (see (122)). Thus the bound in (128) holds for (129).

Next consider (130). The ideas involved are the same as those used for (129). In obtaining (132) we used (56). Now we use (57), with $\beta_n = \gamma_n$ and with $f(x_0, x_1, x_2, x_3) = f_0(x_0) f_1(x_1) f_0(x_2) f_1(x_3)$ (and λ_1, λ_2, λ_3 and λ_4 as in (58)). Then, in exactly the same way as above, we see that, using $\left|\widehat{f}(\lambda)\right| \leq C|\lambda|$,

$$\begin{aligned} &(2\pi)^4 \left|E\left[f(S_l) f(S_{l+r}) f(S_{l+r+q}) f(S_{l+r+q+s})\right]\right| \\ \leq\ & \frac{C}{\gamma_l\gamma_r\gamma_q\gamma_s}\left(\frac{l^\vartheta + l^\vartheta r^\vartheta + l^\vartheta r^\vartheta q^\vartheta}{\gamma_l} + \frac{r^\vartheta + r^\vartheta q^\vartheta}{\gamma_r}\right)\left(\frac{r^\vartheta + r^\vartheta q^\vartheta}{\gamma_r} + \frac{q^\vartheta}{\gamma_q}\right)\left(\frac{q^\vartheta}{\gamma_q} + \frac{1}{\gamma_s}\right)\frac{1}{\gamma_s}. \end{aligned}$$

Using this bound and using (122), it easy to see, in the same way as in (132), that the sum (130) is bounded by the right hand side of (128). This completes the verification of (R3).

Verification of (R*3) is done similar to the proof of the second part of Lemma 24, using the preceding ideas of the verification of (R3) together with those in the proof of the second part of Lemma 24. For this reason we omit the details. ■

7 Appendix: A martingale CLT and the proof of Lemma 23

We begin with the reduction of Lemma 23 to an explicit version of a martingale CLT. First, for convenience, we extend the random variables $\zeta'_{m_n k}$ defined in Section 5 for $1 \leq k < \infty$ to all $-\infty < k < \infty$ by taking

$$\zeta'_{nk} = 0 \text{ for } k \leq 0.$$

The random variables χ_{nk}, as well as the σ-fields $\mathcal{F}_{nk}$, for $-\infty < k < \infty$, are as in Section 5. The stopping times $\tau_n(t)$ are also as in Section 5, in particular $\{\tau_n(t) \leq k\} \in \mathcal{F}_{n,k-1}$. Recall that $W_n(t) = \sum_{k=1}^{\tau_n(t)} \zeta'_{nk}$ for $t > 0$ and $W_n(t) = 0$ for $t \leq 0$.

First consider **the case** $0 < \alpha < 2$. We have $\sum_{k=-m_n(l+1)}^{[m_n t]} \chi_{nk} = \frac{1}{n^{1/\alpha}\delta_n} \sum_{j=-nl}^{\left[n\frac{[m_n t]}{m_n}\right]} \xi_j$. Hence, using (1) and noting that l is an integer,

$$\sum_{k=-m_n(l+1)}^{[m_n t]} \chi_{nk} - \frac{1}{n^{1/\alpha}\delta_n} \sum_{j=-nl}^{[nt]} \xi_j = \frac{1}{n^{1/\alpha}\delta_n} \sum_{j=\left[n\frac{[m_n t]}{m_n}\right]}^{\left[n\frac{[m_n t]}{m_n}\right]} \xi_j \xrightarrow{p} 0. \quad (t \text{ is fixed})$$

Therefore, letting

$$Z_n(t) = \sum_{k=-m_n(l+1)}^{[m_n t]} \chi_{nk},$$

we need to show that $(Z_n(t), W_n(t)) \overset{fdd}{\Longrightarrow} (Z_\alpha(t) - Z_\alpha(-l), W(t))$ with $Z_\alpha(t)$ and $W(t)$ independent, where $Z_\alpha(t)$ is the stable process as before and $\{W(t), 0 \leq t < \infty\}$ is a standard Brownian motion (note that $W(t) = 0$ for $t \leq 0$). This means, for each finite $t_0 < t_1 < ... < t_{q-1} < t_q < t_{q+1} < ... < t_{q+r} < \infty$ with

$$t_0 = -l \;\text{ and }\; t_q = 0,$$

and for reals $u_1, ...u_{q+r}$, $v_1, ..., u_{q+r}$, we need to show that

$$\sum_{j=1}^{q+r} \left(u_j \left(Z_n(t_j) - Z_n(t_{j-1})\right) + v_j \left(W_n(t_j) - W_n(t_{j-1})\right)\right)$$
$$\Longrightarrow \sum_{j=1}^{q+r} \left(u_j \left(Z(t_j) - Z(t_{j-1})\right) + v_j \left(W(t_j) - W(t_{j-1})\right)\right), \tag{133}$$

with $Z_\alpha(t)$ and $W(t)$ independent. Here it is important to note that the marginal convergencies of $\sum_{j=1}^{q+r} u_j \left(Z_n(t_j) - Z_n(t_{j-1})\right)$ and $\sum_{j=1}^{q+r} v_j \left(W_n(t_j) - W_n(t_{j-1})\right)$ are well known. The deeper part is that the limits $Z_\alpha(t)$ and $W(t)$ are independent.

First note that, by (1), $Z_n(t) \overset{fdd}{\Longrightarrow} Z_\alpha(t) - Z_\alpha(-l)$. Also, for each $n \geq 1$, $\{\chi_{nk}, -\infty < k < \infty\}$ is an array of iid random variables. In addition, because $\frac{m_n}{n} \to 0$,

$$\sup_{-m_n(l+1) \leq k \leq [m_n M]} P\left[|\chi_{nk}| > \varepsilon\right] = P\left[|\chi_{n1}| > \varepsilon\right] \to 0.$$

Therefore, the following conditions (134) - (137) hold, where $\Psi(x)$ is the Levy measure corresponding to the stable random variable $Z_\alpha(1)$, $0<\alpha<2$, see Loève (1963, Section 22.4, Central Convergence Criterion, page 311). (The detailed form of $\Psi(x)$ and of the function $A(\tau)$ in (136) below are not essential for what follows.)

For every $s<t$,

$$\sum_{k=[m_n s]+1}^{[m_n t]} P[\chi_{nk}\le x] = ([m_n t]-[m_n s])\, P[\chi_{n1}\le x] \to (t-s)\,\Psi(x) \quad \text{for all } x<0 \tag{134}$$

and

$$\sum_{k=[m_n s]+1}^{[m_n t]} P[\chi_{nk}> x] = ([m_n t]-[m_n s])\, P[\chi_{n1}> x] \to (t-s)\,\Psi(x) \quad \text{for all } x>0, \tag{135}$$

for some $\tau>0$, there is a constant $A(\tau)$ such that

$$\sum_{k=[m_n s]+1}^{[m_n t]} E\left[\chi_{nk}\mathbb{I}_{\{|\chi_{nk}|<\tau\}}\right] = ([m_n t]-[m_n s])\, E\left[\chi_{n1}\mathbb{I}_{\{|\chi_{n1}|<\tau\}}\right] \to (t-s)\,A(\tau), \tag{136}$$

and

$$\begin{aligned} &\sum_{k=[m_n s]+1}^{[m_n t]} \left(E\left[\left|\chi_{nk}^2\right|\mathbb{I}_{\{||\chi_{nk}||<\varepsilon\}}\right] - \left(E\left[\chi_{nk}\mathbb{I}_{\{|\chi_{nk}|<\varepsilon\}}\right]\right)^2\right) \\ =\; & ([m_n t]-[m_n s])\left(E\left[\chi_{n1}^2\mathbb{I}_{\{|\chi_{n1}|<\varepsilon\}}\right] - \left(E\left[\chi_{n1}\mathbb{I}_{\{|\chi_{n1}|<\varepsilon\}}\right]\right)^2\right) \to 0 \end{aligned} \tag{137}$$

as $n\to\infty$ first and then $\varepsilon\to 0$.

In addition, in view of (103) and according to (105), for any $0<s<t$, we have

$$\sum_{k=\tau_n(s)}^{\tau_n(t)} E_{n,k-1}\left[\left|\zeta'_{nk}\right|^2\mathbb{I}_{\left\{\left|\zeta'_{nk}\right|>\varepsilon\right\}}\right] \xrightarrow{p} 0 \quad\text{and}\quad \sum_{k=\tau_n(s)}^{\tau_n(t)} E_{n,k-1}\left[\left|\zeta'_{nk}\right|^2\right] \xrightarrow{p} t-s, \tag{138}$$

where we now use the notations $P_{k-1}[.] = P[.|\digamma_{n,k-1}]$ and $E_{k-1}[.] = E[.|\digamma_{n,k-1}]$ (instead of $P_{n,k-1}[.]$ and $E_{n,k-1}[.]$ as in Section 5).

Note that the sums $Z_n(t)$ and $W_n(t)$ involve respectively the time scales $[m_n t]$ and $\tau_n(t)$. To proceed further we need to rewrite them as sums involving a common time scale, which becomes possible because one of the time scales is the natural time scale $[m_n t]$. For this purpose, let

$\tau_n(t_{q+r+1}) = \max(\tau_n(t_{q+r}),\ [m_n t_{q+r}])$, $\tau_n(t_{q-1}) = [nt_0]$. Also $\tau_n(t_q)=0$ because $t_q=0$.

Define, with the reals $u_1, ...u_{q+r}, v_1, ..., u_{q+r}$ as in (133),

$$U_{nk} = \begin{cases} u_j \chi_{nk} & \text{if } [nt_{j-1}] < k \leq [nt_j], \ j = 1, ..., q+r, \\ 0 & \text{if } [m_n t_{q+r}] < k \leq \tau_n(t_{q+r+1}) \end{cases}$$

and (recall that $\zeta'_{nk} = 0$ for $k \leq 0$)

$$V_{nk} = \begin{cases} 0 & \text{if } [nt_0] < k \leq 0, \\ v_j \zeta'_{nk} & \text{if } \tau_n(t_{j-1}) < k \leq \tau_n(t_j), \ j = q+1, ..., q+r, \\ 0 & \text{if } \tau_n(t_{q+r}) < k \leq \tau_n(t_{q+r+1}). \end{cases}$$

Then, the left hand side of (133) takes the form

$$\sum_{k=[nt_0]}^{\tau_n(t_{q+r+1})} (U_{nk} + V_{nk}),$$

where the array $\{(U_{nk}, V_{nk}), [nt_0] < k \leq \tau_n(t_{q+r+1})\}$ is now viewed as the array

$$\left\{ \left(U_{n.\tau_n(t_{j-1})+1}, V_{n.\tau_n(t_{j-1})+1} \right), ..., \left(U_{n.\tau_n(t_j)}, V_{n.\tau_n(t_j)} \right), \ j = q, ..., q+r+1 \right\}$$

adapted to the array $\left\{ \mathcal{F}_{m_n.\tau_n(t_{j-1})+1}, ..., \mathcal{F}_{m_n.\tau_n(t_j)}, \ j = q, ..., q+r+1 \right\}$.

With these preliminaries, the main step in obtaining (133) will consist of verifying the following conditions (139) - (142) from (134) - (138). With $\Psi(x)$ and $A(\tau)$ as in (134) - (136),

$$\sum P_{k-1}[U_{nk} + V_{nk} \leq x] \xrightarrow{p} \sum_{j=0}^{q+r} (t_j - t_{j-1}) \Psi\left(\frac{x}{u_j}\right) \quad \text{for all } x < 0 \tag{139}$$

and

$$\sum P_{k-1}[U_{nk} + V_{nk} > x] \xrightarrow{p} \sum_{j=0}^{q+r} (t_j - t_{j-1}) \Psi\left(\frac{x}{u_j}\right) \quad \text{for all } x > 0, \tag{140}$$

$$\sum E_{k-1}\left[(U_{nk} + V_{nk}) \mathbb{I}_{\{|U_{nk}+V_{nk}|<\tau\}}\right] \xrightarrow{p} \sum_{j=0}^{q+r} (t_j - t_{j-1}) A\left(\frac{\tau}{u_j}\right) \quad \text{for some } \tau > 0, \tag{141}$$

and, as $n \to \infty$ first and then $\varepsilon \to 0$,

$$\sum \left(E_{k-1}\left[|U_{nk} + V_{nk}|^2 \mathbb{I}_{\{|U_{nk}+V_{nk}|<\varepsilon\}}\right] - \left(E_{k-1}\left[(U_{nk} + V_{nk}) \mathbb{I}_{\{|U_{nk}+V_{nk}|<\varepsilon\}}\right]\right)^2 \right)$$
$$\xrightarrow{p} \sum_{j=q+1}^{q+r} v_j^2 (t_j - t_{j-1}). \tag{142}$$

Above and below, we make the convention that the notation $\sum$ *stands for* $\sum_{k=[nt_0]}^{\tau_n(t_{q+r+1})}$, *unless otherwise specified.*

Now, we have

(**a**). (137) entails, as $n \to \infty$ first and then $\varepsilon \to 0$,

$$\sum \left(E_{k-1} \left[|U_{nk}|^2 \mathbb{I}_{\{|U_{nk}|<\varepsilon\}} \right] - \left(E_{k-1} \left[U_{nk} \mathbb{I}_{\{|U_{nk}|<\varepsilon\}} \right] \right)^2 \right)$$
$$= \sum_{j=1}^{q+r} \sum_{k=[nt_{j-1}]}^{[nt_j]} \left(E \left[|u_l \chi_{nk}|^2 \mathbb{I}_{\{|u_l \chi_{nk}|<\varepsilon\}} \right] - \left(E \left[u_l \chi_{nk} \mathbb{I}_{\{|u_l \chi_{nk}|<\varepsilon\}} \right] \right)^2 \right) \to 0. \quad (143)$$

(**b**). Similarly, (139) - (141) above with $U_{nk} + V_{nk}$ in the left hand side replaced by U_{nk} are implied respectively by (134) - (136).

(**c**). (138) implies

$$\sum E_{k-1} \left[|V_{nk}|^2 \mathbb{I}_{\{|V_{nk}|>\varepsilon\}} \right] \xrightarrow{p} 0 \quad \text{and} \quad \sum E_{k-1} \left[|V_{nk}|^2 \right] \xrightarrow{p} \sum_{j=q+1}^{q+r} v_l^2 \left(t_l - t_{l-1} \right). \quad (144)$$

Theorem A. *Assume that the preceding conditions* (**a**) - (**c**) *(each of them involve either* U_{nk}*'s only or* V_{nk}*'s only) are satisfied. Then the conditions (139) - (142) involving* $U_{nk} + V_{nk}$*'s are satisfied.*

As a consequence

$$E \left[e^{iw \sum (U_{nk}+V_{nk})} \right] \to e^{-\frac{1}{2} w^2 \sum_{j=q+1}^{q+r} v_l^2 (t_l - t_{l-1})} E \left[e^{iw \sum_{l=1}^{q+r} u_l (Z(t_l) - Z(t_{l-1}))} \right] \quad \text{for all real } w. \quad (145)$$

It is well known that (139) - (142) imply the convergence in distribution of $\sum (U_{nk} + V_{nk})$ to a suitable infinitely divisible distribution determined by the limits in (139) - (142). We in particular obtain (145), where the specified form of the limit follows from the forms of the limits in (139) - (142). See for instance Jeganathan (1983) for the details. Therefore it only remains to obtain (139) - (142).‘

Verification of (139) - (142). First consider (139). Because $V_{nk} = 0$ for $[nt_0] < k \le 0$,

$$\sum_{k=[nt_0]}^{0} P_{k-1} \left[U_{nk} + V_{nk} \le x \right] = \sum_{k=[nt_0]}^{0} P_{k-1} \left[U_{nk} \le x \right] = \sum_{j=1}^{q} \sum_{k=[nt_{j-1}]}^{[nt_j]} P \left[u_j \chi_{nk} \le x \right]$$
$$\to \sum_{j=0}^{q} (t_j - t_{j-1}) \Psi \left(\frac{x}{u_j} \right) \quad \text{for all } x < 0, \quad (146)$$

using (134). Next, using the fact $\{U_{nk} + V_{nk} \le x, |V_{nk}| \le \varepsilon\} \subset \{U_{nk} \le x + \varepsilon\}$, we have

$$\sum_{k=1}^{\tau_n(t_{q+r+1})} P_{k-1} \left[U_{nk} + V_{nk} \le x \right] \le \sum_{k=1}^{\tau_n(t_{q+r+1})} P_{k-1} \left[U_{nk} \le x + \varepsilon \right] + \sum P_{k-1} \left[|V_{nk}| > \varepsilon \right],$$

and similarly

$$\sum_{k=1}^{\tau_n(t_{q+r+1})} P_{k-1}\left[U_{nk}+V_{nk}\le x\right] \ge \sum_{k=1}^{\tau_n(t_{q+r+1})} P_{k-1}\left[U_{nk}\le x-\varepsilon\right] - \sum P_{k-1}\left[|V_{nk}|>\varepsilon\right].$$

Now note that, similar to (146), $\sum_{k=1}^{\tau_n(t_{q+r+1})} P_{k-1}\left[U_{nk}\le x+\varepsilon\right]$ is nonrandom and converges to $\sum_{j=0}^{r}\left(t_{q+j}-t_{q+j-1}\right)\Psi\left(\frac{x+\varepsilon}{u_{q+j}}\right)$ if $x+\varepsilon<0$, using (134). The same holds when $x-\varepsilon$ is involved in place of $x+\varepsilon$. Thus, because $\Psi\left(\frac{x\pm\varepsilon}{u_{q+j}}\right)\to\Psi\left(\frac{x}{u_{q+j}}\right)$ as $\varepsilon\to 0$, and taking into account the first part in (144), we have

$$\sum_{k=1}^{\tau_n(t_{q+r+1})} P_{k-1}\left[U_{nk}+V_{nk}\le x\right] \xrightarrow{p} \sum_{j=0}^{r}\left(t_{q+j}-t_{q+j-1}\right)\Psi\left(\frac{x}{u_{q+j}}\right) \quad \text{if } x<0.$$

This together with (146) gives (139). In the same way (140) holds also.

To proceed further, we next obtain

$$\sum E_{k-1}\left[\left(|V_{nk}|+V_{nk}^2\right)\left(\mathbb{I}_{\{|U_{nk}|\ge\tau\}}+\mathbb{I}_{\{|U_{nk}+V_{nk}|\ge\tau\}}\right)\right] \xrightarrow{p} 0. \tag{147}$$

To see that this is true, consider for instance, for all $0<\varepsilon<\frac{\tau}{2}$,

$$\begin{aligned} E_{k-1}\left[|V_{nk}|^2\,\mathbb{I}_{\{|U_{nk}+V_{nk}|\ge\tau\}}\right] &\le E_{k-1}\left[|V_{nk}|^2\,\mathbb{I}_{\{|U_{nk}|>\tau-\varepsilon,|V_{nk}|\le\varepsilon\}}\right]+E_{k-1}\left[|V_{nk}|^2\,\mathbb{I}_{\{|V_{nk}|>\varepsilon\}}\right] \\ &\le \varepsilon^2 P\left[|U_{nk}|>\frac{\tau}{2}\right]+\frac{1}{\varepsilon}E_{k-1}\left[|V_{nk}|^2\,\mathbb{I}_{\{|V_{nk}|>\varepsilon\}}\right], \end{aligned}$$

where note that $\sum_{k=1}^{\tau_n(t_{q+r+1})} P\left[|U_{nk}|>\frac{\tau}{2}\right]$ is non random and is bounded by (134) and (135), similar to (146). Thus, taking (144) into account further, $\sum\left|E_{k-1}\left[|V_{nk}|^2\,\mathbb{I}_{\{|U_{nk}+V_{nk}|\ge\tau\}}\right]\right| \xrightarrow{p} 0$. In the same way the remaining parts in (147) are obtained.

In addition to (147), we also have

$$\sum E_{k-1}\left[\left(|U_{nk}|+U_{nk}^2\right)\left|\mathbb{I}_{\{|U_{nk}+V_{nk}|<\tau\}}-\mathbb{I}_{\{|U_{nk}|<\tau\}}\right|\right] \xrightarrow{p} 0. \tag{148}$$

To see this, note $\left|\mathbb{I}_{\{|U_{nk}+V_{nk}|<\tau\}}-\mathbb{I}_{\{|U_{nk}|<\tau\}}\right| \le \mathbb{I}_{\{|U_{nk}+V_{nk}|\ge\tau,|U_{nk}|<\tau\}}+\mathbb{I}_{\{|U_{nk}+V_{nk}|<\tau,|U_{nk}|\ge\tau\}}$. We have

$$E_{k-1}\left[|U_{nk}|\,\mathbb{I}_{\{|U_{nk}+V_{nk}|\ge\tau,|U_{nk}|<\tau\}}\right] \le \tau P_{k-1}\left[|U_{nk}+V_{nk}|\ge\tau,|U_{nk}|<\tau\right],$$

where

$$\begin{aligned} & P_{k-1}\left[|U_{nk}+V_{nk}|\ge\tau,|U_{nk}|<\tau\right] \\ \le\ & P_{k-1}\left[|U_{nk}+V_{nk}|\ge\tau,|U_{nk}|<\tau-\varepsilon\right]+P_{k-1}\left[\tau-\varepsilon\le|U_{nk}|<\tau\right] \\ \le\ & P_{k-1}\left[|V_{nk}|\ge\varepsilon\right]+P_{k-1}\left[\tau-\varepsilon\le|U_{nk}|<\tau\right], \end{aligned}$$

using $\{|U_{nk}+V_{nk}|\ge\tau, |U_{nk}|<\tau-\varepsilon\} \subset \{|V_{nk}|\ge\varepsilon\}$. Here $\sum P_{k-1}[|V_{nk}|>\varepsilon] \to 0$ by (144). Further, $\sum P_{k-1}[\tau<|U_{nk}|\le\tau+\varepsilon]$ is nonrandom and converges to 0 as $n\to\infty$ first and then $\varepsilon\to 0$, according to (134) and (135). Thus $\sum E_{k-1}\left[|U_{nk}|\,\mathbb{I}_{\{|U_{nk}+V_{nk}|\ge\tau,|U_{nk}|<\tau\}}\right] \xrightarrow{p} 0$.

Similarly $\sum E_{k-1}\left[(U_{nk}+V_{nk})\,\mathbb{I}_{\{|U_{nk}+V_{nk}|<\tau,|U_{nk}|\ge\tau\}}\right] \xrightarrow{p} 0$, because we have already verified (139) and (140). Hence, taking (147) into account, $\sum E_{k-1}\left[|U_{nk}|\,\mathbb{I}_{\{|U_{nk}+V_{nk}|<\tau,|U_{nk}|\ge\tau\}}\right] \xrightarrow{p} 0$. Thus (148) holds.

Now consider (141). We have, $\left|E_{k-1}\left[V_{nk}\mathbb{I}_{\{|U_{nk}+V_{nk}|<\tau\}}\right]\right| \le E_{k-1}\left[|V_{nk}|\,\mathbb{I}_{\{|U_{nk}+V_{nk}|\ge\tau\}}\right]$ because

$$\left|E_{k-1}\left[V_{nk}\mathbb{I}_{\{|U_{nk}+V_{nk}|<\tau\}}\right]\right| = \left|E_{k-1}\left[V_{nk}\mathbb{I}_{\{|U_{nk}+V_{nk}|\ge\tau\}}\right]\right|, \quad \text{using } E_{k-1}[V_{nk}]=0.$$

Hence, using (147), we have $\sum\left|E_{k-1}\left[V_{nk}\mathbb{I}_{\{|U_{nk}+V_{nk}|<\tau\}}\right]\right| \xrightarrow{p} 0$. Hence, taking (148) into account further, (141) follows from (143).

It remains to obtain (142). For this purpose let

$$U^*_{nk} = U_{nk} - E_{k-1}\left[U_{nk}\mathbb{I}_{\{|U_{nk}+V_{nk}|<\varepsilon\}}\right], \qquad V^*_{nk} = V_{nk} - E_{k-1}\left[V_{nk}\mathbb{I}_{\{|U_{nk}+V_{nk}|<\varepsilon\}}\right].$$

Then the left hand side of (142) takes the form

$$\sum E_{k-1}\left[|U^*_{nk}+V^*_{nk}|^2\,\mathbb{I}_{\{|U_{nk}+V_{nk}|<\varepsilon\}}\right]. \tag{149}$$

Consider

$$\begin{aligned} &\sum E_{k-1}\left[|V^*_{nk}|^2\,\mathbb{I}_{\{|U_{nk}+V_{nk}|<\varepsilon\}}\right] \\ =\ &\sum E_{k-1}\left[|V_{nk}|^2\,\mathbb{I}_{\{|U_{nk}+V_{nk}|<\varepsilon\}}\right] - \sum\left(E_{k-1}\left[V_{nk}\mathbb{I}_{\{|U_{nk}+V_{nk}|<\varepsilon\}}\right]\right)^2, \end{aligned} \tag{150}$$

where, using (147), $\sum\left(E_{k-1}\left[V_{nk}\mathbb{I}_{\{|U_{nk}+V_{nk}|<\varepsilon\}}\right]\right)^2 = \sum\left(E_{k-1}\left[V_{nk}\mathbb{I}_{\{|U_{nk}+V_{nk}|\ge\varepsilon\}}\right]\right)^2 \xrightarrow{p} 0$ and using (147) again $\sum E_{k-1}\left[|V_{nk}|^2\,\mathbb{I}_{\{|U_{nk}+V_{nk}|\ge\varepsilon\}}\right] \xrightarrow{p} 0$. Thus

$$\sum E_{k-1}\left[|V^*_{nk}|^2\,\mathbb{I}_{\{|U_{nk}+V_{nk}|<\varepsilon\}}\right] - \sum E_{k-1}\left[|V_{nk}|^2\right] \xrightarrow{p} 0. \tag{151}$$

Next consider

$$\sum E_{k-1}\left[|U^*_{nk}|^2\,\mathbb{I}_{\{|U_{nk}+V_{nk}|<\varepsilon\}}\right] = \sum E_{k-1}\left[|U_{nk}|^2\,\mathbb{I}_{\{|U_{nk}+V_{nk}|<\varepsilon\}}\right] - \sum\left(E_{k-1}\left[U_{nk}\mathbb{I}_{\{|U_{nk}+V_{nk}|<\varepsilon\}}\right]\right)^2.$$

Here

$$\begin{aligned} &\sum\left|\left(E_{k-1}\left[U_{nk}\mathbb{I}_{\{|U_{nk}+V_{nk}|<\tau\}}\right]\right)^2 - \left(E_{k-1}\left[U_{nk}\mathbb{I}_{\{|U_{nk}|<\tau\}}\right]\right)^2\right| \\ \le\ &\max_k E_{k-1}\left[|U_{nk}|\left(\mathbb{I}_{\{|U_{nk}|<\tau\}} + \mathbb{I}_{\{|U_{nk}+V_{nk}|<\tau\}}\right)\right] \sum E_{k-1}\left[|U_{nk}|\left|\mathbb{I}_{\{|U_{nk}+V_{nk}|<\tau\}} - \mathbb{I}_{\{|U_{nk}|<\tau\}}\right|\right] \\ &\xrightarrow{p} 0. \end{aligned}$$

Here, $\xrightarrow{p} 0$ holds because of (148) and because $E_{k-1}\left[|U_{nk}|\,\mathbb{I}_{\{|U_{nk}+V_{nk}|<\tau\}}\right] \le E_{k-1}\left[|U_{nk}|\,\mathbb{I}_{\{|U_{nk}|<\tau\}}\right] + o_p(1)$ using (148), and $E_{k-1}\left[|U_{nk}|\,\mathbb{I}_{\{|U_{nk}|<\tau\}}\right] \le \tau$. Combining this with (148), we then see that the difference between $\sum E_{k-1}\left[|U^*_{nk}|^2\,\mathbb{I}_{\{|U_{nk}+V_{nk}|<\varepsilon\}}\right]$ and the left hand side of (143) converges to 0 in probability, and hence, by (143) again,

$$\sum E_{k-1}\left[|U^*_{nk}|^2\,\mathbb{I}_{\{|U_{nk}+V_{nk}|<\varepsilon\}}\right] \xrightarrow{p} 0 \quad \text{as } n\to\infty \text{ first and then } \varepsilon\to 0. \tag{152}$$

Now, the difference between (149) and (150) is bounded by

$$\begin{aligned} &\sum E_{k-1}\left[|U^*_{nk}|\,|U^*_{nk}+2V^*_{nk}|\,\mathbb{I}_{\{|U_{nk}+V_{nk}|<\varepsilon\}}\right] \\ \le\ & \sqrt{\sum E_{k-1}\left[|U^*_{nk}|^2\,\mathbb{I}_{\{|U_{nk}+V_{nk}|<\varepsilon\}}\right]}\sqrt{\sum E_{k-1}\left[|U^*_{nk}+2V^*_{nk}|^2\,\mathbb{I}_{\{|U_{nk}+V_{nk}|<\varepsilon\}}\right]} \xrightarrow{p} 0, \end{aligned}$$

using (152) and (151) together with (144). This means, in view of (151), the difference between (149) and $\sum E_{k-1}\left[|V_{nk}|^2\right]$ converges to 0 in probability. Thus in view of the second part of (144), (142) is verified.

Now consider **the case** $\alpha = 2$**.** In this case recall that $E[\xi_1]=0$ and $E[\xi_1^2]<\infty$, see (8). Therefore, taking $\delta_n \equiv \sqrt{\frac{2}{E[\xi_1^2]}}$ in (1), we have $\max_k E[\chi^2_{nk}] = E[\chi^2_{n1}] \to 0$ and

$$\sum_{k=[m_n s]+1}^{[m_n t]} E\left[\chi^2_{nk}\right] \to (t-s)\,2 \quad \text{for all } s<t.$$

Further, by (1) for $\alpha = 2$, $Z_n(t) \overset{fdd}{\Longrightarrow} (Z_2(t) - Z_2(-l))$, where recall that $Z_2(t)$ is a Brownian motion with variance 2. Therefore, because the Lindeberg condition is both sufficient and necessary for the preceding normal convergence, we have

$$\sum_{k=[m_n s]+1}^{[m_n t]} E\left[\chi^2_{nk}\mathbb{I}_{\{|\chi_{nk}|\ge\varepsilon\}}\right] \to 0 \quad \text{for all } s<t.$$

The above conditions (together with $E[\chi_{n1}]=0$) now replace (134) - (137). Taking into account (104) and (144) further, it then follows, as can be easily seen, that

$$\sum E_{k-1}\left[|U_{nk}+V_{nk}|^2\right] \xrightarrow{p} 2\sum_{j=1}^{q+r} u_j^2\,(t_j - t_{j-1}) + \sum_{j=q+1}^{q+r} v_j^2\,(t_j-t_{j-1})$$

and

$$\sum E_{k-1}\left[|U_{nk}+V_{nk}|^2\,\mathbb{I}_{\{|U_{nk}+V_{nk}|>\varepsilon\}}\right] \xrightarrow{p} 0 \text{ for all } \varepsilon>0.$$

Therefore, taking into account $E_{k-1}[U_{nk}+V_{nk}]=0$, (145) still holds, by applying a suitable version of Martingale CLT for the sum $\sum U_{nk}+V_{nk}$. This completes the proof of the Lemma 23.

See Jeganathan (2006a) more general statements related to Theorem A and the preceding convergence.

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