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# Distal actions and shifted convolution property

C. R. E. RAJA AND R. SHAH

Indian Statistical Institute, Bangalore Centre  
8th Mile Mysore Road, Bangalore, 560059 India



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C. R. E. Raja and R. Shah

To Professor S. G. Dani on his 60th birthday

## Abstract

A locally compact group  $G$  is said to have shifted convolution property (abbr. as SCP) if for every regular Borel probability measure  $\mu$  on  $G$ , either  $\sup_{x \in G} \mu^n(Cx) \rightarrow 0$  for all compact subsets  $C$  of  $G$ , or there exist  $x \in G$  and a compact subgroup  $K$  normalised by  $x$  such that  $\mu^n x^{-n} \rightarrow \omega_K$ , the Haar measure on  $K$ . We first consider distality of factor actions of distal actions. It is shown that this holds in particular for factors under compact groups invariant under the action and for factors under the connected component of identity. We then characterize groups having SCP in terms of a readily verifiable condition on the conjugation action (point-wise distality). This has some interesting corollaries to distality of certain actions and Choquet Deny measures which actually motivated SCP and point-wise distal groups. We also relate distality of actions on groups to that of the extensions on the space of probability measures.

## 1 Introduction

Let  $X$  be a Hausdorff space and  $\Gamma$  be a (topological) semigroup acting continuously on  $X$  by continuous self-maps. We say that the action of  $\Gamma$  on  $X$  is *distal* if for any two distinct points  $x, y \in X$ , the closure of  $\{(\alpha(x), \alpha(y)) \mid \alpha \in \Gamma\}$  does not intersect the diagonal  $\{(a, a) \mid a \in X\}$  and we say that the action of  $\Gamma$  on  $X$  is *point-wise distal* if for each  $\gamma \in \Gamma$ ,  $\{\gamma^n\}_{n \in \mathbb{N}}$ -action on  $X$  is distal. The notion of distality was introduced by Hilbert (cf. [15]) and studied by many in different contexts: see [15], [17] and [24] and the references cited therein.

Let  $G$  be a locally compact (Hausdorff) group and let  $e$  denote the identity of  $G$ . Let  $\Gamma$  be a semigroup acting continuously on  $G$  by endomorphisms. Then  $\Gamma$ -action on  $G$  is distal if and only if  $e \notin \overline{\Gamma x}$  for all  $x \in G \setminus \{e\}$ . The group  $G$  itself is said to be *distal* (resp. *point-wise distal*) if conjugacy action of  $G$  on  $G$  is distal (resp. point-wise distal).

It can easily be seen that the class of distal groups is closed under compact extensions. Abelian groups, discrete groups and compact groups are obviously distal. Nilpotent groups, connected Lie groups of type  $R$  and connected groups of polynomial growth are distal (cf. [35]) and p-adic Lie groups of type  $R$  and p-adic Lie groups of polynomial growth are point-wise distal (cf. [30] and [33]): point-wise distal groups are called non-contracting in [30] and [35].

Clearly, distal groups are point-wise distal but there are point-wise distal groups which are not distal (see [24] and [35] for instance).

For a locally compact group  $G$ , let  $P(G)$  denote the space of all regular Borel probability measures on  $G$  with weak\* topology. For  $\mu, \lambda \in P(G)$ , let  $\mu\lambda$  denote the convolution of  $\mu$  and  $\lambda$  and for  $x \in G$ , let  $\delta_x$  denote the Dirac measure at  $x$  and  $x\mu$  (resp.  $\mu x$ ) denote the measure  $\delta_x\mu$  (resp.  $\mu\delta_x$ ). For  $n \geq 1$  and  $\mu \in P(G)$ , let  $\mu^n$  denote the  $n$ -th convolution power of  $\mu$ . For any compact subgroup  $K$  of  $G$ , let  $\omega_K$  denote the normalized Haar measure on  $K$ . For  $\mu \in P(G)$ , let  $\check{\mu} \in P(G)$  be the measure defined by  $\check{\mu}(E) = \mu(\{x^{-1} \mid x \in E\})$  for any Borel set  $E$  of  $G$ .

Let  $G$  be a locally compact group and  $\text{Aut}(G)$  denote the group of all bi-continuous automorphisms of  $G$ . If  $\Gamma$  is a group acting continuously on  $G$  (by automorphisms), then this action extends to an action on  $P(G)$  which is given by  $\alpha(\mu)(E) = \mu(\alpha^{-1}(E))$  for any  $\alpha \in \Gamma$ , any  $\mu \in P(G)$  and for any measurable subset  $E$  of  $G$ .

We say that a locally compact group  $G$  has *shifted convolution property* which would be called SCP if for any  $\mu \in P(G)$ , one of the following holds:

- (i) the concentration functions of  $\mu$ ,  $\sup_{g \in G} \mu^n(Kg)$  converges to zero (as  $n \rightarrow \infty$ ) for any compact subset  $K$  of  $G$  (in which case we say that  $\mu$  is *dissipating*);
- (ii) there exists a  $x \in G$  such that  $\mu^n x^{-n} \rightarrow \omega_K$  for some compact subgroup  $K$  of  $G$  with  $xKx^{-1} = K$ .

If a measure  $\mu \in P(G)$  satisfies one of the above two conditions, we say that  $\mu$  has *shifted convolution property* (SCP).

Tortrat [39] proved SCP for groups satisfying certain conditions. Motivated by [39], Eisele [13] introduced a class of groups called Tortrat groups and showed that Tortrat groups have shifted convolution property: a locally compact group  $G$  is called *Tortrat* if for any sequence  $\{x_n\}$  in  $G$  and  $\mu \in P(G)$ , the sequence  $\{x_n \mu x_n^{-1}\}$  has idempotent limit point only if  $\mu$  is an idempotent. Corollary 5.1 and Theorem 5.2 of [13] showed that the class of Tortrat groups strictly contains the class of groups satisfying conditions of [39]. Dani and Raja [9] proved that almost connected groups of polynomial growth, equivalently, almost connected (point-wise) distal groups are Tortrat and Raja [31] showed that distal linear groups are Tortrat. Thus, proving SCP for almost connected distal groups and distal linear groups. It may be noted that Tortrat groups are distal, but the converse is not true in general as we shall see in Example 7.2. Here we show that a locally compact group is point-wise distal if and only if it has SCP. Our proof/techniques rely on dynamics on compact groups, on zero-dimensional groups and on Lie groups over local fields. It may be noted that [2]-[4], [14], [22] and [23] have results on SCP for some classes of groups and measures.

In Section 2, we state and prove some preliminary results which will be used often. In Section 3, we prove some results about factor-actions of distal actions, where the factors are either modulo compact subgroups or the connected component of the identity. In Section 4, we prove the main result for totally disconnected (metrizable) groups. In Section 5, we discuss  $\mathbb{Z}$ -actions on compact metric groups and prove the main result for discrete extension of compact groups. In Section 6, we prove the main result for all locally compact groups and a few interesting Corollaries. In Section 7, we compare actions on groups and its extension on the space of probability measures.

## 2 Preliminaries

The following result is proved in [2] for locally compact  $\sigma$ -compact groups and is valid without the  $\sigma$ -compactness assumption as we can restrict our attention to the closed subgroup generated by the support of  $\mu$  which is  $\sigma$ -compact (see also [12]).

**Lemma 2.1** *Let  $G$  be a locally compact group and  $\mu \in P(G)$ . Suppose  $\mu$  is non-dissipating. Then  $\mu^n \check{\mu}^n \rightarrow \rho \in P(G)$  and  $\mu \rho \check{\mu} = \rho$ .*

We next prove some basic properties of SCP.

**Proposition 2.2** *Let  $G$  be a locally compact group and  $K$  be a compact normal subgroup of  $G$ . Let  $\pi: G \rightarrow G/K$  be the canonical projection. If  $\mu \in P(G)$  has SCP, then  $\pi(\mu) \in P(G/K)$  has SCP and hence, if  $G$  has SCP, so does  $G/K$ .*

**Proof** Let  $\mu \in P(G)$ . Since  $K$  is compact,  $\mu$  is dissipating if and only if  $\pi(\mu)$  is dissipating. Suppose  $\mu$  is non-dissipating and has SCP. Then there exists a compact subgroup  $L$  of  $G$  and  $z \in G$  such that  $\mu^n z^{-n} \rightarrow \omega_L$  and  $zLz^{-1} = L$ . This implies that  $\pi(\mu^n z^{-n}) \rightarrow \omega_{\pi(L)}$  and  $\pi(z)\pi(L)\pi(z)^{-1} = \pi(L)$ .

**Proposition 2.3** *Let  $G$  be a locally compact group and  $\mu \in P(G)$ . Let us assume that each neighbourhood  $U$  of identity contains a compact normal subgroup  $K_U$  and  $\pi_U: G \rightarrow G/K_U$  be the canonical quotient map. If  $\pi_U(\mu)$  has SCP for all  $U$ , then  $\mu \in P(G)$  also has SCP. If  $G/K_U$  has SCP for all  $U$ , then  $G$  also has SCP.*

**Proof** It is easy to see that for any neighbourhood  $U$  of  $e$  in  $G$ ,  $\mu \in P(G)$  is dissipating if and only if  $\mu \in P(G/K_U)$  is dissipating. Suppose  $\mu \in P(G)$  is non-dissipating. Then  $\mu^n \check{\mu}^n \rightarrow \rho \in P(G)$  and  $\mu \in P(G/K_U)$  is also non-dissipating for all  $U$ . If  $\pi_U(\mu)$  has SCP, then  $\{\pi_U(\mu^n \check{\mu}^n)\}$  converges to an idempotent normalized by the support of  $\pi_U(\mu)$ . Thus,  $\pi_U(\rho)$  is an idempotent normalized by the support of  $\pi_U(\mu)$ . This implies that  $\rho\omega_{K_U} = \omega_{K_U}\rho$  and  $x\rho x^{-1}\omega_{K_U} = \rho\omega_{K_U}$  for all  $x$  in the support of  $\mu$ . Since  $\omega_{K_U} \rightarrow \delta_e$  as  $U \rightarrow e$ , we get  $\rho = \rho^2$  and  $x\rho x^{-1} = \rho$  for all  $x$  in the support of  $\mu$  (cf. [18], Theorem 1.2.2). Let  $x$  be in the support of  $\mu$ . Then the equation  $\mu^n \rho \check{\mu}^n = \rho$  implies that  $\{\mu^n x^{-n}\}$  is relatively compact. Suppose  $\nu$  is a limit point of  $\{\mu^n x^{-n}\}$ . Then  $\nu\check{\nu} = \rho$  and  $\nu$  is supported on the support of  $\rho$ . By Lemma 2.1 of [12],  $\nu = \rho z$  for some  $z$  in the support of  $\nu$  and hence  $\nu = \rho$ . Thus,  $\mu^n x^{-n} \rightarrow \rho$  for all  $x$  in the support of  $\mu$ .

**Proposition 2.4** *Let  $G$  and  $H$  be locally compact groups and let  $\phi: H \rightarrow G$  be a continuous injection. If  $\mu \in P(H)$  is non-dissipating and  $\phi(\mu) \in P(G)$  has SCP, then  $\mu$  also has SCP. In particular,  $G$  has SCP implies  $H$  has SCP.*

**Proof** Let  $\mu \in P(H)$  be non-dissipating. Then  $\phi(\mu) \in P(G)$  is non-dissipating. Suppose  $\phi(\mu)$  has SCP. Then there exists a  $x \in G$  such that  $\phi(\mu^n)x^{-n} \rightarrow \omega_K$  for some compact group  $K$  with  $xKx^{-1} = K$ . Since

$\mu \in P(H)$  is non-dissipating, by Lemma 2.1,  $\mu^n \check{\mu}^n \rightarrow \rho \in P(H)$ , but  $\phi(\mu)^n \phi(\check{\mu})^n \rightarrow \omega_K \in P(G)$ . Thus,  $\phi(\rho) = \omega_K$ . Since  $\phi: H \rightarrow G$  is injective, its extension  $\phi: P(H) \rightarrow P(G)$  is also injective. Thus,  $\rho$  is an idempotent. We now prove that  $y\rho y^{-1} = \rho$  for any  $y$  in the support of  $\mu$ . Let  $y$  be in the support of  $\mu$ . Then  $\phi(y)$  is in the support of  $\phi(\mu)$  and hence  $\phi(y)\omega_K\phi(y)^{-1} = \omega_K$ . Since  $\phi$  is injective,  $y\rho y^{-1} = \rho$ . We can now show as in Proposition 2.2 that  $\mu^n y^{-n} \rightarrow \rho$ .

For a locally compact group  $G$  and  $\alpha \in \text{Aut}(G)$ , we define  $C_K(\alpha)$ , the  $K$ -contraction group of  $\alpha$ , for any compact group  $K$  such that  $\alpha(K) = K$ , as follows:

$$C_K(\alpha) = \{x \in G \mid \alpha^n(x)K \rightarrow K \text{ in } G/K \text{ as } n \rightarrow \infty\}.$$

We denote  $C_{\{e\}}(\alpha)$  by  $C(\alpha)$  and it is called the *contraction group* of  $\alpha$ . For any  $g \in G$ , we denote by  $C(g)$ , the contraction group of  $\text{Inn}(g)$ , where  $\text{Inn}(g)$  is the inner automorphism of  $G$  defined by  $g$ . An automorphism  $\alpha$  of a locally compact group  $G$  is said to contract  $G$  if  $C(\alpha) = G$ .

The following lemma about contraction groups will be quite useful.

**Lemma 2.5** *Let  $G$  be a locally compact group and  $\alpha \in \text{Aut}(G)$ . Let  $C(\alpha)$  be a non-trivial group. Suppose there is a topology on  $C(\alpha)$  which turns  $C(\alpha)$  into a locally compact group  $\tilde{C}(\alpha)$  such that  $\alpha$  is a bi-continuous automorphism of  $\tilde{C}(\alpha)$  with  $\alpha^n(x) \rightarrow e$  as  $n \rightarrow \infty$  for all  $x \in \tilde{C}(\alpha)$  and the canonical inclusion is a continuous map of  $\tilde{C}(\alpha)$  into  $G$ . Then for every probability measure  $\lambda \neq \delta_e$  on  $C(\alpha)$  such that  $\lambda$  is supported on a compact set of  $\tilde{C}(\alpha)$ ,  $\{\prod_{i=0}^n \alpha^i(\lambda)\}_{n \in \mathbb{N}}$  is convergent in  $P(G)$  and its limit is not an idempotent invariant under  $\alpha$ . (Here,  $\alpha^0 = I$ , the identity map).*

When  $G$  is a real or a p-adic Lie group, locally compact topology on  $C(\alpha)$  are possible (cf. [37]) in which case the Lemma can be proved using Theorem 5 of [10] – this case is sufficient for our purpose. Siebert [38] has results on topologies of  $C(\alpha)$  that turn  $C(\alpha)$  into a locally compact group and cases where such topologies are not possible.

**Proof of Lemma 2.5** The convergence of the product  $\{\prod_{i=0}^n \alpha^i(\lambda)\}_{n \in \mathbb{N}}$  to, say,  $\nu$  in  $\tilde{C}(\alpha)$  and hence in  $G$  follows from “(3)  $\Rightarrow$  (2)” of Proposition 4.3 of [22]. This implies that  $\nu = \lambda\alpha(\nu)$ . Since  $\lambda \neq \delta_e$ ,  $\nu \neq \delta_e$ . Moreover,  $\tilde{C}(\alpha)$  does not have any nontrivial compact subgroup invariant under  $\alpha$ . Hence  $\nu$  is not an idempotent invariant under  $\alpha$ .

**Remark 2.6** Given  $\alpha$  and  $\tilde{C}(\alpha)$  as in Lemma 2.5 above, there exist measures  $\lambda$  on  $C(\alpha)$  such that  $\{\lambda_n = \prod_{i=0}^n \alpha^i(\lambda)\}$  is convergent, but its limit is not an idempotent. For any compactly supported nontrivial measure  $\lambda$  on  $\tilde{C}(\alpha)$ ,  $\{\lambda_n\}$  is convergent and for its limit point  $\nu$ , we have  $\nu = \lambda\alpha(\nu)$  and if it were an idempotent, say  $\omega_K$  for some compact group  $K$ , then  $\alpha(K) \subset K$  and  $\lambda$  is supported on  $K$ . Here,  $K \neq \{e\}$  since  $\lambda \neq \delta_e$ . In this case,  $\alpha(K)$  is strictly contained in  $K$  and we take  $\lambda'$  to be any measure on  $K$  such that  $\lambda' = \lambda'\omega_{\alpha(K)} \neq \omega_{\alpha(K)}$  and  $\lambda' \neq \omega_K$ . Then  $\prod_{i=0}^n \alpha^i(\lambda') = \lambda'$ , for all  $n \in \mathbb{N}$ .

The following type of groups occurs in our study often. Let  $\alpha$  be an automorphism of a locally compact group  $G$ . Then  $\mathbb{Z} \ltimes_{\alpha} G$  denotes the semidirect product of  $\mathbb{Z}$  with  $G$  where the  $\mathbb{Z}$ -action is given by  $\alpha$  and is a locally compact group with  $G$  as an open subgroup.

**Proposition 2.7** *Let  $G$  be a real Lie group. If  $G$  has SCP, then  $G$  is point-wise distal. In particular, if  $\alpha \in \text{Aut}(G)$  is such that  $\mathbb{Z} \ltimes_{\alpha} G$  has SCP, then  $\{\alpha^n\}_{n \in \mathbb{Z}}$ -action is distal on  $G$ .*

**Proof** The second assertion trivially follows from the first. Suppose  $G$  is a real Lie group with SCP. Let  $g \in G$ . It is easy to see that  $C(g) \subset G^0$ , the connected component of  $e$  in  $G$  as  $G/G^0$  is discrete. Thus,  $C(g)$  has a topology that turns  $C(g)$  into a locally compact group  $\tilde{C}(g)$  and the identity map from  $\tilde{C}(g) \rightarrow C(g)$  is continuous. Also, conjugacy action of  $g$  contracts  $\tilde{C}(g)$ . It is obvious that  $\{g^n \mid n \in \mathbb{Z}\} \cap C(g) = \{e\}$ . Let  $\alpha$  denote the inner automorphism defined by  $g$  restricted to  $C(g)$ . This implies that there is a continuous injection from  $\mathbb{Z} \ltimes_{\alpha} \tilde{C}(g)$  into  $G$  given by  $(n, h) \mapsto hg^n$ . By, Proposition 2.4,  $\mathbb{Z} \ltimes_{\alpha} \tilde{C}(g)$  has SCP.

If  $C(g) = \tilde{C}(g) \neq \{e\}$ , then by Lemma 2.5, there exists a  $\lambda \in P(\tilde{C}(g))$  such that  $e$  is in the support of  $\lambda$  and  $\{\lambda_n = \prod_{i=0}^n \alpha^i(\lambda)\}$  converges but its limit is not an idempotent invariant under  $\alpha$ . Now for  $\mu = \lambda\delta_1 \in P(\mathbb{Z} \ltimes \tilde{C}(g))$ ,  $\mu^n \delta_{-n} = \lambda_{n-1}$  converges but its limit is not an idempotent invariant under  $\alpha$ . This is a contradiction to the above assertion that  $\mathbb{Z} \ltimes_{\alpha} \tilde{C}(g)$  has SCP (cf. [12], Theorem 4.3). Thus,  $C(g) = \{e\}$ . In a similar way one can show that  $C(g^{-1}) = \{e\}$ . This implies that the inner automorphism of  $g$  is distal on the Lie algebra of  $G$  (cf. [6]). Now, Theorem 1.1 of [1] implies that the conjugacy action of  $G$  on  $G^0$  is point-wise distal. Since  $G/G^0$  is discrete, we get the result.



### 3 Factor actions of distal actions

The following simple result about the factor actions of distal actions modulo compact groups will be very useful.

**Theorem 3.1** *Let  $G$  be a Hausdorff group and let  $\Gamma$  be a semigroup of bi-continuous automorphisms of  $G$ . Suppose  $K$  is a compact subgroup of  $G$  such that  $\gamma(K) = K$ , for all  $\gamma \in \Gamma$ . Then the following are equivalent:*

- (1)  $\Gamma$ -action on  $G$  is distal;
- (2)  $\Gamma$ -actions on both  $K$  and  $G/K$  are distal.

Note that if  $G$  is compact and metrizable then the above follows from Theorem 3.3 of Furstenberg [17].

**Proof** It is sufficient to prove the only non-trivial implication that (1)  $\Rightarrow$  (2). Suppose  $\Gamma$ -action on  $G$  is distal. Then  $\Gamma$ -action on  $K$  is distal. Let  $E$  be the closure of  $\Gamma$  in  $K^K$ , the set of all functions from  $K$  into  $K$ . Then  $E$  is a group (cf. [15], Theorem 1) and it is a compact subset of  $K^K$ .

Let  $\{U_d\}$  be a neighbourhood basis at  $e$  in  $G$  such that each  $U_d$  is  $K$ -invariant, i.e.  $kU_dk^{-1} = U_d$ , for all  $k \in K$ ; this is possible since  $K$  is a compact group. Suppose  $(aK, aK)$  is in the closure of  $\{(\gamma(x)K, \gamma(y)K) \mid \gamma \in \Gamma\}$  for some  $x, y \in G$ . Then for any neighbourhood  $U_d$  of  $e$ ,  $\gamma_d(x)K, \gamma_d(y)K \in aU_dK = aKU_d$ . This implies that  $\gamma_d(x) = au_dk_d$  and  $\gamma_d(y) = au'_dk'_d$  for  $u_d, u'_d \in U_d$  and  $k_d, k'_d \in K$ . Now  $\gamma_d(x)k_d^{-1} = au_d$ , since  $K$  is compact,  $\{k_d\}$  (resp.  $\{k'_d\}$ ) is relatively compact and there exists a subnet (we denote it by the same) such that  $k_d \rightarrow k$  (resp.  $k'_d \rightarrow k'$ ); subnet is possible as  $G$  is a topological group. Here,  $u_d \rightarrow e$ . Let  $\gamma$  be a limit point of  $\gamma_d$  in  $K^K$ . Let  $k_1 = \gamma^{-1}(k^{-1})$  and  $k'_1 = \gamma^{-1}((k')^{-1})$ . Then passing to a subnet, we get that  $\gamma_d(k_1)$  converges to  $\gamma(k_1) = k^{-1}$  and  $\gamma_d(xk_1) = au_dk_d\gamma_d(k_1)$ , which converges to  $akk^{-1} = a$ . Similarly,  $\gamma_d(yk'_1)$  converges to  $a$ , therefore  $xk_1 = yk'_1$  and hence  $xK = yK$ .

The next result follows easily from the above.

**Corollary 3.2** *Let  $G$  be a locally compact group and let  $\alpha \in \text{Aut}(G)$  be such that  $\{\alpha^n\}_{n \in \mathbb{N}}$ -action is distal on  $G$ . Then for any compact subgroup  $K$  such that  $\alpha(K) = K$ , we have*

$$C_K^w(\alpha) = \{x \in G \mid \text{along a subsequence } \alpha^n(xK) \text{ converges to } K\} = K.$$

**Theorem 3.3** *Let  $G$  be a locally compact group and let  $\Gamma$  be a semigroup in  $\text{Aut}(G)$ . Let  $G^0$  denote the connected component of the identity  $e$  in  $G$ . Then  $\Gamma$ -action on  $G$  is distal if and only if  $\Gamma$ -action on both  $G^0$  and  $G/G^0$  is distal.*

**Proof** The proof of the “if” statement is obvious. Now we prove the “only if” statement. Assume that  $\Gamma$ -action on  $G$  is distal. It is obvious that the  $\Gamma$ -action on  $G^0$  is distal. We need to prove that the  $\Gamma$ -action on  $G/G^0$  is distal.

**Step 1:** Since  $G^0$  is a connected group it contains a maximal compact normal subgroup  $C$  such that  $G^0/C$  is a real Lie group (cf. [29]) and since  $C$  is characteristic,  $\gamma(C) = C$  for all  $\gamma \in \Gamma$  and  $\Gamma$ -action on  $G/C$  is distal (cf. Theorem 3.1). Also  $G/G^0$  is isomorphic to  $(G/C)/(G^0/C)$ . Hence, without loss of any generality, we may assume that  $G^0$  is a real Lie group and it has no nontrivial compact normal subgroup. Let  $H$  be an open subgroup of  $G$  containing  $G^0$  such that  $H/G^0$  is a compact open subgroup of  $G/G^0$ . Then  $H$  contains a compact normal subgroup  $K$  such that  $H/K$  is a real Lie group. This implies that  $KG^0$  is an open subgroup of  $H$ . Moreover,  $K \cap G^0$  being a compact normal subgroup of  $G^0$ , is trivial. Thus,  $KG^0 \simeq K \times G^0$ . In particular,  $K \subset Z(G^0)$ , the centraliser of  $G^0$  in  $G$ , which is a closed normal subgroup.

**Step 2:** If  $\Gamma$ -action on  $G/G^0$  is not distal, then we show that the  $\Gamma$ -action on  $G$  is not distal. This would lead to a contradiction. Suppose the identity of  $G/G^0$  belongs to the closure of  $\Gamma(x)G^0$  for some  $x \notin G^0$ . Let  $\{\gamma_d(x)G^0\}$  in  $G/G^0$  is a net converging to the identity in  $G/G^0$ .

We show that there exists an element  $k$  in a compact totally disconnected subgroup  $L$  centralising  $G^0$  such that  $L \cap G^0 = \{e\}$ ,  $kG^0 = xG^0$  and  $\{\gamma_d(k)\}$  converges to  $e$ .

There exists a neighbourhood basis  $\{U_d = K_d \times U'_d\}$ , where  $\{K_d\}$  is a basis consisting of compact open subgroups contained in  $K$  and  $U'_d$  is a neighbourhood of identity in  $G^0$ . We may assume that  $\gamma_d(x) \in U_d G^0 = K_d \times G^0$ . Let  $\gamma_d(x) = k_d g_d = g_d k_d$  for some  $k_d \in K_d$  and  $g_d \in G^0$ .

Choose  $\gamma$  to be some fixed  $\gamma_d$ , then  $\gamma(x) \in KG^0$ . Let  $\gamma(x) = k'g'$ ,  $k' \in K$ ,  $g' \in G^0$ . Then  $x = kg$ , where  $k = \gamma^{-1}(k') \in \gamma^{-1}(K) \subset Z(G^0)$  and  $g = \gamma^{-1}(g') \in G^0$  as  $G^0$  and  $Z(G^0)$  are characteristic. Let  $L = \gamma^{-1}(K)$  which is a compact totally disconnected subgroup in  $Z(G^0)$  and  $L \cap G^0 = \{e\}$  and  $k \in L$ .

It is enough to show that  $\gamma_d(k) \rightarrow e$ . In fact we show that  $\gamma_d(k) = k_d \in K_d$  for each  $d$ .

$$\gamma_d(x) = \gamma_d(k)\gamma_d(g) = \gamma_d(g)\gamma_d(k) = k_d g_d = g_d k_d.$$

Let  $a_d = \gamma_d(k)k_d^{-1} = \gamma_d(g^{-1})g_d$ . Then  $a_d \in Z(G^0) \cap G^0$ , the centre of  $G^0$ . In particular, this implies  $k_d$  and  $a_d$ , and hence,  $k_d$  and  $\gamma_d(k)$  commute. Therefore,  $a_d$  generates a compact group, which is contained in the center of  $G^0$ , and hence, is trivial, i.e.  $a_d = e$  and hence,  $\gamma_d(k) = k_d \rightarrow e$ . This completes the proof.

The following corollary follows easily from the above theorem.

**Corollary 3.4** *Let  $G$  be a locally compact group. Then  $G$  is (point-wise) distal if and only if  $G/G^0$  is (point-wise) distal and the  $G$ -action on  $G^0$  is (point-wise) distal.*

## 4 Totally disconnected groups

In this section we apply Poisson boundary and Choquet-Deny Theorem to show that SCP implies point-wise distal for totally disconnected groups. A probability measure  $\mu$  on a locally compact group  $G$  is said to be a Choquet-Deny measure if the bounded continuous functions satisfying the equation

$$f(g) = \int f(gh) d\mu(h), \quad g \in G \tag{1}$$

are constants on the cosets of the smallest closed subgroup generated by the support of  $\mu$ . Let  $H_\mu$  denote the space of bounded functions satisfying the equation

$$f(g) = \int f(gh) d\mu(h), \quad g \in G \tag{2}$$

with  $L^\infty$ -norm. If  $G$  is a locally compact second countable group, then there exists a (compact metric)  $G$ -space  $X$  with a  $\sigma$ -finite quasi-invariant measure  $\nu$  and an equivariant isometry  $\Phi$  of  $L^\infty(X, \nu)$  onto  $H_\mu$  given by the Poisson formula

$$\Phi(f)(g) = \int f(gx) d\rho(x)$$

where  $\rho$  is a  $\mu$ -stationary probability measure on  $X$ , that is,  $\mu * \rho = \rho$  (cf. [21]). The  $G$ -space  $X$  is called the  $\mu$ -boundary: see [21] and [24] for further

details on  $\mu$ -boundary and Choquet-Deny Theorem. It may be noted that  $\mu$  is Choquet-Deny if and only if the  $\mu$ -boundary is trivial.

Jaworski et al [25] proved that any Choquet-Deny measure has SCP. Here we prove the converse which will be useful in characterizing groups with SCP.

**Proposition 4.1** *Let  $G$  be a locally compact group. If  $\mu \in P(G)$  is non-dissipating and has shifted convolution property, then  $\mu$  is a Choquet-Deny measure.*

**Proof** Let  $\mu \in P(G)$  is non-dissipating and has SCP. Let  $G_\mu$  be the closed subgroup generated by the support of  $\mu$ . We may assume that  $G_\mu$  is not compact (cf. [24], Lemma 4.1). Then  $\mu \in P(G_\mu)$  is also non-dissipating. By Proposition 2.4,  $\mu \in P(G_\mu)$  also has shifted convolution property. Hence, using Lemma 4.1 of [24] and replacing  $G$  by  $G_\mu$ , we may assume that  $G$  is the closed subgroup generated by the support of  $\mu$ .

Let us first consider the case when  $G$  is second countable. Since  $\mu$  has SCP, there is a compact subgroup  $K$  and a  $z \in G$  such that  $\mu^n z^{-n} \rightarrow \omega_K$  with  $K = zKz^{-1}$ . Then  $\mu$  is supported on  $zK = Kz$ . This implies that  $G/K \simeq \mathbb{Z}$  as  $G$  is not compact. Thus,  $G$  is the semidirect product of  $\mathbb{Z}$  and  $K$  where  $\mathbb{Z}$ -action is given by the inner automorphism of  $z$  restricted to  $K$ . Also, there is a probability measure  $\lambda$  on  $K$  such that  $\mu = \lambda \delta_z$ .

By Theorem 4.2 of [JR], the boundary of  $\mu$  is a homogeneous space of  $G$  and  $K$  acts transitively on the boundary of  $\mu$ . Thus, there exists a closed subgroup  $H$  of  $G$  such that  $G/H$  is the boundary of  $\mu$  and  $G = KH = HK$  as  $K$  is normal in  $G$ . Under the natural isomorphism between  $G/H$  and  $K/(K \cap H)$ , the canonical action of  $G$  on  $G/H$  defines an action of  $G$  on  $K/(K \cap H)$  by  $g \odot x(K \cap H) = ahxh^{-1}(K \cap H)$  where  $g = ah$  for  $a \in K$  and  $h \in H$ . Let  $T: K \rightarrow K$  be defined by  $T(x) = zxz^{-1}$  for all  $x \in K$ . Let  $\eta$  be the Poisson kernel in  $K/(K \cap H)$  and  $\pi: K \rightarrow K/(K \cap H)$  be the canonical projection. Let  $\rho \in P(K)$  be such that  $\pi(\rho) = \eta$  and for  $n \geq 1$ , let  $z^n = a_n h_n$  for some  $a_n \in K$  and  $h_n \in H$ . Then  $\eta = \mu^n \odot \eta = \mu^n z^{-n} a_n h_n \odot \eta = \pi(\mu^n z^{-n} T^n(\rho) a_n)$  for all  $n \geq 1$ . Since  $\mu^n z^{-n} \rightarrow \omega_K$ , it is easy to see that  $\mu^n z^{-n} T^n(\rho) a_n \rightarrow \omega_K$  and hence  $\eta = \pi(\omega_K)$ . This implies that  $\eta$  is  $G$ -invariant and hence the boundary is trivial. This shows that  $\mu$  is a Choquet-Deny measure.

Suppose  $G$  is any locally compact group. Since  $G$  is the closed subgroup generated by the support of  $\mu$ ,  $G$  is  $\sigma$ -compact. Thus, by Theorem 8.7 of [19], every neighbourhood  $U$  of  $e$  contains a compact normal subgroup  $K_U$  such

that  $G/K_U$  is second countable. By Proposition 2.2,  $\mu$  is non-dissipating and has shifted convolution property in  $G/K_U$  and hence  $\mu$  is a Choquet-Deny measure in  $G/K_U$ . Now, Lemma 3.1 of [24] implies that  $\mu$  is a Choquet-Deny measure in  $G$ .

**Theorem 4.2** *Let  $G$  be a locally compact metrizable group and  $G^0$  be the connected component of identity in  $G$ . Suppose  $G$  has SCP. Then  $G/G^0$  is point-wise distal. In particular, a totally disconnected locally compact metrizable group with SCP is point-wise distal.*

**Proof** Since  $G^0$  is a connected group it contains a maximal compact normal characteristic subgroup  $C$  such that  $G^0/C$  is a real Lie group (cf. [29]) and  $G/C$  also has SCP (cf. Proposition 2.2). Also,  $G/G^0$  is isomorphic to  $(G/C)/(G^0/C)$ . Hence, we may assume that  $G^0$  is a real Lie group and  $G^0$  has no nontrivial compact normal subgroup. Suppose  $G/G^0$  is not point-wise distal. Then there exists  $g \in G$  such that  $\{g^n\}_{n \in \mathbb{N}}$ -action on  $G/G^0$  is not distal. Let  $\alpha = \text{Inn}(g)$ . Then by Proposition 2.1 of [24], there exists  $x \in G$ , such that  $\alpha^n(x)G^0 \rightarrow G^0$ . Now by Step 2 of Theorem 3.3, for  $\gamma_n = \alpha^n$ , there exists  $k \in L$ , a compact totally disconnected group in  $Z(G^0)$  such that  $L \cap G^0 = \emptyset$  and  $\alpha^n(k) \rightarrow e$ . In particular,  $C(\alpha) = C(g) \neq \{e\}$ .

Now we show that  $\overline{C(g)} \cap G^0 = \{e\}$ . Since  $\{g^n\}_{n \in \mathbb{N}}$ -action on  $G/G^0$  is not distal,  $\{g^n\}_{n \in \mathbb{Z}}$  generates a discrete infinite cyclic group modulo  $G^0$ , hence,  $Z \rtimes_\alpha G^0$  is a closed subgroup of  $G$  and it has SCP (cf. Proposition 2.4). Also, since  $G^0$  is a Lie group, by Proposition 2.7,  $C(g) \cap G^0 = \{e\}$ . Let  $K$  be a subgroup of  $G$  as in the proof of Theorem 3.3, such that  $KG^0 = K \times G^0$  and  $KG^0$  is an open subgroup of  $G$ .

First we show that  $(K \times G^0) \cap C(g) = K \cap C(g)$ . Let  $b \in C(g)$  be such that  $b = k'x$ , for some  $k' \in K$  and  $x \in G^0$ . Then  $g^n b g^{-n} \rightarrow e$ . Arguing as in Step 2 of the proof of Theorem 3.3, we get that  $g^n k' g^{-n} \rightarrow e$  and hence  $g^n x g^{-n} \rightarrow e$ . The above implies that  $x = e$  and  $b = k' \in K$ . Let  $b_n \in C(g)$  be such that  $b_n \rightarrow b \in G^0$ . Since  $K \times G^0$  is an open neighbourhood of  $b$ ,  $b_n \in K \times G^0$  for all large  $n$ . Since  $b_n \in C(g)$ ,  $b_n \in K$  and hence  $b \in K \cap G^0 = \{e\}$ . Thus,  $\overline{C(g)} \cap G^0 = \{e\}$ .

Now we have that  $\overline{C(g)}$  is totally disconnected subgroup normalised by  $g$ . Also, the subgroup  $\mathbb{Z} \rtimes_\alpha \overline{C(g)}$  is a (Borel) subgroup of  $G$ . Clearly, the group  $\mathbb{Z} \rtimes_\alpha \overline{C(g)}$  is totally disconnected. By Lemma 3.5 of [24], the measure  $\mu = \delta_g \nu$  is not a Choquet-Deny measure where  $\nu = p\delta_x + (1-p)\delta_e$  for any  $0 < p < 1$  and  $p \neq \frac{1}{2}$  for some  $x \in C(g) \setminus \{e\}$ . By Lemma 3.3 of [24],

$\mu$  is non-dissipating and hence by Proposition 4.1,  $\mu$  does not have SCP in  $\mathbb{Z} \ltimes_{\alpha} \overline{C(g)}$ . By Lemma 3.4 of [24], the map  $(n, x) \mapsto xg^n$  is (continuous) injective from  $\mathbb{Z} \ltimes_{\alpha} \overline{C(g)}$  into  $G$  and hence  $\mu$  does not have SCP in  $G$  also (cf. Proposition 2.4).

## 5 $\mathbb{Z}$ -actions on compact groups

In this section we apply dynamics of  $\mathbb{Z}$ -actions on compact groups to prove the main theorem for  $\mathbb{Z} \ltimes K$ . We first recall some dynamical notions.

Let  $K$  be a compact group and  $\alpha \in \text{Aut}(K)$ . Then  $\alpha$  is said to be *ergodic* (on  $K$ ) if for any  $\alpha$ -invariant Borel set  $E$  of  $K$ ,  $\omega_K(E) = 0$  or  $1$ . We say that the  $\{\alpha^n\}_{n \in \mathbb{Z}}$ -action on  $K$  has *descending chain condition* which would be called DCC if any decreasing sequence  $\{K_n\}$  of closed  $\alpha$ -invariant subgroups is finite, that is, there exists a  $k \geq 1$  such that  $K_n = K_m$  for all  $n, m \geq k$ ; see [27] and [36] for details on DCC.

**Lemma 5.1** *Let  $L$  be any non-trivial compact group and  $\tau: L^{\mathbb{Z}} \rightarrow L^{\mathbb{Z}}$  be the shift automorphism given by  $\tau((g_i)) = (g_{i+1})$  for any  $(g_i) \in L^{\mathbb{Z}}$ . Then the group  $\mathbb{Z} \ltimes_{\tau} L^{\mathbb{Z}}$  does not have SCP.*

The above Lemma follows from Theorem 4.2 if  $L$  is not connected. Here, we give a simple proof in the general case.

**Proof of Lemma 5.1** Let  $M = \{(x_i) \in L^{\mathbb{Z}} \mid x_i = e \text{ for } i > 0\}$ . Then  $\tau(M) \subset M$ . Let  $\mu = \omega_M \delta_1$ . Then  $\mu^n \delta_{-n} = \prod_{i=0}^{n-1} \tau^i(\omega_M) = \omega_M$  for any  $n \geq 1$ . Thus,  $\mu$  is non-dissipating and  $\mu^n \check{\mu}^n \rightarrow \omega_M$ . Suppose  $\mu$  has shifted convolution property. Then there exists a compact subgroup  $N$  of  $\mathbb{Z} \ltimes_{\tau} L^{\mathbb{Z}}$  and  $g \in \mathbb{Z} \ltimes_{\tau} L^{\mathbb{Z}}$  such that  $\mu^n g^{-n} \rightarrow \omega_N$  with  $gNg^{-1} = N$ . Then  $\mu^n \check{\mu}^n \rightarrow \omega_N$  and hence  $M = N$ . Since  $g$  normalizes  $N$ , the closed subgroup generated by  $g$  and  $M$  contains the support of  $\mu$  and hence  $M$  is normalized by the support of  $\mu$ . Since  $1$  is in the support of  $\mu$ ,  $M$  is  $\tau$ -invariant. This is a contradiction unless  $L$  is trivial.

**Lemma 5.2** *Let  $K$  be a compact metrizable group and  $\alpha$  be an ergodic automorphism of  $K$ . Suppose  $\mathbb{Z} \ltimes_{\alpha} K$  has SCP and the  $\{\alpha^n\}_{n \in \mathbb{Z}}$ -action on  $K$  has DCC. Then  $K$  is a compact connected abelian group of finite-dimension.*

**Proof** By Lemma 5.1,  $K$  contains no  $\alpha$ -invariant subgroup of the form  $L^{\mathbb{Z}}$ , where  $\alpha$ -action is given by a shift. Now using this fact, the assertion

essentially follows from the proof of Theorem 10.6 of [36], but for the sake of clarity, we give a sketch of proof along the same lines.

Since  $\{\alpha^n\}_{n \in \mathbb{Z}}$ -action on  $K$  has DCC, there exists a compact Lie group  $G$  such that  $K \subset G^{\mathbb{Z}}$  and the action of  $\alpha$  is given by shift on  $G^{\mathbb{Z}}$  (cf. [27]). For  $k \in \mathbb{Z}$ , let  $F_H(k) = \{g_k : (g_n)_{n \in \mathbb{Z}} \in K, g_0 = e\}$  be closed subgroups of  $G$ . Let  $\eta_k : G \rightarrow G_k = G/F_H(k)$  be the natural projection and let  $\boldsymbol{\eta}_k : K \rightarrow (G_k)^{\mathbb{Z}}$  be the map induced by  $\eta_k$  as follows:  $\boldsymbol{\eta}_k((g_n)_{n \in \mathbb{Z}}) = (\eta_k(g_n))_{n \in \mathbb{Z}}$ . Then by the proof of Theorem 10.6 of [36] (see also [28] and [40]), there exists  $N \geq 1$  such that  $\eta_k(G) = G/F_H(k)$  is isomorphic to  $\mathbb{T}^n$ , the  $n$ -dimensional torus, for all  $k \geq N$ , for some fixed  $n$  and  $\boldsymbol{\eta}_N(K)$  is isomorphic to  $X^A$ , for some  $A \in GL_n(\mathbb{Q})$ , where  $X^A = \{(\pi(y_n)) : y_{n+1} = A(y_n), n \in \mathbb{Z}\}$  and  $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^n/\mathbb{Z}^n \simeq \mathbb{T}^n$  is the canonical quotient map. Hence  $\boldsymbol{\eta}_N(K)$  is a connected abelian finite dimensional group.

It remains to show that  $\boldsymbol{\eta}_N$  is an isomorphism, i.e.  $\ker \boldsymbol{\eta}_N$  is trivial. Let  $Y_k = \ker \boldsymbol{\eta}_k$ ,  $k \in \mathbb{N}$ . Then  $Y_1 = \Lambda_1^{\mathbb{Z}}$  is a subgroup of  $K$  (cf. Proposition 10.2 of [36]), which has SCP. By Lemma 5.1  $\Lambda_1$  is trivial and hence,  $Y_1$  is trivial. This shows that  $\boldsymbol{\eta}_1$  is injective, and hence, an isomorphism. Now  $\boldsymbol{\eta}_1(Y_2) = \Lambda_2^{\mathbb{Z}}$  (cf. [27], Proposition 5.7 (2)) which is a closed subgroup of  $\boldsymbol{\eta}_1(K)$ . Thus,  $\boldsymbol{\eta}_1(Y_2) = \Lambda_2^{\mathbb{Z}}$  has SCP and hence by Lemma 5.1,  $\boldsymbol{\eta}_1(Y_2)$  is trivial. This implies that  $\boldsymbol{\eta}_2$  is also injective. Arguing inductively, we get that all  $\boldsymbol{\eta}_k$ ,  $k \in \mathbb{N}$ , are injective. This completes the proof.

**Lemma 5.3** *Let  $K$  be a non-trivial compact metrizable connected finite-dimensional abelian group and  $\alpha$  be an ergodic automorphism of  $K$ . Then  $\mathbb{Z} \ltimes_{\alpha} K$  does not have SCP.*

**Proof** Let  $m > 0$  be such that  $\mathbb{Z}^m \subset \hat{K} \subset \mathbb{Q}^m$ . Let  $B_m$  be the dual of  $\mathbb{Q}^m$ . Then  $K$  is a quotient of  $B_m$ . It can easily be shown that any automorphism of  $K$  lifts to an automorphism of  $B_m$  (see [34]). We denote the lift of  $\alpha$  also by  $\alpha$ .

For any prime number  $p$ , let  $\mathbb{Q}_p$  be the field of  $p$ -adic numbers. Then  $\alpha$  is in  $GL_m(\mathbb{Q}_p)$  for all  $p$  and  $\alpha \in GL_m(\mathbb{R})$ . We now show that there exists a prime  $p$  for which  $\alpha$  has an eigenvalue of  $p$ -adic absolute value different from one or  $\alpha \in GL_m(\mathbb{R})$  has an eigenvalue of absolute value different from one. Suppose for every prime  $p$ , eigenvalues of  $\alpha \in GL_m(\mathbb{Q}_p)$  are of  $p$ -adic absolute value one. Let  $f$  be the characteristic polynomial of  $\alpha$ . Then the leading coefficient of  $f$  is one and all coefficients are rational. Since the coefficients of  $f$  are elementary symmetric functions of eigenvalues of  $\alpha$ , for any prime

$p$ ,  $p$ -adic absolute value of the coefficients are less than or equal one. This implies that all coefficients of  $f$  are integers and the leading coefficient is one. Thus, eigenvalues of  $\alpha \in GL_m(\mathbb{R})$  are algebraic integers. Since  $\alpha$  is ergodic, no eigenvalue of  $\alpha$  is a root of unity. Thus, by a classical result of Kronecker (see [16]), we get that  $\alpha \in GL_m(\mathbb{R})$  has an eigenvalue of absolute value different from one. Since  $\mathbb{Z} \rtimes_{\alpha} K$  and  $\mathbb{Z} \rtimes_{\alpha^{-1}} K$  are isomorphic, we can assume that an eigenvalue of  $\alpha \in GL_m(\mathbb{Q}_p)$  or  $\alpha \in GL_m(\mathbb{R})$  is of absolute value less than one.

Let  $F$  be  $\mathbb{Q}_p$  or  $\mathbb{R}$  such that  $\alpha \in GL_m(F)$  has an eigenvalue of absolute value less than one. Since  $\mathbb{Q}^m \subset F^m$  and  $\mathbb{Q}^m$  is dense, there exists a continuous injection  $\psi: F^m \rightarrow B_m$ . It can easily be verified that  $\psi\alpha = \alpha\psi$ . Now, let  $V = \{v \in F^m \mid \alpha^n(v) \rightarrow 0 \text{ as } n \rightarrow \infty\}$ . Then  $V$  is a non-trivial closed subspace. By Lemma 2.5 we get a measure  $\lambda \in P(F^m)$  that is supported on  $V$  such that the support of  $\lambda$  contains 0 and  $\{\lambda\alpha(\lambda) \cdots \alpha^n(\lambda)\}_{n \in \mathbb{N}}$  converges but the limit point is not an idempotent invariant under  $\alpha$ . Let  $\pi: B_m \rightarrow K$  be the canonical projection. Let  $\rho \in P(F^m)$  be the limit point of  $\{\lambda\alpha(\lambda) \cdots \alpha^n(\lambda)\}_{n \in \mathbb{N}}$ .

Let  $(1, \pi\psi(\lambda))$  denote the probability measure on  $\mathbb{Z} \rtimes_{\alpha} K$  defined by

$$(1, \pi\psi(\lambda))(A \times B) = \delta_1(A)\pi\psi(\lambda)(B)$$

for any Borel sets  $A$  in  $\mathbb{Z}$  and  $B$  in  $K$ . We show that this measure does not have SCP. If possible, suppose  $(1, \pi\psi(\lambda))$  has SCP. Then  $\pi\psi(\rho)$  is an idempotent invariant under  $\alpha$  (cf. [12], Theorem 4.3). Let  $L$  be the compact subgroup of  $K$  such that  $\pi\psi(\rho) = \omega_L$ . Since  $\lambda$  is supported on  $V$  and  $V$  is  $\alpha$ -invariant,  $\rho(V) = 1$ . Since  $\pi\psi(V)$  is a Borel subgroup of  $K$ ,  $\pi\psi\rho(\pi\psi(V)) = 1$  and hence  $\omega_L(\pi\psi(V)) = 1$ . Since  $\omega_L$  is  $L$ -invariant,  $\omega_L(x\pi\psi(V)) = 1$  for any  $x \in L$ . This implies that  $x\pi\psi(V) \cap \pi\psi(V) \neq \emptyset$  and hence  $x \in \pi\psi(V)$  for all  $x \in L$ , that is,  $L \subset \pi\psi(V)$ . Since  $\alpha$  contracts  $V$ ,  $\alpha$  contracts  $L$  as well. Since  $L$  is a compact group,  $L$  is trivial. Thus,  $\psi(\lambda)$  is supported on the kernel of  $\pi$ , say,  $M$ .

Let  $H = \psi^{-1}(M) \cap V$ . Then  $H$  is a  $\alpha$ -invariant closed subgroup of  $V$ . Let  $\hat{M}$  be the dual of  $M$ . Then  $\hat{M} \simeq \mathbb{Q}^m / \hat{K}$  has only elements of finite order. Let  $V'$  be the maximal vector subspace contained in  $H$ . Then  $V'$  is  $\alpha$ -invariant. Now  $\psi$  restricted to  $V'$  defines a continuous homomorphism  $\hat{\psi}: \hat{M} \rightarrow V'$ . Since  $V'$  has no element of finite order,  $\hat{\psi}$  and hence  $\psi$  is trivial. This implies that  $H$  contains no vector subspace. Thus,  $H$  is compact or discrete. Since  $\alpha$  contracts  $H$ ,  $H$  is trivial. Since  $\lambda$  is supported on  $H$ ,  $\lambda = \delta_e$ . Then for all



$n \in \mathbb{N}$ ,  $\lambda\alpha(\lambda) \cdots \alpha^n(\lambda) = \delta_e$  an idempotent which is  $\alpha$ -invariant, this leads to a contradiction. Therefore,  $(1, \pi\psi(\lambda))$ , and hence,  $\mathbb{Z} \ltimes_{\alpha} K$  does not have SCP.

**Theorem 5.4** *Let  $K$  be a compact group and  $\alpha$  be an automorphism of  $K$ . Suppose the group  $\mathbb{Z} \ltimes_{\alpha} K$  has SCP. Then  $\{\alpha^n\}_{n \in \mathbb{Z}}$ -action is distal on  $K$ .*

**Proof** Let us first consider the case when  $K$  is second countable. Assume that the group  $\mathbb{Z} \ltimes_{\alpha} K$  has SCP. By Proposition 2.1 of [34], it is enough to show that  $\alpha$  is not ergodic on  $H$  for any non-trivial  $\Gamma$ -invariant closed subgroup  $H$  of  $K$ . But since  $\mathbb{Z} \ltimes_{\alpha} H$  also has SCP, it is enough to show this for  $H = K$ .

Suppose  $\alpha$  is ergodic on  $K$ . By Theorem 3.16 of [27], there exists a decreasing sequence  $\{K_i\}$  of closed normal  $\alpha$ -invariant subgroups such that  $\cap K_i = \{e\}$  and the action of  $\alpha$  on  $K/K_i$  has DCC. It is easy to see that the action of  $\alpha$  on  $K/K_i$  is ergodic. By Proposition 2.2,  $\mathbb{Z} \ltimes_{\alpha} K/K_i$  has SCP. Thus, Lemma 5.2 implies that  $K/K_i$  is a compact connected abelian group of finite-dimension. Now by Lemma 5.3,  $K = K_i$ . Since  $\cap K_i = \{e\}$ ,  $K$  is trivial.

Now consider the case when  $K$  is not necessarily second countable. Suppose  $\mathbb{Z} \ltimes_{\alpha} K$  has SCP. Since  $\mathbb{Z} \ltimes_{\alpha} K$  is  $\sigma$ -compact, each neighbourhood  $U$  of  $e$  in  $K$  contains a compact normal subgroup  $K_U$  of  $\mathbb{Z} \ltimes_{\alpha} K$  such that  $(\mathbb{Z} \ltimes_{\alpha} K)/K_U$  is second countable (cf. [19], Theorem 8.7). This implies that  $K_U$  is a normal subgroup of  $K$  and is  $\alpha$ -invariant. By Proposition 2.2,  $\mathbb{Z} \ltimes_{\alpha} (K/K_U)$  has SCP and hence  $\{\alpha^n\}_{n \in \mathbb{Z}}$ -action is distal on  $K/K_U$ . Since  $U$  is an arbitrary neighbourhood of  $e$  in  $K$ ,  $\{\alpha^n\}_{n \in \mathbb{Z}}$ -action is distal on  $K$ .

## 6 Locally compact groups

**Theorem 6.1** *A locally compact group  $G$  is point-wise distal if and only if the group  $G$  has SCP.*

**Proof** We first assume that  $G$  is second countable and hence metrizable. Suppose  $G$  is point-wise distal. Let  $\mu \in P(G)$  be non-dissipating. It is enough to show that  $\mu$  has SCP. Without loss of any generality, we may assume that  $G = G_{\mu}$ , the closed subgroup generated by the support of  $\mu$ . Let  $N_{\mu}$  be the smallest closed normal subgroup of  $G$  such that a coset of it

contains the support of  $\mu$ . Then there exists  $z \in G$ , a compact subgroup  $H$  of  $N_\mu$  and an increasing sequence  $\{k_n\}$  such that  $z$  normalises  $H$  and there exists a Borel subgroup  $N_1 \subset \{g \in G \mid z^{-k_n} g z^{k_n} H \rightarrow H\}$  with  $N_\mu = \overline{N_1}$  (cf. [23] or [25]). Let  $\alpha: G \rightarrow G$  be  $\alpha(g) = z^{-1} g z$  for all  $g \in G$ . By Theorem 3.1,  $\{\alpha^n\}_{n \in \mathbb{Z}}$ -action on  $G/H$  is distal and hence  $N_1 \subset H \subset N_\mu$ . Thus,  $H = N_\mu$ . Now by Lemma 3.8 of [23],  $\mu^n z^{-n} \rightarrow \omega_H$ , i.e.  $\mu$  has SCP.

Conversely, suppose  $G$  has SCP. It is enough to show that  $G/G^0$  is point-wise distal and  $G$ -action on  $G^0$  is point-wise distal. It follows from Theorem 4.2 that  $G/G^0$  is point-wise distal. Now we show that  $G$ -action on  $G^0$  is point-wise distal, which in turn would imply that  $G$  is point-wise distal. Let  $K$  be the maximal compact normal subgroup of  $G^0$  such that  $G^0/K$  is a real Lie group. Then  $K$  is characteristic in  $G^0$  and hence  $K$  is a normal subgroup of  $G$ . By Theorem 5.4  $G$ -action on  $K$  is point-wise distal. Hence it is enough to show that the  $G$ -action on  $G^0/K$ , or equivalently, the  $G/K$ -action on  $G^0/K$  is point-wise distal. Also since  $G$  has SCP so does  $G/K$  (cf. Proposition 2.2). Now replacing  $G$  by  $G/K$ , we may assume that  $G^0$  is a Lie group without any nontrivial compact normal subgroup. Let  $x \in G$  and  $\alpha_x: G^0 \rightarrow G^0$  be defined by  $\alpha_x(g) = x g x^{-1}$  for all  $g \in G^0$ . Here, the closed subgroup generated by  $x G^0$  is either a discrete or a compact subgroup in  $G/G^0$ . First suppose that it is discrete. Then the map  $(n, g) \mapsto g x^n$  defines a continuous injection of  $\mathbb{Z} \ltimes_{\alpha_x} G^0$  into  $G$ . By Proposition 2.4, the group  $\mathbb{Z} \ltimes_{\alpha_x} G^0$  has SCP. It follows from Proposition 2.7 that  $G$ -action on  $G^0$  is point-wise distal.

Now suppose  $x G^0$  generates a relatively compact subgroup in  $G/G^0$ . Let  $G_x$  be the closed subgroup generated by  $G^0$  and  $x$ . Then it is an almost connected locally compact group having SCP. Then  $G_x^0 = G_0$  is a Lie group with SCP and by Proposition 2.7, it is point-wise distal. Therefore,  $G^0$  is distal. Now  $G_x$ , being a compact extension of  $G^0$ , is also distal. This implies that  $\{\alpha_x^n\}_{n \in \mathbb{Z}}$ -action is distal on  $G^0$ . Thus, the  $G$ -action on  $G^0$  is point-wise distal.

Now assume that  $G$  is any locally compact group. Suppose  $G$  is point-wise distal. Let  $\mu \in P(G)$  be non-dissipating. Let  $G_\mu$  be the closed subgroup generated by the support of  $\mu$  in  $G$ . Then  $G_\mu$  is  $\sigma$ -compact and it is point-wise distal. This implies that every neighbourhood  $U$  of the identity  $e$  in  $G_\mu$  has a compact normal subgroup  $K_U$  such that  $G_\mu/K_U$  is second countable (cf. [19], Theorem 8.7). By Theorem 3.1,  $G_\mu/K_U$  is also point-wise distal and hence it has SCP. This implies that the image of  $\mu$  on  $G_\mu/K_U$  has SCP. By Proposition 2.3,  $\mu$  itself has SCP.

Conversely, assume that  $G$  has SCP. Let  $x \in G$  and  $g \in G$  be such that  $e$  is a limit point of  $\{x^n g x^{-n} \mid n \in \mathbb{Z}\}$ . Let  $H$  be the closed subgroup generated by  $x$  and  $g$ . Then  $H$  is  $\sigma$ -compact and  $H$  also has SCP. Then every neighbourhood  $U$  of  $e$  in  $H$  contains a compact normal subgroup  $K_U$  of  $H$  such that  $H/K_U$  is second countable (cf. [19], Theorem 8.7). By Proposition 2.2,  $H/K_U$  has SCP and hence point-wise distal. This implies  $g \in K_U \subset U$ . Since  $U$  is an arbitrary neighbourhood of  $e$ ,  $g = e$ . Thus,  $G$  is point-wise distal.

**Remark 6.2** It is clear from Theorem 3.1 and the above proof that point-wise distal implies SCP is valid even if  $G$  is any metrizable group.

We now prove a few interesting consequences of Theorem 6.1.

**Corollary 6.3** *Let  $G$  be a locally compact point-wise distal group. If  $x \in G$  and  $K$  is a compact group such that  $xKx^{-1} \subset K$ , then  $xKx^{-1} = K$ .*

This result is known in case  $G$  is a real Lie group or  $G$  is a Torsion group (see Lemma 2.2 of [13]).

**Proof** Let  $\lambda = \omega_K$ . Then  $\lambda x \lambda x^{-1} = \lambda$ . Thus,  $\prod_{k=0}^n x^k \lambda x^{-k} = \lambda$ . Let  $\mu = \lambda x$ . Then  $\mu^n x^{-n} = \prod_{k=0}^{n-1} x^k \lambda x^{-k} = \lambda$  for all  $n \geq 1$ . Thus,  $\mu$  is non-dissipating. Since  $G$  is point-wise distal, by Theorem 6.1 we get that  $\{\mu^n * \check{\mu}^n\}$  converges to an idempotent whose support (group) is normalized by the support of  $\mu$ . But  $\mu^n * \check{\mu}^n = \lambda \check{\lambda} = \omega_K$  and since,  $x$  is in the support of  $\mu$ , we have  $xKx^{-1} = K$ .

We next obtain the Krengel-Lin decomposition for measures on point-wise distal groups. Let  $G$  be a locally compact group with right Haar measure  $m$ . Let  $L^2(G)$  be the Hilbert space of all square integrable functions on  $G$  with respect to  $m$  with norm  $\|\cdot\|$ . For  $\mu \in P(G)$ , define  $P_\mu: L^2(G) \rightarrow L^2(G)$  by

$$P_\mu(f)(x) = \int f(xy^{-1}) d\mu(y)$$

for all  $x \in G$ . Then  $P_\mu$  is a contraction on  $L^2(G)$ .

We recall that a  $\mu \in P(G)$  is called *adapted* if the closed subgroup generated by the support of  $\mu$  is all of  $G$ .

It is shown in [11] that adapted  $\mu \in P(G)$  is dissipating if and only if  $L^2(G) = \{f \in L^2(G) \mid \|P_\mu^n(f)\| \rightarrow 0\} = E_0$ , (say).

When  $\mu$  is adapted and is non-dissipating,  $E_0 \neq L^2(G)$  and in this situation Krengel-Lin decomposition is about determining the orthogonal complement of  $E_0$  in  $L^2(G)$  and showing that the orthogonal complement of  $E_0$  in  $L^2(G)$  is equal to  $L^2(G, \Sigma_d)$  where  $\Sigma_d$  is the deterministic  $\sigma$ -algebra of  $P_\mu$  consisting of Borel sets  $A$  in  $G$  such that for each  $n \geq 1$  there exists a Borel set  $B_n$  with  $P_\mu(1_A) = 1_{B_n}$ : see [32] and references cited therein. We now prove the Krengel-Lin decomposition for measures on point-wise distal groups which is a continuation of [2], [26] and [32].

**Corollary 6.4** *Let  $G$  be a non-compact locally compact point-wise distal group and  $\mu$  be an adapted probability measure on  $G$ . Suppose  $\mu$  is non-dissipating. Then there exists a compact normal subgroup  $K$  such that:*

1.  $L^2(G) = E_0 \oplus L^2(G, \Sigma_d)$  and the deterministic  $\sigma$ -algebra  $\Sigma_d$  is generated by  $\{x^n K \mid n \in \mathbb{Z}\}$  for any  $x$  in the support of  $\mu$ ;
2.  $(L^2(G, \Sigma_d), P_\mu)$  is isomorphic to the bilateral shift on  $l^2(\mathbb{Z})$ .

**Proof** The result follows from Theorem 6.1 and Proposition 3.1 of [32].

The following is a generalization of implication (a)  $\Rightarrow$  (c) of Theorem 3.6 of [24].

**Corollary 6.5** *Let  $G$  be a locally compact group. If every  $\mu \in P(G)$  is Choquet-Deny, then  $G$  is point-wise distal.*

**Proof** Suppose  $\mu \in P(G)$  is Choquet-Deny. Then by Theorem 2.25 of [25]  $\mu$  has SCP. Now Theorem 6.1 implies that  $G$  is point-wise distal.

## 7 Actions on spaces of measures

In this section we consider actions on connected Lie groups and show that distality of group actions carries over to the actions on spaces of measures under a certain condition. For a connected Lie group  $G$  with Lie algebra  $\mathcal{G}$ ,  $\text{Aut}(G)$  can be realised as a subgroup of  $GL(\mathcal{G})$ , by identifying each automorphism with its derivative on  $\mathcal{G}$ . By an *almost algebraic subgroup* of  $\text{Aut}(G)$  we shall mean a subgroup of  $\text{Aut}(G)$  which is a subgroup of finite index or equivalently open subgroup in an algebraic subgroup of  $GL(\mathcal{G})$ , under the identification. Since subgroups of  $GL(\mathcal{G})$  have algebraic closure, subgroup of

an almost algebraic group in  $\text{Aut}(G)$  has almost algebraic closure, that is, the smallest almost algebraic group containing the subgroup.

**Theorem 7.1** *Let  $G$  be a connected Lie group having no compact central subgroup of positive dimension. Let  $\Gamma \subset \text{Aut}(G)$  be a subgroup. Then the following are equivalent.*

- (1)  $\Gamma$ -action on  $G$  is point-wise distal;
- (2)  $\Gamma$ -action on  $G$  is distal;
- (3)  $\Gamma$ -action on  $P(G)$  is distal;
- (4)  $\Gamma$ -action on  $P(G)$  is point-wise distal.

**Proof** The implication that (1)  $\Rightarrow$  (2) follows from Theorem 1.1 of [1] and [6] and that (3)  $\Rightarrow$  (4)  $\Rightarrow$  (1) are easy to verify. We now prove that (2)  $\Rightarrow$  (3).

Assume that  $\Gamma$ -action on  $G$  is distal. Let  $\mathcal{G}$  be the Lie algebra of  $G$ . Since  $\Gamma$ -action on  $G$  is distal, by Theorem 1.1 of [1] we get that  $\Gamma$ -action on  $\mathcal{G}$  is also distal. By Theorem 1 of [6], almost algebraic closure of  $\Gamma$  is a compact extension of a unipotent subgroup of  $GL(\mathcal{G})$ . Since  $G$  has no compact central subgroup of positive dimension, Theorem 1 of [7] implies that  $\text{Aut}(G)$  is an almost algebraic subgroup of  $GL(\mathcal{G})$ . It follows that almost algebraic closure of  $\Gamma$  is contained in  $\text{Aut}(G)$ . Thus, we may assume that  $\Gamma$  is a unipotent subgroup of  $\text{Aut}(G)$ .

Since  $\Gamma$  is a unipotent subgroup of  $\text{Aut}(G) \subset GL(\mathcal{G})$ , if  $\Phi$  defines the conjugacy action of the group  $\tilde{G} = \text{Aut}(G) \ltimes G$ , then  $\Phi(\Gamma)$  is also a unipotent subgroup of  $\text{Aut}(\tilde{G})$  and hence its action on  $\tilde{G}$  is distal. Thus, replacing  $G$  by  $\tilde{G}$  and considering the conjugacy action, we may assume that  $\alpha(x) = x$  for all  $x$  in the center of  $G$  and all  $\alpha \in \Gamma$ .

Suppose  $\{\alpha_n\}$  is in  $\Gamma$  such that  $\lim \alpha_n(\mu) = \lim \alpha_n(\lambda)$  for some  $\mu$  and  $\lambda$  in  $P(G)$ . Let  $\text{Ad}: G \rightarrow GL(\mathcal{G})$  be the adjoint representation and  $\theta: \Gamma \rightarrow GL(\text{End}(\mathcal{G}))$  be defined by

$$\theta(\alpha)(v) = (d\alpha)v(d\alpha)^{-1}$$

for any  $\alpha \in \Gamma$  and any  $v \in \text{End}(\mathcal{G})$ . Then  $\theta(\alpha)\text{Ad}(g) = \text{Ad}(\alpha(g))$  for any  $\alpha \in \Gamma$  and any  $g \in G$ . Thus,  $\lim \theta(\alpha_n)\text{Ad}(\lambda) = \lim \theta(\alpha_n)\text{Ad}(\mu)$ .

Now, by passing to subsequence we may assume by Lemma 2.1 of [8] that  $\theta(\alpha_n)(w)$  converges for all  $w$  in the support of  $\text{Ad}(\mu)$  and in the support of  $\text{Ad}(\lambda)$ . Let  $W = \{w \in \text{End}(\mathcal{G}) \mid \{\theta(\alpha_n)(w)\} \text{ converges}\}$ . Then  $W$  is a subalgebra of  $\text{End}(\mathcal{G})$  containing the supports of  $\text{Ad}(\lambda)$  and  $\text{Ad}(\mu)$ . By Lemma 2.2 of [8], there exist sequences  $\{\beta_n\}$  and  $\{\beta'_n\}$  such that  $\{\theta(\beta'_n)\}$  converges and  $\theta(\beta_n)(v) = v$  for all  $v \in W$  and  $\theta(\alpha_n) = \theta(\beta'_n\beta_n)$  for all  $n \geq 1$ . Again by passing to a subsequence and replacing  $\beta_n$  and  $\beta'_n$  by an element from  $\beta_n \text{Ker}(\theta)$  and  $\beta'_n \text{Ker}(\theta)$  respectively, we may assume that  $\{\beta'_n\}$  converges and  $\alpha_n = \beta'_n\beta_n$  with  $\theta(\beta_n)(w) = w$  for all  $w \in W$  and all  $n \geq 1$ . This implies that  $\lim \beta_n(\mu) = \lim \beta_n(\lambda)$  and  $\beta_n \in \Gamma' = \{\alpha \in \Gamma \mid \theta(\alpha(v)) = v \text{ for all } v \in W\}$ . Then  $\Gamma'$  is a unipotent subgroup of  $\text{Aut}(G)$ .

Let  $\tilde{H}$  be the smallest almost algebraic subgroup containing the supports of  $\text{Ad}(\mu)$  and  $\text{Ad}(\lambda)$ . Now for  $\alpha \in \Gamma'$  and  $x \in \text{Ad}^{-1}(\tilde{H}) = H$ , say,  $x^{-1}\alpha(x)$  is in the center of  $G$  and  $H$  contains the center of  $G$ . Thus, using Lemma 3.1 of [8] we see that the conditions of Proposition 3.2 of [8] are verified and hence by Proposition 3.2 of [8], there exist sequences  $\{T_n\}$  and  $\{S_n\}$  in  $\Gamma'$  such that  $\{T_n\}$  is relatively compact,  $S_n(x) = x$  for all  $x$  in the supports of  $\mu$  and  $\lambda$  and  $\beta_n = T_n S_n$  for all  $n \geq 1$ . If  $T$  is a limit point of  $\{T_n\}$ , then  $T(\mu) = T(\lambda)$ . Thus,  $\lambda = \mu$ . This proves that the action of  $\Gamma$  on  $P(G)$  is distal.

**Example 7.2** We will now show that the assumption on the center of  $G$  in Theorem 7.1 can not be relaxed. Let  $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$  be the circle group and  $K = \mathbb{T}^2$  and  $\alpha: K \rightarrow K$  be given by

$$\alpha(w, z) = (wz, z)$$

for all  $(w, z) \in K$ . Then  $\alpha$  is a continuous automorphism of  $K$  and let  $\Gamma$  be the group generated by  $\alpha$ . Since the eigenvalues of  $\alpha$  are one, the action of  $\Gamma$  on  $K$  is distal. Let  $L = \{(1, z) \in K \mid z \in \mathbb{T}\}$  and  $\mu \in P(K)$  be invariant under  $L$ . We will now show that there exists a subsequence  $\{k_n\}$  such that  $\alpha^{k_n}(\mu) \rightarrow \omega_K$ .

Since  $K$  is monothetic, there exists  $x, y$  in  $\mathbb{T}$  such that the closed subgroup generated by  $(x, y)$  is  $K$ . This implies that the closed subgroup generated by  $y$  is the circle group  $\mathbb{T}$ . Let  $\{k_n\}$  be such that  $y^{k_n} \rightarrow x$ . By passing to a subsequence of  $\{k_n\}$ , we may assume that  $\lim \alpha^{k_n}(\mu) = \lambda \in P(K)$  exists. Now,  $(x, y)\lambda = \lim (y^{k_n}, y)\alpha^{k_n}(\mu) = \lim \alpha^{k_n}((1, y)\mu) = \lim \alpha^{k_n}(\mu) = \lambda$  as  $\mu$  is invariant under  $L$ . Thus,  $\lambda$  is  $(x, y)$ -invariant. Since  $K$  is the closed subgroup generated by  $(x, y)$ , we get that  $\lambda$  is  $K$ -invariant and hence  $\lambda$  is the normalized Haar measure on  $K$ . This shows that if  $\mu$  is invariant under

$L$ , then the closure of the orbit  $\Gamma(\mu)$  contains  $\omega_K$ . Thus, the action of  $\Gamma$  on  $P(K)$  is not distal. This also shows that the group  $\mathbb{Z} \ltimes_{\alpha} K$  is distal but not Tortrat.

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C. R. E. Raja  
 Stat-Math Unit  
 Indian Statistical Institute (ISI)  
 8th Mile Mysore Road  
 Bangalore 560 059, India  
 creraja@isibang.ac.in

R. Shah  
 School of Physical Sciences (SPS)  
 Jawaharlal Nehru University (JNU)  
 New Delhi 110 067, India  
 rshah@mail.jnu.ac.in  
 riddhi@math.tifr.res.in