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On the existence of non-abelian representations of slim dense near hexagons having big quads

Binod Kumar Sahoo and N. S. Narasimha Sastry

Indian Statistical Institute, Bangalore Centre
8th Mile Mysore Road, Bangalore, 560059 India

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Binod Kumar Sahoo

Department of Mathematics
National Institute of Technology
Rourkela - 769008, India
E-mail: binodkumar@gmail.com

and

N. S. Narasimha Sastry

Statistics & Mathematics Unit
Indian Statistical Institute
R.V. College Post
Bangalore - 560059, India
E-mail: nsastry@isibang.ac.in

Abstract. It is known that [21] if a slim dense near hexagon S admits a non-abelian representation in a group R, then $|R| = 2^{\beta}$, $1 + n(S) \leq \beta \leq 1 + dimV(S)$, where $n(S)$ and $dimV(S)$ are as defined in Section 1. In this paper, we show that, among the slim dense near hexagons admitting big quads (see (1.1) for notation), $DH_6(2^2)$, $\mathbb{E}_3$ and $\mathbb{G}_3$ do not admit non-abelian representations and the remaining ones, $Q_6^-(2) \otimes Q_6^-(2)$, $Q_6^-(2) \times \mathbb{L}_3$, $DW_6(2)$, $\mathbb{H}_3$, $W_4(2) \times \mathbb{L}_3$ and $Q_4^+(2) \times \mathbb{L}_3$, admit a non-abelian representation in an extraspecial 2-group of order $2^{1+n(S)}$ (Theorem 1.3).

Key words. Near polygons, generalized quadrangles, non-abelian representations, extraspecial 2-groups

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1. **Introduction**

Let $S = (P, L)$ be a partial linear space with lines of size three. For distinct points $x, y \in P$, we write $x \sim y$ if they are collinear, otherwise we write $x \nsim y$. If $x, y \in P$ and $x \sim y$, then we denote by xy the unique line containing x and y and define $x * y$ by $xy = \{x, y, x * y\}$. For unexplained terminology, see [20] and [21].

Definition 1.1 ([14], p.525)**.** *A representation* (R, ψ) *of* S *with representation group* R *is a mapping* $\psi : x \mapsto \langle r_x \rangle$ *from* P *into the set of subgroups of* R *of order* 2 *such that the following hold:*

(*i*) $R = \langle r_x : x \in P \rangle$.
(*ii*) *For each line* $\{x, y, x * y\}$ *of* S, $\{1, r_x, r_y, r_{x*y}\}$ *is a Klein four group.*

A representation (R, ψ) of S is *faithful* if ψ is injective and is *abelian* or *non-abelian* according as R is abelian or not. Note that, in [14], 'non-abelian representation' means that 'the representation group is not necessarily abelian'. For an abelian representation, the representation group can be considered as a vector space over the field F_2 with two elements. If S is connected (that is; the collinearity graph $\Gamma(P)$ of S is connected), then there exists a unique abelian representation, called the *universal abelian representation*, of S with the property that any other abelian representation of S is a composition of it with a linear mapping [18]. The F_2-vector space $V(S)$ underlying the universal abelian representation of S is called the *universal representation module* of S.

We refer to [15] for more on universal abelian representations of point-line geometries; and [13] and ([20], Sections 1 and 2) for more on non-abelian representations of partial linear spaces with $p+1$ points per line, p a prime.

A *near polygon* [22] here is a connected partial linear space $S = (P, L)$ with at least three points per line and of finite diameter (that is, the diameter of $\Gamma(P)$ is finite) such that: for each point-line pair $(x, l) \in P \times L$, x is nearest to exactly one point of l. Here, the distance $d(x, y)$ between two points x and y of S is measured in $\Gamma(P)$. If the diameter of S is n, then S is called a *near* $2n$*-gon.* If $n = 2$, then S is a *generalized quadrangle* (GQ, for short) and if $n = 3$, then it is called a *near hexagon.* For more on near polygons, see [10].

Let $S = (P, L)$ be a near $2n$-gon. For $x \in P$ and $A \subseteq P$, we define $x^\perp = \{x\} \cup \{y \in P : x \sim y\}$ and $A^\perp = \bigcap_{x \in A} x^\perp$. A subset of P is a *subspace* of S if any line of S containing at least two of its points is contained in it. For a subset A of P, the *subspace* $\langle A \rangle$ *generated by* A is the intersection of all subspaces of S containing A. A *geometric hyperplane* of S is a subspace of S, different from the empty set and

P, that meets each line of S non-trivially. A subspace C of P is *convex* if every geodesic in $\Gamma(P)$ between two points of C is entirely contained in C. A *quad* of S is a convex subspace of P of diameter 2 such that no point of it is adjacent to all other points of it. If $x_1, x_2 \in P$ with $d(x_1, x_2) = 2$ and $|\{x_1, x_2\}^\perp| \geq 2$, then x_1 and x_2 are contained in a unique quad, denoted by $Q(x_1, x_2)$, and this quad $Q(x_1, x_2)$ is a generalized quadrangle ([22], Proposition 2.5, p.10). We say that S is *dense* if every pair of points at distance 2 is contained in a quad.

Let $S = (P, L)$ be a dense near $2n$-gon. Then, the number, $t+1$, of lines containing a point of S is independent of the point ([5], Lemma 19, p.152). Let $t_2 = \{|\{x, y\}^\perp| - 1 : x, y \in P, d(x, y) = 2\}$. We say that S has *parameters* (s, t, t_2) if each line of S contains $s + 1$ points, each point is contained in $t + 1$ lines and t_2 is as above. If $n = 2$, then $t_2 = \{t\}$, though t_2 may have more than one element in general. A near 4-gon with parameters $(s, t, \{t\})$ is written as a (s, t)-GQ. We say that a quad of S is of *type* (s, t') if it is a (s, t')-GQ. If Q is a quad of S, then for $x \in P \setminus Q$, either

- (i) there is a unique point $y \in Q$ (depending on x) collinear with x and $d(x, z) = d(x, y) + d(y, z)$ for all $z \in Q$; or
- (ii) $d(x, Q) = 2$ and the set $\mathcal{O}_x = \{y \in Q : d(x, y) = 2\}$ is an ovoid of Q.

([22], Proposition 2.6, p.12). We say that the quad Q is *big* if (i) holds for each $x \in P \setminus Q$. If Q is a big quad of S and $x \in P \setminus Q$, then we denote by x_Q the unique point y as in (i).

A near polygon is *slim* if each of its lines contains exactly three points. Let $S = (P, L)$ be a slim dense near $2n$-gon, $n \geq 1$. If $n = 1$, then $S \simeq \mathbb{L}_3$, a line of size 3. If $n = 2$, then S is a $(2, t)$-GQ. In that case, P is finite, $t = 1, 2$ or 4 and for each value of t there exists a unique $(2, t)$-GQ, up to isomorphism ([7], Theorem 7.3, p.99). Thus, S is isomorphic to one of the classical generalized quadrangles $Q_4^+(2)$, $W_4(2) \simeq Q_5(2)$ and $Q_6^-(2)$ for $t = 1, 2$ and 4, respectively. From ([4], Theorem 1.1, p.349), there are 11 possibilities for S, up to isomorphism, when $n = 3$. Further, each of these slim dense near hexagons is uniquely determined by its parameters $s = 2$, t and t_2 (see [2], [3], [4] and [22]). For other classification results about slim dense near polygons, see [23] and [10].

Let $S = (P, L)$ be a slim dense near hexagon having big quads. Since a (2,4)-GQ admits no ovoids, it follows that every quad of S of type $(2, 4)$ is big. Let Q be a quad of S. If Q is of type $(2, t')$, then $|P| \geq |Q|(1 + 2(t - t'))$ and equality holds if and only if Q is big (see [4], p.359). In particular, if a quad of S of type $(2, t')$ is big, then so are all quads of S of that type. If Q is big and $\{a, b, c\}$ is a line of S disjoint from Q, then $\{a_Q, b_Q, c_Q\}$ is a line of Q ([5], Lemma 5, p.148).

Lemma 1.2 (see [4], Proposition 4.3, p.354)**.** *Let Q_1 and Q_2 be two disjoint big quads of S. Let τ be the map from Q_1 to Q_2 defined by $\tau(x) = x_{Q_2}$, $x \in Q_1$. Then*

- (i) *τ is an isomorphism from Q_1 to Q_2.*
- (ii) *The set $Q_1 * Q_2 = \{x * x_{Q_2} : x \in Q_1\}$ is a big quad of S.*

Further, $Y = Q_1 \cup Q_2 \cup Q_1 * Q_2$ is a subspace of S isomorphic to the near hexagon $Q_1 \times \mathbb{L}_3$, a direct product of Q_1 and $\mathcal{L}_3$ (see Section 2).

Let B be the collection of all big quads of S and L_B be the collection of subsets $\{Q_1, Q_2, Q_1 * Q_2\}$ of B, where Q_1 and Q_2 are disjoint. Then $\mathcal{F} = (B, L_B)$ is a Fischer space (see [4], Corollary 4.4, p.354). That is, $\mathcal{F}$ is a partial linear space satisfying the following conditions:

(i) Each line of $\mathcal{F}$ contains exactly three points.
(ii) The subspace generated by any two intersecting lines of $\mathcal{F}$ is isomorphic to the affine plane of order three or the dual affine plane of order two.

(see [1], p.92). The partial linear space $\mathcal{F}$ is called the *Fischer space on big quads of S* which is used here in the study of non-abelian representations of S. We also remark that some of the slim dense near hexagons are constructed in ([4], p.352) using Fischer spaces.

We list below the slim dense near hexagons admitting big quads with their parameters (see [4], Theorem 1.1, p.349).

(1.1)

Near hexagon	$\lvert P\rvert$	t	t_2	$dimV(S)$	$n(S)$
$DH_6(2^2)$	891	20	$\{4^\star\}$	22	20
$\mathbb{E}_3$	567	14	$\{2, 4^\star\}$	21	20
$\mathbb{G}_3$	405	11	$\{1, 2, 4^\star\}$	20	20
$Q_6^-(2) \otimes Q_6^-(2)$	243	8	$\{1, 4^\star\}$	18	18
$Q_6^-(2) \times \mathbb{L}_3$	81	5	$\{1, 4^\star\}$	12	12
$DW_6(2)$	135	6	$\{2^\star\}$	15	8
$\mathbb{H}_3$	105	5	$\{1, 2^\star\}$	14	8
$W_4(2) \times \mathbb{L}_3$	45	3	$\{1, 2^\star\}$	10	8
$Q_4^+(2) \times \mathbb{L}_3$	27	2	$\{1^\star\}$	8	8

Here, $dimV(S)$ denotes the dimension of the universal representation module of S and $n(S)$ denotes the F_2-rank of the matrix $A_3 : P \times P \longrightarrow \{0, 1\}$ defined by $A_3(x, y) = 1$ if $d(x, y) = 3$ and zero otherwise. We add a star in column t_2 if and only if the corresponding quads are big.

In ([21], Theorem 1.6), it is proved that the representation group R for a non-abelian representation (R, ψ) of S is a finite group of order 2^β, where $1 + n(S) \leq \beta \leq 1 + dimV(S)$.

In this paper, we prove

Theorem 1.3. *Let $S = (P, L)$ be a slim dense near hexagon having big quads.*

(i) *If $S = DH_6(2^2), \mathbb{E}_3$ or $\mathbb{G}_3$, then every representation of S is abelian.*
(ii) *If S is one of the near hexagons $Q_6^-(2) \otimes Q_6^-(2)$, $Q_6^-(2) \times \mathbb{L}_3$, $DW_6(2)$, $\mathbb{H}_3$, $W_4(2) \times \mathbb{L}_3$ and $Q_4^+(2) \times \mathbb{L}_3$, then S admits a non-abelian representation such that the representation group is an extraspecial 2-group of order $2^{1+n(S)}$.*

The structure of the Fischer space $\mathcal{F}$ on big quads of S is used in proving Theorem 1.3(i) and in the construction of non-abelian representation for $Q_6^-(2) \otimes Q_6^-(2)$.

In Section 2, we give a construction for each of the slim dense near hexagons having big quads. In Section 3, we study the structure of the near hexagon relative to a line in the Fischer space $\mathcal{F}$. We prove Theorem 1.3(i) in Section 4 and Theorem 1.3(ii) in Section 5.

2. **Constructions**

$DH_6(2^2)$ and $DW_6(2)$: Let $S = (P, L)$ be a polar space of rank $n \geq 2$ (see [6]). The partial linear space, whose point set is the collection of all maximal singular subspaces of S and lines are the collections of all maximal singular subspaces of S containing a specific singular subspace of co-dimension 1 in each of them, is called the *dual polar space of rank* n associated with S. Cameron characterized these geometries in terms of points and lines and proved that dual polar spaces of rank n are dense near $2n$-gons ([8], Theorem 1, p.75). The neat hexagons $DH_6(2^2)$ and $DW_6(2)$ are the unitary and symplectic dual polar spaces of rank 3, respectively. All slim dense near hexagons having big quads are subspaces of $DH_6(2^2)$ (see [4], p.353).

$Q_4^+(2) \times \mathbb{L}_3$, $W_4(2) \times \mathbb{L}_3$ and $Q_6^-(2) \times \mathbb{L}_3$: Let $S_1 = (P_1, L_1)$ and $S_2 = (P_2, L_2)$ be two partial linear spaces. Then, their *direct product* $S_1 \times S_2$ is the partial linear space whose point set is $P_1 \times P_2$ and the line set consists of all subsets of $P_1 \times P_2$ projecting to a single point in P_i for each i and projecting in P_j onto an element of L_j, where $\{i, j\} = \{1, 2\}$. A direct product of near polygons is again a near polygon ([5], Theorem 1, p.146). The slim dense near hexagons on 81, 45 and 27 points are direct products of a $(2, t)$-GQ with $\mathbb{L}_3$ for $t = 4, 2, 1$, respectively.

$Q_6^-(2) \otimes Q_6^-(2)$: The following description of this near hexagon is taken from [12]. Let $S = (P, L)$ be a (2,4)-GQ, $T = \{l_1 \cdot\cdot\cdot, l_9\} \subset L$ be a spread of S and l be an arbitrary line in T. Let $\phi_j : P \longrightarrow l_j$ be the map taking each $x \in P$ to the unique point of l_j nearest to x in S. Let $\mathcal{G} = \mathcal{G}(S, T, l)$ be the graph with vertex set $l \times T \times T$. Two distinct vertices (x, l_i, l_j) and (y, l_m, l_n) are *adjacent* whenever at least one of the following two conditions holds:

(1) $j = n$ and $\phi_i(x)$ and $\phi_m(y)$ are collinear points in S;
(2) $i = m$ and $\phi_j(x)$ and $\phi_n(y)$ are collinear points in S.

Note that if $i = m$ and $j = n$, then (1) and (2) both are satisfied. Any two adjacent vertices of $\mathcal{G}$ are contained in a unique maximal clique of size three. The points and the lines of $Q_6^-(2) \otimes Q_6^-(2)$ are, respectively, the vertices and the maximal cliques of $\mathcal{G}$.

$\mathbb{G}_3$: Let the vector space $\mathbb{F}_4^6$ with base $\{e_0, \cdots, e_5\}$ be equipped with the non-singular hermitian form $(x, y) = x_0 y_0^2 + x_1 y_1^2 + \cdots + x_5 y_5^2$. Let H denote the corresponding hermitian variety in $PG(5, 2^2)$. The *support* S_α of a point $\alpha = \mathbb{F}_4 x$ of $PG(5, 2^2)$ is the set of all $i \in \{0, 1, \cdots, 5\}$ for which $(x, e_i) \neq 0$ and its cardinality is called the *weight* of α. A point of $PG(5, 2^2)$ belongs to H if and only if its weight is even. A subspace of $PG(5,2^2)$ contained in H is said to be *good* if it is generated by a set of points whose supports are pairwise disjoint. Then, $\mathbb{G}_3$ is a subspace of $DH_6(2^2)$ whose point set consists of all good subspaces of H of dimension 2 (see [11]).

$\mathbb{E}_3$: A non-empty set $\mathcal{O}$ of points of a partial linear space S is called a *hyperoval* if every line of S intersects $\mathcal{O}$ in zero or two points. The unitary polar space $H_6(2^2)$ of rank three has two isomorphism classes of hyperovals [16]. The hyperovals of one class contain 126 points and the hyperovals of the other class contain 162 points. Let $\mathcal{O}$ be a hyperoval of $H_6(2^2)$ of size 126. Each maximal singular subspace of $H_6(2^2)$ has zero or six points in common with $\mathcal{O}$. Then, $\mathbb{E}_3$ is the subspace of $DH_6(2^2)$ consisting of all points intersecting $\mathcal{O}$ in six points (see [10], p.159).

$\mathbb{H}_3$: Let X be a set of size 8. The point set of $\mathbb{H}_3$ is the set of all partitions of X into 2-subsets; and the line set consists of all triples of these partitions sharing two 2-subsets of X (see [4], p.355).

The following alternate construction of $\mathbb{H}_3$ and $DW_6(2)$ [19] is used in Section 5.

Proposition 2.1. *Let $S=(P,L)$ and $S^1=(P^1,L^1)$ be two (2,2)-GQs and let $\pi: x \mapsto x^1$, $x\in P, x^1\in P^1$, denote an isomorphism from S to S^1. Let*

$\mathcal{P} = \{(x,y^1)\in P\times P^1 : y^1 \in {x^1}^{\perp}\}$;

$\mathcal{L} = \{\{(x,u^1),(y,v^1),(z,w^1)\} : \{x,y,z\}$ *is a line or a complete triad of points of S and* $\{x^1,y^1,z^1\}^{\perp} = \{u^1,v^1,w^1\}$ *in* $S^1\}$;

$\mathbb{P} = \mathcal{P}\cup P\cup P^1$;

$\mathbb{L} = \mathcal{L}\cup\mathcal{L}^1$, *where* $\mathcal{L}^1 = \{\{x,(x,u^1),u^1\} : (x,u^1)\in\mathcal{P}\}$.

Then $\mathcal{S}=(\mathcal{P},\mathcal{L})\simeq\mathbb{H}_3$ and $\mathbb{S}=(\mathbb{P},\mathbb{L})\simeq DW_6(2)$ ([19], Theorems 2.1 and 2.2). Clearly, $\mathbb{H}_3$ is a geometric hyperplane of $DW_6(2)$.

3. **Preliminaries**

Let $S=(P,L)$ be a slim dense near hexagon having big quads. Fix two disjoint big quads Q_1 and Q_2 of S. Let $Q_3 = Q_1 * Q_2$ and $Y = Q_1\cup Q_2\cup Q_3$. By Lemma 1.2, Y is a subspace of S isomorphic to $Q_4^+(2)\times\mathbb{L}_3, W_4(2)\times\mathbb{L}_3$ or $Q_6^-(2)\times\mathbb{L}_3$ according as Q_1 and Q_2 are of type (2,1), (2,2) or (2,4). Now, fix a big quad Q of S disjoint from Y. Let $\{i,j,k\}=\{1,2,3\}$. For $x\in P\setminus Y$, we define $x^j = x_{Q_j}$ and, for $x\in Q_i$, we define $z_x^j = x_{Q_j}$. Thus, for $x\in Q_i$, $\{x,z_x^j,z_x^k\}$ is a line of Y meeting each of Q_i,Q_j and Q_k. For a line $l=\{a,b,c\}$ of S, we set $l_Q=\{a_Q,b_Q,c_Q\}$ if $l\cap Q$ is empty, and $l^j=\{a^j,b^j,c^j\}$ if $l\cap Q_j$ is empty. We denote by τ_j the isomorphism from Q to Q_j defined by $\tau_j(x)=x^j$, $x\in Q$ and by τ_{ij} the isomorphism from Q_i to Q_j defined by $\tau_{ij}(x)=z_x^j$, $x\in Q_i$ (see Lemma 1.2(i)). For $x\in P\setminus(Y\cup Q)$, we denote by x_Q^i the point $(x_Q)^i$ in Q_i. Similarly, for a line l disjoint from both Y and Q, we denote by l_Q^i the line $(l_Q)^i$ in Q_i.

Lemma 3.1. *Let $x\in P\setminus Y$. Then:*

(i) $d(z_{x^i}^j, x^j)=1$ *and* $d(x^i,x^j)=2$.
(ii) $\{x^i, z_{x^j}^i, z_{x^k}^i\}$ *is a line in* Q_i.

Proof. (i) Since $x \in \Gamma_1(x^i) \cap \Gamma_1(x^j)$, $d(x^i, x^j) = 2$. Further, $d(x^i, x^j) = d(x^i, z^j_{x^i}) + d(z^j_{x^i}, x^j)$. So $d(z^j_{x^i}, x^j) = 1$.

(ii) By (i), $x^i \sim z^i_{x^j}$ and $x^i \sim z^i_{x^k}$. We show that $z^i_{x^j} \sim z^i_{x^k}$. The quad $Q(x^j, x^k)$ of Y is of type (2,1) and $\{x^j, x^k\}^\perp = \{z^k_{x^j}, z^j_{x^k}\}$ in Y. Now, from the parallel lines $\{x^j, z^i_{x^j}, z^k_{x^j}\}$ and $\{x^k, z^i_{x^k}, z^j_{x^k}\}$ in $Q(x^j, x^k)$, it follows that $z^i_{x^j} \sim z^i_{x^k}$. □

Lemma 3.2. *Let $l = \{a, b, c\}$ be a line of S intersecting Y at c.*

- (i) *If $c \in Q_i \cup Q_j$, then $d(a^i, b^j) = 2$.*
- (ii) *If $c \in Q_k$, then $d(a^i, b^j) = 1$. In fact, $a^i = z^i_{b^j}$.*

Proof. (i) Let c be in, say, Q_i. Since $a^i = b^i = c$, $d(a^i, b^j) = d(b^i, b^j) = 2$ by Lemma 3.1(i).

(ii) We have $a^k = b^k = c$. Since l is disjoint from Q_i, $l^i = \{a^i, b^i, c^i = z^i_c\}$ is a line of Q_i. By Lemma 3.1(ii), $\{b^i, z^i_{b^j}, z^i_{b^k} = z^i_c\}$ is also a line of Q_i. So, $a^i = z^i_{b^j}$ and $d(a^i, b^j) = 1$. □

Lemma 3.3. *Let l be a line of S disjoint from Y and $x, y \in l$ with $x \neq y$.*

- (i) *If $l^j = x^j z^j_{x^i}$ in Q_j, then $(y^i, y^j) = (z^i_{x^j}, x^j * z^j_{x^i})$ or $(x^i * z^i_{x^j}, z^j_{x^i})$. In particular, $l^i = x^i z^i_{x^j}$ in Q_i, $l^j = x^j z^j_{x^i}$ in Q_j and $\tau_{ij}(l^i) = l^j$ are equivalent.*
- (ii) *$d(x^i, y^j) \leq 2$ if and only if $l^i = x^i z^i_{x^j}$ in Q_i.*

Proof. (i) If $l^j = x^j z^j_{x^i}$, then $y^j \in \{z^j_{x^i}, x^j * z^j_{x^i}\}$. Assume that $y^j = x^j * z^j_{x^i}$. Since $\tau_{ji}(x^j z^j_{x^i}) = x^i z^i_{x^j}$, $z^i_{y^j} = x^i * z^i_{x^j}$ and so $y^i \sim x^i * z_{x^j}$ (Lemma 3.1(i)). Since $y^i \sim x^i$ also, y^i is a point in the line $x^i z^i_{x^j}$. Now, $d(y^i, y^j) = 2$ implies that $y^i = z^i_{x^j}$.

If $y^j = z^j_{x^i}$, then applying the above argument to $(x * y)^j = x^j * z^j_{x^i}$, we get $(x * y)^i = z^i_{x^j}$ and so, $y^i = x^i * z^i_{x^j}$.

(ii) If $l^i = x^i z^i_{x^j}$ in Q_i, then $\tau_{ij}(l^i) = l^j$ by (i) and it follows that $d(x^i, y^j) \leq 2$. Suppose that $l^i \neq x^i z^i_{x^j}$ in Q_i. By (i), $l^j \neq x^j z^j_{x^i}$ in Q_j. So $y^j \nsim z^j_{x^i}$, and $d(x^i, y^j) = d(x^i, z^j_{x^i}) + d(z^j_{x^i}, y^j) = 1 + 2 = 3$. □

Lemma 3.4. *For every $x \in Q$, there exists a unique line l in Q containing x such that $\tau_{ij}(l^i) = l^j$. In particular, $l^i = \{x^i, z^i_{x^j}, z^i_{x^k}\}$.*

Proof. Since τ_i is an isomorphism from Q to Q_i, there exists a line l of Q containing x such that $l^i = x^i z^i_{x^j}$. By Lemma 3.3(i), $\tau_{ij}(l^i) = l^j$. The line l in Q through x such that $\tau_{ij}(l^i) = l^j$ is unique because, for any other line $\overline{l}$ of Q containing x, $\tau_{ij}(\overline{l}^i)$ and $\overline{l}^j$ are two disjoint lines in Q_j containing $z^j_{x^i}$ and x^j, respectively. Now, $l^i = x^i z^i_{x^j} = \{x^i, z^i_{x^j}, z^i_{x^k}\}$ (see Lemma 3.1(ii)). □

Notation 3.5. *For $x \in Q$, we denote by ζ_x the unique line l in Q containing x as in Lemma 3.4 and we write $T_Q = \{\zeta_x : x \in Q\}$.*

Corollary 3.6. *T_Q is a spread of Q.*

Proof. This follows because, $\zeta_x = \zeta_y$ for $x \in Q$ and $y \in \zeta_x$, by Lemma 3.4. □

Let $l = \{a, b, c\}$ be a line of Q. First, let $l \in T_Q$. Set $T^l = l^i \cup l^j \cup l^k$ and $T^l_{jk} = l^i \cup \tau_{ij}(l^i) \cup \tau_{ik}(l^i)$. The set T^l_{jk} is a quad of Y of type (2,1) whose lines are the rows and the columns of the matrix

$$T^l_{jk} = \begin{bmatrix} a^i & z^j_{a^i} & z^k_{a^i} \\ b^i & z^j_{b^i} & z^k_{b^i} \\ c^i & z^j_{c^i} & z^k_{c^i} \end{bmatrix}. \tag{3.2}$$

Since $l \in T_Q$, Lemma 3.4 implies that T^l coincides with T^l_{jk}. So T^l is a quad of Y of type $(2,1)$ whose lines are the rows and columns of one of the matrices

$$T^l = \begin{bmatrix} a^i & c^j & b^k \\ b^i & a^j & c^k \\ c^i & b^j & a^k \end{bmatrix}; \text{ or } T^l = \begin{bmatrix} a^i & b^j & c^k \\ b^i & c^j & a^k \\ c^i & a^j & b^k \end{bmatrix}. \tag{3.3}$$

Note that if $b^k \sim a^i$, then the line containing them is $\{a^i, c^j, b^k\}$.

Now, let $l \notin T_Q$. Then, $\tau_{ij}(l^i)$ and l^j are disjoint lines in Q_j. The set $T^l_i = l^i \cup \tau_{ji}(l^j)\tau_{ki}(l^k)$ form a $(2,1)$-GQ in Q_i. In fact, we can write

$$T^l_i = \begin{bmatrix} a^i & b^i & c^i \\ z^i_{a^j} & z^i_{b^j} & z^i_{c^j} \\ z^i_{a^k} & z^i_{b^k} & z^i_{c^k} \end{bmatrix}; T^l_j = \begin{bmatrix} z^j_{a^i} & z^j_{b^i} & z^j_{c^i} \\ a^j & b^j & c^j \\ z^j_{a^k} & z^j_{b^k} & z^j_{c^k} \end{bmatrix}; \text{ and } T^l_k = \begin{bmatrix} z^k_{a^i} & z^k_{b^i} & z^k_{c^i} \\ z^k_{a^j} & z^k_{b^j} & z^k_{c^j} \\ a^k & b^k & c^k \end{bmatrix}. \tag{3.4}$$

Each row as well as each column in T^l_i (respectively, T^l_j, T^l_k) is a line of Q_i (respectively, Q_j, Q_k). Further, the (m,n)-th entries from T^l_i, T^l_j and T^l_k form a line of Y.

As a consequence of the above, we have

Corollary 3.7. *Let l be a line of Q. For distinct $a, b \in l$, $d(a^i, b^j) \leq 2$ or $d(a^i, b^j) = 3$ according as $l \in T_Q$ or not.*

4. **Proof of Theorem** 1.3(i)

A finite 2-group G is *extraspecial* if its Frattini subgroup $\Phi(G)$, the commutator subgroup G' and the center $Z(G)$ coincide and have order 2. An extraspecial 2-group is of exponent 4 and of order 2^{1+2m} for some integer $m \geq 1$ and the maximum of the orders of its abelian subgroups is 2^{m+1} (see [9], section 20, p.78,79). An extraspecial 2-group G of order 2^{1+2m} is a central product of either m copies of the dihedral group D_8 of order 8 or $m-1$ copies of D_8 with a copy of the quaternion group Q_8 of order 8. In the first case, G possesses a maximal elementary abelian subgroup of order 2^{1+m} and we write $G = 2^{1+2m}_+$. If the later holds, then all maximal abelian subgroups of G are of type $2^{m-1} \times 4$ and we write $G = 2^{1+2m}_-$.

Let $S = (P, L)$ be a slim dense near hexagon having big quads of type (2,4) and (R, ψ) be a non-abelian representation of S. For $x, y \in P$ with $d(x, y) \leq 2$, $[r_x, r_y] =$

1 : if $d(x,y)=2$, we apply ([20], Theorem 1.5(i), p.55) to the restriction of ψ to the quad $Q(x,y)$. From ([20], Theorem 2.9, p.58, see [20], Example 2.2, p.56) and ([21], Theorem 1.6), we have

Proposition 4.1. *The following hold:*

(i) *For $x,y \in P$, $[r_x,r_y] \neq 1$ if and only if $d(x,y)=3$. In that case, $\langle r_x,r_y\rangle$ is a dihedral group of order 8.*
(ii) *$|R'|=2$ and $R'=\Phi(R) \subseteq Z(R)$.*
(iii) *$r_x \notin Z(R)$ for each $x \in P$, and ψ is faithful.*
(iv) *R is of order 2^β, where $1+n(S) \leq \beta \leq 1+dimV(S)$.*
(v) *If $\beta = 1+n(S)$, then R is an extraspecial 2-group. In that case, $R = 2_+^{1+n(S)}$ except for the near hexagon $Q_6^-(2) \otimes Q_6^-(2)$, in which case $R = 2_-^{1+n(S)}$.*

We repeatedly use Proposition 4.1(i), mostly without mention.

Let Q_1 and Q_2 be two disjoint big quads of S and Y be the subspace of S generated by them. Then, $Y \simeq Q_6^-(2)\times \mathbb{L}_3$ and $Y = Q_1\cup Q_2 \cup Q_3$, where $Q_3 = Q_1 * Q_2$ (see Lemma 1.2(ii)). Let (R,ψ) be a non-abelian representation of S. Set $M=\langle\psi(Y)\rangle$ and $N = C_R(M)$. Then $M \simeq 2_+^{1+12}$ (Theorem 4.1(v)) and R is a central product of M and N, written as $R = M\circ N$. In the following, we use the notation of Section 3.

Lemma 4.2 ([21], Proposition 5.3)**.** *For each $x \in P\setminus Y$, r_x has a unique decomposition as $r_x = r_{z^1_{x^2}} r_{z^2_{x^1}} n_x$, where n_x is an involution in $N\setminus Z(R)$.*

Lemma 4.3 ([21], Corollary 5.5)**.** *Let Q be a big quad of S disjoint from Y and $I_2(N)$ be the set of involutions in N. Let δ be the map from Q to $I_2(N)$ defined by $\delta(x)=n_x, x\in Q$. Then:*

(i) *δ is one-one.*
(ii) *For $x,y\in Q$, $[\delta(x),\delta(y)]=1$ if and only if $x=y$ or $x\sim y$.*
(iii) *There exists a spread T in Q such that for $x,y \in Q$ with $x \sim y$,*

$$\delta(x*y) = \begin{cases} \delta(x)\delta(y) & \text{if } xy \in T \\ \delta(x)\delta(y)\theta & \text{if } xy \notin T \end{cases};$$

where $R'=\{1,\theta\}$ (see Proposition 4.1(ii)).

The proof of the above two lemmas is mainly based on the fact that big quads of S are of type (2,4).

Proposition 4.4. *Let $S=(P,L)$ be a slim dense near hexagon having big quads of type $(2,4)$. Suppose that the Fischer space on big quads of S contains a subspace H isomorphic to the dual affine plane of order 2. Then, every representation of S is abelian.*

Proof. Let $H=\{Q_1,Q_2,Q_3,Q,T_1,T_2\}$ with lines $\{Q_1,Q_2,Q_3\}$, $\{Q_1,Q,T_1\}$, $\{T_1,T_2,Q_3\}$ and $\{Q,T_2,Q_2\}$. Then, $Y=Q_1\cup Q_2\cup Q_3$ is isomorphic to $Q_6^-(2)\times\mathbb{L}_3$ and Q is a big quad of S disjoint from Y.

Suppose that (R,ψ) is a non-abelian representation of S and let M and N be as above. Let $l=\{a,b,c\}$ be a line of S meeting T_1 at a, T_2 at b and Q_3 at c. We show that $n_a=n_b, n_a=n_{a_Q}$ and $n_b=n_{b_Q}$. Since $a_Q\neq b_Q$, $n_{a_Q}=n_{b_Q}$ would contradict Lemma 4.3(i), thus completing the proof.

For $m\in\{1,2\}$, l is disjoint from Q_m, so $l^m=\{a^m,b^m,c^m=z_c^m\}$ is a line of Q_m. By Lemma 3.2(ii), $(a^1,b^1)=(z^1_{b^2},z^1_{a^2})$ and $(a^2,b^2)=(z^2_{b^1},z^2_{a^1})$. So $r_a=r_{z^1_{a^2}}r_{z^2_{a^1}}n_a=r_{b^1}r_{b^2}n_a$ by Lemma 4.2. Similarly, $r_b=r_{a^1}r_{a^2}n_b$. Now, $r_ar_b=(r_{b^1}r_{b^2})(r_{a^1}r_{a^2})n_an_b=(r_{b^1}r_{a^1})(r_{b^2}r_{a^2})n_an_b=r_{c^1}r_{c^2}n_an_b$. The second equality holds since $d(a^1,b^2)=1$ by Lemma 3.2(ii). Since $c^1=z_c^1,c^2=z_c^2$ and $\{c,z_c^1,z_c^2\}$ is a line of Y, we get $r_ar_b=r_cn_an_b$. But $r_ar_b=r_c$ by the definition of a representation. So $n_a=n_b$.

Now, consider the line $l_a=\{a,a_Q,a^1=a^1_Q\}$ meeting T_1 at a, Q at a_Q and Q_1 at $a^1=a^1_Q$. We have $r_ar_{a_Q}=r_{a^1}$. Since l_a is disjoint from Q_2, $l_a^2=\{a^2,a^2_Q,z^2_{a^1}=z^2_{a^1_Q}\}$ is a line of Q_2. Now, $r_ar_{a_Q}=r_{z^1_{a^2}}r_{z^2_{a^1}}r_{z^1_{a^2_Q}}r_{z^2_{a^1_Q}}n_an_{a_Q}$. By Lemma 3.2($i$), $d(a^1,a^2_Q)=2$ and so, $[r_{z^2_{a^1}},r_{z^1_{a^2_Q}}]=1$. Since $a^1=a^1_Q$, we get $r_ar_{a_Q}=r_{z^1_{a^2}}r_{z^1_{a^2_Q}}n_an_{a_Q}$. Since the line l^2_a is disjoint from Q_1, its projection on Q_1 is the line $\{a^1=a^1_Q,z^1_{a^2},z^1_{a^2_Q}\}$. So $r_ar_{a_Q}=r_{a^1}n_an_{a_Q}$. Thus, $n_a=n_{a_Q}$. Similarly, considering the line $l_b=\{b,b_Q,b^2=b^2_Q\}$ disjoint from Q_1, the above argument yields that $n_b=n_{b_Q}$. This completes the proof. □

Proof of Theorem 1.3(i)**.** Let $S=(P,L)$ be either $\mathbb{E}_3$ or $\mathbb{G}_3$. Let Δ_S be the graph on big quads of S, two distinct big quads being adjacent when they have non-empty intersection. If $S=\mathbb{G}_3$, then $\Delta_{\mathbb{G}_3}$ is the 3-coclique extension of the (2,2)-GQ, and if $S=\mathbb{E}_3$, then $\Delta_{\mathbb{E}_3}$ is locally the collinearity graph of the (2,4)-GQ (see [4], p.361). In either case, it follows that for two adjacent vertices V_1 and V_2 of Δ_S, there exists a vertex V of Δ_S which is not adjacent to both V_1 and V_2. Consider the Fischer space $\mathcal{F}$ on big quads of S. Since V_1 and V_2 are not collinear in $\mathcal{F}$, the subspace H of $\mathcal{F}$ generated by the two intersecting lines $\{V,V_1,V*V_1\}$ and $\{V,V_2,V*V_2\}$ is isomorphic to the dual affine plane of order 2. So, by Proposition 4.4, every representation of S is abelian. Since S is a subspace of $DH_6(2^2)$ (see [4], p.353), Proposition 4.1(i) implies that every representation of $DH_6(2^2)$ is abelian. □

5. **Proof of Theorem** 1.3(ii)

In this section, we construct non-abelian representations for each of the near hexagons in Theorem 1.3(ii).

5.1. $Q_6^-(2)\times\mathbb{L}_3$**,** $W_4(2)\times\mathbb{L}_3$ **and** $Q_4^+(2)\times\mathbb{L}_3$**.** Let $R=2_+^{1+2k}$, $k\in\{4,6\}$, $R'=\{1,\theta\}$ and $V=R/R'$. We consider V as a vector space over F_2. The map $f:V\times V\longrightarrow F_2$ taking (xR',yR') to 0 or 1 according as $[x,y]=1$ or not, is a non-degenerate symplectic bilinear form on V. Write V as an orthogonal direct sum of k hyperbolic planes K_i $(1\leq i\leq k)$ in V and let H_i be the inverse image of K_i in R. Then, H_i is generated by two elements x_i and x_i^1 such that $[x_i,x_i^1]=\theta$. Let $M=\langle x_i:1\leq i\leq k\rangle$

and $M^1 = \langle x_i^1 : 1 \leq i \leq k\rangle$. Then, M and M^1 are elementary abelian 2-subgroups of R of order 2^k each. Further, M, M^1 and $Z(R)$ pairwise intersect trivially and $R = MM^1Z(R)$.

Let $F = (Q, B)$ be a $(2, t)$-GQ in M with $M = \langle Q\rangle$. Then, $(k, t) = (4, 1), (4, 2)$ or (6,4). (If $(k, t) = (4, 2)$, then F is of symplectic type.) For each $m \in Q$, the subgroup $H_m = \langle z \in Q : z \in m^{\perp}\rangle$ of M is of index 2 in M ([17], 4.2.4, p.68). The centralizer of H_m in M^1 is a subgroup $\langle \kappa_m^1\rangle$ of M^1 of order 2. Then, $M^2 = \langle m\kappa_m^1 : m \in Q\rangle$ is an elementary abelian 2-subgroup of R of order 2^k intersecting each of M, M^1 and $Z(R)$ trivially. We set

$$\begin{aligned} Q^1 &= \{\kappa_m^1 \in M^1 : m \in Q\}, \\ Q^2 &= \{m\kappa_m^1 \in M^2 : m \in Q\}, \\ B^1 &= \{\{\kappa_a^1, \kappa_b^1, \kappa_c^1\} : \{a, b, c\} \in B\}, \\ B^2 &= \{\{a\kappa_a^1, b\kappa_b^1, c\kappa_c^1\} : \{a, b, c\} \in B\}. \end{aligned}$$

Then, $F^1 = (Q^1, B^1)$ and $F^2 = (Q^2, B^2)$ are $(2, t)$-GQs in M^1 and M^2, respectively. Now, take

$$\begin{aligned} \mathsf{Q} &= Q \cup Q^1 \cup Q^2, \\ \mathsf{B} &= B \cup B^1 \cup B^2 \cup \{\{m, m\kappa_m^1, \kappa_m^1\} : m \in Q\}. \end{aligned}$$

Then, $\mathsf{S} = (\mathsf{Q}, \mathsf{B})$ is a partial linear space, isomorphic to $Q_4^+(2) \times \mathbb{L}_3$ if $(k, t) = (4, 1)$; $W_4(2) \times \mathbb{L}_3$ if $(k, t) = (4, 2)$; and $Q_6^-(2) \times \mathbb{L}_3$ if $(k, t) = (6, 4)$. Note that F, F^1 and F^2 are the only big quads in the last two cases. Thus we get non-abelian representation for $Q_6^-(2) \times \mathbb{L}_3, W_4(2) \times \mathbb{L}_3$ and $Q_4^+(2) \times \mathbb{L}_3$.

5.2. $\mathbb{H}_3$ **and** $DW_6(2)$**.** Let $R = 2_+^{1+8}$. Let F and F^1 be the symplectic (2,2)-GQs for $(k, t) = (4, 2)$ as in Subsection 5.1. The map $\sigma : Q \longrightarrow Q^1$ defined by $m \mapsto \kappa_m^1, m \in Q$, is an isomorphism from F to F^1. We set

$$\mathcal{Q} = \{mn^1 : m \in Q, n^1 \in Q^1, [m, n^1] = 1\}.$$

We define collinearity in $\mathcal{Q}$. For distinct $m_1n_1^1, m_2n_2^1 \in \mathcal{Q}$ with $m_1, m_2 \in Q$ and $n_1^1, n_2^1 \in Q^1$, we say that $m_1n_1^1 \sim m_2n_2^1$ if and only if $[m_1, n_2^1] = [m_2, n_1^1] = 1$ and $(m_1m_2)(n_1^1n_2^1) \in \mathcal{Q}$. The second condition implies that $m_1 \neq m_2$ and $n_1^1 \neq n_2^1$. The line containing $m_1n_1^1$ and $m_2n_2^1$ is $\{m_1n_1^1, m_2n_2^1, (m_1m_2)(n_1^1n_2^1)\}$. Let $\mathcal{B}$ be the set of all such lines in $\mathcal{Q}$. Set

$$\mathbb{Q} = Q \cup Q^1 \cup \mathcal{Q}, \text{ and } \mathbb{B} = \mathcal{B} \cup \mathcal{B}^1,$$

where $\mathcal{B}^1 = \{\{m, mn^1, n^1\} : mn^1 \in \mathcal{Q}\}$. Using the constructions of $\mathbb{H}_3$ and $DW_6(2)$ in Proposition 2.1, we now show that $\mathcal{F} = (\mathcal{Q}, \mathcal{B}) \simeq \mathbb{H}_3$ and $\mathbb{F} = (\mathbb{Q}, \mathbb{B}) \simeq DW_6(2)$, thus giving non-abelian representation for $\mathbb{H}_3$ and $DW_6(2)$.

Let $S = (P, L)$, $S^1 = (P^1, L^1)$, $\mathcal{S} = (\mathcal{P}, \mathcal{L})$, $\mathbb{S} = (\mathbb{P}, \mathbb{L})$ and the map π be as in Proposition 2.1. Let $\alpha : P \longrightarrow Q$ be an isomorphism from S to F and $\beta : P^1 \longrightarrow Q^1$ be the isomorphism from F^1 to Q^1 such that the following diagram commute:

$$\begin{array}{ccc} P & \xrightarrow{\pi} & P^1 \\ \alpha\downarrow & & \downarrow\beta \\ Q & \xrightarrow[\sigma]{} & Q^1. \end{array}$$

Thus, $\beta(u^1) = \sigma\alpha\pi^{-1}(u^1)$, $u^1 \in P^1$. We show that, if $x \in P$ and $u^1 \in P^1$, then $(x, u^1) \in \mathcal{P}$ if and only if $\alpha(x)\beta(u^1) \in \mathcal{Q}$. First, assume that $(x, u^1) \in \mathcal{P}$ and $u \in P$ be such that $\pi(u) = u^1$. Since $(x, u^1) \in \mathcal{P}$, $x \in u^\perp$ and $\alpha(x) \in \alpha(u)^\perp$. This implies that $[\alpha(x), \sigma(\alpha(u))] = 1$, since $\kappa^1_{\alpha(u)} = \sigma(\alpha(u))$. But $[\alpha(x), \sigma(\alpha(u))] = [\alpha(x), \sigma\alpha\pi^{-1}(u^1)] = [\alpha(x), \beta(u^1)]$. So $\alpha(x)\beta(u^1) \in \mathcal{Q}$. Reversing the argument we conclude that $(x, u^1) \in \mathcal{P}$ when $\alpha(x)\beta(u^1) \in \mathcal{Q}$.

Let the map $\rho : \mathbb{P} \longrightarrow \mathbb{Q}$ be equal to α on P, β on P^1 and $\rho((x, u^1)) = \alpha(x)\beta(u^1)$ for $(x, u^1) \in \mathcal{P}$. Then, ρ induces a bijection from $\mathcal{L}$ to $\mathcal{B}$ and from $\mathcal{L}^1$ to $\mathcal{B}^1$. For the injectivity on $\mathcal{L}$, we use the fact that if $\{u, v, w\}$ is either a line or a complete triad in Q or Q^1, then $uvw = 1$ ([21], Proposition 3.5). So $\mathbb{S} \simeq \mathbb{F}$. Further, the restriction of ρ to $\mathcal{P}$ is an isomorphism from $\mathcal{S}$ to $\mathcal{F}$.

[If $m_1n_1^1$ and $m_2n_2^1$ are distinct points of $\mathcal{Q}$ with $m_1, m_2 \in Q$ and $n_1^1, n_2^1 \in Q^1$, then the following hold:

(1) $d(m_1n_1^1, m_2n_2^1) = 1$ if and only if $m_1 \neq m_2, n_1^1 \neq n_2^1$ and $[m_1, n_2^1] = [m_2, n_1^1] = 1$.
(2) $d(m_1n_1^1, m_2n_2^1) = 2$ if and only if one of the following occur:
  (*i*) $m_1 = m_2, n_1^1 \neq n_2^1$;
  (*ii*) $m_1 \neq m_2, n_1^1 = n_2^1$;
  (*iii*) $m_1 \neq m_2, n_1^1 \neq n_2^1$ and $[m_1, n_2^1] = [m_2, n_1^1] \neq 1$.
(3) $d(m_1n_1^1, m_2n_2^1) = 3$ if and only if $m_1 \neq m_2, n_1^1 \neq n_2^1$ and one of the following occur:
  (*i*) $[m_1, n_2^1] = 1$ and $[m_2, n_1^1] \neq 1$;
  (*ii*) $[m_1, n_2^1] \neq 1$ and $[m_2, n_1^1] = 1$.]

5.3. $Q_6^-(2) \otimes Q_6^-(2)$**.** Let $S = (P, L)$ be the near hexagon $Q_6^-(2) \otimes Q_6^-(2)$. We refer to ([4], p.363) for the description of the Fischer space on the set of the 18 big quads of S. This set partitions into two families $\mathcal{F}_1$ and $\mathcal{F}_2$ of size 9 each such that each $\mathcal{F}_i$ defines a partition of the point set P of S. Let $\mathcal{U}_i$, $i = 1, 2$, be the partial linear space whose points are the big quads of $\mathcal{F}_i$; every pair of distinct points of $\mathcal{U}_i$ are collinear. Further, if Q_1 and Q_2 are two distinct points of $\mathcal{U}_i$, then the line containing them is $\{Q_1, Q_2, Q_3\}$, where $Q_3 = Q_1 * Q_2$ (Lemma 1.2(*ii*)). Then, $\mathcal{U}_i$ is an affine plane of order 3.

Consider the affine plane $\mathcal{U}_1$. Fix an affine line $\{Q_1, Q_2, Q_3\}$ in $\mathcal{U}_1$. Then, $Y = Q_1 \cup Q_2 \cup Q_3$ is isomorphic to $Q_6^-(2) \times \mathbb{L}_3$. Fix an affine point Q in $\mathcal{U}_1$ such that $Q \cap Y$ is empty. Taking $\{i, j, k\} = \{1, 2, 3\}$, we make use of the notation and the results of Section 3 in the rest of this section.

Let $l = \{a, b, c\}$ be a line of S not contained in Y. If l meets Y at some point c, say, and is disjoint from Q, then exactly one of the lines aa_Q and bb_Q meet Y. If l

meets Q at some point and is disjoint from Y, then l corresponds to the affine line of $\mathcal{U}_1$ containing Q and parallel to $\{Q_1, Q_2, Q_3\}$. Further, if $x \in l \setminus (l \cap Q)$, then the line xx^i is disjoint from Q. Now, let l be disjoint from both Y and Q. Then l is contained in a point of $\mathcal{U}_1$ different from Q and Q_i, $i \in \{1,2,3\}$; or it corresponds to the affine line of $\mathcal{U}_1$ not containing Q and parallel to $\{Q_1, Q_2, Q_3\}$. So, the lines aa_Q, bb_Q and cc_Q either meet Y or all have empty intersection with Y. In the first case, if $xx_Q \cap Y = \{x_Y\}$ for $x \in l$ and $l_Y = \{x_Y : x \in l\}$, then l_Y is a line of Q_i for some $i \in \{1,2,3\}$; or $|l_Y \cap Q_i| = 1$ for each $i \in \{1,2,3\}$ (l_Y need not be a line in this case).

Lemma 5.1. *Let $l = \{a, b, c\}$ be a line of S disjoint from $Y \cup Q$ such that the line xx_Q meets Y at x_Y for each $x \in l$. Let $m, n \in \{1,2,3\}$, $m \neq n$.*

- (i) *If l is contained in a point of $\mathcal{U}_1$, then $d(a^m, b^n) \leq 2$ or $d(a^m, b^n) = 3$ according as $l_Q \in T_Q$ or not.*
- (ii) *If l corresponds to the affine line of $\mathcal{U}_1$ not containing Q and parallel to $\{Q_1, Q_2, Q_3\}$, then $l_Q \notin T_Q$ and $d(a^m, b^n) = 3$.*

Proof. (i) Let $l_Y = \{a_Y, b_Y, c_Y\}$. Then, l_Y is a line of Q_i, Q_j or Q_k, say Q_i. (If K is the affine point of $\mathcal{U}_1$ containing l, then $Q_i = K * Q$.) Let $x \in l$. Then $x^i = x^i_Q = x_Y \in Q_i$, so $l^i = l^i_Q$. The line $l_x = \{x, x_Q, x_Y\}$ is disjoint from Q_j and Q_k. So $l^j_x = \{x^j, x^j_Q, z^j_{x_Y} = z^j_{x^i_Q}\}$ and $l^k_x = \{x^k, x^k_Q, z^k_{x_Y} = z^k_{x^i_Q}\}$ are lines of Q_j and Q_k respectively.

If $l_Q \in T_Q$, then $l^j_Q = \{x^j_Q, z^j_{x^i_Q}, z^j_{x^k_Q}\}$ by Lemma 3.4. Since $|l^j_Q \cap l^j_x| \geq 2$, we get $l^j_Q = l^j_x$. Thus $x^j \in l^j_Q$ for each $x \in l$ and so, $l^j = l^j_Q$. Similarly, $l^k = l^k_Q$. Now, Corollary 3.7 completes the proof of (i) in this case.

If $l_Q \notin T_Q$, then consider (3.4) for the line l_Q and the lines l^j_x and l^k_x above. Then, l^j_x and l^k_x are the lines corresponding to the x-column in $T^{l_Q}_j$ and $T^{l_Q}_k$, respectively. So $z^j_{x^k_Q} = x^j$ and $z^k_{x^j_Q} = x^k$ and (i) in this case follows from Corollary 3.7.

(ii) Here l_Y meets each of Q_i, Q_j and Q_k. We may assume that $a_Y \in Q_i$, $b_Y \in Q_j$ and $c_Y \in Q_k$. Then $a^i = a^i_Q = a_Y$, $b^j = b^j_Q = b_Y$ and $c^k = c^k_Q = c_Y$. Suppose that $l_Q \in T_Q$. Since $\tau_{ik}(l^i_Q) = l^k_Q$ (Lemma 3.4), we may assume that $b^k_Q \sim a^i_Q$ (see (3.3)). Then, $z^j_{a^i_Q} = c^j_Q$. The line $l_a = \{a, a_Q, a_Y = a^i_Q\}$ is disjoint from Q_j. So $l^j_a = \{a^j, a^j_Q, z^j_{a^i_Q} = c^j_Q\}$ is a line in Q_j. But $l^j_Q = \{a^j_Q, b^j = b^j_Q, c^j_Q\}$ is a line in Q_j, and so $a^j = b^j$, a contradiction to the fact that $\{a^j, b^j, c^j\}$ is a line in Q_j. So $l_Q \notin T_Q$. Since $a^i = a^i_Q, b^j = b^j_Q, c^k = c^k_Q$, (3.4) applied to the line l_Q together with Corollary 3.7 implies (ii). □

Lemma 5.2. *Let x be a point in $P \setminus (Y \cup Q)$ such that the line xx_Q is disjoint from Y. Let $\zeta_{x_Q} = \{x_Q, a_x, b_x\} \in T_Q$ and $xx_Q = \{x, x_Q, y\}$. Then $\{(x^1, x^2, x^3), (y^1, y^2, y^3)\} = \{(a^1_x, a^2_x, a^3_x), (b^1_x, b^2_x, b^3_x)\}$.*

Proof. Let $l = xx_Q$. If $x^i \in \zeta^i_{x_Q}$, then $\zeta^i_{x_Q} = l^i$. By definition of ζ_{x_Q} and Lemma 3.4, $\tau_{ij}(\zeta^i_{x_Q}) = \zeta^j_{x_Q}$. So $z^j_{x^i} \in \zeta^j_{x_Q}$. Since $x^j \sim z^j_{x^i}$ and $x^j \sim x^j_Q$ in the line $\zeta^j_{x_Q}$, it follows that $x^j \in \zeta^j_{x_Q}$. So $l^i = x^i z^i_{x^j}$. Then, $\tau_{ij}(l^i) = l^j$ (Lemma 3.3(i)). So $l^j = \zeta^j_{x_Q}$ and the result follows (see (3.3)). Thus, it is enough we show that $x^i \in \zeta^i_{x_Q}$.

Suppose that $x^i \notin \zeta^i_{x_Q}$. Let $\bar{l} = \{x, x^i, w\}$ be the line xx^i of S. Then, $\bar{l}$ is disjoint from Q. Consider the line $\bar{l}_Q = \{x_Q, (x^i)_Q, w_Q\}$ of Q. Since $(x^i)_Q \notin \zeta_{x_Q}$, $\bar{l}_Q \neq \zeta_{x_Q}$ and $\zeta_{x_Q} \cap \bar{l}_Q = \{x_Q\}$. The line ww_Q meets either Q_j or Q_k, say Q_k. Since $\bar{l}$ is disjoint from both Q_j and Q_k, $\bar{l}^j = \{x^j, z^j_{x^i}, w^j\}$ and $\bar{l}^k = \{x^k, z^k_{x^i}, w^k = w^k_Q\}$ are lines of Q_j and Q_k, respectively. Applying Lemma 3.2(ii) to $\bar{l}$, we get $w^j \sim x^k$ and $w^k \sim x^j$.

Now, $d(x^k, x_Q) = d(x^k, x) + d(x, x_Q) = 2$ and $d(x^k, w_Q) = d(x^k, w^k) + d(w^k, w_Q) = 2$. So, $d(x^k, (x^i)_Q) = 1$. Again, $d(x^j, x_Q) = d(x^j, x) + d(x, x_Q) = 2$ and $d(x^j, w_Q) = d(x^j, w^k) + d(w^k, w_Q) = 2$ (since $w^k \sim x^j$). So, $d(x^j, (x^i)_Q) = 1$. Let $c = (x^i)_Q$. Then, $c^j = x^j$ and $c^k = x^k$. Now, $\bar{l}^k_Q = c^k w^k_Q = c^k w^k = c^k z^k_{x^j} = c^k z^k_{c^j}$. Applying Lemma 3.3($i$) to $\bar{l}_Q$, we get $\tau_{kj}(\bar{l}^k_Q) = \bar{l}^j_Q$. So $\bar{l}_Q \in T_Q$ (see Lemma 3.4). But $\zeta_{x_Q} \in T_Q$ and $\zeta_{x_Q} \cap \bar{l}_Q = \{x_Q\}$. This leads to a contradiction to the fact that T_Q is a spread of Q (Corollary 3.6). So $x^i \in \zeta^i_{x_Q}$. □

In view of Lemma 4.3, we prove the following.

Lemma 5.3. *Let $N = 2^{1+6}_-$ with $N' = \{1, \theta\}$ and let $I_2(N)$ be the set of involutions in N. There exists a map δ from Q to $I_2(N)$ satisfying the following:*

(i) *δ is one-one.*
(ii) *For $x, y \in Q$, $[\delta(x), \delta(y)] = 1$ if and only if either $x = y$ or $x \sim y$.*
(iii) *If $x, y \in Q$ and $x \sim y$, then*

$$\delta(x * y) = \begin{cases} \delta(x)\delta(y) & \text{if } xy \in T_Q \\ \delta(x)\delta(y)\theta & \text{if } xy \notin T_Q \end{cases}.$$

Proof. We use the following model for Q ([17], 6.1.1, p.122): Let $\Omega = \{1, 2, 3, 4, 5, 6\}$ and $\Omega' = \{1', 2', 3', 4', 5', 6'\}$. A *factor* of Ω is a set of three pair-wise disjoint 2-subsets of Ω. Let $\mathcal{E}$ be the set of all 2-subsets of Ω and $\mathcal{F}$ be the set of all factors of Ω. Then, the point set of Q is $\mathcal{E} \cup \Omega \cup \Omega'$ and the line set is $\mathcal{F} \cup \{\{i, \{i, j\}, j'\} : 1 \leq i \neq j \leq 6\}$. We may assume that the spread T_Q of Q consists of the following lines:

$$l_1 = \{\{1, 2\}, \{3, 4\}, \{5, 6\}\};\ l_2 = \{\{1, 4\}, 1, 4'\};\ l_3 = \{\{2, 6\}, 2, 6'\};$$
$$l_4 = \{\{1, 6\}, \{2, 4\}, \{3, 5\}\};\ l_5 = \{\{1, 5\}, 1', 5\};\ l_6 = \{\{2, 3\}, 2', 3\};$$
$$l_7 = \{\{1, 3\}, \{2, 5\}, \{4, 6\}\};\ l_8 = \{\{3, 6\}, 3', 6\};\ l_9 = \{\{4, 5\}, 4, 5'\}.$$

We write N as a central product $N = \langle x_1, y_1\rangle \circ \langle x_2, y_2\rangle \circ Q_8$, where x_i, y_i are involutions, $\langle x_i, y_i\rangle$ is isomorphic to the dihedral group D_8 of order 8, and Q_8 is the quaternion group of order 8. Let $Q_8 = \{1, \theta, i, j, k, i^3, j^3, k^3\}$, where $i^2 = j^2 = k^2 = \theta, ij = k$ and $ji = k^3 = k\theta$. We define $\delta : Q \longrightarrow I_2(N)$ as follows:

$\delta(l_1) = \{x_1, x_2, x_1x_2\}$;

$$\begin{aligned}
\delta(l_2) &= \{x_1y_1y_2i, x_2y_2j, x_1x_2y_1k\theta\};\\
\delta(l_3) &= \{x_1y_1i\theta, x_1x_2y_2k, x_2y_1y_2j\theta\};\\
\delta(l_4) &= \{y_1, y_1y_2, y_2\};\\
\delta(l_5) &= \{x_1x_2y_1i, x_2y_2k\theta, x_1y_1y_2j\};\\
\delta(l_6) &= \{x_2y_1y_2i\theta, x_1x_2y_2j\theta, x_1y_1k\};\\
\delta(l_7) &= \{x_1x_2y_1y_2\theta, x_2y_1\theta, x_1y_2\theta\};\\
\delta(l_8) &= \{x_1x_2y_2i\theta, x_1y_1j\theta, x_2y_1y_2k\};\\
\delta(l_9) &= \{x_2y_2i, x_1x_2y_1j, x_1y_1y_2k\theta\}.
\end{aligned}$$

Here, if $l_i = \{a, b, c\}$, then $\delta(l_i) = \{\delta(a), \delta(b), \delta(c)\}$ preserving the order. It can be verified that δ satisfies the conditions $(i), (ii)$ and (iii) of the lemma. □

Consider the map $\delta : Q \longrightarrow I_2(N)$ in Lemma 5.3. We now extend δ to $P \setminus Y$. For $x \in P \setminus (Y \cup Q)$, let $\zeta_{x_Q} = \{x_Q, a_x, b_x\} \in T_Q$. If the line xx_Q intersects Y, then we define $\delta(x) = \delta(x_Q)$. If xx_Q is disjoint from Y, let $(b_x^1, b_x^2, b_x^3) = (x^1, x^2, x^3)$ (see Lemma 5.2). In that case, we define $\delta(x) = \delta(a_x)$. That is; for $x \in P \setminus (Y \cup Q)$,

$$\delta(x) = \begin{cases} \delta(x_Q) & \text{if } xx_Q \text{ intersects } Y \\ \delta(a_x) & \text{if } xx_Q \cap Y \text{ is empty and } (x^1, x^2, x^3) = (b_x^1, b_x^2, b_x^3) \end{cases}.$$

We now construct a non-abelian representation of S. Let $R = 2_-^{1+18}$ with $R' = \{1, \theta\}$. We write R as a central product $R = M \circ N$, where $M = 2_+^{1+12}$ and N_-^{1+6}. Let (M, λ) be a non-abelian representation of Y (see Subsection 5.1). Define a map $\beta : P \longrightarrow R$ as follows:

$$\beta(x) = \begin{cases} \lambda(x) & \text{if } x \in Y \\ \lambda(z_{x^2}^1)\lambda(z_{x^1}^2)\delta(x) & \text{if } x \in P \setminus Y \end{cases}.$$

For $x \in P \setminus Y$, Lemma 3.1(i) implies that $d(z_{x^1}^2, z_{x^2}^1) = 2$. So $[\lambda(z_{x^2}^1), \lambda(z_{x^1}^2)] = 1$ and $\beta(x)$ is an involution.

Proposition 5.4. *(R, β) is a non-abelian representation of S.*

Proof. Only condition (ii) of Definition 1.1 needs to be verified. Let $l = \{u, v, w\}$ be a line of S. We assume that l is not contained in Y and that $l \cap Y = \{w\}$ if l intersects Y. We show that $\beta(u)\beta(v) = \beta(w)$. We have

$$\beta(u)\beta(v) = \lambda(z_{u^2}^1)\lambda(z_{v^2}^1)\lambda(z_{u^1}^2)\lambda(z_{v^1}^2)\delta(u)\delta(v)r', \tag{5.5}$$

where $r' = [\lambda(z_{u^1}^2), \lambda(z_{v^2}^1)] \in R'$.

Case (I) *Let l intersects Y at w.* Then Lemma 3.2 yields that $r' = 1$. If $w \in Q_1$, then $u^1 = v^1 = w$ and $\beta(u)\beta(v) = \lambda(z_{u^2}^1)\lambda(z_{v^2}^1)\delta(u)\delta(v) = \lambda(w)\delta(u)\delta(v)$. The last equality holds because $\{z_{u^2}^1, z_{v^2}^1, w\}$ is a line of Q_1. Similarly, $\beta(u)\beta(v) = \lambda(w)\delta(u)\delta(v)$ if $w \in Q_2$. If $w \in Q_3$, then $\{z_{u^2}^1, z_{v^2}^1, z_w^1\}$ and $\{z_{u^1}^2, z_{v^1}^2, z_w^2\}$ are lines of Q_1 and Q_2 respectively. So, $\beta(u)\beta(v) = \lambda(z_w^1)\lambda(z_w^2)\delta(u)\delta(v) = \lambda(w)\delta(u)\delta(v)$. The last equality holds because $\{z_w^1, z_w^2, w\}$ is a line of Y. Since $\beta(w) = \lambda(w)$, we get $\beta(u)\beta(v) = \beta(w)\delta(u)\delta(v)$. Thus, we need to prove that $\delta(u) = \delta(v)$.

If l intersects Q, say $l \cap Q = \{v\}$, then $u_Q = v$ and so, $\delta(u) = \delta(v)$. Let $l \cap Q$ be empty. Exactly one of the lines uu_Q and vv_Q, say uu_Q, meets Y. So $\delta(u) = \delta(u_Q)$. Let $l_{v_Q} = \{v_Q, a_v, b_v\}$. By Lemma 5.2, we assume that $(v^1, v^2, v^3) = (b_v^1, b_v^2, b_v^3)$. Then $\delta(v) = \delta(a_v)$. Since $w \in \{v^1, v^2, v^3\}$, it follows that $b_v \sim w$. So $w_Q = b_v$ and $u_Q = a_v$. Thus, $\delta(u) = \delta(u_Q) = \delta(a_v) = \delta(v)$.

Case (II) *Let l be disjoint from Y.* Since $\{z^1_{u^2}, z^1_{v^2}, z^1_{w^2}\}$ and $\{z^2_{u^1}, z^2_{v^1}, z^2_{w^1}\}$ are lines of Q_1 and Q_2 respectively, we get $\beta(u)\beta(v) = \lambda(z^1_{w^2})\lambda(z^2_{w^1})\delta(u)\delta(v)r'$. To complete the proof, we need to show that either $r' = 1$ and $\delta(u)\delta(v) = \delta(w)$ or $r' = \theta$ and $\delta(u)\delta(v) = \delta(w)\theta$. This holds by Corollary 3.7 and Lemma 5.3(*iii*) if $l \subset Q$.

Assume that l intersects Q at a point, say w. Let $\zeta_w = \{w, a, b\} \in T_Q$. Applying Lemma 5.2, we get $\zeta_w^j = l^j$ in Q_j and $\{\delta(u), \delta(v)\} = \{\delta(a), \delta(b)\}$. This, together with $\zeta_w \in T_Q$, yields that $\delta(u)\delta(v) = \delta(w)$ (Lemma 5.3(*iii*)) and $r' = 1$ (Corollary 3.7).

Now, assume that $l \cap Q$ is empty. If the lines uu_Q, vv_Q and ww_Q meet Y, then Lemmas 5.1 and 5.3(*iii*) complete the proof. So, we may assume that none of uu_Q, vv_Q and ww_Q meet Y. First, let $l_Q \in T_Q$. Then $l_Q = \zeta_{u_Q} = \zeta_{v_Q} = \zeta_{w_Q}$. Applying Lemma 5.2 to the lines xx_Q, $x \in l$, it follows that $l_Q^j = l^j$ in Q_j and $(\delta(u), \delta(v), \delta(w)) = (\delta(w_Q), \delta(u_Q), \delta(v_Q))$ or $(\delta(v_Q), \delta(w_Q), \delta(u_Q))$. This implies that $\delta(u)\delta(v) = \delta(w)$ (Lemma 5.3(*iii*)) and $r' = 1$ (Corollary 3.7).

Now, let $l_Q \notin T_Q$. For $x \in l$, let $\zeta_{x_Q} = \{x_Q, a_x, b_x\}$. We may assume, by Lemma 5.2, that $(x^1, x^2, x^3) = (a_x^1, a_x^2, a_x^3)$. So, $\delta(x) = \delta(b_x)$. For distinct $x, y \in l$, $a_x^i = x^i \sim y^i = a_y^i$ in Q_i. Thus, $l_a = \{a_u, a_v, a_w\}$ and $l_b = \{b_u, b_v, b_w\}$ are lines of Q. Since $l_b \notin T_Q$, $\delta(u)\delta(v) = \delta(b_u)\delta(b_v) = \delta(b_w)\theta = \delta(w)\theta$. Again, $l_a \notin T_Q$ implies that $d(u^1, v^2) = d(a_u^1, a_v^2) = 3$ (Corollary 3.7) and so, $r' = \theta$. This completes the proof. $\square$

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