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A note on a paper of Ercan and Önal

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ABSTRACT. In this short note we formulate and prove a Banach space version of the Banach-Stone theorem, obtained recently ([2]) for the case of lattice-valued continuous functions.

1. Introduction

Let X, Y be compact Hausdorff spaces and E a Banach lattice and F be an abstract M-space with unit. Let $\pi : C(X, E) \to C(Y, F)$ be a Riesz isomorphism such that $0 \notin f(X)$ if and only if $0 \notin \pi(f)(Y)$ for each $f \in C(X, E)$. Ercan and Önal have proved in [2] that E is Riesz isomorphic to F and X is homeomorphic to Y. In this paper we show that similar conclusion can be drawn when E is a complex Banach space and π is a surjective isometry.

For a Banach space E, let E_1 denote the closed unit ball and $\partial_e E_1$ denote the set of extreme points.

We recall that $e \in E_1$ is a strong extreme point if $e_k \in E$, $\|e \pm e_k\| \to 1 \implies e_k \to 0$. It is easy to see that $1 \in C(X)$ is a strong extreme point.

2. Main Result

We recall that from [4] (Chapter 1) that by the Kakutani' representation theorem F is isometric to $C(N)$ for a compact Hausdorff space N. In what follows we use the well-known identification of $C(Y, C(N))$ with $C(Y \times N)$. A Banach space E is said to be an L^1-predual if E^* is isometric to $L^1(\mu)$ for a positive measure μ. See [4] Chapter 7 for properties of these spaces. For a compact Hausdorff space K, $C(K)$ is an L^1-predual.

Theorem 1. *Let E be a Banach space and let $\pi : C(X, E) \to C(Y, C(N))$ be a surjective isometry such that $0 \notin f(X)$ if and only if $0 \notin \pi(f)(Y)$ for*

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each $f \in C(X, E)$. Then there exists a compact Hausdorff space M such that E is isometric to $C(M)$ and X, M are homeomorphic to Y, N respectively.

Proof. Since E is isometric to the range of a projection of norm one in $C(X, E)$ it follows from the results in Chapter 6 in [4] that E is an L^1-predual space. To show that E is isometric to a $C(M)$, in view of the results from [5], (see [6] for the complex versions) we need to show that $\partial_e E_1 \neq \emptyset$ and $\partial_e E_1^*$ is a weak*-closed set.

Since $1 \in C(Y, C(N))$ is a strong extreme point, we have that $C(X, E)_1$ has a strong extreme point, say f. It follows from [1] that $f(k) \in \partial_e E_1$ for all $k \in K$. Since π is a surjective isometry, we have that $\partial_e C(X, E)_1^*$ is a weak*-closed set. Now let $\{e_\alpha^*\}_{\alpha \in \Delta} \subset \partial_e E_1^*$ be a net such that $e_\alpha^* \to e^*$, in the weak*-topology. For $x \in X$, as $\{\delta(x) \otimes e_\alpha^*\} \subset \partial_e C(X, E)_1^*$ and $\delta(x) \otimes e_\alpha^* \to \delta(x) \otimes e^*$ in the weak*-topology, we get that $\delta(x) \otimes e^* \in \partial_e C(X, E)_1^*$. Therefore $e^* \in \partial E_1^*$.

Hence E is isometric to $C(M)$ for a compact Hausdorff space M. After identifying $C(X, E)$ with $C(X \times M)$, it follows from the classical Banach-Stone theorem that there is a homeomorphism $\sigma : Y \times N \to X \times M$ such that $\pi(f) = \pi(1) f \circ \sigma$ for all $f \in C(X \times M)$.

Since $|\pi(1)| \equiv 1$, it is easy to see that the hypothesis implies that for each $y \in Y$ there exists a unique $x \in X$ such that, $\sigma(\{y\} \times N) = \{x\} \times M$. It therefore follows from Lemma 5 of [2] that X, Y are homeomorphic to M, N respectively.

□

We next state a more general version for injective tensor product spaces. This can be proved using arguments similar to the ones given during the proof of the above Theorem. For the extremal arguments, instead of the results from [1], one uses the results from [3]. The arguments will be symmetric for the component spaces. In what follows, we use the well-known fact that the injective product space, $E \otimes_\epsilon F$ can be identified as a subspace of the space of compact operators, $\mathcal{K}(E^*, F)$.

Theorem 2. *Let $\pi : E \otimes_\epsilon F \to C(Y, C(N))$ be a surjective isometry such that for $T \in E \otimes_\epsilon F$, $0 \notin \pi(T)(Y) \iff 0 \notin T^*(\partial_e F_1^*)$. Then there exist compact Hausdorff spaces X, M such that E, F are isometric to $C(X), C(M)$ respectively and X, M are homeomorphic to Y, N respectively.*

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