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Recurrent random walks on homogeneous spaces of p-adic algebraic groups of polynomial growth

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Abstract

Let G be a p-adic algebraic group of polynomial growth and H be a closed subgroup of G. We prove the growth conjecture for the homogeneous space G/H, that is, G/H supports a recurrent random walk if and only if G/H has polynomial growth of degree atmost two.

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Let G be a locally compact separable group. A probability measure μ on G is called *adapted* if the closed subgroup generated by the support of μ is G and μ is called *spread-out* if μ^k is not singular with respect to the Haar measure on G for some $k \geq 1$.

Let (X_n) be a sequence of iid random variables on G with common law μ. Assume that μ is adapted and spread-out. For $g \in G$, let $S_n^g = X_n \cdots X_1 \cdot g$ be the left random walk on G starting at $g \in G$ defined by μ. Let H be a closed subgroup of G and $\pi\colon G \to G/H$ be the canonical quotient map. In this note we consider the induced random walk $Z_n = \pi(S_n^e) = X_n \cdots X_1 \cdot H$ on the homogeneous space G/H.

Let A be a Borel subset of G/H. Then define for $x = \pi(g) \in G/H$,

$$R_A^x = \{w \mid \sum_{n=0}^{\infty} 1_A(\pi(S_n^g)) = \infty\}$$

and

$$h_A(x) = P(R_A^x).$$

An element $x \in G/H$ is called *recurrent* if $h_V(x) = 1$ for all neighbourhood V of x in G/H and x is called *transient* if there exists a neighbourhood V of x in G/H such that $h_V(x) = 0$. Théorème 2 of [4] proves the dichotomy that all states are recurrent or all states are transient. We say that the random walk (Z_n) on G/H is *recurrent* if all states $x \in G/H$ are recurrent: see [3] and [4] for various results on random walks on homogeneous spaces.

Let us further assume that the homogeneous space G/H has a G-invariant measure m: Theorem 2.49 of [1] gives a necessary and sufficient conditions for the homogeneous space G/H to have a G-invariant measure. We say that G/H has polynomial growth if to each compact neighbourhood V of e in G, there is an integer $k \geq 0$ and a constant $c > 0$ such that

$$m(\pi(V^n)) \leq cn^k$$

for all $n \geq 1$.

It can be easily seen that G has polynomial growth implies that for any closed subgroup H of G, G/H also has polynomial growth and the converse is true if H is compact: if G has polynomial growth, then for any closed subgroup H, G/H has a G-invariant measure as G and H are unimodular. Here we show the following generalization: this result is proved in I.1.2 of [3] for Lie groups, here we present a modified version suitable for our purpose.

Lemma 1 *Let G be a locally compact separable group and H_1 be a closed subgroup of G. Suppose H is a closed subgroup of H_1 and H_1/H is compact. Then the growth of G/H is same as the growth of G/HK.*

Proof Let $\pi: G \to G/H$ and $\tilde{\pi}: G \to G/H_1$ be the canonical quotient map. Let $\phi: G/H \to G/H_1$ be defined by $\phi(aH) = aH_1$ for all $a \in G$. Then ϕ is a continuous G-equivariant map. Let m be a G-invariant measure on G/H. Define $\tilde{m}$ on G/H_1 by $\tilde{m}(E) = m(\phi^{-1}(E))$ for any mesurable set E in G/H_1. Then $\tilde{m}$ is a G-invariant measure on G/H_1. Since H_1/H is compact, there exists a compact set K in H_1 such that $H_1 = KH$. Let V be a compact neighbourhood of e in G. Then $xH \in \phi^{-1}\tilde{\pi}(V)$ if and only if $\phi(xH) = \tilde{\pi}(y)$ for some $y \in V$ if and only if $y^{-1}x \in H_1 = KH$ for some $y \in V$ if and only if $y'^{-1}x \in H$ for some $y' \in VK$ if and only if $xH \in \pi(VK)$. Thus, $\phi^{-1}(\tilde{\pi}(V^n)) = \pi(V^nK)$ and hence $\tilde{m}(\tilde{\pi}(V^n)) = m(\pi(V^nK))$. Thus, G/H and G/H_1 have same type of growth.

Guivarch and Keane [2] formulated the natural growth conjecture which classifies all groups that admit recurrent random walks and the precise conjecture is that a locally compact group admits recurrent random walks if and only if it has polynomial growth of degree atmost two. Recently [6] proved the conjecture for p-adic Lie groups. [3], [4] and [8] proved a similar conjecture for homogneous spaces of certain connected Lie groups of type R. Thus, motivated by these considerations we prove a similar conjecture for homogeneous spaces of p-adic algebraic groups of polynomial growth.

It can be easily seen that if the random walk on G defined by μ is recurrent on $G \simeq G/(e)$, then for any closed subgroup H of G, the induced random walk on the homogeneous space G/H defined by μ is recurrent. We next obtain the following general form.

Lemma 2 *Let G be a locally compact separable group and H be a closed subgroup of G. Let π be the canonical projection of G onto G/H. Let N be a closed subgroup of H and $\tilde{\pi}$ be the canonical projection of G onto G/N. Let μ be an adapted probability measure on G and (S_n) be the random walk on G defined by μ. Suppose the induced random walk $(\tilde{Z}_n = \tilde{\pi}(S_n))$ on G/N is recurrent. Then the induced random walk $(Z_n = \pi(S_n))$ on G/H is also recurrent.*

Proof Let $x \in G/H$ and $g \in G$ be such that $\pi(g) = x$. Let $y = \tilde{\pi}(g) \in G/N$. Let V be a neighbourhood of x. Then there exists a neighbourhood W of g containing H such that $\pi(W) = V$. Let $\tilde{V} = \tilde{\pi}(W)$. Then $\tilde{V}$ is a neighbourhood of $y \in G/N$. Since $\tilde{Z}_n$ is recurrent, for almost all ω, $\tilde{Z}_n^y = \tilde{\pi}(S_n^g) \in \tilde{V}$ infinitely often. Let $\phi\colon G/N \to G/H$ be the map defined by $\phi(aN) = aH$ for all $a \in G$. Then $\phi(\tilde{\pi}(a)) = \pi(a)$ for all $a \in G$ and π is a G-equivariant continuous map. This implies that $\phi(\tilde{V}) = V$ and $\phi(\tilde{Z}_n^y) = Z_n^x = \pi(S_n^g)$ as ϕ is G-equivariant. Thus, for almost all ω, $Z_n^x \in V$ infinitely often. Hence x is recurrent.

The next result is proved in II.1.1 of [3] for Lie groups and the same proof works in the general case also as shown below.

Lemma 3 *Let G be a locally compact separable group and H_1 be closed subgroup of G. Suppose H is a closed subgroup of H_1 such that H_1/H is compact. Let $\pi\colon G \to G/H$ and $\pi_1\colon G \to G/H_1$ be the canonical projections. Suppose H is a subgroup of H_1 such that H/H_1 is compact. Let μ be an adapted spread-out probability measure on G and (S_n) be the random walk on G defined by μ. Then the induced random walk $(W_n = \pi_1(S_n))$ on G/H_1 is recurrent if and only if the induced random walk $(Z_n = \pi(S_n))$ on G/H is also recurrent.*

Proof Let $\eta\colon G/H \to G/H_1$ be $\eta(gH) = gH_1$ for all $g \in G$. Then η is a continuous, open and closed G-equivariant map. Let $x \in G/H$ and $g \in G$ be such that $\pi(g) = x$. Let $y = \pi_1(g) \in G/H_1$. Then for any set C in G/H_1, since H_1/H is compact, C is compact in G/H_1 if and only if $\eta^{-1}(C)$ is compact in G/H and since η is G-equivariant, $W_n y = \pi_1(S_n)y \in C$ if and only if $Z_n x = \pi(S_n)x \in \eta^{-1}(C)$ as $\eta(\pi(g)) = \pi_1(g)$ for any $g \in G$. Thus proving the lemma.

We now consider p-adic algebraic groups. Let $\mathbb{Q}_p$ be the field of p-adic numbers: see [9] for some details on $\mathbb{Q}_p$. We say that G is a *p-adic algebraic group* if G is the group of $\mathbb{Q}_p$-rational points in $\mathbb{G}$ for some algebraic group $\mathbb{G}$ defined over $\mathbb{Q}_p$.

Example The following locally compact second countable groups are p-adic algebraic groups.

1. the additive group $\mathbb{Q}_p$ and the multiplicative group $\mathbb{Q}_p^*$ of non-zero p-adic numbers, known as one-dimensional split torus.

2. The groups $GL_n(\mathbb{Q}_p)$ the group of all invertible matrices over $\mathbb{Q}_p$.

3. The special linear group $SL_n(\mathbb{Q}_p) = \{A \in GL_n(\mathbb{Q}_p) \mid \det\,(A) = 1\}$.

4. The p-adic affine group $\mathbb{Q}_p^* \ltimes \mathbb{Q}_p$. More generally, $H \ltimes \mathbb{Q}_p^n$ where H is a p-adic algebraic subgroup of $GL_n(\mathbb{Q}_p)$.

5. the group of all upper triangular matrices $UT_n(\mathbb{Q}_p) = \{A = (a_{ij}) \in GL_n(\mathbb{Q}_p) \mid a_{ij} = 0 \;\text{ if }\; j < i\}$ and the group of unipotent matrices $U_n(\mathbb{Q}_p) = \{A = (a_{ij}) \in UT_n(\mathbb{Q}_p) \mid a_{ii} = 1\}$.

The growth properties of these examples also can be obtained using results in [6]. For instance, the group $U_n(\mathbb{Q}_p)$ has polynomial growth of degree zero and the group $\mathbb{Q}_p^*$ has polynomial growth of degree one whereas the groups $GL_n(\mathbb{Q}_p)$, $SL_n(\mathbb{Q}_p)$ and the affine group $\mathbb{Q}_p^* \ltimes \mathbb{Q}_p$ have exponential growth.

We now prove the growth conjecture for homogeneous spaces of p-adic algebraic groups of polynomial growth.

Theorem 1 *Let G be a p-adic algebraic group of polynomial growth and H be a closed subgroup of G. Then the following are equivalent:*

(1) G/H supports a recurrent random walk;

(2) G/H has polynomial growth of degree at most two.

Proof We first analyze the structure of G and H. Let U be the unipotent radical of G. By Corollary 2.1 of [6], $G = UK\mathbb{Z}^k$, where K is a compact group, $\mathbb{Z}^k$ is central and k is the degree of growth of G. Let $H_1 = HUK$. As $\mathbb{Z}^k$ centralizes U, U has a basis of compact open subgroups normalized by $\mathbb{Z}^k K$. Let V_1 be a compact open subgroup in U normalized by $\mathbb{Z}^k K$. Now, $H_1 = D_1 UK$ where $D_1 = H_1 \cap \mathbb{Z}^k$ as $G = \mathbb{Z}^k UK$. Let $H_2 = (H \cap \mathbb{Z}^k KV_1)KV_1$. Then $H_2 = D_2 KV_1$ where $D_2 = H_2 \cap \mathbb{Z}^k$. Thus, $D_2 \subset D_1$. For $x \in D_1$, there exists a $u \in U$ and $y \in K$ such that $xyu \in H$. Now $(xyu)^n = x^n y^n u_n \in H$ where $u_n = y^{-n+1} u y^{n-1} \cdots y^{-1} u y u$ for any $n \geq 1$. Since U is a union of K-invariant compact open subgroups, we get that $u_n u_m^{-1} \in V_1$ for some $n \neq m$. This implies that $x^n y^n u_n u_m^{-1} y^{-m} x^{-m} \in (H \cap \mathbb{Z}^k KV_1)$ and hence $x^k \in H_2 = (H \cap \mathbb{Z}^k KV_1)KV_1$ for some $k \neq 0$. So, $x^k \in D_2$. This shows that D_1/D_2 is a finitely generated torsion abelian group and hence it is finite. This implies that D_1 and D_2 have the same degree of growth.

Suppose G/H supports a recurrent random walk. Then by Lemma 2, $G/H_1 \simeq (G/UK)/(H_1/UK)$ supports a recurrent random walk. Since G/UK is an abelian group, we get that $(G/UK)/(H_1/UK)$ has polynomial growth of degree at most two (see [7]). As the isomorphism $(G/UK)/(H_1/UK) \simeq G/H_1$ is G-equivariant, we get that G/H_1 also has polynomial growth of degree at most two. This implies that $\mathbb{Z}^k/D_1$ has rank atmost two. Let $V = FKV_1$ where F is a finite symmetric generating subset in $\mathbb{Z}^k$. Let N be the open subgroup of G generated by V. Since $N/H_2 \simeq \mathbb{Z}^k/D_2$, N/H_2 has polynomial growth of degree at most two. Now, $N = \mathbb{Z}^k KV_1$ and so $H \cap N = H \cap \mathbb{Z}^k KV_1 \subset (H \cap \mathbb{Z}^k KV_1)KV_1 = H_2$. This shows that $H_2/(H \cap N)$ is compact. Thus, by Lemma 1, $N/(H \cap N)$ has polynomial growth of degree at most two. Since N is an open subgroup, $x(H \cap N) \mapsto xH$ is a N-equivariant homeomorphism of $N/(H \cap N)$ and the image of N in G/H. Hence G/H has polynomial growth of degree at most two. This proves that (1) implies (2).

Now assume that G/H has polynomial growth of degree at most two. Let V be a compact open subgroup of G containing K. Let $G_1 = \mathbb{Z}^k V$ and $H_1 = G_1 \cap H$. Let $\phi_1 \colon G_1/H_1 \to G/H$ be the canonical quotient map. Then $\phi_1(G_1/H_1)$ is closed and open in G/H and hence we get that ϕ_1 is a G_1-equivariant homeomorphism onto its image. If m is a G-invariant measrue on G/H, then m_1 is a G_1-invariant measure on G_1/H_1 where $m_1(E) = m(\phi_1(E))$ for any borel subset E of G_1/H_1. Thus, G_1/H_1 also has polynomial growth of degree at most two. By Lemma 1, G_1/H_1V also has polynomial growth of degree at most two. Since $G_1 = \mathbb{Z}^k V$, $H_1V = \mathbb{Z}^l V$ where $k-2 \leq l \leq k$. Now, $G/\mathbb{Z}^l$ is a p-adic Lie group of polynomial growth of degree at most two and hence by Theorem 3.1 of [6], $G/\mathbb{Z}^l$ supports a recurrent random walk. Since $\mathbb{Z}^l \subset H_1V$, by Lemma 2, G/H_1V supports a recurrent random walk. Now Lemma 3 implies that G/H_1 supports a recurrent random walk. Since $H_1 \subset H$, using Lemma 2, we get that G/H supports a recurrent random walk. This proves (2) implies (1).

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