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THREE SPACE PROBLEM FOR SOME CLASSES OF L^1-PREDUALS

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Abstract. Following the well-known classification scheme of function spaces whose duals are isometric to $L^1(\mu)$, due to Lindenstrauss, Wulbert and Olsen ([4],[5]), in this paper we study the three space problem for them. We investigate conditions so that a Banach space E is in a specific class if for some M-ideal $M \subset E$, both M and $E|M$ are in that class of function spaces from the classification scheme.

1. Introduction

Let E be a complex Banach space such that E^* is isometric to $L^1(\mu)$ for some positive measure μ or equivalently an abstract L-space in the sense of Kakutani (see [3]). Such spaces are called L^1-preduals or Lindenstrauss spaces. Study of their structure and classification attracted a lot of attention during the 70'. Lindenstrauss and Wulbert ([4]) gave a classification scheme for characterizing several known classes of function spaces among the preduals of L^1. These results were extended to complex Banach spaces by Olsen ([5]). See the monograph [3] for more details. The classes of L^1-preduals that we will be considering here include,the $C(X)$ spaces, the G spaces due to Grothendieck, the C_σ, C_Σ spaces (to be defined later) and the space $A(K)$ of affine continuous functions on a compact Choquet simplex K.

For a Banach space E by $E_1, S(E), \partial_e E_1$ we denote the closed unit ball, the unit sphere and the set of extreme points of the unit ball respectively. We recall from [2] Chapter 1, that a closed subspace $M \subset E$ is said to be an M-ideal, if there exists a linear projection $P : E^* \to E^*$ such that $kerP = M^\perp$ and $\|e^*\| = \|P(e^*)\| + \|e^* - P(e^*)\|$ for all $e^* \in E^*$. In this case range of P is isometric to M^* so that $E^* = M^* \oplus_1 M^\perp$ (ℓ^1-direct sum).

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Also, $P(e^*)$ is the unique norm preserving extension of $e^*|M$. P is a unique projection with this property and we use this notation throughout the paper. If E is an L^1-predual then for any $e^* \in \partial_e E_1^*$, $\ker(e^*)$ is an M-ideal. More generally for any weak*-closed face $F \subset E_1^*$, $M = \{e \in E : e(F) = 0\}$ is an M-ideal (see [1]). It is well-known that being an L-space and hence being an L^1-predual is preserved by ranges of projections of norm one [5]. Thus if E is an L^1-predual and $M \subset E$ is an M-ideal, then as M^* and $M^\perp$ being ranges of projections of norm one in E^*, are abstract L-spaces, we get that both $M, E|M$ are L^1-preduals. Conversely if for an M-ideal, M of a Banach space E, if both $M, E|M$ are L^1-preduals, then as M^* and $M^\perp$ are abstract L-spaces and $E^* = M^* \oplus_1 M^\perp$, we have that E^* is an L-space and hence E is an L^1-predual. Thus it is interesting to ask if M and $E|M$ are in the same class of L^1-preduals from the classification scheme, does E belong to the same class? This turns out to be a hard problem to settle without some further assumptions.

We note that being an L^1-predual is not preserved by closed subspaces or quotients . Also by taking $E = \ell^\infty \oplus_1 \ell^\infty$, we see that a subspace and a quotient by it can be L^1-predual without the space being an L^1-predual. Thus it is more appropriate to consider the three space problems when the subspace is an M-ideal. We show that for various classes in the classification scheme (definitions given in Section 2), the three space property holds (some times under a stronger condition on M), in the sense that if for an M-ideal, $M \subset E$, if $M, E|M$ are in a specific class then does E.

We note that for the sequence spaces $c_0 \subset c$, we have that c_0 is an M-ideal but not a $C(K)$ space since its unit ball has no extreme points. However the G-spaces are closed under M-ideals and quotients by M-ideals.

We show that if $M, E|M$ are $C(X)$ or $A(K)$ spaces, then so is E. If M is a C_Σ space and $E|M$ is a C_σ space, then E is a C_σ-space.

2. Main result

Let $M \subset E$ be an M-ideal. It is easy to see that $\partial_e E_1^* = \partial_e M_1^* \cup (\partial_e M_1^\perp)$. Thus it is natural to use the characterizations of subclasses of L^1-preduals in terms of extreme points. These we now recall from [4] and [5]. We also formally do not define some of the function spaces considered in [4] and [5] but only mention the characterization in terms of extreme points. As some of the definitions are lengthy we refer the interested reader to [4] and [5]. See also page 83 in [2].

Let E be an L^1-predual space. We note that all the function spaces considered below are L^1-preduals. For a compact set X, $C(X)$ denotes the space of complex-valued continuous functions and for a compact convex set K, $A(K)$ denotes the space of complex-valued affine continuous functions on K, both the spaces equipped with the supremum norm.

1. E is isometric to a $C(X)$ space for a compact set X if and only if $\partial_e E_1 \neq \emptyset$ and $\partial_e E_1^*$ is weak*-closed.
2. E is isometric to a C_σ space if $\partial_e E_1^* \cup \{0\}$ is weak*-closed.
3. E is isometric to a C_Σ space if $\partial_e E_1^*$ is weak*-closed.
4. E is isometric to a $A(K)$ space for a compact Choquet simplex K if $\partial_e E_1 \neq \emptyset$.
5. E is isometric to a G-space if for any net $\{e_\alpha^*\} \subset \partial_e E_1^*$, $e_\alpha^* \to e^*$ in the weak*-topology implies that $e^* = 0$ or $\frac{e^*}{\|e^*\|} \in \partial_e E_1^*$.

We recall that $e \in E_1$ is a strong extreme point if $e_k \in E$, $\|e \pm e_k\| \to 1 \implies e_k \to 0$. It is easy to see that $1 \in C(K)$ or $A(K)$ is a strong extreme point of the unit ball.

Theorem 1. *Let E be a Banach space and let $M \subset E$ be an M-ideal. If $M, E|M$ are $C(X)$ or $A(K)$ spaces, then so is E. If M is a C_Σ space and $E|M$ is a C_σ space, then E is a C_σ-space.*

Proof. As already noted, the hypothesis implies that E is an L^1-predual.

Suppose $M, E|M$ are $C(K)$ spaces. Since M is an M-ideal with a strong extreme point in the unit ball, it follows from Proposition II.4.2 and Theorem II.4.4 in [2] that it is not a proper M-ideal, i.e., $E = M \oplus_\infty N$ (ℓ^∞ direct-sum) for some closed subspace N. Also as N is a $C(K)$ space, the conclusion follows.

In the case of $A(K)$ spaces, again as M is an M-ideal with a strong extreme point in the unit ball, we have $E = M \oplus_\infty N$ and since by hypothesis, N_1 also has an extreme point we get that $\partial_e E_1 \neq \emptyset$. Thus E is a $A(K)$ space for a compact Choquet simplex K.

Next suppose both M and $E|M$ are C_σ spaces. Let $\{e_\alpha^*\} \subset \partial_e E_1^*$ be a net such that $e_\alpha^* \to e^* \neq 0$ in the weak*-topology.

Suppose $e^* \notin M^\perp$. Then by weak* convergence, we get a β such that $\alpha \geq \beta$ implies $e_\beta^* \notin M^\perp$. As these are extreme points, we have that $\{e_\beta^*\}_{\beta \geq \alpha} \subset$

$\partial_e M_1^*$. Also this subnet converges to $e^* \neq 0$ in the weak*-topology of M^*. As M is a C_Σ space, we get that $e^* \in \partial_e M_1^* \subset \partial_e E_1^*$.

Now suppose $e^* \in M^\perp$ but not an extreme point of the unit ball. As the set $\partial_e M_1^\perp \cup \{0\}$ is weak*-closed, again there exists a α such that $e_\beta^* \notin \partial_e M_1^\perp$ for $\beta \geq \alpha$. Again as these are extreme points, $\{e_\beta^*\}_{\beta \geq \alpha} \subset \partial_e M_1^*$.

Now again since M is a C_Σ space, there is a further subnet, still denoted by e_α^* such that $e_\alpha^* \to m^*$ in the weak*-topology of M^*, for a $m^* \in \partial_e M_1^*$. Note that $m^* \in \partial_e X_1^*$ is the unique norm-preserving extension of $m^* \in \partial_e M_1^*$. Now a simple weak*-compactness argument, using the uniqueness of norm-preserving extensions gives that for the subnet , $e_\alpha^* \to m^*$ in the weak*-topology of X^*. Thus $e^* = m^*$. A contradiction. Hence $e^* \in \partial_e M^\perp \subset \partial_e E_1^*$.

Thus E is a C_σ space.

□

Remark 2. *For a M-ideal $M \subset E$, since $\partial_e (M^\perp)_1 \subset \partial_e E_1^*$, and $M^\perp$ is a weak*-closed set, it is easy to see that if E is any of C_σ, C_Σ, G spaces then so is $E|M$. If E is a $A(K)$ space, then it is known that $M = \{a \in A(K) : a(F) = 0\}$ for a closed face $F \subset K$. It is easy to see that $E|M$ is isometric to $A(F)$ and as any closed face of a Choquet simplex is a Choquet simplex, we have that $E|M$ is also in this class. As already noted in the introduction M need not be a $C(K)$ or a $A(K)$ space even if E is.*

We do not know if the above theorem is valid for G-spaces. Our argument crucially depended on the fact that a non-zero weak*-limit is a unit vector.

We next note that G-spaces are closed under M-ideals by using the function space definition of a G-space and the description of M-ideals.

Proposition 3. *Let E be a G-space, then for any M-ideal, $M \subset E$, M is a G-space.*

Proof. We write the proof for the real scalar field. By the function space characterization of a G-space, there is a compact set K and a collection of points $\{s_i, t_i\}_{i \in I} \subset K$, $\{\lambda_i\}_{i \in I} \subset R$, such that $E = \{x \in C(K) : x(s_i) = \lambda_i x(t_i) \ for\ all\ i \in I\}$.

Since M is an M-ideal, by Proposition II.5.2 in [2], there is a closed set $D \subset K$ such that $M = \{x \in E : x(D) = 0\}$. Thus M has a function space description similar to E with the index set enlarged by takin points of D and $\lambda = 0$. Thus M is a G-space.

□

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