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Algebraic reflexivity of some subsets of the isometry group

S. DUTTA AND T. S. S. R. K. RAO

Indian Statistical Institute, Bangalore Centre
8th Mile Mysore Road, Bangalore, 560059 India

ALGEBRAIC REFLEXIVITY OF SOME SUBSETS OF THE ISOMETRY GROUP

S. DUTTA AND T. S. S. R. K. RAO

ABSTRACT. Let X be a compact metric space. In this paper we show that the set of isometries of $C(X)$ that are involutions is algebraically reflexive. As a consequence of a recent work of Botelho and Jamison this leads to the conclusion that the set of generalized bi-circular projections on $C(X)$ is also algebraically reflexive. We also consider these questions for the space $C(X, E)$ where E is a uniformly convex Banach space.

1. INTRODUCTION

Let E be a complex Banach space. Let $\mathcal{G}(E)$ denote the group of isometries of E . A non-empty, bounded set $L \subset \mathcal{L}(E)$ is said to be algebraically reflexive, if for any linear map $\Phi : E \rightarrow E$ such that $\Phi(e) = \Phi_e(e)$ for some $\Phi_e \in L$ and for all $e \in E$, implies that $\Phi \in L$. It is convenient to denote the set with the above properties as the algebraic closure \overline{L}^a . Note that when $L \subset \mathcal{G}(E)$, Φ is an into isometry. Algebraic reflexivity of the isometry group of function spaces and spaces of operators has received a lot of attention recently. See the Lecture Notes by Molnar [8] for a very comprehensive account of this theory.

In this paper we are interested in studying the algebraic reflexivity of some special subsets of the isometry group. This study is also related to a recently started or revived study of properties of projections on E that have some special geometric properties. Let \mathbf{T} denote the unit circle. A linear projection $P : E \rightarrow E$ is said to be a generalized bi-circular projection (GBP for short) if for some $\lambda \in \mathbf{T}$, $\Phi = P + \lambda(I - P)$ is an isometry (see [5]). Clearly when $\lambda = -1$, P is a bi-contractive projection, i.e, $\|P\| \leq 1$ and $\|I - P\| \leq 1$. It has been shown by Benau , Lacey [3] and Lima [7] that

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on certain function spaces for any bi-contractive projection P , $\Phi = 2P - I$ is an isometry. In this case the isometry is an involution i.e, $\Phi^2 = I$. These include the space $C(X)$ and the space of affine continuous functions on a Choquet simplex, $A(K)$. It is known that in general these projections are norm Hermitian. Since for a large class of Banach spaces one has precise description of surjective isometries, Botelho and Jamison [4] have initiated the study of structure of GBP based on the corresponding description of isometries. We thank Professor Botelho and Jamison for providing us with a copy of their work and for some clarifications.

Throughout this paper we assume that the isometry group under consideration is algebraically reflexive. Motivated by the above consideration in the first section of the paper we consider the algebraic reflexivity of the set of isometries that are involutions. For a metrizable compact set X , we show that for the space $C(X)$, this set is algebraically reflexive. We extend (with a correct formulation) a result of Botelho and Jamison [4] that characterizes GBP' on the space of vector-valued continuous functions $C(X, E)$, to the space of affine continuous functions $A(K, E)$, where K is a compact Choquet simplex. If the set of extreme points $X = \partial_e K$ is a closed set, then the space $A(K, E)$ coincides with the continuous function space $C(X, E)$. In the case of $A(K)$ it turns out that GBP' precisely bi-contractive projections. Thus as a corollary of our result we get that the set of bi-contractive projections on $C(X)$, for a metrizable X is algebraically reflexive.

2. ALGEBRAIC REFLEXIVITY OF INVOLUTIONS

Let $\mathcal{G}^2(C(X))$ denote the set of all involuting isometries of $C(X)$. In what follows we use the classical Banach-Stone theorem that describes the isometry group of $C(X)$.

Theorem 1. *Let X be compact metric space. Then $\mathcal{G}^2(C(X))$ is algebraically reflexive.*

Proof. We first recall from the Banach-Stone theorem that $U \in \mathcal{G}^2(C(X))$ if and only if there exists a homeomorphism ϕ of X , $\phi^2(x) = x$ for all x and $\tau : X \rightarrow \mathbf{T}$ satisfying $\tau(x)\tau(\phi(x)) = 1$ such that $Uf = \tau f(\phi)$ for all $f \in C(X)$.

Let $T \in \overline{\mathcal{G}^2(C(X))}^a$. Thus for each f there exists a homeomorphism ϕ_f of X , $\phi_f^2(x) = x$ for all $x \in X$, and $\tau_f : X \rightarrow \mathbf{T}$ satisfying $\tau_f(x)\tau_f(\phi_f(x)) = 1$

such that $Tf = \tau_f f(\phi_f)$. It follows from [8, Theorem 3.2.1] that $\mathcal{G}(C(X))$ is algebraically closed and hence T itself is an isometry. Thus there exists a homeomorphism ϕ of X and $\tau : X \rightarrow \mathbf{T}$ such that $Tf = \tau f(\phi)$. To show T is an involution we need to show that $\phi^2(x) = x$ for all x and $\tau(x)\tau(\phi(x)) = 1$.

We first show $\phi^2(x) = x$. If $x = \phi(x)$ then this indeed holds true. So let for some x , $x \neq \phi(x)$. We choose an $f \in C(X)$ such that $0 \leq f(x) \leq 1$, $x \in X$ and $f^{-1}(0) = \{\phi(x)\}$, $f^{-1}(1) = \{x\}$. Evaluating $\tau f(\phi)$ at x we have $0 = \tau(x)f(\phi(x)) = \tau_f(x)f(\phi_f(x))$ and hence $f(\phi_f(x)) = 0$ and by the choice of f we have $\phi_f(x) = \phi(x)$. Now evaluating $\tau f(\phi)$ at $\phi(x)$ we get

$$\begin{aligned} \tau(\phi(x))f(\phi^2(x)) &= \tau_f(\phi(x))f(\phi_f(\phi(x))) = \tau_f(\phi(x))f(\phi_f(\phi_f(x))) \\ &= \tau_f(\phi(x))f(x) = \tau_f(x). \end{aligned}$$

Thus $f(\phi^2(x)) = \overline{\tau(\phi(x))}\tau_f(\phi(x)) \in \mathbf{T}$. But $f(\phi^2(x))$ is real and hence $f(\phi^2(x)) = 1$ which by the choice of f implies $\phi^2(x) = x$.

It remains to show that $\tau(x)\tau(\phi(x)) = 1$. Take $x \in X$. If $\phi(x) = x$, choose $f \in C(X)$ such that $0 \leq f(x) \leq 1$, $x \in X$ and $f^{-1}(1) = \{\phi(x)\}$. If $\phi(x) \neq x$ choose above f with the additional property that $f(x) \neq 0$. As before evaluating $\tau f(\phi)$ at x we have $\tau(x) = \tau_f(x)f(\phi_f(x))$ and since $\tau(x)$ and $\tau_f(x)$ are elements of \mathbf{T} and f is real valued, $\tau(x) = \tau_f(x)$ and $\phi(x) = \phi_f(x)$. Now evaluating at $\phi(x)$ and noting that $\phi^2(x) = x$ we get $\tau(\phi(x))f(x) = \tau_f(\phi(x))f(\phi_f(\phi(x))) = \tau_f(\phi(x))f(x)$. Since $f(x)$ is nonzero we have $\tau(\phi(x)) = \tau_f(\phi(x))\tau(\phi_f(x))$. But $\tau_f(x)\tau(\phi_f(x)) = 1$ and hence same is true for τ . This completes the proof. \square

Before applying this result to the set of bi-contractive projections, we prove a theorem that extends the main result of [4] to the space $A(K, E)$.

We first recall the vector-valued Banach-Stone theorem for the space $A(K, E)$ (see [6]). We note that analogous to the case of $C(X)$, isometries are determined by evaluation at the set of extreme points $\partial_e K$.

Theorem 2. *Let K be a Choquet simplex and let E^* be a strictly convex space. Let $\Phi : A(K, E) \rightarrow A(K, E)$ be a surjective isometry. Then there exists a weight function $\tau : \partial K \rightarrow \mathcal{G}(E)$ and an affine homeomorphism ϕ of K such that $\Phi(a)(k) = \tau(k)(a(\phi(k)))$ for all $k \in \partial_e K$ and $a \in A(K, E)$.*

In the proof of the following theorem we adapt the arguments given during the proof of Theorem 2. 1 [4] to the simplex case.

Theorem 3. *Let K be a compact Choquet simplex and E be such that E^* is strictly convex. Let $P : A(K, E) \rightarrow A(K, E)$ be a GBP. Then the affine homeomorphism ϕ associated with Φ is an involution on $\partial_e K$. If further $\phi \neq I$ then $\lambda = -1$, $P(a)(k) = \frac{1}{2}\{\tau(k)(a(\phi(k)) + a(k))\}$ and the weight function τ satisfies $\tau(k) \circ \tau(\phi(k)) = I$ on $\partial_e K$. If $\phi = I$ then $P_k(e) = \frac{\tau(k)(e) - \lambda e}{(1-\lambda)}$, is a GBP in E for each $k \in \partial_e K$. Also, $P(a)(k) = P_k(a(k))$ for $k \in \partial_e K$ and $a \in A(K, E)$.*

Proof. Let $\Phi = P + \lambda(I - P)$. As Φ is a surjective isometry, from the above theorem, we have an affine homeomorphism ϕ and a weight function $\tau : \partial_e K \rightarrow \mathcal{G}(E)$ such that $\Phi(a)(k) = \tau(k)(a(\phi(k)))$. Since P is a projection we further have $\lambda a(k) - (\lambda + 1)\tau(k)(a(\phi(k))) + \tau(k)(\tau(a(\phi(k)))) = 0$, for all $a \in A(K, E)$ and $k \in \partial_e K$. Suppose $k \in \partial_e K$ is such that $\phi^2(k) \neq k$ and $\phi(k) \neq k$. As $\phi(k), \phi^2(k) \in \partial_e K$ and K is a simplex, convex hull of $\{\phi(k), \phi^2(k)\}$ is a face. Therefore by [1, Corollary II.5.20] there exist a $a_0 \in A(K)$ such that $a_0(k) = 1$ and $a_0(\phi(k)) = 0 = a_0(\phi^2(k))$. Evaluating the above formula at this k and a_0 we get that $\lambda = 0$, a contradiction. Thus $\phi(k) = k$ or $\phi^2(k) = k$. In either case $\phi^2(k) = k$ on $\partial_e K$. If $\phi \neq I$, then for some $k_o \in \partial_e K$, using a separation arguments once again in the above equation, we get that $\lambda = -1$ and $\tau(k) \circ \tau(\phi(k)) = I$ for $k \in \partial_e K$. Thus $P(a)(k) = \frac{1}{2}\{\tau(k)(a(\phi(k)) + a(k))\}$.

If $\phi = I$, then as in the proof of [4, Theorem 2.1], using [4, Lemma 2.1] we get that $P_k(e) = \frac{\tau(k)(e) - \lambda e}{(1-\lambda)}$, is a GBP in E for each $k \in \partial_e K$. Hence the conclusion follows. □

From the above theorem we have that for a Choquet simplex K , bi-contractive projections on $A(K)$ coincide with GBP'. This in particular is also the case for $C(X)$.

Remark 4. *A remarkable feature of the above description is the additional requirement on the function τ , $\tau(k)\tau(\phi(k)) = I$ on $\partial_e K$. Since by the Banach-Stone theorem, $\tau : \partial_e K \rightarrow \mathcal{G}(E)$ is continuous when the range space is equipped with the strong operator topology, $\tau' : \partial_e K \rightarrow \mathcal{G}(E)$ defined by $\tau'(k) = \tau(k)^{-1} = \tau(\phi(k))$ is also continuous. While considering the algebraic reflexivity question, we have been able to overcome the difficulty induced by this requirement on τ only in the scalar-valued case.*

Corollary 5. *Let X be a compact metric space. Then the set of GBP's on $C(X)$ is also algebraically reflexive.*

Proof. Let the set of all GBPs on $C(X)$ be denoted by \mathcal{P} . Let $T \in \overline{\mathcal{P}}^a$. Then for each $f \in C(X)$, there exists $P_f \in \mathcal{P}$ such that $Tf = P_f f$. Thus by [6], for each f there exists a homeomorphism ϕ_f of X , $\phi_f^2(x) = x$ for all $x \in X$ and $\tau_f : X \rightarrow \mathbf{T}$ satisfying $\tau_f(x)\tau_f(\phi_f(x)) = 1$ such that $Tf = \frac{1}{2}(f + \tau_f f(\phi_f))$.

Thus for each f , $(2T - Id)f = \tau_f f(\phi_f)$ is given by an involuting isometry. The conclusion follows from Theorem 1. \square

We recall [7] that a closed subspace $J \subseteq E$ is called a semi L-summand if for every $e \in E$ there exists a unique $j \in J$ such that $\|e - j\| = \text{dist}(e, J)$ and moreover j satisfies $\|e\| = \|j\| + \|e - j\|$. In [7, Theorem 4.4] it was proved that if E is such that for every $e^* \in \partial_e E_1^*$ $\text{span}\{e^*\}$ is a semi L-summand then $P : E \rightarrow E$ is a bi contractive projections if and only if $U = 2P - I$ is an isometry and in this case $U^2 = I$.

The following lemma gives more example of $C(X, E)$ spaces where GBPs are given by $\lambda = -1$ and hence they are bi-contractive.

Lemma 6. *Let E be such that for every $e^* \in \partial_e E_1^*$ $\text{span}\{e^*\}$ is a semi L-summand. Then the same is true for $C(X, E)$.*

Proof. Let $\tau \in \partial_e C(X, E)_1^*$. Then $\tau = \delta(x) \otimes e^*$ for some $x \in X$ and $e^* \in \partial_e E_{*1}$. By hypotheses $\text{span}\{e^*\}$ is a semi L-summand. Since $C(X, E)^* = (\text{span}\{\delta(x)\} \otimes E^*) \oplus_1 N$, for a closed subspace N , it follows that $\text{span}\{\tau\}$ is a semi L-summand in $C(X, E)^*$. \square

Suppose the isometries of $C(X, E)$ are described by generalized Banach-Stone Theorem, that is, if $U \in \mathcal{G}(C(X, E))$ then there exists a map $\tau_U : X \rightarrow \mathcal{G}(E)$ continuous in s.o.t. and a homeomorphism ϕ_U of X such that $Uf(x) = \tau(x)f(\phi(x))$, for all $f \in C(X, E)$. Let $\mathcal{G}(C(X, E))$ be algebraically reflexive. Take \mathcal{I} the subset of $\mathcal{G}(C(X, E))$ which are described by involuting homeomorphism of X , that is,

$$\mathcal{I} = \{U \in \mathcal{G}(C(X, E)) : \phi_U^2(x) = x, \forall x \in X\}.$$

Now if $T \in \overline{\mathcal{I}}^a$ then by algebraic reflexivity of $\mathcal{G}(C(X, E))$, T is an isometry and hence $Tf(x) = \tau(x)f(\phi(x))$ where $\tau(x) \in \mathcal{G}(E)$ for each x and ϕ is an homeomorphism of X . Now if $\phi(x) \neq x$ for some x , choose $h \in C(X)$ such that $0 \leq h(x) \leq 1$, $x \in X$ and $h^{-1}(1) = \{x\}$, $h^{-1}(0) = \{\phi(x)\}$. Fix a

norm one vector $e \in E$ and take $f = h \otimes e$. Evaluating $\tau f(\phi) = \tau_f f(\phi_f)$ at x and $\phi(x)$ one can show as in the proof of Theorem 1 that $\phi^2(x) = x$.

The description of isometries of $C(X, E)$ given by the generalized Banach-Stone Theorem, holds when E has trivial centralizer. This is the case, in particular, when E or E^* is strictly convex. It was shown in [6] that if E is uniformly convex and $\mathcal{G}(E)$ is algebraically reflexive then $\mathcal{G}(C(X, E))$ is algebraically reflexive. Combining this with our preceding discussion we obtain the following.

Proposition 7. *Let E be a uniformly convex Banach space such that $\mathcal{G}(E)$ is algebraically reflexive. Let*

$$\mathcal{I} = \{U \in \mathcal{G}(C(X, E)) : \phi_U^2(x) = x, \forall x \in X\}.$$

Then \mathcal{I} is algebraically reflexive.

Using Proposition 7 we can describe the algebraic closure of GBPs in $C(X, E)$ for some class of spaces E . Note that for finite dimensional spaces E GBPs were studied in [5]. We also recall [6, Theorem 6] that for a finite dimensional space E with trivial centralizer and X metrizable isometry group of $C(X, E)$ is algebraically reflexive.

Proposition 8. *Let E be a uniformly convex Banach space such that $\mathcal{G}(E)$ is algebraically reflexive. Assume further that E does not have non-trivial GBP. Let \mathcal{P} denote the GBPs in $C(X, E)$. If $T \in \overline{\mathcal{P}}^a$ then there exists a $\tau : X \rightarrow \mathcal{G}(E)$ continuous in s.o.t. and a homeomorphism ϕ of X satisfying $\phi^2(x) = x$ for all $x \in X$ such that for all $f \in C(X, E)$, $Tf(x) = \tau(x)f(\phi(x))$.*

Remark 9. *Even under some very generous assumptions on E , when the isometry group is non-trivial, we have not been able to establish the algebraic reflexivity of GBP' for $C(X, E)$. An independent question of interest here is to examine the algebraic reflexivity of the set of isometries which are give by a weight function τ such that $\tau' : X \rightarrow \mathcal{G}(E)$ defined by $\tau'(x) = \tau(x)^{-1}$ is continuous.*

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(Dutta) DEPARTMENT OF MATHEMATICS AND STATISTICS, INDIAN INSTITUTE OF TECHNOLOGY, KANPUR, U.P.,INDIA, *E-mail* : sudipta@iitk.ac.in

(T. S. S. R. K. Rao) STAT-MATH UNIT, INDIAN STATISTICAL INSTITUTE, R. V. COLLEGE P.O., BANGALORE 560059, INDIA, *E-mail* : tss@isibang.ac.in