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Super Critical Age Dependent Branching Markov Processes

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SUPER CRITICAL AGE DEPENDENT BRANCHING MARKOV PROCESSES

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Abstract. This paper studies the long time behaviour of the empirical distribution of age and normalised position of an age dependent critical branching Markov process conditioned on non-extinction;

1. Introduction

Consider an age dependent branching Markov process where i) each particle lives for a random length of time and during its lifetime moves according to a Markov process and ii) upon its death it gives rise to a random number of offspring. We assume that the system is super-critical, i.e. the mean of the offspring distribution is strictly greater than one.

We study two aspects of such a system. First, at time t, conditioned on non-extinction (as such systems die out w.p. 1) we consider a randomly chosen individual from the population. We show that asymptotically (as $t \to \infty$), the joint distribution of the position (appropriately scaled) and age (unscaled) of the randomly chosen individual decouples (See Theorem 2.1). Second, it is shown that conditioned on non-extinction at time t, the empirical distribution of the age and the normalised position of the population converges as $t \to \infty$ in law to (See Theorem 2.2).

The rest of the paper is organised as follows. In Section 2.1 we define the branching Markov process precisely and in Section 2.2 we state the main theorems of this paper and make some remarks on various possible generalisations of our results.

In Section 4 we prove four propositions on age-dependent Branching processes which are used in proving Theorem 2.1 (See Section 5). In Section 4 we also show that the joint distribution of ancestoral times for a sample of $k \geq 1$ individuals chosen at random from the population at time t converges as $t \to \infty$ (See Theorem 4.4). This result is of independent interest and is a key tool that is needed in proving Theorem 2.2 (See Section 6).

2. Statement of Results

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2.1. **The Model.**

Each particle in our system will have two parameters, age in $\mathbb{R}_+$ and location in $\mathbb{R}$. We begin with the description of the particle system.

(i) **Lifetime Distribution** $G(\cdot)$**:** Let $G(\cdot)$ be a cumulative distribution function on $[0, \infty)$, with $G(0) = 0$. Let $\mu = \int_0^\infty s dG(s) < \infty$.

(ii) **Offspring Distribution p :** Let $\mathbf{p} \equiv \{p_k\}_{k \geq 0}$ be a probability distribution such that $p_0 = 0$, $m = \sum_{k=0}^\infty k p_k > 1$ and that $\sigma^2 = \sum_{k=0}^\infty k^2 p_k - 1 < \infty$. Let α be the Malthusian parameter defined by the equation $m \int_0^\infty e^{-\alpha x} dG(x) = 1$.

(iii) **Motion Process** $\eta(\cdot)$**:** Let $\eta(\cdot)$ be a $\mathbb{R}$ valued Markov process starting at 0.

Branching Markov Process $(G, \mathbf{p}, \eta)$**:** Suppose we are given a realisation of an age-dependent branching process with offspring distribution $\mathbf{p}$ and lifetime distribution G (See Chapter IV of [5] for a detailed description). We construct a branching Markov process by allowing each individual to execute an independent copy of η during its lifetime τ starting from where its parent died.

Let N_t be the number of particles alive at time t and

$$\mathcal{C}_t = \{(a_t^i, X_t^i) : i = 1, 2, \ldots, N_t\} \tag{2.1}$$

denote the age and position configuration of all the individuals alive at time t. Since $m = 1$ and $G(0) = 0$, there is no explosion in finite time (i.e. $P(N_t < \infty) = 1$) and consequently $\mathcal{C}_t$ is well defined for each $0 \leq t < \infty$ (See [5]).

Let $\mathcal{B}(\mathbb{R}_+)$ (and $\mathcal{B}(\mathbb{R})$) be the Borel σ-algebra on $\mathbb{R}_+$ (and $\mathbb{R}$). Let $M(\mathbb{R}_+ \times \mathbb{R})$ be the space of finite Borel measures on $\mathbb{R}_+ \times \mathbb{R}$ equipped with the weak topology. Let $M_a(\mathbb{R}_+ \times \mathbb{R}) := \{\nu \in M(\mathbb{R}_+ \times \mathbb{R}) : \nu = \sum_{i=1}^n \delta_{a_i, x_i}(\cdot, \cdot), n \in \mathbb{N}, a_i \in \mathbb{R}_+, x_i \in \mathbb{R}\}$. For any set $A \in \mathcal{B}(\mathbb{R}_+)$ and $B \in \mathcal{B}(\mathbb{R})$, let $Y_t(A \times B)$ be the number of particles at time t whose age is in A and position is in B. As pointed out earlier, $m < \infty$, $G(0) = 0$ implies that $Y_t \in M_a(\mathbb{R}_+ \times \mathbb{R})$ for all $t > 0$ if Y_0 does so. Fix a function $\phi \in C_b^+(\mathbb{R}_+ \times \mathbb{R})$, (the set of all bounded, continuous and positive functions from $\mathbb{R}_+ \times \mathbb{R}$ to $\mathbb{R}_+$), and define

$$\langle Y_t, \phi \rangle = \int \phi \, dY_t = \sum_{i=1}^{N_t} \phi(a_t^i, X_t^i). \tag{2.2}$$

Since $\eta(\cdot)$ is a Markov process, it can be seen that $\{Y_t : t \geq 0\}$ is a Markov process and we shall call $Y \equiv \{Y_t : t \geq 0\}$ *the* $(G, \mathbf{p}, \eta)$*- branching Markov process.*

Note that $\mathcal{C}_t$ determines Y_t and conversely. The Laplace functional of Y_t, is given by

$$L_t \phi(a, x) := E_{a,x}[e^{-\langle \phi, Y_t \rangle}] \equiv E[e^{-\langle \phi, Y_t \rangle} \mid Y_0 = \delta_{a,x}]. \tag{2.3}$$

From the independence intrinsic in $\{Y_t : t \geq 0\}$, we have:

$$E_{\nu_1 + \nu_2}[e^{-\langle \phi, Y_t \rangle}] = (E_{\nu_1}[e^{-\langle \phi, Y_t \rangle}])(E_{\nu_2}[e^{-\langle \phi, Y_t \rangle}]), \tag{2.4}$$

for any $\nu_i \in M_a(\mathbb{R}_+ \times \mathbb{R})$ where $E_{\nu_i}[e^{-\langle \phi, Y_t \rangle}] := E[e^{-\langle \phi, Y_t \rangle} \mid Y_0 = \nu_i]$ for $i = 1, 2$. This is usually referred to as the branching property of Y and can be used to define the process Y as the unique measure valued Markov process with state space $M_a(\mathbb{R}_+ \times \mathbb{R})$ satisfying $L_{t+s}\phi(a, x) = L_t(L_s(\phi))(a, x)$ for all $t, s \geq 0$.

2.2. **The Results.**

In this section we describe the main results of the paper. Let A_t be the event $\{N_t > 0\}$, where N_t is the number of particles alive at time t. As $p_0 = 0$, $P(A_t) = 1$ for all $0 \leq t < \infty$ provided $P(N_0 = 0) = 0$.

Theorem 2.1. (Limiting behaviour of a randomly chosen particle)
Let (a_t, X_t) be the age and position of a randomly chosen particle from those alive at time t. Assume that $\eta(\cdot)$ is such that for all $0 \leq t < \infty$

$$E(\eta(t)) = 0, v(t) \equiv E(\eta^2(t)) < \infty, \ \sup_{0 \leq s \leq t} v(s) < \infty, \tag{2.5}$$

$$\text{and } \psi \equiv \int_0^\infty v(s)G(ds) < \infty.$$

Then, $(a_t, \frac{X_t}{\sqrt{t}})$ converges as $t \to \infty$, to (U, V) in distribution, where U and V are independent with U a strictly positive absolutely continuous random variable with density proportional to $(1 - G(\cdot))$ and V is normally distributed with mean 0 and variance $\frac{\psi}{\mu}$.

Next consider the scaled empirical measure $\tilde{Y}_t \in M_a(\mathbb{R}_+ \times \mathbb{R})$ given by $\tilde{Y}_t(A \times B) = Y_t(A \times \sqrt{t}B)$, $A \in \mathcal{B}(\mathbb{R}_+), B \in \mathcal{B}(\mathbb{R})$.

Theorem 2.2. (Empirical Measure)
Assume (2.5). Then the scaled empirical measures $\tilde{Y}_t$ converges in distribution to a deterministic measure ν such that $\nu(A \times B) = P(U \in A, V \in B)$ for $A \in \mathcal{B}(\mathbb{R}_+)$ and $B \in \mathcal{B}(\mathbb{R})$.

2.3. **Remarks.**

(a) If $\eta(\cdot)$ is not Markov then $\tilde{C}_t = \{a_t^i, X_t^i, \tilde{\eta}_{t,i} \equiv \{\eta_{t,i}(u) : 0 \leq u \leq a_t^i\} : i = 1, 2 \ldots, N_t\}$ is a Markov process where $\{\tilde{\eta}_{t,i}(u) : 0 \leq u \leq a_t^i\}$ is the history of $\eta(\cdot)$ of the individual i during its lifetime. Theorem 2.1 and Theorem 2.2 extends to this case.

(b) Most of the above results also carry over to the case when the motion process is $\mathbb{R}^d$ valued $(d \geq 1)$ or is Polish space valued and where the offspring distribution is age-dependent.

(c) Theorem 2.1 and Theorem2.2 can also be extended to the case when $\eta(L_1)$, with $L_1 \stackrel{d}{=} G$, is in the domain of attraction of a stable law of index $0 < \alpha \leq 2$.

3. Results from Renewal Theory

Let $\{X_i : i \geq 1\}$ be an i.i.d. sequence of positive random varibles with cummulative distribution function G. Let $S_0 = 0, S_n = \sum_{i=1}^n X_i, n \geq 1$. For $t \geq 0$ let $Z(t) = k$ if $S_k \leq t \leq S_{k+1}, k \geq 0$. Further let $P_t = t - S_{Z(t)}$ and $R_t = S_{Z(t)+1} - t$. Let $\mu = \int_0^\infty x dG(x)$.

Lemma 3.1. *Let $P_t, Z(t), R_t$ and μ be as above. Then:*
(i) $\frac{Z(t)}{t} \to \frac{1}{\mu}$
(ii) Let $\theta \in \mathbb{R}$ and $g(\theta) = \int_0^\infty e^{\theta t}(1 - G(t))dt < \infty$ and $\mu < \infty$. Then

$$\lim_{t\to\infty} E(e^{\theta P_t}) = \lim_{t\to\infty} E(e^{\theta R_t}) = \frac{g(\theta)}{\mu}, \tag{3.1}$$

and for any $0 < l < \infty$

$$\lim_{t\to\infty} E(e^{\theta P_t} : P_t > l) = \lim_{t\to\infty} E(e^{\theta R_t} : R_t > l) = \frac{1}{\mu}\int_l^\infty e^{\theta t}(1 - G(t))dt \tag{3.2}$$

and hence

$$\lim_{l\to\infty}\lim_{t\to\infty} E(e^{\theta R_t} : R_t > l) = 0. \tag{3.3}$$

Proof : For $t \geq 0$, $S_{Z(t)} \leq t \leq S_{Z(t)+1}$. So,

$$\frac{S_{Z(t)}}{Z(t)} \leq \frac{t}{Z(t)} \leq \frac{S_{Z(t)+1}}{Z(t)+1} \cdot \frac{Z(t)+1}{Z(t)}.$$

Now $Z(t) \to \infty$ a.e. as $t \to \infty$ and by the strong law of large numbers $\frac{S_n}{n} \to \mu$. Consequently (i) follows.

For $t \geq 0$, $\theta \in \mathbb{R}$ let $f(t,\theta) = E(e^{\theta P_t})$ and $g(t,\theta) = E(e^{\theta R_t})$. It is easy to see that,

$$\begin{aligned} f(t,\theta) &= e^{\theta t}(1 - G(t)) + \int_0^t f(\theta, t-u)dG(u) \\ g(t,\theta) &= h(\theta,t)(1 - G(t)) + \int_0^t g(\theta, t-u)dG(u), \end{aligned}$$

where $h(\theta,t) = E(e^{\theta(X-t)}|X > t)$ with $X \stackrel{d}{=} G$. By the Key renewal theorem,

$$f(\theta,t) \to \frac{\int_0^\infty e^{\theta t}(1 - G(t))dt}{\mu},$$

$$\begin{aligned} g(\theta,t) &\to \frac{\int_0^\infty g(\theta,t)(1 - G(t))dt}{\mu} \\ &= \frac{\int_0^\infty E(e^{\theta(X-t)}; X > t)dt}{\mu} \\ &= \frac{E(\int_0^X e^{\theta u})du}{\mu} = \frac{\int_0^\infty e^{\theta u}P(X > u)du}{\mu} \\ &= \frac{\int_0^\infty e^{\theta u}(1 - G(u))du}{\mu}. \end{aligned}$$

This proves (3.1). Consequently $P_t \xrightarrow{d} P_\infty$, $R_t \xrightarrow{d} R_\infty$ and $R_\infty \overset{d}{=} P_\infty$. So

$$E(e^{\theta R_t}; R_t > l) \to E(e^{\theta R_\infty}; R_\infty > l) = \frac{1}{\mu}\int_l^\infty e^{\theta t}(1-G(t))dt,$$

which proves (3.2) and as $\int_0^\infty e^{\theta t}(1-G(t))dt < \infty$ (3.3) follows easily from (3.2). □

Lemma 3.2. *Let $\{X_i\}_{i\geq 1}$ be i.i.d. positive random variables with cummulative distribution function G and $G(0)=0$. Let $0<\alpha<\infty$ be the Malthusian parameter given by $m\int_0^\infty e^{-\alpha x}dG(x) = 1$. Let $\{\tilde{X}_i\}_{i\geq 1}$ be i.i.d. positive random variables with cummulative distribution function $\tilde{G}(x) = m\int_0^x e^{-\alpha y}dG(y)$. Then*

(i) *for any $k \geq 1$, and bounded Borel measurable function $\phi : \mathbb{R}^k \to \mathbb{R}$,*

$$E(\phi(\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_k)) = E(e^{-\alpha S_k} m^k \phi(X_1, X_2, \ldots, X_k)) \tag{3.4}$$

where $S_k = \sum_{i=1}^k X_i$.

(ii) *for any $t \geq 0$, $k \geq 0$, $c \in \mathbb{R}$ and Borel measurable function h,x*

$$\begin{aligned} & E(e^{\alpha R_t} e^{-\alpha S_{k+1}} m^{k+1} I(Z(t) = k, |\frac{1}{k}\sum_{i=1}^k h(X_i) - c| > \epsilon)) \\ = \;& E(e^{\alpha \tilde{R}_t} I(\tilde{Z}(t) = k, |\frac{1}{k}\sum_{i=1}^k h(\tilde{X}_i) - c| > \epsilon)), \end{aligned} \tag{3.5}$$

and

$$\lim_{l\to\infty}\lim_{t\to\infty} E(e^{\theta \tilde{R}_t} : \tilde{R}_t > l) = 0. \tag{3.6}$$

Proof: From the definition of $\tilde{G}$, for any Borel sets $B_i, i = 1, 2, \ldots, k$

$$\begin{aligned} P(\tilde{X}_i \in B_i, i = 1, 2, \ldots, k) &= \prod_{i=1}^k m\int_{B_i} e^{-\alpha x}dG(x) \\ &= E(e^{-\alpha S_k} m^k I(X_i \in B_i)). \end{aligned}$$

So (3.4) follows and (3.5) follows from it. Now,

$$\begin{aligned} \int_0^\infty e^{\alpha x}(1-\tilde{G}(x))dx &= m\int_0^\infty e^{\alpha x}\left(\int_x^\infty e^{-\alpha y}dG(y)\right)dx \\ &= m\int_0^\infty \left(\int_0^y e^{\alpha x}dx\right) e^{-\alpha y}dG(y) \\ &= m\int_0^\infty \frac{e^{\alpha u}-1}{\alpha} e^{-\alpha y}dG(y) \\ &= \frac{m}{\alpha}(1-\frac{1}{m}) < \infty \end{aligned}$$

So (3.6) follows as in Lemma 3.1

4. Results on Branching Processes

Let $\{N_t : t \geq 0\}$ be an age-dependent branching process with offspring distribution $\{p_k\}_{k\geq 0}$ and lifetime distribution G (see [5] for detailed discussion). Let $\{\zeta_k\}_{k\geq 0}$ be the embedded discrete time Galton-Watson branching process with ζ_k being the size of the kth generation, $k \geq 0$. Let A_t be the event $\{N_t > 0\}$. On this event, choose an individual uniformly from those alive at time t. Let M_t be the generation number and a_t be the age of this individual.

Proposition 4.1. *(Law of large numbers) Let $\epsilon > 0$ be given. For the randomly chosen individual at time t, let $\{L_{ti} : 1 \leq i \leq M_t\}$, be the lifetimes of its ancestors. Let $h : [0, \infty) \to \mathbb{R}$ be Borel measurable such that $m \int_0^\infty |h(x)|e^{-\alpha x} dG(x) < \infty$. Then, as $t \to \infty$*

$$P(|\frac{1}{M_t}\sum_{i=1}^{M_t} h(L_{ti}) - c| > \epsilon) \to 0,$$

where $c = m \int_0^\infty h(x)e^{-\alpha x} dG(x)$.

Proof : Let $\{\zeta_k\}_{k\geq 0}$ be the embedded Galton-Watson process. For each $t > 0$ and $k \geq 1$ let ζ_{kt} denote the number of lines of descent in the k-th generation alive at time t (i.e. the successive life times $\{L_i\}_{i\geq 1}$ of the individuals in that line of descent satisfying $\sum_{i=1}^{k} L_i \leq t \leq \sum_{i=1}^{k+1} L_i$). Denote the lines of descent of these individuals by $\{\zeta_{ktj} : 1 \leq j \leq \zeta_{kt}\}$. Call ζ_{ktj} *bad* if

$$|\frac{1}{k}\sum_{i=1}^{k} h(L_{ktji}) - E(h(L_1)))| > \epsilon, \tag{4.1}$$

where $\{L_{ktji}\}_{i\geq 1}$ are the successive lifetimes in the line of descent ζ_{ktj} starting from the ancestor. Let $\zeta_{kt,b}$ denote the cardinality of the set $\{\zeta_{ktj} : 1 \leq j \leq \zeta_{kt}$ and ζ_{ktj} is bad$\}$. Then

$$\begin{aligned} P(|\frac{1}{M_t}\sum_{i=1}^{M_t} h(L_{ti}) - c| > \epsilon) &= E(\frac{\sum_{j=0}^\infty \zeta_{jt,b}}{N_t}) \\ &= E(\frac{\sum_{j=0}^\infty \zeta_{jt,b}}{N_t}; N_t < e^{\alpha t}\eta) + E(\frac{\sum_{j=0}^\infty \zeta_{jt,b}}{N_t}; N_t \geq e^{\alpha t}\eta) \\ &\leq P(N_t < e^{\alpha t}\eta) + \frac{1}{\eta e^{\alpha t}} E(\sum_{j=0}^\infty \zeta_{jt,b}), \qquad (4.2) \end{aligned}$$

where $\eta > 0$. Using X_i replaced with L_i in Section 3, notation therein, and Lemma 3.2 and (3.5) we have

$$\begin{aligned}
\sum_{j=0}^{\infty}\frac{1}{\eta}e^{-\alpha t}E(\zeta_{jt,b}) &= \frac{1}{\eta}\sum_{j=0}^{\infty}e^{-\alpha t}P(\sum_{i=1}^{j}L_i \le t < \sum_{i=1}^{j+1}L_i, \frac{1}{j}|\sum_{i=1}^{j}h(L_i)-c|>\epsilon)\\
&= \frac{1}{\eta}\sum_{j=0}^{\infty}E(e^{-\alpha t}m^j I(S_j\le t<S_{j+1}), |\bar{Y}_j|>\epsilon)\\
&= \frac{1}{\eta m}\sum_{j=0}^{\infty}E(e^{\alpha R_t}e^{-\alpha S_{j+1}}m^{j+1}I(S_j\le t<S_{j+1}), |\bar{Y}_j|>\epsilon)\\
&= \frac{1}{\eta m}\sum_{j=0}^{\infty}E(e^{\alpha\tilde{R}_t}I(\tilde{Z}(t)=j, |\tilde{\bar{Y}}_j|>\epsilon))\\
&= \frac{1}{\eta m}E(e^{\alpha\tilde{R}_t}I(|\tilde{\bar{Y}}_{\tilde{Z}(t)}|>\epsilon))\\
&\le \frac{e^{\alpha l}}{\eta m}P(|\tilde{\bar{Y}}_{\tilde{Z}(t)}|>\epsilon)+\frac{1}{\eta m}E(e^{\alpha\tilde{R}_t};\tilde{R}_t>l)
\end{aligned}$$

By the strong law of large numbers, $|\tilde{\bar{Y}}_{\tilde{Z}(t)}| \xrightarrow{a.e.} 0$ and consequently $\limsup_{t\to\infty} P(|\tilde{\bar{Y}}_{\tilde{Z}(t)}| > \epsilon) = 0$. This and (3.6) along with (4.2) imply that

$$\begin{aligned}
&\limsup_{t\to\infty} P(|\frac{1}{M_t}\sum_{i=1}^{M_t}h(L_{ti})-c|>\epsilon)\\
&\le \limsup_{t\to\infty}\left(P(N_t<e^{\alpha t}\eta)+\frac{e^{\alpha l}}{\eta m}P(|\tilde{\bar{Y}}_{\tilde{Z}(t)}|>\epsilon)+\frac{1}{\eta m}E(e^{\alpha\tilde{R}_t};\tilde{R}_t>l)\right)\\
&= \limsup_{t\to\infty} P(e^{-\alpha t}N_t<\eta)
\end{aligned}$$

Since $\sigma^2 < \infty$, $e^{-\alpha t}N_t \xrightarrow{a.e.} W$ with $\lim_{\eta\downarrow 0} P(W < \eta) = 0$(See chapter 11, [5]). Therefore we have the result.

□

Proposition 4.2. *Assume (2.5) holds. Let $\{L_i\}_{i\ge1}$ be i.i.d G and $\{\eta_i\}_{i\ge1}$ be i.i.d copies of η and independent of the $\{L_i\}_{i\ge1}$. For $\theta\in\mathbb{R}, t\ge 0$ define $\phi(\theta,t) = Ee^{i\theta\eta(t)}$. Then there exists an eventd D, with $P(D) = 1$ and on D for all $\theta\in\mathbb{R}$,*

$$\prod_{j=1}^{n}\phi\left(\frac{\theta}{\sqrt{n}}, L_j\right)\to e^{\frac{-\theta^2\psi}{2}}, \qquad as\ n\to\infty,$$

where ψ is as in (2.5).

Proof: Recall from (2.5) that $v(t) = E(\eta^2(t))$ for $t \ge 0$. Consider

$$X_{ni} = \frac{\eta_i(L_i)}{\sqrt{\sum_{j=1}^{n}v(L_j)}} \text{ for } 1\le i\le n$$

and $\mathcal{F} = \sigma(L_i : i \ge 1)$. Given $\mathcal{F}$, $\{X_{ni} : 1 \le i \le n\}$ is a triangular array of independent random variables such that for $1 \le i \le n$, $E(X_{ni}|\mathcal{F}) = 0$, $\sum_{i=1}^{n} E(X_{ni}^2|\mathcal{F}) = 1$.

Let $\epsilon > 0$ be given. Let

$$L_n(\epsilon) = \sum_{i=1}^{n} E\left(X_{ni}^2\,; X_{ni}^2 > \epsilon|\mathcal{F}\right).$$

By the strong law of large numbers,

$$\frac{\sum_{j=1}^{n} v(L_j)}{n} \to \psi \qquad \text{w.p. } 1. \tag{4.3}$$

Let D be the event on which (4.3) holds. Then on D

$$\begin{aligned}
\limsup_{n\to\infty} L_n(\epsilon) &\leq \limsup_{n\to\infty} \frac{\psi}{2n}\sum_{i=1}^{n} E(|\eta_i(L_i)|^2) :| \eta_i(L_i)|^2 > \frac{\epsilon n\psi}{2}|\mathcal{F}) \\
&\leq \limsup_{k\to\infty} \frac{\psi}{2} E(|\eta_1(L_1)|^2 :| \eta_1(L_1)\ |^2 > k) \\
&= 0.
\end{aligned}$$

Thus the Linderberg-Feller Central Limit Theorem (see [4]) implies, that on D, for all $\theta \in \mathbb{R}$

$$\prod_{i=1}^{n} \phi\left(\frac{\theta}{\sqrt{\sum_{j=1}^{n} v(L_j)}}, L_j\right) = E(e^{i\theta\sum_{j=1}^{n} X_{nj}}|\mathcal{F}) \to e^{\frac{-\theta^2}{2}}.$$

Combining this with (4.3) yields the result. □

Proposition 4.3. *For the randomly chosen individual at time t, let $\{L_{ti}, \{\eta_{ti}(u) : 0 \leq u \leq L_{ti}\} : 1 \leq i \leq M_t\}$, be the lifetimes and motion processes of its ancestors. Let $Z_{t1} = \frac{1}{\sqrt{M_t}}\sum_{i=1}^{M_t} \eta_{ti}(L_{ti})$, and $\mathcal{L}_t = \sigma\{M_t, L_{ti} : 1 \leq i \leq M_t\}$. Then*

$$E\left(|E(e^{i\theta Z_{t1}}|\mathcal{L}_t) - e^{-\frac{\theta^2\psi}{2}}|\right) \to 0 \tag{4.4}$$

Proof: Fix $\theta \in \mathbb{R}, \epsilon_1 > 0$ and $\epsilon > 0$. Replace the definition of "bad" in (4.1) by

$$|\prod_{i=1}^{k} \phi(\frac{\theta}{\sqrt{k}}, L_{ktji}) - e^{-\frac{\theta^2\psi}{2}}| > \epsilon \tag{4.5}$$

By Proposition 4.2 we have,

$$\lim_{k\to\infty} P(\sup_{j\geq k} |\prod_{i=1}^{j} \phi(\frac{\theta}{\sqrt{j}}, L_i) - e^{-\frac{\theta^2\psi}{2}}| > \epsilon) = 0. \tag{4.6}$$

Using this imitating the proof of Proposition 4.1, (since the details mirror that proof we avoid repeating them here), we obtain that for t sufficiently large

$$P(|\prod_{i=1}^{M_t} \phi(\frac{\theta}{\sqrt{M_t}}, L_{ti}) - e^{-\frac{\theta^2\psi}{2}}| > \epsilon_1) < \epsilon. \tag{4.7}$$

Now for all $\theta \in \mathbb{R}$,

$$E(e^{i\theta Z_{t1}}|\mathcal{L}_t) = \prod_{i=1}^{M_t} \phi(\frac{\theta}{\sqrt{M_t}}, L_{ti}).$$

So,

$$\begin{aligned}
&\limsup_{t\to\infty} E(|E(e^{i\theta \frac{1}{\sqrt{M_t}} \sum_{i=1}^{M_t} \eta_i(L_{ti})}|\mathcal{L}_t) - e^{-\frac{\theta^2\psi}{2}}|) \\
&= \limsup_{t\to\infty} E(|\prod_{i=1}^{M_t} \phi(\frac{\theta}{\sqrt{M_t}}, L_{ti}) - e^{-\frac{\theta^2\psi}{2}}|) \\
&< \epsilon_1 + 2\limsup_{t\to\infty} P(|\prod_{i=1}^{M_t} \phi(\frac{\theta}{\sqrt{M_t}}, L_{ti}) - e^{-\frac{\theta^2\psi}{2}}| > \epsilon_1) \\
&= \epsilon_1 + 2\epsilon.
\end{aligned}$$

Since $\epsilon > 0, \epsilon_1 > 0$ are arbitrary we have the result. □

The above four Propositions will be used in the proof of Theorem 2.1. For the proof of Theorem 2.2 we will need a result on coalescing times of the lines of descent.

Theorem 4.4. *At time $t \geq 0$ choose two individuals alive at time t and trace their lines of descents backwards in time to find the time τ_t of their last common ancestor. Then for $0 < s < \infty$,*

$$\lim_{t\to\infty} P(\tau_t < s) = H(s) \text{exists and } \lim_{s\to\infty} H(s) = 1. \tag{4.8}$$

Proof : For $s \geq 0$ and $t \geq s$ let $\{N_{t-s,i} : t \geq s\}$ for $i = 1, 2, \dots, N_s$ denote the branching processes initiating from the N_s individuals at time s. Then

$$P(\tau_t < s) = E\left(\frac{\sum_{i\neq j=1}^{Z_s} Z_{t-s,i}Z_{t-s,j}}{Z_t(Z_t-1)}\right) \tag{4.9}$$

As $\sigma^2 < \infty$, (see chapter 11 [5]), $e^{-\alpha(t-s)}Z_{t-s,i} \xrightarrow{a.e.} W_i$ as $t \to \infty$ for all $i = 1, \dots, Z_s$ with $W_i < \infty$ a.e and $E(W_i) = 1$. Hence,

$$\frac{\sum_{i\neq j=1}^{Z_s} Z_{t-s,i}Z_{t-s,j}}{Z_t(Z_t-1)} \xrightarrow{a.e.} \frac{\sum_{i\neq j=1}^{Z_s} W_iW_j}{\left(\sum_{i=1}^{Z_s} W_i\right)^2} \tag{4.10}$$

So by the Bounded convergence theorem, $\lim_{t\to\infty} P(\tau_t < s) = E(\phi(Z_s)) \equiv H(s)$ with $\phi : \mathbb{N} \to \mathbb{N}$ given by

$$\phi(k) = E\left(\frac{\sum_{i\neq j=1}^{k} W_iW_j}{\left(\sum_{i=1}^{k} W_i\right)^2}\right) = 1 - E\left(\frac{\sum_{i=1}^{k} W_i^2}{\left(\sum_{i=1}^{k} W_i\right)^2}\right)$$

Since $P(W_i > 0) = 1$, $\exists 0 < \lambda < \infty$ such that $P(\sum_{i=1}^{k} W_i \leq k\lambda) \to 0$ as $k \to \infty$. Also

$$P(\frac{\max_{i=1}^{k} W_i}{k} > \epsilon) = 1 - P(W_1 \leq k\epsilon)^k \leq 1 - (1 - \frac{E(W_1^2)}{k^2\epsilon^2})^k$$

As $E(W_1)^2 < \infty$, we have that $\frac{\max_{i=1}^k W_i}{\sum_{i=1}^k W_i} \xrightarrow{p} 0$. As

$$0 \leq \frac{\sum_{i=1}^k W_i^2}{\sum_{i=1}^k W_i} \leq \frac{\max_{i=1}^k W_i}{\sum_{i=1}^k W_i}$$

we have that $\phi(k) \to 1$ as $k \to \infty$.

□

A similar argument to the above leads to the following corollary.

Corollary 4.5. *Suppose r individuals are chosen at time t by simple random sampling without replacement. Let $\tau_{r,t}$ be the last time they have a common ancestor. Then*

$$\lim_{t\to\infty} P(\tau_{r,t} < s) = H_r(s) \text{ exists and } \lim_{s\to\infty} H_r(s) = 1. \tag{4.11}$$

5. Proof of Theorem 2.1

For the individual chosen, let (a_t, X_t) be the age and position at time t. As in Proposition 4.3, let $\{L_{ti}, \{\eta_{ti}(u), 0 \leq u \leq L_{ti}\} : 1 \leq i \leq M_t\}$, be the lifetimes and the motion processes of the ancestors of this individual and $\{\eta_{t(M_t+1)}(u) : 0 \leq u \leq t - \sum_{i=1}^{M_t} L_{ti}\}$ be the motion this individual. Let $\mathcal{L}_t = \sigma(M_t, L_{ti}, 1 \leq i \leq M_t)$. It is immediate from the construction of the process that:

$$a_t = t - \sum_{i=1}^{M_t} L_{ti},$$

whenever $M_t > 0$ and is equal to $a + t$ otherwise; and that

$$X_t = X_0 + \sum_{i=1}^{M_t} \eta_{ti}(L_{ti}) + \eta_{t(M_t+1)}(a_t).$$

Rearranging the terms, we obtain

$$(a_t, \frac{X_t}{\sqrt{t}}) = (a_t, \sqrt{\frac{1}{\mu}} Z_{t1}) + (0, \left(\sqrt{\frac{M_t}{t}} - \sqrt{\frac{1}{\mu}}\right) Z_{t2}) + (0, \frac{X_0}{\sqrt{t}} + Z_{t2}),$$

where $Z_{t1} = \frac{\sum_{i=1}^{M_t} \eta_{ti}(L_{t_i})}{\sqrt{M_t}}$ *and* $Z_{t2} = \frac{1}{\sqrt{t}} \eta_{t(M_t+1)}(a_t)$. Let $\epsilon > 0$ be given.

$$\begin{aligned} P(|Z_{t2}| > \epsilon) &\leq P(|Z_{t2}| > \epsilon, a_t \leq k) + P(|Z_{t2}| > \epsilon, a_t > k) \\ &\leq P(|Z_{t2}| > \epsilon, a_t \leq k) + P(a_t > k) \\ &\leq \frac{E(|Z_{t2}|^2 I_{a_t \leq k})}{\epsilon^2} + P(a_t > k) \end{aligned}$$

By Proposition 3.1 and the ensuing tightness, for any $\eta > 0$ there is a k_η

$$P(a_t > k) < \frac{\eta}{2}.$$

for all $k \geq k_\eta, t \geq 0$. Next,

$$\begin{aligned} E(|Z_{t2}|^2 I_{a_t \le k_\eta}) &= E(I_{a_t \le k_\eta} E(|Z_{t2}|^2 | \mathcal{L}_t)) \\ &= E(I_{a_t \le k_\eta} \frac{v(a_t)}{t}) \\ &\le \frac{\sup_{u \le k_\eta} v(u)}{t}. \end{aligned}$$

Hence,

$$P(|Z_{t2}| > \epsilon) \quad \le \quad \frac{\sup_{u \le k_\eta} v(u)}{t\epsilon^2} + \frac{\eta}{2}$$

Since $\epsilon > 0$ and $\eta > 0$ are arbitrary this shows that as $t \to \infty$

$$Z_{t2} \xrightarrow{d} 0, \tag{5.1}$$

Now, for $\lambda > 0, \theta \in \mathbb{R}$, as a_t is $\mathcal{L}_t$ measurable we have

$$\begin{aligned} E(e^{-\lambda a_t} e^{-i\frac{\theta}{\sqrt{\mu}} Z_{t1}}) &= E(e^{-\lambda a_t}(E(e^{-i\theta Z_{t1}} | \mathcal{L}_t) - e^{-\frac{\theta^2 \psi}{2\mu}})) + \\ &\quad + ^{-\frac{\theta^2 \psi}{2\mu}} E(e^{-\lambda a_t}) \end{aligned}$$

Proposition 4.2 shows that the first term above converges to zero and using Proposition 3.1 we can conclude that as $t \to \infty$

$$(a_t, \frac{1}{\sqrt{\mu}} Z_{t1}) \xrightarrow{d} (U, V) \tag{5.2}$$

As X_0 is a constant, by Proposition 3.1, (5.2), (5.1) and Slutsky's Theorem, the proof is complete. □

6. Proof of Theorem 2.2

Let $\phi \in C_b(\mathbb{R} \times \mathbb{R}_+)$. Observe that

$$E(\frac{< \tilde{Y}_t, \phi >}{N_t}) \to E(\phi(U, V)), \tag{6.1}$$

from Theorem 2.1 and the bounded convergence theorem. We shall show that

$$E(\frac{< \tilde{Y}_t, \phi >^2}{N_t^2}) - \left(E(\frac{< \tilde{Y}_t, \phi >}{N_t})\right)^2 \to 0 \tag{6.2}$$

converges as $t \to \infty$.

Pick two individuals C_1, C_2 at random (i.e. by simple random sampling without replacement) from those alive at time t. Let the age and position of the two individuals be denoted by $(a_t^i, X_t^i), i = 1, 2$. Let $\tau_t = \tau_{C_1, C_2, t}$ be the birth time of their common ancestor, say D, whose position we denote by $\tilde{X}_{\tau_t}$. Let the net displacement of C_1 and C_2 from D be denoted by $X_{t-\tau_t}^i, i = 1, 2$ respectively. Then $X_t^i = \tilde{X}_{\tau_t} + X_{t-\tau_t}^i, i = 1, 2$.

Next, conditioned on this history up to the birth of $D (\equiv \mathcal{G}_t)$, the random variables $(a_t^i, X_{t-\tau_t}^i), i = 1, 2$ are independent. By Theorem 4.4, $\frac{\tau_t}{t} \xrightarrow{d} 0$. Also by

Theorem 2.1 conditioned on $\mathcal{G}_t$, $\{(a_t^i, \frac{X^i_{t-\tau_t}}{\sqrt{t-\tau_t}}), i = 1,2\}$ converges in distribution to $\{(U_i, V_i), i = 1,2\}$ which are i.i.d. with distribution (U,V) as in Theorem 2.1. Also $\frac{\tilde{X}_{\tau_t}}{\sqrt{\tau_t}}$ conditioned on A_{τ_t} converges in distribution to a random variable S distributed as V.

Combining these one can conclude that $\{(a_t^i, \frac{X_t^i}{\sqrt{t}}), i = 1,2\}$ converges in distribution to $\{(U_i, V_i), i = 1,2\}$. Thus for any $\phi \in C_b(\mathbb{R}_+ \times \mathbb{R})$ we have, by the bounded convergence theorem,

$$\lim_{t\to\infty} E(\prod_{i=1}^{2} \phi(a_t^i, \frac{X_t^i}{\sqrt{t}})|A_t) = E\prod_{i=1}^{2}\phi(U_i, V_i) = (E\phi(U,V))^2$$

Now,

$$\begin{aligned} E\left(\frac{\tilde{Y}_t(\phi)}{N_t}\right)^2 &= E(\frac{(\phi(a_t, \frac{X_t}{\sqrt{t}}))^2}{N_t}|A_t) \\ &\quad + E(\prod_{i=1}^{2}\phi(a_t^i, \frac{X_t^i}{\sqrt{t}})\frac{N_t(N_t-1)}{N_t^2}|A_t) \end{aligned}$$

Using the fact that ϕ is bounded and $e^{-\alpha t}N_t$ converges in distribution (as described earlier) we have

$$\lim_{t\to\infty} E(\frac{\tilde{Y}_t(\phi)}{N_t})^2 \to (E\phi(U,V))^2.$$

This along with (6.1) implies (6.2) and we have proved the result. □

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