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Central extensions of p -adic algebraic groups by finite p -groups

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Abstract

Some problems on algebraic groups over global fields like the congruence subgroup problem involve the determination of topological central extensions of the adelic group which, in turn, leads naturally to the study of topological central extensions of p -adic Lie groups by finite groups like the group of roots of unity in the p -adic field. Moreover, central extensions of semisimple p -adic Lie groups often come from a subgroup of small rank like SL_2 which has the interesting property that abstract central extensions of the locally compact group $SL_2(k)$ for a p -adic field k , by a finite group turn out to be automatically topological. Let k be a nonarchimedean local field and let A be a finite, abelian group. We use the corresponding Tits building to prove that, for a semisimple, simply connected k -isotropic algebraic group G , the group $H^2(G(k), A)$ of abstract central extensions of $G(k)$ by any group A maps injectively under restriction maps, into a direct sum of $H^2(H(k), A)$ over k -rank 1 subgroups H when the group A is a finite, abelian p -group, where p is the characteristic of the residue field of k .

The aim of this paper is to prove the following theorem.

Theorem.

Let k be a nonarchimedean local field and A a finite, abelian p -group, where p is the characteristic of the residue field of k . Let G be an absolutely almost simple, simply-connected algebraic group defined over k with k -rank(G) = $r \geq 2$. Then there exist semisimple k -subgroups G_1, \dots, G_r without k -anisotropic factors and, each of k -rank equal to k -rank(G) - 1 and semisimple k -subgroups G_{ij} of $G_i \cap G_j$ such that the 'restriction' map

$$H^2(G(k), A) \rightarrow \bigoplus_{i=1}^r H^2(G_i(k), A)$$

of abstract central extensions is injective, and injects into

$$\text{Ker}\left(\bigoplus_{i \leq r} H^2(G_i(k), A) \rightarrow \bigoplus_{i < j} H^2(G_{ij}(k), A)\right).$$

Remarks.

If $G = SL_{n,D}$ for a k -central division algebra with $n \geq 3$, the theorem produces G_1, \dots, G_{n-1} each of which is k -isomorphic to $SL_{n-1,D}$.

1 Modus operandi - using the Tits building

In proving the theorem, we adapt the technique from [PR] - especially from chapter 4 there. Basically, the method we use to compute $H^2(G(k), A)$ is by using a natural resolution of the $G(k)$ -module A provided by the Tits building associated to G over k . In [PR], the Bruhat-Tits building is used to compute topological central extensions. We crucially use a result due to Solomon & Tits asserting that the Tits building of G over k has the homotopy type of a bouquet of spheres, each of dimension $r - 1$. In contrast, the Bruhat-Tits building is contractible.

In order to define the Tits building of G over k and to recall basic properties we need from [BS], let us recall the information provided by the Borel-Tits structure theory. Let k, G, r, A be as above and we consider the trivial action of $G(k)$ on A . Let S be a maximal k -split torus of G and $T \supset S$, a maximal k -torus of G . Let Δ denote the corresponding set of simple k -roots and Φ^+ be the positive k -roots. For a subset Θ of Δ , write $S^\Theta = (\bigcap_{\theta \in \Theta} \text{Ker } \theta)^\circ$ and $M^\Theta = C_G(S^\Theta)$. The latter is a connected reductive k -group in which S^Θ is the maximal k -split central torus. Note that $[M^\Theta, M^\Theta]$ is a semisimple, simply-connected k -group. Further, if $\mathcal{U}_\Theta = \sum_{\alpha \in \Phi^+ - \langle \Theta \rangle} \mathfrak{g}_\alpha$, and if U_Θ is the corresponding connected unipotent group normalized by T , then Observe that M^Θ normalizes U_Θ . Also, $P^\Theta := M^\Theta U_\Theta$ is a parabolic k -subgroup of G with U_Θ as its unipotent radical. Note that $\Theta = \emptyset$ corresponds to a minimal parabolic k -subgroup and the above correspondence between the set of subsets of Δ and the set of k -parabolics containing P^\emptyset is bijective and inclusion-preserving.

In what follows, $H^n(G(k), A)$ stands for the abstract group cohomology for the trivial action. We recall the definition and properties of the Tits building of G over k following Borel-Serre ([BS]). This is a simplicial complex of dimension $r - 1$ (where $r = k\text{-rank}(G)$) whose vertices are maximal parabolic k -subgroups. A set $\{P_1, \dots, P_d\}$ of vertices forms a simplex if and only if the intersection $\bigcap_{i=1}^d P_i$ is a parabolic k -subgroup - this parabolic is precisely the stabilizer of the simplex. $G(k)$ acts on parabolic k -subgroups by con-

jugation. As recalled above, once a simple system Δ of k -roots has been fixed (equivalently, an r -simplex of the Tits complex is fixed) the parabolic k -subgroups can be defined in terms of the subsets of this set Δ . In order to parametrize the simplices by parabolic k -subgroups, it is more convenient to work with $P_\Theta = P^{\Delta-\Theta}$ and $M_\Theta := M^{\Delta-\Theta}$, $U^\Theta := U_{\Delta-\Theta}$. Then, the set of s -dimensional simplices in the Tits building is $G(k)$ -equivariantly parametrized by $\bigcup_{|\Theta|=s+1} G(k)/P_\Theta(k)$. Since $G(k)$ acts simplicially on the Tits building, we have a complex of $G(k)$ -modules

$$0 \rightarrow A \rightarrow C^0(A) \rightarrow C^1(A) \rightarrow \dots \rightarrow C^{r-1}(A) \rightarrow 0$$

where $C^i(A)$ is the group of simplicial i -cochains of the Tits building, with coefficients in A . Therefore, $C^i(A) = \bigoplus_{|\Theta|=i+1} \text{Ind}_{P_\Theta(k)}^{G(k)}(A)$ as a $G(k)$ -module. Here, $\text{Ind}_{P_\Theta(k)}^{G(k)}(A)$ stands for the $G(k)$ -module induced by the trivial action of $P_\Theta(k)$ on A . By a theorem of Solomon and Tits (see [G], Appendix II), the Tits building of G over k is of the homotopy type of a bouquet of spheres, each of dimension $r - 1$. Indeed, this was proved in [G] for any Tits system with finite Weyl group; when the Weyl group of a Tits system is infinite, the corresponding Tits building is even contractible. Thus, the Bruhat-Tits building of G over k (which we have not defined as we don't need it here) is contractible whereas the Tits building of G over k is not. Therefore, the above $G(k)$ -complex is exact except at the last stage. In fact,

$$0 \rightarrow A \rightarrow C^0(A) \cdots \rightarrow C^{r-1}(A) \rightarrow St(A) \rightarrow 0$$

is exact, where $St(A)$ is called the Steinberg module with values in A ([BS], 5.10). The associated spectral sequence which computes $H^*(G(k), A)$ has its $E_2^{i,j}$ -term to be the i -th cohomology of the complex

$$0 \rightarrow H^j(G(k), C^0(A)) \rightarrow H^j(G(k), C^1(A)) \rightarrow \dots \rightarrow 0.$$

Using Shapiro's lemma, this is just the complex

$$0 \rightarrow \bigoplus_{|\Theta|=1} H^j(P_\Theta(k), A) \cdots \rightarrow H^j(P_\Delta(k), A) \rightarrow H^j(G(k), St(A)) \rightarrow 0.$$

2 Key step : Reduction to parabolic k -subgroups.

The proof of the theorem proceeds in steps - each step reducing the computation of the relevant cohomology groups to a computation for subgroups

of a particular kind like parabolic subgroups, then their Levi subgroups and finally to groups of smaller k -ranks. The crucial step is really the following proposition.

Proposition 1.

We claim that $E_\infty^{1,1} = 0 = E_\infty^{2,0}$. Further,

$$H^2(G(k), A) \hookrightarrow \text{Ker}\left(\bigoplus_{|\Theta|=1} H^2(P_\Theta(k), A) \rightarrow \bigoplus_{|\Theta|=2} H^2(P_\Theta(k), A)\right)$$

Proof.

Look at the complex whose i -th cohomology for $i > 0$ computes the $E_2^{i,0}$ -th term of the spectral sequence is

$$0 \rightarrow A \rightarrow \bigoplus_{|\Theta|=1} A \rightarrow \cdots \rightarrow \bigoplus_{|\Theta|=r} A \rightarrow \text{St}(A)^{G(k)} \rightarrow 0.$$

As the part

$$0 \rightarrow A \rightarrow \bigoplus_{|\Theta|=1} A \rightarrow \cdots \rightarrow \bigoplus_{|\Theta|=r} A \rightarrow 0$$

is simply the augmented cochain complex of an $(r - 1)$ -simplex, it is exact. Further, using the description of $\text{St}(A)$ in $[BS]$, it can be shown that $\text{St}(A)^{G(k)} = 0$. Therefore, we get that the i -th cohomology $E_2^{i,0} = 0$ for all $i > 0$. In particular, we have $E_2^{2,0} = 0$. Thus,

$$E_\infty^{2,0} = E_3^{2,0} = E_2^{2,0} / \text{Im}(E_2^{0,1} \rightarrow E_2^{2,0}) = 0.$$

Towards proving $E_\infty^{1,1} = 0$, we need some information about $E_2^{1,1}$. As this involves $H^1(P(k), A)$ for parabolic k -subgroups P , we start by observing a special property of parabolic k -subgroups in our case when A is a p -group.

Claim 1.

For any parabolic k -subgroup $P = P_\Theta$, consider the Levi subgroup $M = M_\Theta$ as above and let $D = [D, D]$. Then, $D(k)/[D(k), D(k)]$ is a finite abelian group of order prime to p and $\text{Hom}(P(k), A) \cong \text{Hom}((P/[P, P])(k), A)$.

Proof.

We will use here the assumption that G is simply-connected. Look at the decomposition $P = M \ltimes U$ where U is the unipotent radical. We know that M, U are defined over k . Moreover, $D = [M, M]$ is a semisimple, simply-connected k -group.

We shall first show that $P(k) \rightarrow (P/[P, P])(k)$ is a surjection. In other words, looking at the Galois cohomology sequence

$$1 \rightarrow [P, P](k) \rightarrow P(k) \rightarrow (P/[P, P])(k) \rightarrow H^1(k, [P, P])$$

corresponding to

$$1 \rightarrow [P, P] \rightarrow P \rightarrow P/[P, P] \rightarrow 1,$$

we shall show that $H^1(k, [P, P]) = 0$.

Since $P = M \cdot U$, we have $[P, P] = [M, M] \cdot U$.

Now $P \rightarrow P/[P, P]$ has the kernel $[P, P] = [M, M]U$. We crucially use the fact that $D = [M, M]$ is a simply-connected group. Firstly, a result due to Kneser (and due to Bruhat & Tits when $\text{char } k > 0$) on Galois cohomology implies that $H^1(k, D) = 0$. Moreover, $H^1(k, U) = 0$, as U is a k -split unipotent group. So, we obtain $H^1(k, D \cdot U) = 0$; that is, $H^1(k, [P, P]) = 0$. Thus, we have shown that $P(k) \rightarrow (P/[P, P])(k)$ is a surjection.

On the other hand, the kernel of the above map $P(k) \rightarrow (P/[P, P])(k)$ is

$$[P, P](k) = [M, M](k) \cdot U(k) \supseteq [P(k), P(k)].$$

Now, we use the validity of the Kneser-Tits conjecture for the simply-connected, semisimple k -isotropic group $D = [M, M]$.

Note that - being simply-connected - this group is a direct product of its k -simple factors and each k -isotropic k -simple factor H satisfies $H(k) = [H(k), H(k)]$ by the validity of the Kneser-Tits conjecture ([M], 2.3.2(b)). For a k -anisotropic factor J , we have $J(k)/[J(k), J(k)]$ to be a subquotient of F^* where F is the residue field of k . Therefore, in particular, $D(k)/[D(k), D(k)]$ is a finite, Abelian group of order prime to p .

Finally, since $[P(k), P(k)]$ contains $U(k)$ as well as $[D(k), D(k)]$, we have that $[P, P](k)/[P(k), P(k)]$ is a quotient of $D(k)/[D(k), D(k)]$.

Hence $[P, P](k)/[P(k), P(k)]$ is also a finite, Abelian group of order prime to p . The latter group being the kernel of $P(k)/[P(k), P(k)] \rightarrow (P/[P, P])(k)$ and, with A having only p -power torsion, we get

$$\text{Hom}((P/[P, P])(k), A) \cong \text{Hom}(P(k)/[P(k), P(k)], A) = \text{Hom}(P(k), A).$$

This proves the claim.

Remarks.

(a) Note that the claim above is analogous (but dual !) to the situation in chapter 4 of [PR]. There, the groups are over the residue field of k and have p -power order while the coefficients are considered with prime-to- p torsion.

(b) In view of this claim, it will be convenient to use the following notations in the proof of the proposition. For any set Θ of simple k -roots, let $\overline{P_\Theta}$ denote the abelian group $(P_\Theta)_{ab}(k)$. For any $\alpha \in \Delta$, if we let P_α^* denote the

abelian group $\text{Hom}((P_\alpha)_{ab}(k), A) = \text{Hom}(P_\alpha(k), A)$. Then, for any set Θ of simple k -roots, we may identify $\overline{P_\Theta}$ with $\prod_{\alpha \in \Theta} \overline{P_\alpha}$ and $\text{Hom}(P_\Theta(k), A)$ with the direct product $\prod_{\alpha \in \Theta} P_\alpha^*$ in view of the following result which is similar to [PR], 4.6.:

Claim 2.

For each set Θ of simple k -roots, the map $\overline{P_\Theta} \rightarrow \prod_{\alpha \in \Theta} \overline{P_\alpha}$ is an isomorphism. More generally, for two disjoint subsets Θ, Θ' of Δ , the map

$$P_{\Theta \cup \Theta'} / [P_{\Theta \cup \Theta'}, P_{\Theta \cup \Theta'}] \rightarrow P_\Theta / [P_\Theta, P_\Theta] \times P_{\Theta'} / [P_{\Theta'}, P_{\Theta'}]$$

is a k -isomorphism of k -algebraic groups.

Proof.

Since the latter map is defined over k , it suffices to prove that it is an isomorphism over \bar{k} . The idea of the proof is to produce tori R_Θ (defined over \bar{k}) inside the k -parabolics P_Θ which are evidently seen to have the asserted isomorphism property and, which map isomorphically onto the abelianization under the natural map $P_\Theta \rightarrow P_\Theta / [P_\Theta, P_\Theta]$. We shall consider the various corresponding subgroups over \bar{k} which we did over k earlier. Let T be a maximal k -torus of G containing S and let Δ_T denote the set of simple roots with respect to T . For a subset $\Theta \subseteq \Delta_T$, let us write $T^\Theta = (\bigcap_{\alpha \in \Theta} \text{Ker}(\alpha))^0$, and $N_\Theta = C_G(T^\Theta)$. Now, the torus $T_\Theta := (T \cap [N_\Theta, N_\Theta])^0$ has dimension equal to $|\Theta|$ and moreover, (as G is simply-connected) is a direct product of all T_α 's for $\alpha \in \Theta$. In particular, T itself is isomorphic to the direct product of all T_α as α runs over Δ_T . Now, if $\alpha \in \Delta$ (that is, it is a simple k -root), look at the set of all $\tilde{\alpha} \in \Delta_T$ such that $\tilde{\alpha}|_S = \alpha$. Let R_α be the subtorus of T generated by all such $T_{\tilde{\alpha}}$'s. More generally, for any set Θ of simple k -roots, we have a subtorus R_Θ of T and evidently $R_{\Theta \cup \Theta'} \cong R_\Theta \times R_{\Theta'}$. Returning to our k -parabolic subgroups P_Θ 's, we note that R_Θ is a maximal torus of M_Θ which intersects $[M_\Theta, M_\Theta]$ only trivially (and hence also $[P_\Theta, P_\Theta]$ only trivially). So, the quotient homomorphism from P_Θ to its abelianization is injective on R_Θ and maps surjectively onto $P_\Theta / [P_\Theta, P_\Theta]$. The claim follows.

In view of the lemma, it is meaningful to write P_Θ^ for*

$$\prod_{\alpha \in \Theta} P_\alpha^* = \prod_{\alpha \in \Theta} \text{Hom}((P_\alpha)_{ab}(k), A).$$

Completion of proof of proposition 1.

We are trying to prove here that $E_\infty^{1,1} = 0 = E_\infty^{2,0}$ and further, that

$$H^2(G(k), A) \hookrightarrow \text{Ker} \left(\bigoplus_{|\Theta|=1} H^2(P_\Theta(k), A) \rightarrow \bigoplus_{|\Theta|=2} H^2(P_\Theta(k), A) \right)$$

holds. We have already shown that $E_\infty^{2,0} = 0$. Now, $E_2^{i,1} = i$ -th homology of the complex

$$0 \rightarrow \bigoplus_{|\Theta|=1} H^1(P_\Theta(k), A) \rightarrow \cdots \rightarrow \bigoplus_{|\Theta|=r} H^1(P_\Theta(k), A) \rightarrow H^1(G(k), St(A)) \rightarrow 0$$

which is

$$0 \rightarrow \bigoplus_{|\Theta|=1} P_\Theta^* \rightarrow \cdots \rightarrow \bigoplus_{|\Theta|=r} P_\Theta^* \rightarrow H^1(G(k), St(A)) \rightarrow 0.$$

By the two claims, we have $P_\Theta^* = \prod_{\alpha \in \Theta} P_\alpha^*$.

We write this complex B as the direct sum of complexes $B_\alpha, \alpha \in \Delta$ as follows. Consider the $(r-1)$ -simplex whose vertices are the elements of Δ and the coefficients are considered in the abelian group P_α^* . Then, excepting the last term, we see that the above complex B is just the direct sum of the relative cochain complexes B_α of this $(r-1)$ -simplex relative to the $(r-2)$ -dimensional face gotten by throwing out the vertex α . Hence, each B_α is exact and, so is B except at the last term. So, we have $E_2^{i,1} = 0$ for all $i \leq r-1$. In particular,

$$E_\infty^{1,1} = E_3^{1,1} = \text{Ker}(E_2^{1,1} \rightarrow E_2^{3,0}) \subseteq E_2^{1,1} = 0.$$

Using also the fact that $E_\infty^{2,0} = 0$, we will have

$$E_\infty^{0,2} = E_4^{0,2} \subseteq E_3^{0,2} \subseteq E_2^{0,2}.$$

That is,

$$H^2(G(k), A) \hookrightarrow \text{Ker} \left(\bigoplus_{|\Theta|=1} H^2(P_\Theta(k), A) \rightarrow \bigoplus_{|\Theta|=2} H^2(P_\Theta(k), A) \right).$$

Thus, the proposition is proved.

3 Reduction to Levi parts of parabolic subgroups

Proposition 2.

$$H^2(G(k), A) \hookrightarrow \text{Ker} \left(\bigoplus_{|\Theta|=1} H^2(M_\Theta(k), A) \rightarrow \bigoplus_{|\Theta|=2} H^2(M_\Theta(k), A) \right).$$

Proof.

We first claim that for any parabolic k -subgroup $P = M \cdot U$, the restriction map gives an isomorphism.

$$H^2(P(k), A) \cong H^2(M(k), A).$$

We use the Hochschild-Serre spectral sequence for $U \leq P$. We claim that

$$H^i(M, H^{2-i}(U, A)) = 0 \quad ; \quad i = 0, 1.$$

Now the central torus of $M (= M_\Theta$ say) acts non-trivially on $U (= U_\Theta)$ as S^Θ has no fixed points on U^Θ ; that is, there exists t in the central torus of M such that $t-1$ acts as an automorphism on $\text{Hom}(U, A)$. By ([R], P.121), for any group Γ and a Γ -module V , if there exists $t \in Z(\Gamma)$ so that $t-1$ acts as automorphisms on V , then $H^i(\Gamma, V) = 0$ for all i . Hence, we conclude in our case that $H^1(M, H^1(U, A)) = 0$. Thus, to complete the proof of the proposition, we are left to showing that $H^2(U, A)^M = 0$.

If $U = U_\Theta$, then we shall apply induction using Hochschild-Serre to the connected unipotent groups corresponding to the eigenspaces $\mathfrak{g}_\alpha; \alpha \in \Phi^+ - \langle \Theta \rangle$. It suffices to show $H^2(U_1, A)^M = 0$ if U_1 is 1-dimensional. But, if U_1 corresponds to the root α , the central torus of M acts on $H^2(U_1, A) = \text{Ext}^2(U_1, A)$ through the character 2α . Since we can choose t in the central torus of M such that $(2\alpha)(t) \neq 1$, it follows that $H^2(U_1, A)^M = \{0\}$. To recapitulate, we have shown

$$H^i(M, H^{2-i}(U, A)) = 0 \quad ; \quad i = 0, 1.$$

Therefore, we have shown

$$H^2(G(k), A) \hookrightarrow \text{Ker} \left(\bigoplus_{|\Theta|=1} H^2(M_\Theta(k), A) \rightarrow \bigoplus_{|\Theta|=2} H^2(M_\Theta(k), A) \right).$$

4 Reduction to semisimple subgroups of less rank

Proposition 3.

For $\emptyset \neq \Theta \subseteq \Delta$, let $D_\Theta = [M_\Theta, M_\Theta]$. Then, D_Θ 's are semisimple, simply-connected k -groups having k -ranks lower than that of G and satisfy

$$H^2(G(k), A) \hookrightarrow \text{Ker} \left(\bigoplus_{|\Theta|=1} H^2(D_\Theta(k), A) \rightarrow \bigoplus_{|\Theta|=2} H^2(D_\Theta(k), A) \right).$$

Proof.

We shall again use the Hochschild-Serre spectral sequence - this time corresponding to $D_\Theta \leq M_\Theta$. Recall that we have a canonical isomorphism $\overline{P_\Theta} \cong \bigoplus_{\alpha \in \Theta} (P_\alpha/[P_\alpha, P_\alpha])(k)$. Further, they are isomorphic with $M_\Theta(k)/D_\Theta(k)$ - we have again used the vanishing of the Galois cohomology $H^1(k, D_\Theta)$. Since $H^1(D_\Theta(k), A) = 0$, we have an exact sequence coming from the inflation and restriction maps :

$$0 \rightarrow H^2(\overline{P_\Theta}, A) \rightarrow H^2(M_\Theta(k), A) \rightarrow H^2(D_\Theta(k), A).$$

Consider the following commutative diagram :

$$\begin{array}{ccccccc} 0 & \rightarrow & \bigoplus_{|\Theta|=1} H^2(\overline{P_\Theta}, A) & \rightarrow & \bigoplus_{|\Theta|=1} H^2(M_\Theta(k), A) & \rightarrow & \bigoplus_{|\Theta|=1} H^2(D_\Theta(k), A) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & \bigoplus_{|\Theta|=2} H^2(\overline{P_\Theta}, A) & \rightarrow & \bigoplus_{|\Theta|=2} H^2(M_\Theta(k), A) & \rightarrow & \bigoplus_{|\Theta|=2} H^2(D_\Theta(k), A) \end{array}$$

When $\Theta = \{\alpha, \beta\}$, the natural maps $(M_\Theta/D_\Theta)(k) \rightarrow (M_\alpha/D_\alpha)(k)$ and $(M_\Theta/D_\Theta)(k) \rightarrow (M_\beta/D_\beta)(k)$ are simply the projections of $\overline{P_\Theta}$ to $\overline{P_\alpha}$ and $\overline{P_\beta}$ respectively. So, the left-most vertical arrow in the diagram is injective. Therefore, it follows from the inclusion

$$H^2(G(k), A) \hookrightarrow \text{Ker} \left(\bigoplus_{|\Theta|=1} H^2(M_\Theta(k), A) \rightarrow \bigoplus_{|\Theta|=2} H^2(M_\Theta(k), A) \right)$$

of proposition 2, that we have

$$H^2(G(k), A) \hookrightarrow \text{Ker} \left(\bigoplus_{|\Theta|=1} H^2(D_\Theta(k), A) \rightarrow \bigoplus_{|\Theta|=2} H^2(D_\Theta(k), A) \right).$$

The proposition is proved.

5 Reduction to subgroups without k -anisotropic factors

Finally, we want to reduce the above injectivity to one where D_Θ 's are replaced by connected normal k -subgroups which have no k -anisotropic factors. Now, since D_Θ is semisimple and simply connected, it is a direct product of its connected k -simple normal subgroups; in particular, D_Θ is a direct product of its maximal connected normal k -anisotropic subgroup H_Θ and a connected normal k -subgroup G_Θ which has no k -anisotropic factors. If $\Theta \subseteq \Theta' \subseteq \Delta$, then H_Θ is a direct factor of $H_{\Theta'}$. Indeed, recall that the Tits index of H_Θ over k is gotten from that of G by removing all orbits

corresponding to Θ and the edges containing them. Hence $G_{\Theta'} \leq G_{\Theta}$ for $\Theta \subseteq \Theta'$. Now, if Θ_1, Θ_2 are disjoint subsets of Δ , then

$$H_{\Theta_1} \leq G_{\Theta_2} ; H_{\Theta_2} \leq G_{\Theta_1}.$$

So, the subgroups $H_{\Theta_1}, H_{\Theta_2}$ commute element-wise and intersect trivially if $\Theta_1 \cap \Theta_2 = \emptyset$. Thus, $H_{\alpha}H_{\beta}$ is a direct factor of $H_{\{\alpha, \beta\}}$ for all $\alpha, \beta \in \Delta$. Now, $D_{\Theta}(k)$ is the direct product $H_{\Theta}(k) \cdot G_{\Theta}(k)$ for each $\Theta \subseteq \Delta$.

If $\Theta \subseteq \Theta' \subseteq \Delta$, then consider the homomorphism

$$D_{\Theta'}(k)/G_{\Theta'}(k) \rightarrow D_{\Theta}(k)/G_{\Theta}(k)$$

induced by the inclusions $D_{\Theta'} \subseteq D_{\Theta}, G_{\Theta'} \subseteq G_{\Theta}$. It is just the natural projection of $H_{\Theta'}(k)$ onto the factor $H_{\Theta}(k)$, when we identify $D_{\Theta}(k)/G_{\Theta}(k)$ with $H_{\Theta}(k)$. Moreover, $H^1(G_{\Theta}(k), A) = 0 = H^1(D_{\Theta}(k)/G_{\Theta}(k), A)$ since $D_{\Theta}(k)/G_{\Theta}(k) \cong H_{\Theta}(k)$ and, since (as we observed in the course of the proof of claim 1) the abelianization is finite, or order prime to p .

Look at the commutative diagram

$$\begin{array}{ccccccc} 0 \rightarrow & \bigoplus_{|\Theta|=1} H^2(H_{\Theta}(k), A) & \rightarrow & \bigoplus_{|\Theta|=1} H^2(D_{\Theta}(k), A) & \rightarrow & \bigoplus_{|\Theta|=1} H^2(G_{\Theta}(k), A) \\ & \downarrow & & \downarrow & & \downarrow \\ 0 \rightarrow & \bigoplus_{|\Theta|=2} H^2(H_{\Theta}(k), A) & \rightarrow & \bigoplus_{|\Theta|=2} H^2(D_{\Theta}(k), A) & \rightarrow & \bigoplus_{|\Theta|=2} H^2(G_{\Theta}(k), A) \end{array}$$

Since $H_{\alpha}H_{\beta}$ is a direct factor of $H_{\alpha, \beta}$, we have the injectivity of the left-most vertical arrow; so, the kernel of the middle vertical arrow injects into the kernel of the right-most vertical arrow; that is,

$$H^2(G(k), A) \hookrightarrow \text{Ker}\left(\bigoplus_{\alpha \in \Delta} H^2(G_{\alpha}(k), A) \rightarrow \bigoplus_{|\Theta|=2} H^2(G_{\Theta}(k), A)\right).$$

Thus, we have proved the theorem.

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