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# Ball remotal subspaces of Banach spaces

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# BALL REMOTAL SUBSPACES OF BANACH SPACES

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ABSTRACT. In this paper, we study Banach spaces  $X$  with subspaces  $Y$  whose unit ball is densely remotal in  $X$ . We show that for several classes of Banach spaces, the unit ball of the space of compact operators is densely remotal in the space of bounded operators. We also show that for several classical Banach spaces, the unit ball is densely remotal in the duals of higher even order. We show that for a separable remotal set  $E \subseteq X$ , the set of Bochner integrable functions with values in  $E$  is a remotal set in  $L^1(\mu, X)$ .

## 1. INTRODUCTION

For a closed and bounded set  $K$  in a real Banach space  $X$ , the farthest distance map  $\rho$  is defined as  $\rho(x, K) = \sup\{\|z - x\| : z \in K\}$ ,  $x \in X$ . For  $x \in X$ , we define the farthest point map as  $F_K(x) = \{z \in K : \|z - x\| = \rho(x, K)\}$ , *i.e.*, the set of points of  $K$  farthest from  $x$ . Note that this set may be empty. Let  $R(K, X) = \{x \in X : F_K(x) \neq \emptyset\}$ . We will write simply  $R(K)$  if the space  $X$  is understood. Call a closed and bounded set  $K$  remotal if  $R(K, X) = X$  and densely remotal if  $R(K, X)$  is norm dense in  $X$ . A sequence  $\{z_n\} \subseteq K$  such that  $\lim \|z_n - x\| = \rho(x, K)$  is called a maximizing sequence for  $x$ .

Let  $Y \subseteq X$  be a closed subspace. In this paper, we study when is the closed unit ball  $B_Y$  of  $Y$  remotal or densely remotal in  $X$ . From known results on densely remotal sets, we have that :

- (a) if  $Y \subseteq X$  is finite dimensional, then  $B_Y$  is remotal in  $X$ .
- (b) if  $Y$  is a reflexive subspace of any Banach space, then  $B_Y$  is densely remotal [6].
- (c) if  $X^*$  is an Asplund space with a LUR dual norm, then for any closed subspace  $Y$  of  $X^*$ ,  $B_Y$  is densely remotal [13]. In particular,

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any space  $X$  with  $X^{**}$  separable has an equivalent renorming with the above property. Note that such a space need not be reflexive.

- (d) if  $X$  has the Radon-Nikodým Property (RNP), then for any  $w^*$ -closed subspace  $Y$  of  $X^*$ ,  $B_Y$  is densely remotal [2, Proposition 3].

Our main emphasis in this paper is to study these concepts in spaces of operators and in function spaces. We exhibit several spaces of operators on classical Banach spaces and function spaces whose unit balls are densely remotal in an appropriate superspace. Among other things, we show that

- (1) for any closed subspace  $Y$  of a Hilbert space  $X$ ,  $B_Y$  is remotal in  $X$ .
- (2)  $B_{\ell_1}$  is remotal in  $\ell_1^{**}$ .
- (3)  $B_{c_0}$  is densely remotal, but not remotal, in  $\ell_\infty$ .
- (4) Suppose  $X$  is strictly convex and has the RNP, then  $B_{\mathcal{K}(L^1(\mu), X)}$  is densely remotal in  $\mathcal{L}(L^1(\mu), X)$ , where  $\mu$  is the Lebesgue measure on  $[0, 1]$ .
- (5) For a large class of Banach spaces that include all reflexive spaces,  $C(K)$  spaces and  $L^1(\mu)$  spaces,  $B_{\mathcal{K}(X)}$  is densely remotal in  $\mathcal{L}(X)$ .
- (6) For a reflexive Banach space  $X$ ,  $B_{C(K, X)}$  is densely remotal in  $WC(K, X)$ .

We also study the dense remotality of  $X$  in  $X^{**}$ . This investigation leads us to study the largeness of the set of vectors in  $X^{**}$  which attain their norm at a norm attaining element of  $X^*$ . In some of these cases, we also give an explicit description of the set  $R(B_Y, X)$ . Another interesting geometric property that we use in this study is that of all the extreme points in  $B_{X^*}$  attaining their norm.

We also obtain several stability results. In particular, we show that for any separable remotal set  $E \subseteq X$ , the set of Bochner integrable functions,  $L^1(\mu, E)$  is remotal in  $L^1(\mu, X)$ , where  $\mu$  is the Lebesgue measure on  $[0, 1]$ , producing a correct proof, under weaker assumptions, of [5, Theorem 2.3]. We also prove that if  $E \subseteq X$  is densely remotal and  $K$  is a compact totally disconnected space, then  $C(K, E)$  is densely remotal in  $C(K, X)$ .

The closed unit ball and the unit sphere of  $X$  will be denoted by  $B_X$  and  $S_X$  respectively. We will identify any  $x \in X$  with its canonical image in  $X^{**}$ . We will denote by  $NA(X)$  the set of all  $x^* \in X^*$  which attain their norm on  $B_X$ . For a closed bounded convex set  $C$ ,  $ext(C)$  will denote the set of extreme points of  $C$ . Any unexplained terminology can be found in [4].

## 2. (\*)-SUBSPACES

**Definition 2.1.** We say that  $A \subseteq B_{X^*}$  is a norming set for  $X$  if  $\|x\| = \sup\{|x^*(x)| : x^* \in A\}$  for all  $x \in X$ .

Let us say that a subspace  $Y$  of  $X$  is a (\*)-subspace of  $X$  if the set

$$A_Y = \{x^* \in S_{X^*} : \|x^*|_Y\| = 1\}$$

is a norming set for  $X$ .

**Definition 2.2.** A subspace  $Y$  of a Banach space  $X$  is said to be an ideal in  $X$  if there is a projection  $P$  of norm 1 on  $X^*$  with  $\ker(P) = Y^\perp$ .

$Y$  is said to be a  $M$ -ideal in  $X$  if there is a projection  $P$  on  $X^*$  with  $\ker(P) = Y^\perp$  and for all  $x^* \in X^*$ ,  $\|x^*\| = \|Px^*\| + \|x^* - Px^*\|$ .

A Banach space  $X$  is said to be  $M$ -embedded if  $X$  is a  $M$ -ideal in  $X^{**}$  under the canonical embedding.

See [4, Chapters III and VI] for several important examples of  $M$ -embedded spaces and their geometric and topological properties.

Here are some natural examples of (\*)-subspaces.

**Example 2.3.** (a)  $X$  is a (\*)-subspace of  $X^{**}$ .

(b) If  $Y \subseteq Z \subseteq X$  and  $Y$  is a (\*)-subspace of  $X$ , then  $Z$  is a (\*)-subspace of  $X$  and  $Y$  is a (\*)-subspace of  $Z$ .

(c) For any two Banach spaces  $X$  and  $Y$ ,  $\mathcal{K}(X, Y)$ , the space of all compact operators from  $X$  to  $Y$ , is a (\*)-subspace of  $\mathcal{L}(X, Y)$ , the space of all bounded operators from  $X$  to  $Y$ .

(d) For a compact Hausdorff space  $K$ ,  $C(K, X)$ , the space of continuous functions from  $K$  to  $X$ , is a (\*)-subspace of  $WC(K, X)$ , the space of continuous functions when  $X$  has the weak topology, with the sup norm.

(e) More generally, if  $Y$  is an ideal in  $X$  and satisfies the conditions of [8, Lemma 1(i)], then  $Y$  is a (\*)-subspace of  $X$ . See [8] for details.

**Proposition 2.4.** *If  $Y$  is a (\*)-subspace of  $X$ , then*

$$(1) \quad \rho(x, B_Y) = \|x\| + 1 \quad \text{for any } x \in X.$$

*Proof.* Clearly,  $\rho(x, B_Y) \leq \|x\| + 1$ . Now, let  $\varepsilon > 0$ . Since  $A_Y$  is norming for  $X$ , there exists  $x^* \in A_Y$  such that  $x^*(x) > \|x\| - \varepsilon/2$ .

Since  $\|x^*|_Y\| = 1$ , there exists  $y \in S_Y$  such that  $-x^*(y) > 1 - \varepsilon/2$ . Therefore,  $\|x - y\| \geq x^*(x - y) > \|x\| + 1 - \varepsilon$ .  $\square$

**Remark 2.5.** It follows that if every point of  $S_Y$  is a rotund point of  $B_X$  (see [1]), then no point of  $X \setminus Y$  has a farthest point in  $B_Y$ . This happens in particular when  $X$  is strictly convex.

In the special case of  $X$  in  $X^{**}$ , this yields the following result :

If  $X$  is wALUR (see [1]), in particular, if  $X^*$  is smooth, then no point of  $X^{**} \setminus X$  has a farthest point in  $B_X$ .

Apropos Example 2.3(b) above, we have, for  $x \in X$ ,  $1 + \|x\| = \rho(x, B_Z) = \rho(x, B_Y)$ . Thus if  $B_Y$  is remotal in  $X$ , then  $B_Y$  is remotal in  $Z$  and  $B_Z$  is remotal in  $X$ . Similarly, if  $B_Y$  is densely remotal in  $X$ , then  $B_Z$  is densely remotal in  $X$ . But we do not know in general if  $B_Y$  is densely remotal in  $Z$ .

It is immediate from Proposition 2.4 that if  $Y$  is a  $(*)$ -subspace of  $X$ , then any farthest point in  $B_Y$  comes from  $S_Y$ . We now observe that this actually is always true.

**Proposition 2.6.** *Let  $Y$  be a subspace of a Banach space  $X$ . Let  $x_0 \in X$ . Then either  $F_{B_Y}(x_0) \subseteq S_Y$  or  $F_{B_Y}(x_0) = B_Y$ .*

*Proof.* Suppose there exists  $y_0 \in F_{B_Y}(x_0)$  such that  $\|y_0\| < 1$ . Then, there exists  $\varepsilon > 0$  such that  $y_0 + \varepsilon y \in B_Y$  for all  $y \in B_Y$ .

Let  $x^* \in S_{X^*}$  such that  $x^*(x_0 - y_0) = \|x_0 - y_0\| = \rho(x_0, B_Y)$ . Then for any  $y \in B_Y$ ,

$$\begin{aligned} \|x_0 - y_0\| &\geq \|x_0 - y_0 - \varepsilon y\| \geq x^*(x_0 - y_0 - \varepsilon y) \\ &= x^*(x_0 - y_0) - \varepsilon x^*(y) = \|x_0 - y_0\| - \varepsilon x^*(y) \end{aligned}$$

Thus  $x^*(y) \geq 0$  for all  $y \in B_Y$ , whence  $x^*|_Y \equiv 0$ . It follows that for any  $z \in B_Y$

$$\|x_0 - z\| \geq x^*(x_0 - z) = x^*(x_0) = x^*(x_0 - y_0) = \|x_0 - y_0\| = \rho(x_0, B_Y).$$

Therefore,  $z \in F_{B_Y}(x_0)$ . □

**Proposition 2.7.** *Suppose  $Y \subseteq X$  is a subspace of a wLUR Banach space  $X$ . If  $x \in X$  is such that  $\rho(x, B_Y) = \|x\| + 1$ , then  $x \in Y$ .*

*Proof.* If  $x \in X$  is such that  $\rho(x, B_Y) = \|x\| + 1$ , then for any maximizing sequence  $\{y_n\} \subseteq B_Y$ ,  $\|x - y_n\| \rightarrow \|x\| + 1$ . Since  $X$  is wLUR,  $y_n \rightarrow -x/\|x\|$  weakly and thus  $x \in Y$ . □

It follows that a wLUR Banach space has no proper  $(*)$ -subspace. This produces one of the easiest proof of the following well-known fact.

**Corollary 2.8.** *If  $X^{**}$  is wLUR, then  $X$  is reflexive.*

**Proposition 2.9.** *If  $Y \subseteq X$  is a subspace such that the farthest distance formula (1) holds, in particular, if  $Y$  is a  $(*)$ -subspace of  $X$ , then a point  $x_0 \in X$  has a farthest point in  $B_Y$  if and only if there exists  $x^* \in A_Y$  and  $y \in S_Y$  such that*

$$x^*(x_0) = \|x_0\| \text{ and } x^*(y) = 1.$$

*In other words,  $R(B_Y) = \{x \in X : \text{there exists } x^* \in A_Y, y \in S_Y \text{ such that } x^*(x) = \|x\| \text{ and } x^*(y) = 1\}$ .*

*Proof.* If such an  $x^*$  and  $y$  exists then

$$\|x_0 + y\| \geq x^*(x_0 + y) = \|x_0\| + 1.$$

Hence  $-y$  is farthest from  $x_0$ .

Conversely, if  $-y$  is farthest from  $x_0$ , then  $\|x_0 + y\| = \|x_0\| + 1$ . Now, there exists  $x^* \in S_{X^*}$  such that

$$x^*(x_0 + y) = \|x_0 + y\| = \|x_0\| + 1.$$

Since  $x^*(x_0) \leq \|x_0\|$  and  $x^*(y) \leq 1$ , the result follows.  $\square$

**Remark 2.10.** It follows that in case of  $(*)$ -subspaces,  $R(B_Y)$  is closed under scalar multiplications. Is this generally true?

We now specialize to some of the cases listed above.

**Definition 2.11.**  $X$  is said to be weakly Hahn-Banach smooth if every  $x^* \in NA(X)$  has a unique norm-preserving extension to all of  $X^{**}$ .

By [4, Lemma III.2.14],  $x^* \in S_{X^*}$  has a unique norm-preserving extension to all of  $X^{**}$  if and only if  $x^*$  is a  $w^*$ -weak PC of  $B_{X^*}$ , that is, the relative weak and  $w^*$  topologies on  $B_{X^*}$  agree at  $x^*$ .

Let us denote by  $NA_2(X)$  the set  $\{x^{**} \in X^{**} : x^{**}(x^*) = \|x^{**}\| \text{ for some } x^* \in NA(X) \cap S_{X^*}\}$ .

Clearly,  $X \subseteq NA_2(X) \subseteq NA(X^*)$ . The relevance of this set in our context follows from our next result.

**Proposition 2.12.** *For any Banach space  $X$ ,  $R(B_X, X^{**}) \supseteq NA_2(X)$ .*

*If  $B_{X^*}$  is  $w^*$ -sequentially compact and  $X$  is weakly Hahn-Banach smooth, then equality holds.*

*Proof.* If  $x_0^{**} \in NA_2(X)$ , there exists  $x^* \in S_{X^*}$  and  $x \in S_X$  such that  $x_0^{**}(x^*) = \|x_0^{**}\|$  and  $x^*(x) = 1$ , then,  $-x \in B_X$  is farthest from  $x_0^{**}$ .

Conversely, suppose  $x_0^{**} \in S_{X^{**}}$  has a farthest point  $-x \in B_X$ , so that  $\|x_0^{**} + x\| = 2$ .

Let  $\{x_n^*\} \subseteq B_{X^*}$  be such that  $\lim_n(x_0^{**} + x)(x_n^*) = 2$ . It follows that

$$\lim_n x_0^{**}(x_n^*) = 1 = \lim_n x_n^*(x).$$

Now since  $B_{X^*}$  is  $w^*$ -sequentially compact, there is a subsequence  $\{x_{n_k}^*\}$   $w^*$ -converging to some  $x_0^* \in B_{X^*}$ . It then follows that  $x_0^*(x) = 1$  and therefore  $x_0^* \in NA(X)$ . Since  $X$  is weakly Hahn-Banach smooth,  $x_0^*$  is a  $w^*$ -weak PC. Therefore,  $x_{n_k}^* \rightarrow x_0^*$  weakly. It follows that  $x_0^{**}(x_0^*) = 1$ , and hence,  $x_0^{**} \in NA_2(X)$ .  $\square$

**Remark 2.13.** In the above result, instead of assuming  $B_{X^*}$  is  $w^*$ -sequentially compact, we may assume that  $X$  is smooth. In that case, any  $w^*$ -cluster point  $x^* \in B_{X^*}$  of  $\{x_n^*\}$  satisfies  $x^* \in S_{X^*}$  and  $x^*(x) = 1$ . Since  $X$  is smooth,  $\{x_n^*\}$  has a unique  $w^*$ -cluster point, *i.e.*,  $\{x_n^*\}$  is  $w^*$ -convergent. The rest of the argument remains as before.

If  $X$  is  $M$ -embedded, then by [4, Proposition I.1.12 & Corollary III.4.7], it satisfies the hypothesis of the above proposition.

**Corollary 2.14.**  $B_{c_0}$  is densely remotal, but not remotal, in  $\ell_\infty$ .

*Proof.* Observe that

$$R(B_{c_0}, \ell_\infty) = \{(\alpha_n) \in \ell_\infty : \text{there exists } n_0 \geq 1 \text{ such that } |\alpha_{n_0}| = \|(\alpha_n)\|_\infty\}.$$

Indeed, if  $(\alpha_n) \in \text{RHS}$ , then  $-e_{n_0}$  is easily seen to be farthest from it. For the converse, apply the above result, noting that  $c_0$  is  $M$ -embedded and if  $(a_n) \in NA(c_0)$ , then only finitely many  $a_n$ 's are nonzero.

But, these are precisely the points of  $\ell_\infty$  that belong to  $NA(\ell_1)$ , and hence, by the Bishop-Phelps Theorem, is dense in  $\ell_\infty$ .

Clearly, the sequence  $(1 - 1/n) \notin R(B_{c_0}, \ell_\infty)$ .  $\square$

**Remark 2.15.** The above argument can be adapted to show that for any family  $\{X_\alpha\}$  of Banach spaces, the unit ball of  $\oplus_{c_0} X_\alpha^*$  is densely remotal in  $\oplus_{\ell_\infty} X_\alpha^*$ . Indeed, it is easy to see that  $R(B_{\oplus_{c_0} X_\alpha^*}) \supseteq \{(x_\alpha^*) \in \oplus_{\ell_\infty} X_\alpha^* : \text{there exists } \alpha_0 \text{ such that } \|x_{\alpha_0}^*\| = \|(x_\alpha^*)\|_\infty\}$ , note that  $\oplus_{\ell_\infty} X_\alpha^* = (\oplus_{\ell_1} X_\alpha)^*$  and again appeal to the Bishop-Phelps Theorem.

**Question 2.16.** For any family  $\{X_\alpha\}$  of Banach spaces, is  $B_{\oplus_{c_0} X_\alpha}$  is densely remotal in  $\oplus_{\ell_\infty} X_\alpha$ ?

We now give an example of a  $M$ -embedded space  $X$  for which  $R(B_X, X^{**}) = X$ .

**Example 2.17.** Let  $A$  be the disc algebra on the unit circle  $\mathbb{T} \subseteq \mathbb{C}$ . Then it is known that the quotient space  $X = C(\mathbb{T})/A$  is a  $M$ -embedded space and its dual  $H_0^1$  is a smooth space. Thus, by Remark 2.5, no point of  $X^{**} \setminus X$  has a farthest point in  $B_X$ .

We now come to the case of  $\mathcal{K}(X, Y)$  as a subspace of  $\mathcal{L}(X, Y)$ . For simplicity, we will work with  $X = Y$ , but most of our results extend, under natural assumptions, to the general case as well.

**Lemma 2.18.** (a)  $R(B_{\mathcal{K}(X)}) \supseteq \{T \in \mathcal{L}(X) : \text{there exists } x \in S_X \text{ such that } \|Tx\| = \|T\|\}$ .  
 (b)  $R(B_{\mathcal{K}(X)}) \supseteq \{T \in \mathcal{L}(X) : \text{there exists } x^* \in NA(X) \cap S_{X^*} \text{ such that } \|T^*x^*\| = \|T^*\|\}$ .

*Proof.* (a). Let  $T \in S_{\mathcal{L}(X)}$  such that there exists  $x \in S_X$  such that  $\|Tx\| = 1$ . Let  $x^* \in S_{X^*}$  be such that  $x^*(x) = 1$ . Define  $S = x^* \otimes Tx$ , then  $S \in S_{\mathcal{K}(X)}$  and  $Sx = x^*(x) \cdot Tx = Tx$ . It follows that  $\|T+S\| \geq \|Tx+Sx\| = 2\|Tx\| = 2$  and therefore,  $-S$  is farthest from  $T$ .

(b). Let  $T \in S_{\mathcal{L}(X)}$  such that there exists  $x^* \in NA(X)$  such that  $\|T^*x^*\| = \|T^*\| = 1$ . Let  $x \in S_X$  be such that  $x^*(x) = 1$ . Then  $S = -T^*x^* \otimes x$  works.  $\square$

**Theorem 2.19.** Let  $\mu$  be the Lebesgue measure on  $[0, 1]$ . Suppose  $X$  is strictly convex and has the RNP, then  $B_{\mathcal{K}(L^1(\mu), X)}$  is densely remotal in  $\mathcal{L}(L^1(\mu), X)$ .

*Proof.* It follows from arguments similar to (a) above that  $R(B_{\mathcal{K}(L^1(\mu), X)}) \supseteq \{T \in \mathcal{L}(L^1(\mu), X) : \text{there exists } f \in S_{L^1(\mu)} \text{ such that } \|Tf\| = \|T\|\}$ . By [12], this set is dense in  $\mathcal{L}(L^1(\mu), X)$ .  $\square$

**Theorem 2.20.** Suppose  $X$  is a reflexive Banach space, then  $B_{\mathcal{K}(X)}$  is densely remotal in  $\mathcal{L}(X)$ .

*Proof.* Since for a reflexive space,  $NA(X) = X^*$ , by Lemma 2.18(b), we have  $R(B_{\mathcal{K}(X)}) \supseteq \{T \in \mathcal{L}(X) : \text{there exists } x^* \in S_{X^*} \text{ such that } \|T^*x^*\| = \|T^*\|\}$ .

Now by [13, Proposition 4], this RHS set is dense in  $\mathcal{L}(X)$ .  $\square$

**Remark 2.21.** It follows that for any measure space  $(\Omega, \Sigma, \mu)$ ,  $B_{\mathcal{K}(L_p(\mu))}$  is densely remotal in  $\mathcal{L}(L_p(\mu))$  for  $1 < p < \infty$ . It is well-known that in this case  $\mathcal{K}(L_p(\mu))^{**} = \mathcal{L}(L_p(\mu))$ . Thus these are also examples of spaces whose unit ball is densely remotal in the bidual.

Also it is clear from our arguments above that several of the results go through for other subspaces of operators that contain finite rank operators. From the results above, we also have that for  $I \in \mathcal{L}(X)$  farthest points in  $B_{\mathcal{K}(X)}$  (or any other operator ideal) are related to the so-called Daugavet equation  $\|I + T\| = 1 + \|T\|$ .

**Proposition 2.22.** *Suppose  $\mathcal{K}(X)$  is a  $M$ -ideal in  $\mathcal{L}(X)$ , then  $R(B_{\mathcal{K}(X)}) \subseteq \{T \in \mathcal{L}(X) : \text{there exists } x^* \in S_{X^*} \text{ such that } \|T^*x^*\| = \|T^*\|\}$ . In particular, if  $X$  is reflexive, then equality holds.*

*Proof.* Let  $T \in R(B_{\mathcal{K}(X)})$  such that  $\|T\| = 1$  and let  $-S$  be farthest from it. Then  $\|T + S\| = 2$  and there exists  $\tau \in \text{ext}(B_{\mathcal{L}(X)^*})$  such that

$$\tau(T) = 1 \text{ and } \tau(S) = 1.$$

Since  $\mathcal{K}(X)$  is a  $M$ -ideal in  $\mathcal{L}(X)$ ,

$$\mathcal{L}(X)^* = \mathcal{K}(X)^* \oplus_1 \mathcal{K}(X)^\perp.$$

Since  $\tau$  is extreme, and  $\tau(S) \neq 0$ ,  $\tau \in \text{ext}(B_{\mathcal{K}(X)^*})$ . That is,  $\tau = x^{**} \otimes x^*$  for some  $x^{**} \in \text{ext}(B_{X^{**}})$  and  $x^* \in \text{ext}(B_{X^*})$ . It follows that

$$x^{**}(T^*x^*) = x^{**}(S^*x^*) = 1,$$

therefore,

$$\|T^*x^*\| = \|S^*x^*\| = 1.$$

If  $X$  is reflexive, combining with by Lemma 2.18(b), the result follows.  $\square$

It is known that  $X = \ell_p$  for  $1 < p < \infty$  satisfies the above hypothesis. See [4, Chapter VI] for more examples of such spaces.

Another class of  $(*)$ -subspaces that can be treated as a special case of  $\mathcal{K}(X, Y) \subseteq \mathcal{L}(X, Y)$  is  $C(K, X) \subseteq WC(K, X)$ . Note that for any  $f \in C(K, X)$ ,  $T : X^* \rightarrow C(K)$  defined by  $T(x^*)(k) = x^*(f(k))$  is a linear map with  $\|f\| = \|T\|$  and is a compact operator. Thus  $C(K, X) \subseteq \mathcal{K}(X^*, C(K))$  and similarly one can see that  $WC(K, X) \subseteq$  the space  $\mathcal{F}(X^*, C(K))$  of weakly compact operators. It can be seen that in the case of a dual space this association is onto so that  $C(K, X^*)$  is isometric to  $\mathcal{K}(X, C(K))$  and  $WC(K, X^*)$  is isometric to  $\mathcal{F}(X, C(K))$ .

To discuss the dense remotality in this case we need the following lemma which is of independent interest.

**Lemma 2.23.** *Let  $X, Y$  be Banach spaces. If  $T \in \mathcal{L}(X, Y)$  with  $\|T^*\| = \|T^*(y^*)\| = \|y^*\| = 1$ , there is an extreme point in  $y_0^* \in \text{ext}(B_{Y^*})$  such that  $\|T^*(y_0^*)\| = 1$ .*

*Proof.* Let  $F = \{y^* \in B_{Y^*} : \|T^*(y^*)\| = 1\}$ . It is easy to see that  $F$  is an extreme set. We recall that a set is extreme if and only if it is a union of faces (convex, extreme sets). Also  $B_{Y^*} \setminus F = \bigcup \{y^* \in B_{Y^*} : \|T^*(y^*)\| \leq 1 - \frac{1}{n}\}$ . As these sets are  $w^*$ -compact and convex, it follows from [7, Theorem 5.8] that  $F \cap \text{ext}(B_{Y^*}) \neq \emptyset$ .  $\square$

**Theorem 2.24.** *Let  $K$  be a compact set and let  $X$  be a reflexive Banach space. Then  $B_{C(K,X)}$  is densely remotal in  $WC(K, X)$ .*

*Proof.* By Proposition 2.4,  $\rho(f, B_{C(K,X)}) = 1 + \|f\|$  for  $f \in WC(K, X)$ . It is easy to see that  $R(B_{C(K,X)}) \supseteq \{f \in WC(K, X) : \text{there exists } k \in K \text{ such that } \|f(k)\| = \|f\|\}$ .

Since  $X$  is reflexive, we have  $C(K, X) = \mathcal{K}(X^*, C(K))$ ,  $WC(K, X) = \mathcal{L}(X^*, C(K))$ . We further note that if  $T \in \mathcal{L}(X^*, C(K))$  is such that  $T^* : C(K)^* \rightarrow X$  attains its norm, then as  $\text{ext}(B_{C(K)^*}) = \mathbb{T}K$  (via the canonical embedding of  $K$ ), by Lemma 2.23, it attains its norm on  $K$ .

By [13, Proposition 4] again,  $B_{C(K,X)}$  is densely remotal in  $WC(K, X)$ .  $\square$

In the following result, we assume that all elements of  $\text{ext}(B_{X^*})$  attain their norm. Apart from the space  $C(K)$ , as noted above, it is easy to see that any reflexive space, any subspace of  $C(K)$  which contains constants and any  $L^1(\mu)$  space has this property.

**Corollary 2.25.** *Let  $X$  be a Banach space such that  $\text{ext}(B_{X^*}) \subseteq NA(X)$ . Then  $B_{\mathcal{K}(X)}$  is densely remotal in  $\mathcal{L}(X)$ .*

*Proof.* By Lemma 2.23, a  $T \in \mathcal{L}(X)$  with  $T^*$  attaining its norm actually attains it at an extreme point  $x^*$ . Now by our assumption  $x^* \in NA(X)$ . Hence the conclusion follows from Lemma 2.18 and [13, Proposition 4] as before.  $\square$

It is easy to see that the property  $\text{ext}(B_{X^*}) \subseteq NA(X)$  is preserved under  $c_0$ -direct sums. Also if  $X$  has this property then since any extreme point of  $B_{C(K,X)^*}$  is of the form  $\delta(k) \otimes x^*$  for some  $k \in K$  and  $x^* \in \text{ext}(B_{X^*})$ , we see that  $C(K, X)$  also has this property. Our next two proposition shows that under some natural conditions, this property is preserved by subspaces and quotients.

**Proposition 2.26.** *Let  $X$  be a Banach space such that  $\text{ext}(B_{X^*}) \subseteq NA(X)$ . Let  $Y$  be the range of a projection  $P : X \rightarrow X$  of norm one. Suppose any*

$y^* \in Y^*$  have unique norm preserving extension to  $X$ . Then  $\text{ext}(B_{Y^*}) \subseteq NA(Y)$ .

*Proof.* Let  $y^* \in \text{ext}(B_{Y^*})$ . It is well known that there is always a  $x^* \in \text{ext}(B_{X^*})$  extending  $y^*$ . By our hypothesis, there exists a  $x \in S_X$  with  $x^*(x) = 1$ . Now  $P^*(x^*)|_Y = x^*|_Y = y^*$  and thus  $\|P^*(x^*)\| = 1$ , so that it is a norm preserving extension of  $y^*$ . Thus by the uniqueness hypothesis,  $P^*(x^*) = x^*$ . Now  $y^*(P(x)) = x^*(P(x)) = P^*(x^*)(x) = x^*(x) = 1$ . Hence  $\text{ext}(B_{Y^*}) \subseteq NA(Y^*)$ .  $\square$

**Proposition 2.27.** *Let  $\text{ext}(B_{X^*}) \subseteq NA(X)$ . Let  $Y \subseteq X$  be a  $M$ -ideal. Then  $\text{ext}(B_{Y^\perp}) \subseteq NA(X/Y)$ .*

*Proof.* Let  $\pi$  be the quotient map. Let  $y^* \in \text{ext}(B_{Y^\perp})$ . As  $Y$  is a  $M$ -ideal we have  $X^* = Y^\perp \oplus_1 Y^*$ . Thus  $y^* \in \text{ext}(B_{X^*})$ . Therefore by hypothesis there exists a  $x_0 \in S_X$  such that  $y^*(x_0) = 1$ . Now  $y^*(x_0) = 1 = y^*(\pi(x_0)) \leq \|\pi(x_0)\| \leq 1$ . So  $y^* \in NA(X/Y)$ .  $\square$

We next consider the ‘3-space’ property for norm attaining extreme points.

**Proposition 2.28.** *Let  $Y \subseteq X$  be a  $M$ -ideal. If  $\text{ext}(B_{Y^*}) \subseteq NA(Y)$  and  $\text{ext}(B_{Y^\perp}) \subseteq NA(X/Y)$  then  $\text{ext}(B_{X^*}) \subseteq NA(X)$ .*

*Proof.* As before we have  $X^* = Y^* \otimes_1 Y^\perp$ . Now if  $x^* \in \text{ext}(B_{X^*})$  then either  $x^* \in \text{ext}(B_{Y^*})$  or  $x^* \in \text{ext}(B_{Y^\perp})$ . If  $x^* \in \text{ext}(B_{Y^*})$ ,  $x^* \in NA(X)$ . If  $x^* \in \text{ext}(B_{Y^\perp})$  then by our assumption there exists a  $x \in X$  such that  $x^*(x) = 1 = \|\pi(x)\|$ . Now since  $Y$  is a proximal subspace of  $X$  [4, Proposition II.1.1] there exists a  $y \in Y$  such that  $x^*(x - y) = 1 = \|\pi(x)\| = \|x - y\|$ . Thus  $x^* \in NA(X)$ .  $\square$

Our next results show that density of  $NA_2(X)$  in  $X^{**}$  is preserved under some natural direct sums.

**Theorem 2.29.** *Let  $\{X_\alpha : \alpha \in \Gamma\}$  be a family of Banach spaces and let  $X = \bigoplus_{c_0} X_\alpha$ .*

(a) *If for every  $\alpha \in \Gamma$ ,  $NA_2(X_\alpha) = NA(X_\alpha^*)$ , then  $NA_2(X) = NA(X^*)$ .*

(b) *If for every  $\alpha \in \Gamma$ ,  $NA_2(X_\alpha)$  is dense in  $X_\alpha^{**}$ , then  $NA_2(X)$  is dense in  $X^{**}$ .*

*Proof.* For simplicity, we work with  $\Gamma = \mathbb{N}$ .

(a). It is well-known that  $NA(X) = \{\Lambda \in \oplus_1 X_n^* : \Lambda \text{ has only finitely many non-zero coordinates and they attain their norm}\}$ . It is also easy to see that  $\tau \in NA(\oplus_1 X_n^*)$  if and only if  $\tau(\Lambda) = \|\tau\|$  for  $\|\Lambda\| = 1$ ,  $\|\tau\| = \tau(n)(\frac{\Lambda(n)}{\|\Lambda(n)\|}) = \|\tau(n)\|$  whenever  $\Lambda(n) \neq 0$ . Now for  $\tau \in NA(\oplus_1 X_n^*)$ , fix such a  $n_0$ . Then by our hypothesis, there exists a  $f_{n_0} \in NA(X_{n_0})$  such that  $\tau(n_0)(f_{n_0}) = \|\tau(n_0)\| = \|\tau\|$ . Let  $f \in \oplus_1 X_n^*$  be such that  $f(n_0) = f_{n_0}$  and 0 at other coordinates. Then from our observation in the first part of the proof,  $f \in NA(X)$  and  $\tau(f) = \|\tau\|$ , and thus  $\tau \in NA_2(X)$ .

To prove (b), consider  $A = \{n \in \mathbb{N} : \Lambda(n) \neq 0\}$ . For any  $\varepsilon > 0$  and for  $n \in A$ , let  $\tau'(n) \in NA_2(X_n)$  with  $\|\tau(n) - \tau'(n)\| \leq \varepsilon$ . Now define  $\tau' \in \oplus_{\ell^\infty} X_n^{**}$  by taking  $\tau'(n)$  for  $n \in A$  and  $\tau(n)$  at other coordinates. It now follows from the arguments given above that  $\tau' \in NA_2(X)$  and  $\|\tau - \tau'\| \leq \varepsilon$ .  $\square$

**Theorem 2.30.** *Let  $\{X_\alpha : \alpha \in \Gamma\}$  be a family of Banach spaces and let  $X = \oplus_{\ell_p} X_\alpha$ ,  $1 < p < \infty$ .  $NA_2(X)$  is dense in  $X^{**}$  if and only if for every  $\alpha \in \Gamma$ ,  $NA_2(X_\alpha)$  is dense in  $X_\alpha^{**}$ .*

*Proof.* It clearly suffices to show that

$$NA_2(X) = \{(F_\alpha) \in \oplus_{\ell_p} X_\alpha^{**} : \text{either } F_\alpha = 0 \text{ or } F_\alpha \in NA_2(X_\alpha)\}.$$

Let  $1/p + 1/q = 1$ . Suppose  $F = (F_\alpha) \in RHS$  and  $\|F\|_p = 1$ . If  $F_\alpha \neq 0$ , then  $F_\alpha/\|F_\alpha\| \in NA_2(X_\alpha)$ , that is, there exists  $f_\alpha \in S_{X_\alpha^*}$  and  $x_\alpha \in S_{X_\alpha}$  such that  $F_\alpha(f_\alpha) = \|F_\alpha\|$  and  $f_\alpha(x_\alpha) = 1$ . Define  $y = (y_\alpha)$  and  $g = (g_\alpha)$  by

$$y_\alpha = \|F_\alpha\|x_\alpha \quad \text{and} \quad g_\alpha = \|F_\alpha\|^{p-1}f_\alpha \quad \text{for all } \alpha \in \Gamma.$$

It is easy to check that  $\|y\|_p = \|g\|_q = 1$  and  $g(y) = F(g) = 1$ . Thus,  $g \in NA(X)$  and  $F \in NA_2(X)$ .

To see the converse, note that if  $f = (f_\alpha) \in S_{X^*}$  attains its norm at  $x = (x_\alpha) \in S_X$ , then

$$1 = f(x) = \sum f_\alpha(x_\alpha) \leq \sum |f_\alpha(x_\alpha)| \leq \sum \|f_\alpha\| \|x_\alpha\| \leq \|f\|_q \|x\|_p = 1.$$

By the conditions of equality in Hölder's inequality, it follows that  $\|f_\alpha\|^q = \|x_\alpha\|^p$  and  $f_\alpha(x_\alpha) = \|f_\alpha\| \|x_\alpha\|$  for all  $\alpha \in \Gamma$ .

Now if  $F = (F_\alpha) \in NA_2(X)$  and  $\|F\|_p = 1$ , then there exists  $f = (f_\alpha) \in S_{X^*}$  and  $x \in S_X$  such that  $F(f) = f(x) = 1$ . By the above observation,  $\|f_\alpha\|^q = \|F_\alpha\|^p = \|x_\alpha\|^p$ ,  $F_\alpha(f_\alpha) = \|F_\alpha\| \|f_\alpha\|$  and  $f_\alpha(x_\alpha) = \|f_\alpha\| \|x_\alpha\|$  for all  $\alpha \in \Gamma$ . It follows that whenever  $F_\alpha = 0$ , so are  $f_\alpha$  and  $x_\alpha$ . And if  $F_\alpha \neq 0$ ,  $F_\alpha$  attains its norm at  $f_\alpha/\|f_\alpha\|$ , which, in turn, attains its norm at  $x_\alpha/\|x_\alpha\|$ . That is,  $F_\alpha \in NA_2(X_\alpha)$ .  $\square$

## 3. STABILITY RESULTS

This section deals with stability aspects of dense remotality and remotality under various natural formations like direct sums and in spaces of vector-valued functions. Our first result is about certain natural summands.

**Lemma 3.1.** *Suppose  $X = Y \oplus Z$  and there exists a monotone map  $\varphi : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$  such that if  $x = y + z$ , then  $\|x\| = \varphi(\|y\|, \|z\|)$ .*

*Let  $E \subseteq Y$  and  $F \subseteq Z$  be remotal sets, then  $E + F$  is remotal in  $X$ .*

*In particular,  $B_Y$  and  $B_Z$  are remotal in  $X$ .*

*Proof.* Let  $x_0 = y_0 + z_0 \in X$ . Let  $e_0$  be farthest from  $y_0$  in  $E$  and  $f_0$  be farthest from  $z_0$  in  $F$ . Then for any  $e \in E$  and  $f \in F$ ,

$$\begin{aligned} \|y_0 + z_0 - e - f\| &= \varphi(\|y_0 - e\|, \|z_0 - f\|) \\ &\leq \varphi(\|y_0 - e_0\|, \|z_0 - f_0\|) = \|y_0 + z_0 - e_0 - f_0\| \end{aligned}$$

by the monotonicity of  $\varphi$ . □

We note that the above lemma holds with remotal replaced by densely remotal when convergence in  $X$  is equivalent to component-wise convergence. The following corollaries are immediate.

**Corollary 3.2.** *Let  $M$  be a subspace of a Hilbert space  $H$ . Then,  $B_M$  is remotal in  $H$ .*

**Corollary 3.3.** *Let  $E$  be a densely remotal set in  $X$ . Then,  $\oplus_{c_0} E$  is densely remotal in  $\oplus_{c_0} X$ .*

**Question 3.4.** *Let  $E$  be a remotal set in  $X$ . Is  $\oplus_{c_0} E$  remotal in  $\oplus_{c_0} X$ ?*

Since for any measure space  $(\Omega, \Sigma, \mu)$ ,  $L^1(\mu)$  is a  $L$ -summand in its bidual, we also have

**Corollary 3.5.**  *$B_{L^1(\mu)}$  is remotal in  $L^1(\mu)^{**}$ .*

It follows from the lemma above that for any  $M$ -summand  $Y$  in  $X$ ,  $B_Y$  is remotal in  $X$ . But Example 2.17 shows that this is no longer true for  $M$ -ideals. Indeed, one may actually have  $R(B_Y, X) = Y$ . This lemma also leads to easy example of an ideal with a densely remotal unit ball but the formula (1) for  $\rho$  in Proposition 2.4 fails to hold.

**Example 3.6.** Let  $X = Y \oplus_\infty Z$  be a non-trivial direct sum. Then  $Y$  is an ideal,  $B_Y$  is remotal in  $X$ . For an appropriate choice of  $x = y + z$ , we see that  $\rho(x, B_Y) < 1 + \|x\|$ .

**Remark 3.7.** For a  $M$ -embedded non-reflexive Banach space  $X$ , it was shown in [9] that  $X$  is also a  $M$ -ideal, under the canonical embedding, in all the even order duals of  $X$ . This was achieved by showing that  $X$  is a  $M$ -ideal in a component of some  $\ell^\infty$ -decomposition of any higher even order dual. Now let  $X$  be a  $M$ -embedded space with  $B_X$  densely remotal in  $X^{**}$ . It thus follows from above remark that  $B_X$  continues to be a densely remotal set in all the higher even order duals of  $X$ .

Our next result exhibits densely ball remotal subspaces of  $c_0$ -direct sum of reflexive spaces. We recall that a subspace  $Y \subseteq X$  is proximal if there exists a  $y \in Y$  such that  $d(x, Y) = \|x - y\|$  for every  $x \in X$  and is said to be factor reflexive if the quotient space  $X/Y$  is reflexive.

**Theorem 3.8.** *Let  $\{X_i\}_{i \in I}$  be any family of reflexive spaces. Let  $X = \oplus_{c_0} X_i$ . For any factor reflexive proximal subspace  $Y \subseteq X$ ,  $B_Y$  is densely remotal.*

*Proof.* Since  $Y$  is a proximal factor reflexive subspace it follows from [10, Lemma 1.1, page 292] that  $Y^\perp \subseteq NA(X)$ . It follows from our computations in the proof of Theorem 2.29 above, any  $\Lambda \in Y^\perp$  has only finitely many non-zero components. As  $Y^\perp$  is a Banach space, a simple Baire category argument shows that there is a finite set  $F \subseteq I$  such that  $Y^\perp \subseteq \oplus_1 \{X_i^* : i \in F\}$ . Let  $X_1 = \oplus_\infty \{X_i : i \in F\}$  and  $X_2 = \oplus_{c_0} \{X_i : I \setminus F\}$ , so that,  $X = X_1 \oplus_\infty X_2$ . Since  $X^* = X_1^* \oplus_1 X_2^*$ , by duality we have,  $Y = (Y \cap X_1) \oplus_\infty X_2 = Y_1 \oplus X_2$  (say). As only finitely many reflexive spaces occur in  $X_1$ , it is a reflexive space and thus  $B_{Y_1}$  is densely remotal in  $X_1$ . Further,  $B_Y = B_{Y_1} \oplus_\infty B_{X_2}$ . Thus, from Lemma 3.1 we get that  $B_Y$  is densely remotal in  $X$ .  $\square$

We have in particular that for any proximal finite co-dimensional subspace  $Y \subseteq c_0$ ,  $B_Y$  is densely remotal in  $c_0$ .

Coming to the spaces of Bochner integrable functions, we note that

**Lemma 3.9.** *Let  $\mu$  be the Lebesgue measure on  $[0, 1]$ . For any densely remotal set  $E \subseteq X$ ,  $L^1(\mu, E)$  is densely remotal in  $L^1(\mu, X)$ .*

*Proof.* Let  $s = \sum_1^n \chi_{A_i} x_i$  be a simple function in  $L^1(\mu, X)$ . Since  $R(E)$  is dense, given  $\varepsilon > 0$ , we can get a simple function  $s' = \sum_1^n \chi_{A_i} x'_i$  with  $x'_i \in R(E)$  and  $\|s - s'\|_1 < \varepsilon$ . For each  $i$ , let  $e_i \in E$  be farthest from  $x'_i$ , then  $s_0 = \sum_1^n \chi_{A_i} e_i$  is point-wise farthest from  $s'$  and hence, by [5, Theorem 1.1], it is farthest from  $s'$  in  $L^1(\mu, E)$  as well.  $\square$

In general, for a remotal set  $E \subseteq X$ , we do not expect  $L^1(\mu, E)$  to be remotal in  $L^1(\mu, X)$ , however, if  $E$  is separable, the result is given below.

We would like to point out that the proof of [5, Theorem 2.3]—which states that if  $E$  is a closed and bounded set spanning a finite dimensional space, then  $L^1(\mu, E)$  is remotal in  $L^1(\mu, X)$ —is incomplete, since their claim that weakly convergent sequences in  $L^1(\mu, X)$  has point-wise convergent subsequences is incorrect. For example, the well-known Rademacher sequence  $\{r_n\} \subseteq L^1(\mu)$  weakly converges to 0, and  $\|r_n\| = 1$  for all  $n \geq 1$ , so that there is no point-wise convergent subsequence. Since their hypothesis implies  $E$  is separable and remotal, our result below gives a correct proof of their result under weaker hypothesis.

**Theorem 3.10.** *Let  $\mu$  be the Lebesgue measure on  $[0, 1]$ . For any separable remotal set  $E \subseteq X$ ,  $L^1(\mu, E)$  is remotal in  $L^1(\mu, X)$ .*

*Proof.* Let  $f \in L^1(\mu, X)$ . Since the measure space is complete, we may assume without loss of generality that all the functions involved are defined everywhere. In view of [5, Theorem 1.1], we only need to get a  $g \in L^1(\mu, E)$  such that  $g(t) \in E$  is a farthest point for  $f(t)$  for all  $t \in [0, 1]$ .

Let  $\{e_n\}_{n \geq 1} \subseteq E$  be a dense sequence. Define  $F : [0, 1] \rightarrow 2^E$  by  $F(t) = F_E(f(t)) = \{e \in E : \rho(f(t), E) = \|f(t) - e\|\}$ . By hypothesis,  $F$  is a non-empty, closed set-valued map. We now show the set  $Gr(F) = \{(t, e) : e \in F(t)\}$  is a measurable set. Note that  $Gr(F) = \{(t, e) : \rho(f(t), E) = \|f(t) - e\|\} = \bigcap \{(t, e) : \|f(t) - e_n\| \leq \|f(t) - e\|\}$ . Since  $f$  is measurable and  $\|\cdot\|$  is continuous we have that  $Gr(F)$  is measurable.

It thus follows from the von Neumann measurable selection theorem [11, Corollary 5.5.8] that there exists a  $g : [0, 1] \rightarrow E$  measurable such that  $g(t) \in F(t)$ . As  $E$  is separable by the Pettis measurability theorem [3, Chapter 2, Theorem 2], we have that  $g$  is strongly measurable. As  $g$  is bounded, we have,  $g \in L^1(\mu, E)$ .  $\square$

It is not clear how these arguments can be extended to spaces which are ‘rich’ in separable subspaces like the weakly compactly generated Banach spaces (for example reflexive spaces). The usual separable reduction argument based on the fact that Bochner integrable functions are almost everywhere separably valued does not work as remotality is in general not a hereditary property. Thus in the following corollary we make an appropriate assumption to get remotality in spaces of Bochner integrable functions. Hilbert spaces satisfy the hypothesis assumed below.

**Corollary 3.11.** *Let  $\mu$  be the Lebesgue measure on  $[0, 1]$ . Let  $E$  be a remotal set such that every maximizing sequence is contained in a separable remotal set. Then  $L^1(\mu, E)$  is remotal in  $L^1(\mu, X)$ .*

**Remark 3.12.** Now let  $X$  be a reflexive Banach space. It is known that  $L^1(\mu, X)$  is a  $L$ -summand in its bidual  $L^1(\mu, X)^{**}$  (see [4, page 200]). By Lemma 3.9,  $L^1(\mu, B_X)$  is densely remotal in  $L^1(\mu, X)$ , since  $B_X$  is always a remotal subset of  $X$ . Thus by Lemma 3.1,  $L^1(\mu, B_X)$  is also densely remotal in  $L^1(\mu, X)^{**}$ . In particular when  $X$  is also separable we even have that  $L^1(\mu, B_X)$  is remotal in  $L^1(\mu, X)^{**}$ .

We now turn to the question of dense remotality in  $C(K, X)$ . Let  $E \subseteq X$  be a densely remotal set. For any  $f \in C(K, X)$  if  $g \in C(K, E)$  is such that  $g(k)$  is a farthest point from  $f(k)$  for all  $k \in K$ , then it is easy to see that  $g$  is a farthest point from  $f$ .

**Theorem 3.13.** *Let  $E \subseteq X$  be a densely remotal set. Let  $K$  be a compact totally disconnected space. Then  $C(K, E)$  is densely remotal in  $C(K, X)$ .*

*Proof.* Since  $E$  is densely remotal,  $R(E)$  is dense. Let  $s = \sum_1^n \chi_{A_i} x_i$  be a simple function, where  $x_i \in R(E)$  and  $A_i$ 's are clopen sets. We may assume that the  $A_i$ 's are disjoint and that their union is  $K$ . Now let  $s_0 = \sum_1^n \chi_{A_i} e_i$  where  $e_i \in E$  are the farthest points from  $x_i$ . Since  $s_0$  is point-wise farthest from  $s$ ,  $s_0 \in R(C(K, E))$ . It is well-known that  $C(K) \otimes X$  is dense in  $C(K, X)$ . Since  $K$  is totally disconnected, it is easy to see that simple functions of the form  $s' = \sum_1^n \chi_{A_i} x_i$  where  $x_i \in X$ , are dense in  $C(K, X)$ . Finally since  $R(E)$  is dense in  $X$  we have that  $s'$  can be approximated by a simple function of the form  $s$ .  $\square$

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