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Cancellation theorem for injective tensor product of order unit Banach spaces

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CANCELLATION THEOREM FOR INJECTIVE TENSOR PRODUCT OF ORDER UNIT BANACH SPACES

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Abstract. In this paper we formulate and prove an order unit Banach space version of the Banach-Stone theorem, obtained recently ([6]) for the case of lattice-valued continuous functions.

1. Introduction

Let X, Y be compact Hausdorff spaces and E a Banach lattice and F be an abstract M-space with unit. Let $\pi : C(X, E) \to C(Y, F)$ be a Riesz isomorphism such that $0 \notin f(X)$ if and only if $0 \notin \pi(f)(Y)$ for each $f \in C(X, E)$. Ercan and Önal have proved in [6] that E is Riesz isomorphic to F and X is homeomorphic to Y. Identifying the space $C(X, E)$ with the injective tensor product space $C(X) \otimes_\epsilon E$ the above theorem can be interpreted as imposing additional conditions on the Riesz isomorphism to ensure that the component spaces are again Riesz isomorphic. Such results are known as cancellation theorems . It is well-known that an additional condition is necessary on π to ensure this conclusion. See Corollary 7.3 of [3] and the remark following it. Also see [4] for topological conditions which make the component spaces homeomorphic.

Let E be an order unit Banach space with an order unit e and let K be a compact convex set. We follow the notation and terminology of [2] and [1], as these monographs relate to both lattice theory and convexity theory. Let $A(K, E)$ denote the space of affine E-valued continuous functions, equipped with the supremum norm. We equip this space with the point-wise ordering. It is well known that when K is a Choquet simplex, $A(K)$ has the metric approximation property and the space $A(K, E)$ can be identified with $A(K) \otimes_\epsilon E$, see [9]. Also when K is a Choquet simplex with the set

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of extreme points $\partial_e K$, closed, $A(K)$ is isometric (via the restriction map) to $C(\partial_e K)$. Thus it is interesting to consider questions similar to those answered in [6] for the family of Choquet simplexes and order unit Banach spaces. We recall that $S = \{e^* \in E^* : e^*(e) = 1 = \|e^*\|\}$ is called the state space of E. Our argument relies on tensor product theory of convex sets as developed in [9] and [8].

2. Main Result

The following theorem extends the main result of [6]. In order to prove this we recall the Banach-Stone theorem for $A(K)$ spaces, when K is a simplex, due to Lazar, [8]. See [10] for this formulation. We note that analogous to the continuous function space, multiplication of affine functions on the extreme boundary of a simplex is well-defined.

Theorem 1. *Let K_1, K_2 be simplexes. Let $\pi : A(K_1) \to A(K_2)$ be an isometry. Then $|\pi(1)| \equiv 1$ on $\partial_e K_2$ and there exists an affine homeomorphism $\phi : K_2 \to K_1$ such that $\pi(a) = \pi(1)\ a \circ \phi$ on $\partial_e K_2$.*

In particular if π is an order unit isometry then $\pi(1) = 1$.

In the following theorem we impose an additional condition on the component spaces to derive conclusion similar to the one in [6]. We recall that for any closed face F of a convex set, its complementary face is denoted by F' (see [1] Chapter 2.6). The notation involved in the statement here will be clear during the proof of the theorem.

Theorem 2. *Let K_1, K_2 be Choquet simplexes. Let E be an order unit Banach space and let F be an order unit Banach space with the Riesz decomposition property. Let $\pi : A(K_1, E) \to A(K_2, F)$ be an order isometry. Suppose $0 \notin a(\partial_e K_1) \Leftrightarrow 0 \notin \pi(a)(\partial_e K_2)$ for every $a \in A(K_1, E)$. Suppose for any affine continuous surjection ψ from the tensor product of K_2 and the state space of F onto K_1 and for any complementary face H' with $\psi(H') = K_1$, there is a closed face $G \subset H'$ such that $\psi(G) = K_1$. Then K_1 is affine homeomorphic to K_2 and E is order unit isometric to F.*

Proof. Since E is an order unit Banach space, by a theorem of Kadison (Theorem II.1.8 in [1]) it follows that there is a compact convex set K_3 such that E is order unit isometric to $A(K_3)$. Similarly since F also has the Riesz decomposition property it follows from Proposition II.3.3 in [1] that

F is order unit isometric to $A(K_4)$ for a simplex K_4. We also note that K_3 and K_4 are the state spaces of E and F respectively.

Let $BA(K_1 \times K_3)$ denote the space of continuous functions that are affine in each variable. Define $\psi : A(K_1, A(K_3)) \to BA(K_1 \times K_3)$ by $\psi(a)(k_1, k_3) = a(k_1)(k_3)$. It is easy to see that ψ is an order unit isometry. Since K_1 is a simplex, it follows from Corollary 2.6 of [9] that $A(K_1, A(K_3))$ is isometric to $A(K_1 \otimes K_3)$.

On the other hand it can be directly verified that $A(K_2, A(K_4)) = A(K_2 \otimes K_4)$ has the Riesz decomposition property so that $K_2 \otimes K_4$ is a simplex. Thus $K_1 \otimes K_3$ is a simplex and hence by Proposition 2.10 of [9] we have that K_3 is also a simplex.

Thus we may assume without loss of generality that there is an affine homeomorphism $\phi : K_2 \otimes K_4 \to K_1 \otimes K_3$ such that $\pi(a) = a \circ \phi$. We also recall from Theorem 1.2 of [9] that any extreme point s of $K_2 \otimes K_4$ is of the form $s = k_2 \otimes k_4$ for extreme points k_2 and k_4. We further note that treating $\{k_4\}$ as a face of K_4 and hence a simplex, we have clearly that $K_2 \otimes \{k_4\}$ is a face of $K_2 \otimes K_4$. Thus if we can show that there is a unique $k_3 \in \partial_e K_3$ such that $\phi(K_2 \otimes \{k_4\}) = K_1 \otimes \{k_3\}$ then it will follow that K_2 is affine homeomorphic to K_1 . Similar argument will show that K_4 is affine homeomorphic to K_3 so that F is order unit isometric to E.

In what follows by P_i we denote projection from the simplex tensor product to the corresponding component space. We first show that for any $k_4 \in \partial_e K_4$, $P_1\phi(K_2 \otimes \{k_4\}) = K_1$. Clearly it is enough to show that $\partial_e K_1 \subset P_1\phi(K_2 \otimes \{k_4\})$. Suppose an extreme point $k_1 \notin P_1\phi(K_2 \otimes \{k_4\})$. Since $\{k_1\} \otimes K_3$ and $\phi(K_2 \otimes \{k_4\})$ are disjoint closed faces of a simplex, it follows from Lemma 3.1.3 of [2] that there exists a $a \in A(K_1 \otimes K_3)$ such that $a = 1$ on $\phi(K_2 \otimes \{k_4\})$ and $a = 0$ on $\{k_1\} \otimes K_3$. Thus we have $0 \notin \pi(a)(\partial_e K_2)$ but $0 \in a(\partial_e K_1)$. This contradiction shows that $P_1\phi(K_2 \otimes \{k_4\}) = K_1$.

Now in order to complete the proof along the lines of arguments given during the proof of Lemma 5 in [6] we need to show the following. Since $\{k_4\}$ is a split face of the simplex K_4, by Theorem II.6.22 of [1], there exists a complementary face F' such that $K_4 = CO(\{k_4\} \cup F')$ (CO, denotes the convex hull). We now claim that $P_1\phi(K_2 \otimes F') \neq K_1$.

Suppose $P_1\phi(K_2 \otimes F') = K_1$. Note that $P_1\phi : K_2 \otimes K_4 \to K_1$ is an affine continuous onto map. Now by the condition we have assumed on the state spaces, there is a closed face $G \subset K_2 \otimes F'$ such that $P_1\phi(G) = K_1$.

Now $\phi(G)$ and $\phi(K_2 \otimes \{k_4\})$ are disjoint closed faces of $K_1 \otimes K_3$. Thus by a separation argument identical to the one given above we get a contradiction. Thus $P_1\phi(K_2 \otimes F') \neq K_1$. This completes the proof.

$\square$

Remark 3. *It follows from Lemma 4 of* [6] *that in the case of continuous function spaces, the additional condition imposed on the state spaces in the above theorem is always satisfied. We do not know if this additional condition always holds in the category of simplex spaces.*

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