Distal actions and ergodic actions on compact groups

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Abstract

Let K be a compact metrizable group and Γ be a group of automorphisms of K. We first show that each $\alpha \in \Gamma$ is distal on K implies Γ itself is distal on K, a local to global correspondence provided Γ is a generalized $\overline{FC}$-group or K is a connected finite-dimensional group. We show that Γ contains an ergodic automorphism when Γ is nilpotent and ergodic on a connected finite-dimensional compact abelian group K.

1 Introduction

We shall be considering actions on compact groups. By a compact group we shall mean a compact metrizable group and by an automorphism we shall mean a continuous automorphism. For a compact group K, Aut (K) denotes the group of automorphisms of K. An action of a topological group Γ on a compact metrizable group K by automorphisms, is a homomorphism $\phi \colon \Gamma \to \mathrm{Aut}\ (K)$ such that the map $(\alpha, x) \mapsto \phi(\alpha)(x)$ is a continuous map: when only one action is studied or when there is no confusion instead of $\phi(\alpha)(x)$ we write $\alpha(x)$ for $\alpha \in \Gamma$ and $x \in K$. In such cases, the map ϕ is said to define the action of Γ on K and such actions are called algebraic actions.

We shall assume that a topological group Γ acts on a compact metrizable group K. For each $\alpha \in \Gamma$, $(n, a) \mapsto \alpha^n(a)$ defines a $\mathbb{Z}$-action on K and this action on K is called $\mathbb{Z}_\alpha$-action. Suppose $K_1 \supset K_2$ are closed Γ-invariant subgroups of K such that K_2 is normal in K_1. By an action of Γ on K_1/K_2, we mean the canonical action of Γ on K_1/K_2 defined by $\alpha(xK_2) = \alpha(x)K_2$ for all $x \in K_1$ and all $\alpha \in \Gamma$.

Suppose Γ acts on the compact groups K and L. We say that K and L are Γ-isomorphic if there exists a continuous isomorphism $\Phi \colon K \to L$ such that $\Phi(\alpha(x)) = \alpha(\Phi(x))$ for all $\alpha \in \Gamma$ and $x \in K$.

It is interesting to find properties of group actions that hold if the property holds for every $\mathbb{Z}_\alpha$-action. We term any such property a local to global correspondence

as this property holds for the whole group Γ when it holds locally at every point of Γ. We first state the follwing well-known classical local to global correspondence for linear actions on vector spaces proof of which may be found in [5].

Burnside Theorem: Let V be a finite-dimensional vector space over reals and G be a finitely generated subgroup of $GL(V)$, the group of linear transformations on V. If each element of G has finite order, then G itself is a finite group.

Main aim of the note is to exhibit such local to global correspondences for algebraic actions on compact groups.

Definition 1 We say that the action of Γ on K is distal if for any $x \in K \setminus (e)$, e is not in the closure of the orbit $\Gamma(x) = \{\alpha(x) \mid \alpha \in \Gamma\}$. In such case, we say that Γ is distal (on K).

We now introduce a type of action which is obiviously distal.

Definition 2 We say that the action of Γ on K is compact (respectively, finite) if the group $\phi(\Gamma)$ is contained in a compact (respectively, finite) subgroup of Aut (K) where ϕ is the map defining the action of Γ on K.

We now see the notion of ergodic action which is orthogonal to distal action.

Definition 3 Let K be a compact group and ω_K be the normalized Haar measure on K. We say that an (algebraic) action of Γ on K is ergodic if any Γ-invariant Borel set A of K satisfies $\omega_K(A) = 0$ or $\omega_K(A) = 1$.

Definition 4 Let K be a compact group and α be a continuous automorphism of K. If the action of $\mathbb{Z}_\alpha$ on K is distal (respectively, ergodic), then we say that α is a distal (respectively, ergodic) automorphism of K or α is distal (respectively, ergodic) on K.

It is easy to see that Γ is distal implies each $\alpha \in \Gamma$ is distal. In general each $\alpha \in \Gamma$ is distal need not imply Γ is distal (Example 1, [16]). For actions on connected Lie groups [1] and for certains actions on p-adic Lie groups [15] the local to global correspondence, that is, from each $\alpha \in \Gamma$ being distal on K to the whole group Γ being distal on K holds: distal notion has canonical extension to actions on locally compact spaces [6]. Recently [11] showed under certain conditions on K and Γ that each $\alpha \in \Gamma$ is distal and the whole group Γ is distal are equivalent to the action being equicontinuous (that is, having invariant neighbourhoods).

We now introduce a class of groups whose action is one of the main studies in this article.

Definition 5 A locally compact group G is called a generalized $\overline{FC}$-group if G has a series $G = G_0 \supset G_1 \supset \cdots \supset G_n = \{e\}$ of closed normal subgroups such that G_i/G_{i+1} is a compactly generated group with relatively compact conjugacy classes for $i = 0, 1, \cdots, n-1$.

It follows from Theorem 2 of [12] that compactly generated locally compact groups of polynomial growth are generalized $\overline{FC}$-groups and any polycyclic group is a generalized $\overline{FC}$-group.

It can be easily seen that the class of generalized $\overline{FC}$-groups is stable under forming continuous homomorphic images and closed subgroups. If H is a compact normal subgroup of a locally compact group G such that G/H is a generalized $\overline{FC}$-group, then it is easy to see that G is also a generalized $\overline{FC}$-group.

In this article we first invesigate the connection between distal actions and ergodic actions and apply it to prove a local to global correspondence for distal actions if Γ is a generalized $\overline{FC}$-group.

We next consider finite-dimensional compact groups and prove the local to global correspondence for any distal action on compact connected finite-dimensional groups. We also prove a structure theorem for distal actions on compact connected finite-dimensional abelian groups along the lines of [4].

The study of ergodic actions on compact groups is a key tool in proving the afore-stated results. Using these methods we prove the existence of ergodic automorphim in Γ when Γ is nilpotent and ergodic on a compact connected finite-dimensional abelian group: when Γ is abelian this result is shown in [2].

Having explained our results, it is easy to see that only $\phi(\Gamma)$ matters and not all of Γ. So, we may assume that Γ is a group of automorphisms of K.

2   Distal and Ergodic

In this section we explore the connection between distal actions and ergodic actions on compact (metrizable) groups using the dual structure of compact groups.

Let K be a compact group and Γ be a group acting on K. Let $\hat{K}$ be the equivalent classes of continuous irreducible unitary representations of K. If π is a continuous irreducible unitary representation of K, then $[\pi] \in \hat{K}$ denotes the set of all continuous irreducible unitary representations of K that are unitarily equivalent to π. We write $\pi_1 \sim \pi_2$ if $\pi_1, \pi_2 \in [\pi]$ for some $[\pi] \in \hat{K}$. For a continuous irreducible unitary representation π of K and $\alpha \in \Gamma$, $\alpha(\pi)$ is defined by

$$\alpha(\pi)(x) = \pi(\alpha^{-1}(x))$$

for all $x \in K$ and it can be easily verified that $\alpha(\pi)$ is also a continuous irreducible unitary representation of K. If $\alpha \in \Gamma$ and $\pi_1, \pi_2 \in [\pi]$, then $\alpha(\pi_1) \sim \alpha(\pi_2)$. Thus, the map $(\alpha, [\pi]) \mapsto \alpha[\pi] = [\alpha(\pi)]$ is well-defined and is known as the dual of action of Γ on the dual $\hat{K}$ of K. For $k \geq 1$, let $U_k(\mathbb{C})$ be the group of unitaries on $\mathbb{C}^k$ and I_k denote the identity matrix in $U_k(\mathbb{C})$. Then $U_k(\mathbb{C})$ is a compact group and for each $[\pi] \in \hat{K}$, there exists a $k \geq 1$ such that $\pi(x) \in U_k(\mathbb{C})$ for all $x \in K$: see [7] for details on representations of compact groups.

Proposition 2.1 *Let K be a (non-trivial) compact group and Γ be a group of auto-morphisms of K. Then the following are equivalent:*

(1) Γ is distal on K;

(2) for each Γ-invariant non-trivial closed subgroup L of K, action of Γ on L is not ergodic;

(3) for each Γ-invariant non-trivial closed subgroup L of K, there exists a non-trivial continuous irreducible unitary representation π of L such that the orbit $\Gamma[\pi] = \{\alpha[\pi] \mid \alpha \in \Gamma\}$ is finite in $\hat{L}$.

Proof Let L be a non-trivial Γ-invariant closed subgroup of K. If the action of Γ on L is ergodic, then by Theorem 2.1 of [2], $\Gamma(x) = \{\alpha(x) \mid \alpha \in \Gamma\}$ is dense in L for some $x \in L$. Since L is non-trivial, $x \neq e$ and hence e is in the closure of $\Gamma(x)$ for $x \neq e$. Thus, we get that $(1) \Rightarrow (2)$ and that $(2) \Rightarrow (3)$ follows from Theroem 2.1 of [2].

Now assume that (3) holds. Let $x \neq e$ be in K and L be the closed subgroup generated by $\Gamma(x)$. Then L is a non-trivial Γ-invariant closed subgroup of K. Then by assumption there exists a non-trivial $[\pi_1] \in \hat{L}$ such that $\Gamma([\pi_1])$ is finite. Let $\Gamma_0 = \{\alpha \in \Gamma \mid \alpha(\pi_1) \sim \pi_1\}$. Then Γ_0 is a closed subgroup of Γ of finite index. Let $\Gamma_1 = \cap_{\alpha \in \Gamma} \alpha \Gamma_0 \alpha^{-1}$. Then Γ_1 is a normal subgroup of Γ of finite index and Γ_1 is contained in Γ_0. Let $A = \{[\pi] \in \hat{L} \mid \Gamma_1[\pi] = [\pi]\}$. Then A contains π_1. Since Γ_1 is normal in Γ, A is Γ-invariant. Let $L_1 = \cap_{[\pi] \in A} \{x \in L \mid \pi(x) = \pi(e)\}$. Then L_1 is a Γ-invariant closed normal subgroup of L and L_1 is a proper subgroup of L as A is non-trivial. If e is in the closure of $\Gamma(x)$, then since Γ/Γ_1 is finite, e is in the closure of $\Gamma_1(x)$. Let $\alpha_n \in \Gamma_1$ be such that $\alpha_n(x) \to e$ and $[\pi] \in A$. Then there exist $u_n \in U_k(\mathbb{C})$ (k may depend on π) such that

$$u_n^{-1}\pi(g)u_n = \pi(\alpha_n(g))$$

for all $g \in L$. This implies that

$$u_n^{-1}\pi(x)u_n = \pi(\alpha_n(x)) \to \pi(e) = I_k$$

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as $n \to \infty$. Since $U_k(\mathbb{C})$ is compact, $\pi(x) = I_k$. This implies that $x \in L_1$ which is a contradiction as L_1 is a proper Γ-invariant subgroup of L and L is the closed subgroup generated by $\Gamma(x)$. Thus, e is not in the closure of $\Gamma(x)$. Hence (3) $\Rightarrow$ (1).

We now prove a result for compact abelian groups by employing the dual structure of locally compact abelian groups. For a locally compact abelian group G, a continuous homomorphism of G into the circle group $T = \{z \in \mathbb{C} \mid |z| = 1\}$ is known as a character of G and the dual group of G denoted by $\hat{G}$ is defined to be the group of all characters of G. The group $\hat{G}$ is a locally compact abelian group with the standard compact-open topology and the dual of $\hat{G}$ is (isomorphic to) G.

It is known that G is compact if and only if $\hat{G}$ is discrete and there is a one-one correspondence between closed subgroups of G and the quotients of $\hat{G}$: [13] for deatils on duality of locally compact abelian groups.

If G is a group and $A_1, A_2, \cdots, A_n$ are subsets of G, then $< A_1, \cdots, A_n >$ denotes the group generated by $A_1, A_2, \cdots, A_n$ and if any $A_i = \{g\}$, we may write g instead of $\{g\}$.

Lemma 2.1 *Let K be a compact abelian group and Γ be a group of automorphisms of K. Let α be an automorphism of K such that $\alpha\Gamma\alpha^{-1} = \Gamma$. Suppose the action of Γ is not ergodic on K and for each α-invariant proper closed subgroup L of K, the action of $\mathbb{Z}_\alpha$ on K/L is not ergodic. Then there exists a non-trivial character χ on K such that the orbit $\{\beta(\chi) \mid \beta \in< \Gamma, \alpha >\}$ is finite - in other words, the group generated by Γ and α is not ergodic on K.*

Proof We first note that the assumption on α is equivalent to saying that for any α-invariant non-trivial subgroup A of $\hat{K}$ there exists a non-trivial character $\chi \in A$ such that the orbit $\{\alpha^n(\chi) \mid n \in \mathbb{Z}\}$ is finite.

Let $A = \{\chi \in \hat{K} \mid \Gamma(\chi) \text{ is finite}\}$. Since Γ is not ergodic on K, A is non-trivial. Since $\alpha\Gamma\alpha^{-1} = \Gamma$, A is α-invariant. By assumption on α, there exists a non-trivial χ_0 in A such that $\alpha^k(\chi_0) = \chi_0$ for some $k \geq 1$. Then

$$\Gamma\alpha^n(\chi_0) \subset \cup_{i=1}^k \Gamma\alpha^i(\chi_0)$$

for all $n \in \mathbb{Z}$. Since $\chi_0 \in A$ and A is α-invariant, we get that $\{\beta(\chi_0) \mid \beta \in< \Gamma, \alpha >\}$ is finite.

Lemma 2.2 *Let K be a compact abelian group and Γ be a group of automorphisms of K. Suppose Γ is a generalized $\overline{FC}$-group and for each $\alpha \in \Gamma$ and each α-invariant proper closed subgroup L of K, the action of $\mathbb{Z}_\alpha$ on K/L is not ergodic. Then there exists a non-trivial character χ on K such that the corresponding Γ-orbit $\{\alpha(\chi) \mid \alpha \in \Gamma\}$ is finite or equivalently the action of Γ on K is not ergodic.*

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Proof Since K is compact abelian, Aut (K) is totally disconnected and hence by Proposition 2.8 of [11], Γ contains a compact open normal subgroup Δ such that Γ/Δ contains a polycyclic subgroup of finite index. Let Λ be a closed normal subgroup of Γ of finite index containing Δ such that Λ/Δ is polycylic. Let $\Lambda_0 = \Lambda$ and $\Lambda_i = [\Lambda_{i-1}, \Lambda_{i-1}]$ for $i \geq 1$. Then there exists a $k \geq 0$ such that $\Lambda_k \Delta \neq \Delta$ and $\Lambda_{k+1} \Delta = \Delta$. It can be easily seen that each $\Lambda_i \Delta$ is finitely generated modulo Δ. For $0 \leq i \leq k$, let $\alpha_{i,1}, \cdots, \alpha_{i,m}$ be in Λ_i such that $\alpha_{i,1}, \cdots, \alpha_{i,m}$ and $\Delta\Lambda_{i+1}$ generate $\Delta\Lambda_i$. It can be easily seen that $\alpha_{i,j}$ normalizes $< \alpha_{i,1}, \cdots, \alpha_{i,j-1}, \Lambda_{i+1}, \Delta >$ for all i and j. Then repeated applicaton of Lemma 2.1 yields a non-trivial character $\chi \in \hat{K}$ such that the orbit $\Lambda(\chi)$ is finite. Since Λ is a normal subgroup of finite index in Γ, $\Gamma(\chi)$ is also finite.

We next prove a lemma which shows that the (global) distal condition in Proposition 2.1, can be relaxed to the local distal condition provided Γ is a generalized $\overline{FC}$-group.

Lemma 2.3 *Let K be a compact abelian group and Γ be a group of automorphisms of K. Suppose Γ is a generalized $\overline{FC}$-group and each $\alpha \in \Gamma$ is distal on K. Then Γ is not ergodic on K or equivalently there exists a non-trivial character χ on K such that the corresponding Γ-orbit $\{\alpha(\chi) \mid \alpha \in \Gamma\}$ is finite.*

Proof Let $\alpha \in \Gamma$ and L be a α-invariant proper closed subgroup of K. Since α is distal on K, the action $\mathbb{Z}_\alpha$ on K/L is also distal (Corollary 6.10 of [3]). This shows by Proposition 2.1 that the action of $\mathbb{Z}_\alpha$ is not ergodic on K/L. Thus, the result follows from Lemma 2.2.

3   Distal actions

In this section we prove that the distal condition has local to global correspodence for actions on compact groups provided the group of automorphisms is a generalized $\overline{FC}$-group.

Theorem 3.1 *Let K be a compact group and Γ be a group of automorphisms of K. Suppose Γ is a generalized $\overline{FC}$-group. Then the following are equivalent:*

1. *each $\alpha \in \Gamma$ is distal on K;*

2. *the action of Γ on K is distal.*

Proof Suppose each $\alpha \in \Gamma$ is distal on K. Let $x \in K$ be such that e is in the closure of the orbit $\Gamma(x)$. We now claim that $x = e$.

Case (i): Suppose K is abelian. Let L be a non-trivial Γ-invariant closed subgroup of K. It follows from Lemma 2.3 that Γ is not ergodic on L. Since L is arbitrary Γ-invariant closed subgroup, by Proposition 2.1 we get that Γ is distal on K.

Case (ii): Suppose K is connected. Let T be a maximal compact connected abelian subgroup of K containing x (see [10]). Then $\mathrm{Aut}(K) = \mathrm{Inn}(K)\Omega$ where $\Omega = \{\alpha \in \mathrm{Aut}(K) \mid \alpha(T) = T\}$ (see [10]). Let $\Gamma' = \Gamma\mathrm{Inn}(K)$ and $\Omega' = (\Gamma' \cap \Omega)$. Then Γ' and Ω' are also generalized $\overline{FC}$-groups. Since $\mathrm{Aut}(K) = \mathrm{Inn}(K)\Omega$, $\Gamma' = \mathrm{Inn}(K)\Omega'$. Since e is in the closure of $\Gamma(x)$, e is in the closure of $\Gamma'(x)$. Since $\mathrm{Inn}(K)$ is compact, e is in the closure of $\Omega'(x)$. As $x \in T$, applying case (i), we get that $x = e$.

General Case: Let K be any compact group and K_0 be the connected component of e in K. Then K_0 is Γ-invariant and by Corollary 6.10 of [3], each $\alpha \in \Gamma$ is distal on K/K_0. Since K/K_0 is totally disconnected, by Proposition 2.8 and Lemma 2.3 of [11], K/K_0 has arbitrarily small compact open subgroups invariant under Γ. This shows that $x \in K_0$. Now $x = e$ follows from case (ii).

As a consequence of results proved so far, we now prove an initial result on the existence of ergodic automorphisms in Γ when the action of Γ on K is ergodic.

Proposition 3.1 *Let K be a compact group and Γ be a group of automorphisms of K. Suppose Γ is a generalized $\overline{FC}$-group and the action of Γ on K is ergodic. Then we have the following:*

(i) there exist a $\beta \in \Gamma$ and a β-invariant non-trivial closed subgroup L of K such that the action of $\mathbb{Z}_\beta$ on L is ergodic.

(ii) In addition if K is abelian, there exist a $\alpha \in \Gamma$ and a α-invariant proper closed subgroup L of K such that the action of $\mathbb{Z}_\alpha$ on K/L is ergodic;

Proof Suppose for each $\alpha \in \Gamma$ and each α-invariant non-trivial closed subgroup L of K, the action of $\mathbb{Z}_\alpha$ on L is not ergodic. Then by Proposition 2.1, each $\alpha \in \Gamma$ is distal on K. By Theorem 3.1, the action of Γ on K is distal and hence by Proposition 2.1, the action of Γ on K is not ergodic. Thus, (i) is proved.

We now assume that K is abelian. Suppose for every $\alpha \in \Gamma$ and for every proper closed α-invariant subgroup L of K, the action of $\mathbb{Z}_\alpha$ on K/L is not ergodic. By Lemma 2.2, the action of Γ on K is not ergodic. Thus, (ii) is proved.

The following example shows that ergodic action of general, even a commutative group Γ on a compact abelian group need not imply the existence of a non-trivial subgroup or a non-trivial quotient that admits an ergodic $\mathbb{Z}_\alpha$-action for some $\alpha \in \Gamma$.

Example 3.1 Let (F_n) be an strictly increasing sequence of finite groups (one may take $F_n = \prod_{k=1}^n \mathbb{Z}/k\mathbb{Z}$) and $A = \cup F_n$. Let $K = M^A$ where M is a compact abelian

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group. Then K is a compact abelian group whose dual $\hat{K}$ consists of functions $f \colon A \to \hat{M}$ such that $f(A)$ is finite (see Theorem 17 of [13]). We consider the shift action of A on K defined by $ag(x) = g(a^{-1}x)$ for all $g \in K = M^A$ and all $a, x \in A$. Then the dual action of A on the dual $\hat{K}$ is given by $af(b) = f(a^{-1}b)$ for all $f \in \hat{K}$ and all $a, b \in A$. Let $f \in \hat{K}$ be non-trivial. Then there exists a F_n such that $f(a)$ is the trivial character on M for all $a \notin F_n$. For $i \geq 1$, let $a_i \in F_{n+i} \setminus F_{n+i-1}$ and $a \in F_n$ be such that $f(a)$ is not the trivial character. Then $a_j^{-1}a_i a \notin F_n$ if $i \neq j$ and hence $a_i^{-1}a_j f(a) = f(a_j^{-1}a_i a)$ is the trivial character if $i \neq j$. This shows that $a_i f \neq a_j f$ for all $i \neq j$. Thus, the orbit Af is infinite for any non-trivial $f \in \hat{K}$. This implies that the action of A on K is ergodic. Since any $a \in A$ has finite order, the action of $\mathbb{Z}_a$ is never ergodic for any $a \in A$.

4 Finite-dimensional compact groups

In this section we consider finite-dimensional compact groups. Let $\mathbb{Q}_d^r$ be the group $\mathbb{Q}^r$ with discrete topology. We may regard $\mathbb{Q}_d^r$ as a finite-dimensional vector space over $\mathbb{Q}$. Let B_r denote the dual of $\mathbb{Q}_d^r$. Then B_r is a compact connected group of finite-dimension and any compact connected finite-dimensional abelian group is a quotient of B_r for some r: see [13].

We first show that distal condition for algebraic actions on B_r has local to global correspondence with no restriction on the acting group Γ. The dual of any automorphism of B_r is a $\mathbb{Q}$-linear transformation of $\mathbb{Q}_d^r$ onto $\mathbb{Q}_d^r$. It can be easily seen that any group of unipotent transformations of $\mathbb{Q}_d^r$ is distal on B_r. We now show that upto finite extensions these are the only distal actions on B_r which is a structure theorem for distal actions on B_r along the lines of [4].

Proposition 4.1 *Let Γ be a group of automorphisms of B_r. Suppose Γ is distal on B_r. Then B_r has a series*

$$B_r = K_0 \supset K_1 \supset K_2 \supset \cdots \supset K_{n-1} \supset K_n = (e)$$

of closed connected Γ-invariant subgroups such that the action of Γ on K_i/K_{i+1} is finite for any $i \geq 0$. In particular, Γ is a finite extension of a group of unipotent transformations of $\mathbb{Q}_d^r$.

Proof Let Γ act distally on B_r. Then by Proposition 2.1, there exists a non-trivial $\chi_1 \in \mathbb{Q}_d^r$ such that orbit $\Gamma(\chi_1)$ is finite. Let $\tilde{\Gamma}_1 = \{\alpha \in \Gamma \mid \alpha(\chi_1) = \chi_1\}$. Then $\tilde{\Gamma}_1$ is a subgroup of finite index in Γ. Let Γ_1 be a normal subgroup of finite index in Γ and contained in $\tilde{\Gamma}_1$. Let $A_1 = \{\chi \in \mathbb{Q}_d^r \mid \Gamma_1(\chi) = \chi\}$. Then A_1 is a non-trivial Γ-invariant $\mathbb{Q}$-vector subspace of $\mathbb{Q}_d^r$. Let K_1 be the closed subgroup of K such that

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the dual of K/K_1 is A_1. Then K_1 is a proper Γ-invariant closed subgroup of K and the action of Γ on K/K_1 is finite. Since the dual of K_1 is the $\mathbb{Q}$-vector space $\mathbb{Q}_d^r/A_1$, $K_1 \simeq B_{r_1}$ for some $r_1 < r$. If $K_1 \neq (e)$, get K_2 by applying the above process to K_1 in place of K. Since B_r has finite-dimension and each K_i is connected, proceeding this way we obtain a series

$$K_0 = B_r \supset K_1 \supset K_2 \supset \cdots \supset K_{n-1} \supset K_n = (e)$$

of Γ-invaraint closed connected subgroups such that the action of Γ on K_i/K_{i+1} is finite.

Theorem 4.1 *Let Γ be a group of automorphisms of B_r. Suppose each $\alpha \in \Gamma$ is distal on B_r. If the dual action of Γ on $\mathbb{Q}_d^r$ is irreducible, then Γ is finite. In general, Γ is distal on B_r.*

Proof Let $\alpha \in \Gamma$. By considering the dual action of α, we may view α as a linear map on $\mathbb{Q}_d^r$. Then by Proposition 4.1, eigenvalues of α are of absolute value one. If the dual action of Γ on $\mathbb{Q}_d^r$ is irreducible, then let $V = \mathbb{Q}_d^r \otimes \mathbb{R}$ be the corresponding vector space over $\mathbb{R}$. Then V is also Γ-irreducible. By [4], Γ is contained in a compact subgroup of $GL(V)$. By Proposition 4.1, α^k is unipotent for some $k \geq 1$. Since α is in a compact group, we get that α is of finite order. Thus, every element α of Γ has finite order. It follows from Lemma 4.3 of [2] that Γ is finite.

We now proceed to show that ergodic action of Γ on a finite-dimensional compact connected abelian group yields an ergodic automorphism in Γ provided Γ is nilpotent.

Lemma 4.1 *Let Γ be a group of automorphisms of a compact group K and α be an automorphism of K. Suppose Γ and α are distal on K and $\alpha\Gamma\alpha^{-1} = \Gamma$. Then the group generated by Γ and α is distal on K.*

Proof Let Δ be the group generated by Γ and α. Let L be a closed subgroup of K invariant under Δ. Let $A = \{[\pi] \in \hat{L} \mid \Gamma([\pi]) \text{ is finite}\}$. Since Γ is normalized by α, A is α-invariant. Since α and Γ are distal on K, it follows from Proposition 2.1 that there exists a non-trivial $[\pi_0] \in A$ such that $\alpha^k(\pi_0) \sim \pi_0$ for some $k \geq 1$. Now, $\Gamma\alpha^i[\pi_0] \subset \cup_{j=1}^k \Gamma(\alpha^j[\pi_0])$ for any $i \in \mathbb{Z}$. This implies that the orbit $\Delta[\pi_0]$ is finite. This shows by Proposition 2.1 that Δ is distal on K.

Lemma 4.2 *Let α be an ergodic automorphism of B_r and L be a closed connected subgroup of B_r. Then α is ergodic on L.*

Proof It can be easily seen that α is ergodic on B_r if and only if no root of unity is an eigenvalue of α on $\mathbb{Q}_d^r$. Let V be the $\mathbb{Q}$-subspace of $\mathbb{Q}_d^r$ such that the dual of L

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is $\mathbb{Q}_d^r/V$. Since α is ergodic, no root of unity is an eigenvalue for α on $\mathbb{Q}_d^r$ and hence no root of unity is an eigenvalue for α on $\mathbb{Q}_d^r/V$. Thus, α is ergodic on L.

Lemma 4.3 *Let α and β be automorphisms on B_r. Suppose α is contained in a group Γ of automorphisms of B_r such that Γ is distal and β is ergodic and normalizes Γ. Then $\alpha^i\beta^j$ and $\beta^j\alpha^i$ are ergodic for all i and j in $\mathbb{Z}$ with $j \neq 0$.*

Proof It is enough to show that $\alpha\beta$ and $\beta\alpha$ are ergodic. We first prove the case when Γ is finite. Assume Γ is finite. Let χ be a character such that the orbit $\{(\alpha\beta)^n(\chi) \mid n \in \mathbb{Z}\}$ is finite. Since Γ is finite and Γ is normalized by $\alpha\beta$, the orbit $\tilde{\Gamma}(\chi)$ is also finite where $\tilde{\Gamma}$ is the group generated by $\alpha\beta$ and Γ. Since $\beta \in \tilde{\Gamma}$ and β is ergodic, we get that χ is trivial. Thus, $\alpha\beta$ is ergodic.

We now consider the general case. Let $V = \{\chi \in \mathbb{Q}_d^r \mid \Gamma(\chi) \text{ is finite}\}$. Since Γ is distal, V is a nontrivial $\mathbb{Q}$-subspace and V is invariant under β as Γ is normalized by β. Let L be the closed connected subgroup of B_r such that the dual B_r/L is V. Then L is a proper closed connected subgroup invariant under Γ and β and Γ is finite on B_r/L. Then $\alpha\beta$ is ergodic on B_r/L. Since the dual of L is $\mathbb{Q}_d^r/V$, $L \simeq B_s$ for some $s < r$. By Lemma 4.2, β is ergodic on L and hence by induction on dimension of B_r, $\alpha\beta$ is ergodic on L. Thus, $\alpha\beta$ is ergodic on B_r. Similarly we may show that $\beta\alpha$ is also ergodic on B_r.

Lemma 4.4 *Let α be an automorphism of B_r. Then there exists a compact connected subgroup K of B_r isomorphic to B_s for some $s > 0$ such that α is ergodic on K and α is distal on B_r/K. Moreover, if Γ is a nilpotent group of automorphisms of B_r containing α, then K is Γ-invariant.*

Proof Let V_1 be the $\mathbb{Q}$-subspace of $\mathbb{Q}_d^r$ defined by

$$V_1 = \{\chi \in \mathbb{Q}_d^r \mid (\alpha^n(\chi)) \text{ is finite}\}$$

and define V_i inductively by

$$V_i = \{\chi \in \mathbb{Q}_d^r \mid (\alpha^n(\chi) + V_{i-1}) \text{ is finite in } \mathbb{Q}_d^r/V_{i-1}\}$$

for any $i > 1$. Then each V_i is a α-invariant $\mathbb{Q}$-subspace. Since $\mathbb{Q}_d^r$ has finite-dimension over $\mathbb{Q}$, there exists a n such that $V_n = V_{n+i}$ for all $i \geq 0$ and for any non-trivial $\chi \in \mathbb{Q}_d^r/V_n$, the orbit $(\alpha^n(\chi) + V_n)$ is infinite. Let K be a closed subgroup of B_r such that the dual of K is $\mathbb{Q}_d^r/V_n$. Then K is α-invariant and connected. The choice of V_n shows that α is ergodic on K and α is distal on B_r/K.

Suppose Γ is a nilpotent group containing α. We show that V_1 is Γ-invariant by induction on the length of the series $\Gamma = \Gamma_0 \supset \cdots \supset \Gamma_k = [\Gamma, \Gamma_{k-1}] \supset \Gamma_{k+1} = (e)$.

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Now for $\beta \in \Gamma_k$ and $i \in \mathbb{Z}$, $\alpha^i \beta = \beta \alpha^i$ and hence V_1 is Γ_k-invariant. If V_1 is Γ_{k-j}-invariant, then for $\beta \in \Gamma_{k-j-1}$ and $i \in \mathbb{Z}$, $\alpha^i \beta = \beta \alpha^i \beta_i$ for some $\beta_i \in \Gamma_{k-j}$. Since V_1 is Γ_{k-j}-invariant and an iterate of α is trivial on V_1, we get that V_1 is Γ_{k-j-1}-invariant. Hence by induction on k, we get that V_1 is Γ-invariant. Since each V_i/V_{i-1} is the space of all characters whose orbit is finite in $\mathbb{Q}_d^r/V_{i-1}$, we get that V_i is Γ-invariant for any $i \geq 1$.

Lemma 4.5 *Let Γ be a nilpotent group of automorphisms of B_r and $\alpha, \beta \in \Gamma$. Let $\Gamma_0 = \Gamma$ and $\Gamma_i = [\Gamma, \Gamma_{i-1}]$ for $i \geq 1$. Let $k \geq 1$ be such that $\alpha \in \Gamma_{k-1} \setminus \Gamma_k$. Suppose α is ergodic on B_r and Γ_k is distal on B_r. Then there exists a $i \geq 0$ such that $\alpha^i \beta$ is ergodic on B_r.*

Proof We prove the result by induction on the dimension of B_r. If $r = 1$, then we have nothing to prove. So, we may assume that $r > 1$. If $\alpha\beta$ is ergodic on B_r, then we are done. So, we may assume by Lemma 4.4 that there exists a closed connected Γ-invariant proper subgroup K of B_r such that $\alpha\beta$ is ergodic on K and $\alpha\beta$ is distal on B_r/K. Let Δ be the group generated by $\alpha\beta$ and Γ_k. Then by Lemma 4.1, Δ is distal on B_r/K. Since α and β commute modulo Γ_k, we get that α normalizes Δ. By Lemma 4.3, $\alpha^i \beta$ is ergodic on B_r/K for all $i \geq 2$. Since α is ergodic on K, induction hypothesis applied to K in place of B_r and $\alpha^2 \beta$ in place of β, we get that $\alpha^j \beta$ is ergodic on K for some $j \geq 2$. Thus, $\alpha^j \beta$ is ergodic on B_r for some $j \geq 2$.

Lemma 4.6 *Let Γ be a nilpotent group of automorphisms of B_r and $\alpha, \beta \in \Gamma$. Let $\Gamma_0 = \Gamma$ and $\Gamma_i = [\Gamma, \Gamma_{i-1}]$ for $i \geq 1$ and $k \geq 1$ be such that $\alpha \in \Gamma_{k-1} \setminus \Gamma_k$. Let K be a closed Γ-invariant subgroup of B_r isomorphic to B_s for some $s \geq 0$ such that α is ergodic on K and α is distal on B_r/K. If β is ergodic on B_r/K and Γ_k is distal on B_r, then there exists $j \geq 0$ such that $\alpha^j \beta$ is ergodic on B_r.*

Proof Since α normalizes Γ_k, by Lemma 4.1, the group generated by α and Γ_k is distal on B_r/K. Since β centralizes α modulo Γ_k, it follows from Lemma 4.3 that $\alpha^i \beta$ is ergodic on B_r/K for all $i \geq 0$. By Lemma 4.5, $\alpha^j \beta$ is ergodic on K for some $j \geq 0$. This shows that for some $j \geq 0$, $\alpha^j \beta$ is ergodic on B_r.

Lemma 4.7 *Let Γ be a nilpotent group of automorphisms of B_r. Let $\Gamma_0 = \Gamma$ and $\Gamma_i = [\Gamma, \Gamma_{i-1}]$ for $i \geq 1$. Suppose that the action of Γ on B_r is ergodic. Then there exist a series*

$$(e) = K_0 \subset K_1 \subset K_2 \subset \cdots \subset K_{m-1} \subset K_m = B_r$$

of closed connected Γ-invariant subgroups with each $K_i \simeq B_{r_i}$ for some $r_i \geq 0$ and automorphisms $\alpha_1, \alpha_2, \cdots, \alpha_m$ in Γ with the following properties for each $i = 1, 2, \cdots, m$:

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(i) *if k_i is the smallest integer k for which $\alpha_i \notin \Gamma_k$, then the action of Γ_{k_i} on B_r/K_{i-1} is distal;*

(ii) *the action of $\mathbb{Z}_{\alpha_i}$ on K_i/K_{i-1} is ergodic;*

(iii) *the action of $\mathbb{Z}_{\alpha_i}$ on B_r/K_i is distal:*

Proof For each $\alpha \in \Gamma$, if the action of $\mathbb{Z}_\alpha$ is distal on B_r, then by Theorem 3.1, the action of Γ is distal. This is a contradiction to the ergodicity of Γ by Proposition 2.1. Thus, the action of $\mathbb{Z}_\alpha$ is not distal for some $\alpha \in \Gamma$.

Since Γ is nilpotent, there exists a k such that $\Gamma_k \neq (e)$ and $\Gamma_{k+1} = (e)$. Now, choose $\alpha_1 \in \Gamma_{k_1-1} \setminus \Gamma_{k_1}$ such that the action of $\mathbb{Z}_{\alpha_1}$ is not distal on B_r but the action of Γ_{k_1} is distal on B_r. By Lemma 4.4, there exists a non-trivial Γ-invariant closed connected subgroup K_1 of B_r isomorphic to B_{r_1} for some $r_1 > 0$ such that α_1 is ergodic on K_1 and the action of $\mathbb{Z}_{\alpha_1}$ is distal on B_r/K_1.

Let $L = B_r/K_1$. Then the action of Γ on L is ergodic and $L \simeq B_{s_1}$ for $s_1 < r$ as K_1 is non-trivial. By applying induction on the dimension of B_r, we get Γ-invariant closed connected subgroups $(e) = K_0 \subset K_1 \subset K_2 \subset \cdots \subset K_{m-1} \subset K_m = B_r$ and automorphisms $\alpha_2, \cdots, \alpha_m$ satisfying (i) - (iii) for $2 \leq i \leq n$.

We now consider connected finite-dimensional compact abelian groups. Let K be a connected finite-dimensional compact abelian group. Then

$$\mathbb{Z}^r \subset \hat{K} \subset \mathbb{Q}_d^r$$

and K is a quotient of B_r for some $r \geq 1$. Let α be an automorphism of K. Then α is an automorphism of $\hat{K}$. Since $\mathbb{Z}^r \subset \hat{K}$, α has a canonical extension to an invertible $\mathbb{Q}$-linear map on $\mathbb{Q}_d^r$, say $\tilde{\alpha}$. Thus, any automorphism α of K can be lifted to a unique automorphism $\tilde{\alpha}$ of B_r. Let Γ be a group of automorphisms of K and $\tilde{\Gamma}$ be the group consisting of lifts $\tilde{\alpha}$ of automorphisms $\alpha \in \Gamma$. We consider Γ and $\tilde{\Gamma}$ as topological groups with their respective compact-open topologies as automorphism groups of K and B_r. By looking at the dual action, we can see that the topological groups Γ and $\tilde{\Gamma}$ are isomorphic. If $\phi: B_r \to K$ is the canonical quotient map, then for $\alpha \in \Gamma$ and $x \in B_r$, we have

$$\phi(\tilde{\alpha}(x)) = \alpha(\phi(x))$$

where $\tilde{\alpha}$ is the lift of α on B_r.

Proposition 4.2 *Let K be a connected finite-dimensional compact abelian group. Let Γ be a group of automorphisms of K and $\tilde{\Gamma}$ be the corresponding group of automorphisms of B_r. Then Γ is distal (respectively, ergodic) on K if and only if $\tilde{\Gamma}$ is distal (respectively, ergodic) on B_r.*

Proof For $\chi \in \mathbb{Q}_d^r$, there exists $n \geq 1$ such that $n\chi \in \hat{K}$ and since $\hat{K} \subset \mathbb{Q}_d^r$, Γ is ergodic on K if and only if $\tilde{\Gamma}$ is ergodic on B_r. Since K is a quotient of B_r, $\tilde{\Gamma}$ is distal on B_r implies Γ is distal on K (see [3], Corrolary 6.10).

Suppose Γ is distal on K. By Proposition 2.1, there exists a non-trivial character χ_1 in $\hat{K} \subset \mathbb{Q}_d^r$ such that $\Gamma(\chi_1)$ is finite. Let $V_1 = \{\chi \in \mathbb{Q}_d^r \mid \tilde{\Gamma}(\chi)$ is finite$\}$. Then V_1 is a non-trivial $\tilde{\Gamma}$-invariant $\mathbb{Q}$-subspace as $\chi_1 \in V_1$. Let M be a closed subgroup of B_r such that the dual of M is $\mathbb{Q}_d^r/V_1$. Then M is $\tilde{\Gamma}$-invariant and $M \simeq B_s$ for $s < r$. Let $A = V_1 \cap \hat{K}$ and L be a closed subgroup of K such that the dual of L is $\hat{K}/A$. Then L is Γ-invariant. Since $\hat{K}/A \subset \mathbb{Q}_d^r/V_1$, $\hat{K}/A$ has no element of finite order and hence L is connected (see Theorem 30 of [13]). It can be verified that the dimension of L is same as the dimension of M. Hence by induction on the dimension of K we get that $\tilde{\Gamma}$ is distal on M. Since the action of $\tilde{\Gamma}$ on B_r/M is finite, $\tilde{\Gamma}$ is distal on B_r.

Theorem 4.2 *Let K be a compact connected finite-dimensional abelian group and Γ be a nilpotent group of automorphisms of K. Suppose Γ is ergodic on K. Then there exists a $\alpha \in \Gamma$ such that α is ergodic on K.*

Remark This result is another local to global correspondence (that is, no $\alpha \in \Gamma$ is ergodic implies Γ itself is not ergodic).

Proof We first assume that $K = B_r$ for some $r \geq 1$. By Lemma 4.7, there are Γ-invariant closed connected subgroups $(e) = K_0 \subset K_1 \subset K_2 \subset \cdots \subset K_{m-1} \subset K_m = B_r$ with each $K_i \simeq B_{r_i}$ for some $r_i \geq 0$ and automorphisms $\alpha_1, \alpha_2, \cdots, \alpha_m$ in Γ satisfying (i) - (iii) of Lemma 4.7. We may assume that $K_i \neq K_{i-1}$ for $1 \leq i \leq m$. We now prove the result by induction on r. If $r = 1$, we are done. Induction hypothesis applied to the action of Γ on $B_r/K_1 \simeq B_{r-r_1}$ yields $\beta \in \Gamma$ such that β is ergodic on B_r/K_1. By Lemma 4.6, there exists $\alpha \in \Gamma$ such that α is ergodic on B_r.

Now let K be a connected finite-dimensional compact abelian group and r be the dimension of K. Then K is a quotient of B_r. Let $\tilde{\Gamma}$ be the group of lifts of automorphisms of Γ. Then by Proposition 4.2, $\tilde{\Gamma}$ is ergodic on B_r. It follows from the previous case that there exists $\alpha \in \Gamma$ such that the lift $\tilde{\alpha}$ of α is ergodic on B_r. Another application of Proposition 4.2 shows that α itself is ergodic on K.

We now show that distal condition for algebraic actions on connected finite-dimensional compact groups has local to global correspondence with no restriction on the acting group Γ.

Theorem 4.3 *Let Γ be a group of automorphisms of a compact connected finite-dimensional group K. Suppose each $\alpha \in \Gamma$ is distal on K. Then the action of Γ on K is distal.*

Proof If K is abelian, then the result follows from Proposition 4.2 and Theoroem 4.1. Suppose K is any finite-dimensional compact connected group. Let $x \in K$ and (α_n) be a sequence in Γ. Suppose $\alpha_n(x) \to e$.

Let T be a maximal compact connected subgroup of K containing x. Since K is a connected group, $\text{Aut}(K) = \text{Inn}(K)\Omega$ where $\Omega = \{\alpha \in \text{Aut}(K) \mid \alpha(T) = T\}$ (see [10]). Let $\Lambda = \text{Inn}(K)\Gamma$. Since $\text{Inn}(K)$ is a compact normal subgroup, each $\alpha \in \Lambda$ is distal on K. Let $\alpha_n = a_n\beta_n$ where $a_n \in \text{Inn}(K)$ and $\beta_n \in \Omega \cap \Lambda$ for all $n \geq 1$. Since $\text{Inn}(K)$ is compact, by passing to a subsequence, if necessary, we may assume that $\beta_n(x) \to e$. Since T is closed in K which is of finite-dimension, T is also of finite-dimension ([14]) and so $\Omega \cap \Lambda$ is distal on T and hence $x = e$ as $x \in T$ and $\beta_n \in \Omega \cap \Lambda$. Thus, the action of Γ is distal on K.

We now prove a structure theorem for distal actions on compact connected finite-dimensional abelian groups along the lines of [4].

Proposition 4.3 *Let K be a compact connected finite-dimensional abelian group and Γ be a group of automorphisms of K. Suppose each $\alpha \in \Gamma$ is distal on K or equivalently the action of Γ on K is distal. Then K has a finite sequence*

$$K = K_0 \supset K_1 \supset K_2 \supset \cdots \supset K_{n-1} \supset K_n = (e)$$

of closed connected Γ-invariant subgroups such that the action of Γ on K_i/K_{i+1} is finite.

Proof Let K be a compact connected finite-dimensional abelian group. Let r be the dimension of K. Then K is a quotient of B_r. Suppose Γ is a group of automorphisms of K such that the action of Γ on K is distal. Let $\tilde{\Gamma}$ be the group of automorphisms of B_r consisting of lifts of automorphisms in Γ. By Proposition 4.2, $\tilde{\Gamma}$ is distal on B_r.

By Proposition 4.1, B_r has a series

$$B_r = K_0 \supset K_1 \supset K_2 \supset \cdots \supset K_{n-1} \supset K_n = (e)$$

of $\tilde{\Gamma}$-invariant subgroups such that the action of $\tilde{\Gamma}$ on K_i/K_{i+1} is finite for $i \geq 0$. Let $\phi: B_r \to K$ be the canonical projection. Let $L_i = \phi(K_i)$ for $1 \leq i \leq n$. Then each L_i is a closed connected Γ-invariant subgroup of K and $L_i \supset L_{i+1}$ for $i \geq 0$. Since lifting of an automorphism of K to an automorphism of B_r is unique, the action of Γ on L_i/L_{i+1} is finite for $i \geq 0$.

The above can be extended to distal actions on compact Lie groups which is already proved in [1] for finite-dimensional torus.

Corollary 4.1 *Let K be a compact real Lie group and Γ be a group of automorphisms of K. Suppose each $\alpha \in \Gamma$ is distal on K. Then there exist a series*

$$K = K_1 \supset K_2 \supset \cdots \supset K_{n-1} \supset K_n = (e)$$

of Γ-invariant closed connected normal subgroups of K such that the action of Γ on K_i/K_{i+1} is compact for any $i \geq 0$.

Proof Since K has only finitely many connected components, we may assume that K is connected. Since K is a compact connected Lie group, $K = TS$ where T is the connected component of identity in the center of K and $S = [K, K]$ is a compact connected semisimple Lie group (see [9]). Thus, T and S are invariant under Γ. Since T is abelian, by Proposition 4.3, there exists a series

$$T = T_0 \supset T_1 \supset \cdots \supset T_{n-1} \supset T_n = (e)$$

of closed connected Γ-invariant subgroups such that the Γ-action on T_i/T_{i+1} is finite for all i. For $0 \leq i \leq n$, let $K_i = T_iS$ and $K_{n+1} = (e)$. Then each K_i is a Γ-invariant closed connected normal subgroup of K such that $K_i \supset K_{i+1}$ for $0 \leq i \leq n$. For $0 \leq i \leq n-1$, K_i/K_{i+1} is Γ-isomorphic to $T_i/T_i \cap T_{i+1}S$ and hence the Γ-action on K_i/K_{i+1} is finite for $0 \leq i \leq n-1$. Since S is a compact connected semisimple Lie group, Aut (S) is compact and hence the Γ-action on $K_n/K_{n+1} = S$ is compact.

We now provide an example to show that nilpotency assumption on the acting group Γ in Theorem 4.2 can not be relaxed: it may be noted that Theorem 4.2 is true with no restriction on the acting group Γ if the compact group K is a two-dimensional torus.

Example 4.1 Let Γ be a subgroup of $\mathrm{GL}(n, \mathbb{Q})$. Let Γ^+ be the semi-direct product of Γ and $\mathbb{Q}_d^n$ with the canonical action of Γ on $\mathbb{Q}_d^n$. We define an action of Γ^+ on $\mathbb{Q}_d^{n+1}$ by

$$(\alpha, w)(q_1, \cdots, q_n, q_{n+1}) = \alpha(q_1, \cdots, q_n) + wq_{n+1} + (0, \cdots, 0, q_{n+1})$$

for all $(\alpha, w) \in \Gamma^+$ and $(q_1, \cdots, q_n, q_{n+1}) \in \mathbb{Q}_d^{n+1}$: $\mathbb{Q}_d^n$ is identified as a subset of $\mathbb{Q}_d^{n+1}$ via the canonical map $(q_1, \cdots, q_n) \mapsto (q_1, \cdots, q_n, 0)$. Considering the dual action, we get that $\Gamma^+ \subset \mathrm{Aut}(B_{n+1})$. For $z \in \mathbb{Q}_d^n$, $\Gamma^+(z) = \Gamma(z)$ and for $z \in \mathbb{Q}_d^{n+1} \setminus \mathbb{Q}_d^n$, $\Gamma^+(z)$ can be easily seen to be infinite. Thus, Γ is ergodic on B_n if and only if Γ^+ is ergodic on B_{n+1}. For any $\Gamma \subset \mathrm{GL}(n, \mathbb{Q})$, no $\alpha \in \Gamma^+$ is ergodic on B_{n+1}. For $n \geq 1$, take Γ to be the group generated by $\alpha \in \mathrm{GL}(n, \mathbb{Q})$ that is ergodic on B_n. Then Γ^+ is a solvable group and is ergodic on B_{n+1} but no automorphism in Γ^+ is ergodic on B_{n+1}.

5 Compact abelian groups

In this section we obtain a general version of Proposition 4.3 for compact abelian groups and also provide an example to show that the existence of finite sequence in Proposition 4.3 need not be true for connected infinite-dimensional compact abelian groups.

Proposition 5.1 *Let K be a compact abelian group and Γ be a group of automorphisms of K. Suppose Γ is distal. Then there exists a collection (K_i) of Γ-invariant closed subgroups of K such that*

(1) $K_0 = K$;

(2) for $i \geq 0$ either $K_{i+1} = (e)$ or K_{i+1} is a proper subgroup of K_i;

(3) the action of Γ on K_i/K_{i+1} is finite for any $i \geq 0$;

Proof Let $K_0 = K$. Then by Proposition 2.1 there exists a non-trivial character χ_0 on K_0 such that $\{\alpha(\chi_0) \mid \alpha \in \Gamma\}$ is finite. Let $\tilde{\Gamma}_0 = \{\alpha \in \Gamma \mid \alpha(\chi_0) = \chi_0\}$. Then $\tilde{\Gamma}_0$ is a subgroup of finite index in Γ. Let Γ_0 be a normal subgroup of Γ of finite index and $\Gamma_0 \subset \tilde{\Gamma}_0$.

Let $A_1 = \{\chi \in \hat{K}_0 \mid \Gamma_0(\chi) = \chi\}$. Since Γ_0 is normal in Γ, A_1 is a Γ-invaraint non-trivial subgroup of $\hat{K}_0$ and hence there exists a proper closed subgroup K_1 of K_0 such that the dual of K_0/K_1 is A_1. Then K_1 is Γ-invariant and the action of Γ on K/K_1 is finite.

If $K_1 = (e)$, then take $K_n = (e)$ for all $n \geq 1$. If $K_1 \neq (e)$, then get K_2 by applying the above arguements to K_1. Proceeding this way we obtain a collection (K_i) of Γ-invariant closed subgroups of K satisfying conditions (1)-(3).

In contrast to the finite-dimensional case we now show by an example that the sequence (K_i) in Proposition 5.1 need not be finite. Let T_k be the k-dimensional torus, a product of k copies of the circle group. Let α_k be the automorphism of T_k defined by

$$\alpha_k(x_1, x_2, \cdots, x_k) = (x_1 x_2 \cdots x_k, x_2 x_3 \cdots x_k, \cdots, x_{k-1} x_k, x_k)$$

for all $(x_1, x_2, \cdots, x_k) \in T_k$. For $0 \leq j \leq k$, let $M_{k,j} = \{(x_1, x_2, \cdots, x_{k-j}, e, \cdots, e) \in T_k\}$. Then each $M_{k,j}$ is α_k-invariant and α_k is trivial on $M_{k,j}/M_{k,j+1}$ for $j \geq 0$.

We first prove the following fact about T_k and α_k.

Lemma 5.1 *Let T_k and α_k be as above. Suppose there exists a series*

$$T_k = M_0 \supset M_1 \supset \cdots \supset M_{n-1} \supset M_n = (e)$$

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of α_k-invariant closed subgroups such that for $i \geq 0$, the action of $\mathbb{Z}_{\alpha_k}$ on M_i/M_{i+1} is finite and M_i/M_{i+1} is not finite. Then $n = k$.

Proof Let V be the Lie algebra of T_k. We first show that M_{n-1} is one-dimensional. For $0 \leq i < n$, let V_i be the Lie subalgebra of V corresponding to the Lie subgroup M_i. Now, there exists a m such that α_k^m is trivial on M_{n-1}. Suppose $(u_1, u_2, \cdots, u_k) \in V_{n-1}$. Then $\alpha_k^m(u_1, u_2, \cdots, u_k) = (u_1, u_2, \cdots, u_k)$. This implies that for $1 \leq i \leq k-1$, $u_i = u_i + \sum_{j>i}^{k} m_{i,j} u_j$ where $m_{i,j} \in \mathbb{N}$. For $i = k-1$, $u_{k-1} = u_{k-1} + m_{k-1,k} u_k$ and hence $u_k = 0$. If $u_p = 0$ for all $p > q > 1$, then for $i = q-1$, $u_{q-1} = u_{q-1} + \sum_{j \geq q} m_{q-1,j} u_j = u_{q-1} + m_{q-1,q} u_q$ and hence $u_q = 0$. Thus, V_{n-1} is atmost one-dimensional. Since $M_{n-1}/M_n = M_{n-1}$ is not finite, M_{n-1} has dimension one and $V_{n-1} = \{(u_1, 0, \cdots, 0) \mid u_1 \in \mathbb{R}\}$.

It can be seen that $T_k/M_{n-1} \simeq T_{k-1}$ and the action of $\mathbb{Z}_{\alpha_k}$ on T_k/M_{n-1} is same as the action of $\mathbb{Z}_{\alpha_{k-1}}$ on T_{k-1}. Moreover, $M_{i+1}/M_{n-1} \subset M_i/M_{n-1}$ and $\frac{M_i/M_{n-1}}{M_{i+1}/M_{n-1}} \simeq M_i/M_{i+1}$ for $0 \leq i < n-1$ with $M_0/M_{n-1} = T_k/M_{n-1}$ and $M_{n-1}/M_{n-1} = (e)$. By induction on k, we get that $n-1 = k-1$.

Let $K = \prod_{k \in \mathbb{N}} T_k$. Let $\alpha\colon K \to K$ be the automorphism defined by $\alpha(f)(k) = \alpha_k(f(k))$ for all $f \in K$ and all $k \in \mathbb{N}$. Then α is a continuous automorphism and the $\mathbb{Z}$-action defined by α is distal on K.

If there is a finite sequence

$$(e) = K_n \subset K_{n-1} \subset \cdots \subset K_1 \subset K_0 = K$$

of α-invariant closed subgroups such that the action of $\mathbb{Z}_\alpha$ on K_i/K_{i+1} is finite for $i \geq 0$. This implies that each T_k has a finite series

$$(e) = K_{n,k} \subset K_{n-1,k} \subset \cdots \subset K_{1,k} \subset K_{0,k} = T_k$$

of α_k-invariant closed subgroups such that the action of $\mathbb{Z}_{\alpha_k}$ on $K_{i,k}/K_{i+1,k}$ is finite for $i \geq 0$.

It follows from Lemma 5.1 that $k \leq n$. Since $k \geq 1$ is arbitrary, this is a contradiction. Thus, the sequence (K_i) of closed subgroups as in Proposition 5.1 for K and α is not finite.

References

[1] H. Abels, Distal automorphism groups of Lie groups, J. Reine Angew. Math. 329 (1981), 82–87.

[2] D. Berend, Ergodic semigroups of epimorphisms, Trans. Amer. Math. Soc. 289 (1985), no. 1, 393–407.

[3] J. F. Berglund, H. D. Junghenn, Hugo D and P. Milnes, Analysis on semigroups. Function spaces, compactifications, representations. Canadian Mathematical Society Series of Monographs and Advanced Texts. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1989.

[4] J. P. Conze and Y. Guivarch, Remarques sur la distalité dans les espaces vectoriels, C. R. Acad. Sci. Paris Sér. A 278 (1974), 1083–1086.

[5] Ch. Curtis and I. Rainer, Representation Theory of Finite Groups and Associative Algebras [Russian translation], Moscow (1969).

[6] R. Ellis, Distal transformation groups. Pacific J. Math. 8 1958 401–405.

[7] G. B. Folland, A course in abstract harmonic analysis, Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1995.

[8] Y. Guivarch, Croissance polynomiale et périodes des fonctions harmoniques, Bull. Soc. Math. France 101 (1973), 333–379.

[9] G. Hochschild, The structure of Lie groups. Holden-Day, Inc., San Francisco-London-Amsterdam 1965.

[10] K. H. Hofmann and S. A. Morris, The structure of compact groups. A primer for the student—a handbook for the expert. de Gruyter Studies in Mathematics, 25. Walter de Gruyter & Co., Berlin, 1998.

[11] W. Jaworksi and C. R. E. Raja, The Choquet-Deny theorem and distal properties of totally disconnected locally compact groups of polynomial growth, preprint. (avilable at http://www.isibang.ac.in/ statmath/eprints/2007/2.pdf).

[12] V. Losert, On the structure of groups with polynomial growth. II. J. London Math. Soc. (2) 63 (2001), no. 3, 640–654.

[13] S. A. Morris, Pontryagin duality and the structure of locally compact abelian groups, London Mathematical Society Lecture Note Series, No. 29. Cambridge University Press, Cambridge-New York-Melbourne, 1977.

[14] J. Nagata, Modern dimension theory. Revised edition. Sigma Series in Pure Mathematics, 2. Heldermann Verlag, Berlin, 1983.

[15] C. R. E. Raja, On classes of p-adic Lie groups, New York J. Math. 5 (1999), 101–105.

[16] J. Rosenblatt, A distal property of groups and the growth of connected locally compact groups, Mathematika 26 (1979), 94–98.

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