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Commutativity Degree of Wreath Products of Finite Abelian Groups

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Abstract

We compute commutativity degrees of wreath products $A \wr B$ of finite abelian groups A and B . When B is fixed of order n the asymptotic commutativity degree of such wreath products is $1/n^2$. This answers a generalized version of a question posed by P. Lescot. As byproducts of our formula we compute the number of conjugacy classes in such wreath products, and obtain an interesting elementary number-theoretic result.

1. INTRODUCTION

For a finite group G let \mathcal{G} denote the set of pairs of commuting elements of G :

$$\mathcal{G} = \{(g, h) \in G \times G \mid gh = hg\}.$$

The quantity $|\mathcal{G}|/|G|^2$ measures the probability of two random elements of G commuting and is called the *commutativity degree* of G . In [1] Paul Lescot computes the commutativity degree of dihedral groups and shows that it tends to $1/4$ as the order of the group tends to infinity. He then asks whether there are other natural families of groups with the same property. In this paper we show that if B is an abelian group of order n and A is a finite abelian group, then the commutativity degree of the wreath product $A \wr B$ tends to $1/n^2$ as the order of A tends to infinity.

Theorem 1.1 *Let $G = A \wr B$ where A is a finite abelian group and $B = \{b_1, b_2, \dots, b_n\}$ is an abelian group of order n . Then*

$$|\mathcal{G}| = \sum_{s,t=1}^n |A|^{n+\alpha(s,t)} \tag{1}$$

where $\alpha(s, t)$ denotes the index of the subgroup of B generated by b_s and b_t .

The exact value of the quantity $\alpha(s, t)$, of course, depends on the structure of B as an abelian group. We show how to obtain it in §3. Here we just note that when $B = \mathbb{Z}_n = \{1, 2, \dots, n\}$ is a cyclic group of order n , $\alpha(s, t) = (n, s, t)$ (where (n, s, t) denotes the greatest common divisor of n , s , and t). More generally, for a fixed value of n the farther B is away from a cyclic group, the larger the commutativity degree of the wreath product $A \wr B$ is. For example, the commutativity degree of $A \wr \mathbb{Z}_4$ is $1/16 + 3|A|^{-2} + 12|A|^{-3}$, while that of $A \wr (\mathbb{Z}_2 \times \mathbb{Z}_2)$ is $1/16 + 9|A|^{-2} + 6|A|^{-3}$. However, the asymptotic behavior of the commutativity degree of the wreath product $A \wr B$ as $|A| \rightarrow \infty$ does not depend on the structure of B as an abelian group.

Corollary 1.2 *Let A be a finite abelian group and B be an abelian group of order n . Then the commutativity degree of the wreath product $A \wr B$ tends to $1/n^2$ as $|A| \rightarrow \infty$.*

A straightforward computation with indices of centralizers shows that the number of conjugacy classes in a finite group G is equal to $|\mathcal{G}|/|G|$, hence (1) yields the formula for the number of conjugacy classes in wreath products of finite abelian groups.

Corollary 1.3 *Let A and B be as in Theorem 1.1. Then the number of conjugacy classes in the wreath product $A \wr B$ is $\frac{1}{n} \sum_{s,t=1}^n |A|^{\alpha(s,t)}$.*

By taking $B = \mathbb{Z}_n$ in Corollary 1.3, we obtain the following interesting elementary number-theoretic result. We had not been able to find an elementary proof of this fact.

Corollary 1.4 *For any natural number a , the sum $\sum_{s,t=1}^n a^{(n,s,t)}$ is divisible by n . If n is prime, this gives Fermat's little theorem.*

2. NOTATION AND TERMINOLOGY FOR WREATH PRODUCTS

We will use some of the notation from [2]. Let A and B be groups and let A^* be the direct sum of copies of A indexed by elements of B . We will write this as $A^* = \sum_{b \in B} A_b$, where each group A_b is a copy of A . Elements of A^* can be thought of as functions from B to A with finite support. An element $f \in A^*$ such that

$$f(b) = \begin{cases} a & \text{if } b = b_0 \in B, \\ e_A & \text{otherwise} \end{cases}$$

will be denoted by $\sigma_a(b_0)$. In this notation, every element of A^* can be uniquely written in the form

$$\sigma_{a_1}(b_1) \cdots \sigma_{a_s}(b_s),$$

where b_1, \dots, b_s are *distinct* elements of B , and a_1, \dots, a_s are any elements of A . Such a presentation will be called a *canonical word*. Define an action of B on A^* by

$$f^c(b) = f(bc^{-1}), \quad c \in B, b \in B. \quad (2)$$

The (standard restricted) wreath product of A and B , denoted by $A \wr B$, is the semidirect product of A^* and B with the action of B on A^* given by (2). If we denote the elements of a copy of B in $A \wr B$ by $\tau_c, c \in B$, then (2) becomes

$$\tau_c \sigma_a(b) = \sigma_a(bc) \tau_c,$$

whence every element of $A \wr B$ can be uniquely written in the canonical form

$$\sigma_{a_1}(b_1) \cdots \sigma_{a_s}(b_s) \tau_b,$$

where $\sigma_{a_1}(b_1) \cdots \sigma_{a_s}(b_s)$ is a canonical word in A^* . We will work with wreath products where the group B is finite, in which case the restricted wreath product and the complete wreath product are the same.

3. PROOF OF THEOREM 1.1

Since both groups A and B are abelian we will use additive notation for their group operations. To make the proof transparent we first work out in detail the case when $B = \mathbb{Z}_n$ is the cyclic group of order n . We may represent elements of B by arbitrary integers assuming that one takes the residue modulo n to obtain an actual element of \mathbb{Z}_n .

We will count the number of commuting pairs of elements of $G = A \wr \mathbb{Z}_n$ as follows. Fix s and t in $\{1, \dots, n-1, n\}$ and let

$$g = \sigma_{a_0}(0)\sigma_{a_1}(1)\cdots\sigma_{a_{n-1}}(n-1)\tau_{-s}$$

and

$$h = \sigma_{x_0}(0)\sigma_{x_1}(1)\cdots\sigma_{x_{n-1}}(n-1)\tau_{-t}.$$

We then count the number of commuting pairs (g, h) with prescribed values of s and t but allowing a_i 's and x_i 's to be arbitrary elements of A . To do so we think of an element g as being "fixed" and count the number of elements h that commute with every such given g . As we will see shortly, there might be some conditions on a_i 's for g to commute with at least one such h .

We will make a convention that a_u and a_v represent the same element of the group A if u and v are equal modulo n ; same for x_u and x_v . With this notation, the elements g and h as above commute if and only if

$$\begin{aligned} x_0 - x_s &= a_0 - a_t \\ x_1 - x_{s+1} &= a_1 - a_{t+1} \\ &\vdots \\ x_{n-1} - x_{s+(n-1)} &= a_{n-1} - a_{t+(n-1)} \end{aligned}$$

which can be thought of as a "linear system" in unknowns x_0, x_1, \dots, x_{n-1} . Let $d+1$ be the order of s in \mathbb{Z}_n , then $d+1 = n/(n, s)$ and there are (n, s) cosets of the cyclic subgroup $\langle s \rangle$ generated by s in \mathbb{Z}_n .

The above linear system will split into (n, s) independent subsystems in unknowns $\{x_i, x_{i+s}, x_{i+2s}, \dots, x_{i+ds}\}$ where i varies over the representatives of the cosets of $\langle s \rangle$ in \mathbb{Z}_n , say $0 \leq i \leq (n, s) - 1$. The matrix of each such subsystem has rank d , hence for the subsystem to be consistent the "constant" column consisting of differences of a_i 's must add up to zero. This gives the following condition for consistency of the i th subsystem:

$$a_i + a_{i+s} + \cdots + a_{i+ds} = a_{i+t} + a_{i+s+t} + \cdots + a_{i+ds+t}, \quad 0 \leq i \leq (n, s) - 1. \quad (3)$$

If $t \in \langle s \rangle$ then the conditions (3) are automatically satisfied for all i , hence for any choice of the elements a_0, a_1, \dots, a_{n-1} the number of elements h commuting with given g is $|A|^{(n, s)}$ since each subsystem has one free variable.

Suppose now that $t \in j + \langle s \rangle$ for some $j \in \{1, \dots, (n, s) - 1\}$. Let u denote the order of t (= order of j) in the quotient group $\mathbb{Z}_n / \langle s \rangle$. Then $u = (n, s) / (n, s, t)$ and the index

of the subgroup $\langle t \rangle$ in $\mathbb{Z}_n / \langle s \rangle$ is $(n, s)/u = (n, s, t)$; in the notation of Theorem 1.1 this is nothing but $\alpha(s, t)$.

The conditions (3) split into $\alpha(s, t)$ blocks corresponding to the cosets of $\langle t \rangle$ in $\mathbb{Z}_n / \langle s \rangle$. The k th block ($0 \leq k \leq \alpha(s, t) - 1$) looks as follows:

$$\begin{aligned} a_k + a_{k+s} + \cdots + a_{k+ds} &= a_{k+t} + a_{k+t+s} + \cdots + a_{k+t+ds} \\ a_{k+t} + a_{k+t+s} + \cdots + a_{k+t+ds} &= a_{k+2t} + a_{k+2t+s} + \cdots + a_{k+2t+ds} \\ &\vdots \\ a_{k+(u-1)t} + a_{k+(u-1)t+s} + \cdots + a_{k+(u-1)t+ds} &= a_{k+ut} + a_{k+ut+s} + \cdots + a_{k+ut+ds} \end{aligned}$$

But ut is a multiple of s , hence the right hand side of the last equation is equal to the left hand side of the first equation. It follows that exactly one of these u equations is a consequence of the others and each block produces $u - 1$ independent "linear" conditions on a_i 's.

To summarize, among the $|A|^n$ sequences $(a_0, a_1, \dots, a_{n-1})$ of elements of A , there are exactly $|A|^{n-\alpha(s,t)(u-1)} = |A|^{n-(n,s)+\alpha(s,t)}$ sequences for which the original linear system in x_0, x_1, \dots, x_{n-1} is consistent. For each such fixed sequence, the number of sequences $(x_0, x_1, \dots, x_{n-1})$ satisfying the corresponding system is $|A|^{(n,s)}$ since each of the (n, s) (= index of the subgroup of B generated by s) subsystems contributes one free variable. Thus, for fixed s and t the total number of commuting pairs (g, h) of elements of G where the canonical form of g ends in τ_{-s} and the canonical form of h ends in τ_{-t} is $|A|^{n+\alpha(s,t)}$. The formula (1) now follows.

In the general case, when $B = \{b_1, b_2, \dots, b_n\}$ is an arbitrary abelian group, fix $b_s, b_t \in B$ and consider two elements of $G = A \wr B$

$$g = \sigma_{a_1}(b_1)\sigma_{a_2}(b_2) \cdots \sigma_{a_n}(b_n)\tau_{-b_s}$$

and

$$h = \sigma_{x_1}(b_1)\sigma_{x_2}(b_2) \cdots \sigma_{x_n}(b_n)\tau_{-b_t}.$$

Note that the above proof essentially did not use the fact that B was a cyclic group (it was only used to have a convenient way to label the indices of a_i 's and x_i 's). Rather, the computation involves the following quantities:

- the index of the cyclic subgroup of B generated by b_s , say $\beta(s)$;
- the index of the cyclic subgroup of the quotient group $B / \langle b_s \rangle$ generated by the image of b_t , this is precisely $\alpha(s, t)$ in our notation.

The "linear system" which gives conditions for elements g and h to commute then splits into $\beta(s)$ subsystems each of which corresponds to a coset of the cyclic subgroup $\langle b_s \rangle$ of B , hence the same reasoning carries over verbatim to the general case. Further, the conditions on a_i 's will split into $\alpha(s, t)$ blocks each of which corresponds to a coset of the cyclic subgroup generated by the image of b_t in $B / \langle b_s \rangle$.

It follows that among the $|A|^n$ sequences (a_1, a_2, \dots, a_n) of elements of A , there are exactly $|A|^{n-\beta(s)+\alpha(s,t)}$ sequences for which the linear system is consistent. For each such fixed sequence, the number of sequences (x_1, x_2, \dots, x_n) satisfying the

corresponding system is $|A|^{\beta(s)}$. Thus, for fixed s and t the total number of commuting pairs (g, h) of elements of G where the canonical form of g ends in τ_{-b_s} and the canonical form of h ends in τ_{-b_t} is $|A|^{n+\alpha(s,t)}$. This completes the proof of Theorem 1.1.

Finally, we give a formula for $\alpha(s, t)$ which depends on the structure of B as an abelian group. Let $B = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$ and let $s = (s_1, \dots, s_k)$, $t = (t_1, \dots, t_k)$ be two elements of B . Let $\alpha(s, t) = [B : \langle s, t \rangle]$.

Consider the surjective homomorphism $\pi: \mathbb{Z}^k \rightarrow B$ with $\ker \pi = n_1\mathbb{Z} \times \cdots \times n_k\mathbb{Z}$. Let $a, b \in \mathbb{Z}^k$ be such that $\pi(a) = s$ and $\pi(b) = t$. Then $\mathbb{Z}^k/H \cong B/\langle s, t \rangle$ where $H = \ker \pi + \langle a, b \rangle$. We determine the order of \mathbb{Z}^k/H as follows. Write $a = (a_1, \dots, a_k)$ and $b = (b_1, \dots, b_k)$ (thinking of s_i 's and t_j 's as integers one may take $a_i = s_i$ and $b_j = t_j$ for all $i, j \in \{1, \dots, k\}$), then

$$H = \{(n_1m_1 + ua_1 + vb_1, \dots, n_km_k + ua_k + vb_k) \mid m_i, u, v \in \mathbb{Z}\}.$$

If $R: \mathbb{Z}^{k+2} \rightarrow \mathbb{Z}^k$ is a homomorphism given by the $k \times (k+2)$ matrix

$$\begin{bmatrix} n_1 & 0 & \cdots & 0 & a_1 & b_1 \\ 0 & n_2 & \cdots & 0 & a_2 & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & n_k & a_k & b_k \end{bmatrix}$$

then $H = \text{Im } R$. Let $P \in GL_k(\mathbb{Z})$ and $Q \in GL_{k+2}(\mathbb{Z})$ be such that

$$PRQ = \begin{bmatrix} d_1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & d_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & d_k & 0 & 0 \end{bmatrix}$$

where $d_1 \mid d_2 \mid \cdots \mid d_k$ are the elementary divisors of R . We have

$$\mathbb{Z}^k/\text{Im } R \cong P(\mathbb{Z}^k)/PR(\mathbb{Z}^{k+2}) = \mathbb{Z}^k/PRQ(\mathbb{Z}^{k+2})$$

so that

$$\alpha(s, t) = |\mathbb{Z}^k/\text{Im } R| = |d_1d_2 \cdots d_k|.$$

For reader's convenience we recall a well-known method for finding elementary divisors. For $i = 1, \dots, k$, let h_i denote the greatest common divisor of all $i \times i$ minors of R ; then $h_i = d_1d_2 \cdots d_i$. This is because the numbers h_i do not change when multiplied on the left and on the right by elementary matrices and these generate all invertible integer matrices. In particular, note that if $k = 1$ then $\alpha(s, t) = (n, s, t)$.

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