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Boundary Harnack Principle for Positive Solutions of Semilinear Elliptic Equations

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Abstract

We study whether the Boundary Harnack Principle holds for positive solutions for the equation $\Delta u = u^p$ in D with boundary value $u = \phi$ on ∂D. When D is a bounded Lipschitz domain and ϕ is continuous we show that the answer is affirmative if $p \geq 1$. Furthermore there is a $p_0 \in (-\infty, 0]$ such that if $p < p_0$ the problem does not even have positive solution. For a bounded $C^{1,1}$ domain we show that $p_0 = -1$.

1 Introduction

The boundary Harnack principle (BHP) is a key tool in obtaining many results in classical potential theory. Suppose u and v are two positive harmonic functions on $D \subset \mathbb{R}^n$ that vanish on a subset Γ of ∂D. The principle (stated precisely in Section 2) says that, under certain regularity assumption on D, u and v tend to zero at Γ at the same rate. Over the past three decades, there has been a lot of research devoted to proving that positive harmonic functions satisfy the principle in a large class of domains [4].

Our discussion starts with another natural question: When do positive solutions to the semilinear elliptic Dirichlet problem,

$$\begin{cases} \Delta u = f(u) & \text{in} \quad D, \\ u = \phi & \text{on} \quad \partial D, \end{cases} \tag{1.1}$$

with $\phi = 0$ on Γ, satisfy the BHP? One quickly observes that, in general, subharmonic functions do not satisfy BHP (see for example Remark 1.4(ii)), but subharmonic functions bounded below by a positive harmonic function do satisfy the principle.

Under certain regularity conditions on $D \subset \mathbb{R}^n$ and ϕ, where $n \geq 3$, the existence of solutions to (1.1) bounded below by a positive harmonic function was established in [7] when f satisfies the condition that $-u \leq f(u) \leq u$ for $|u| < \varepsilon$ for some $\varepsilon > 0$, and in [1] the case when $0 \leq f(u) \leq u^{-\alpha}$ for some $\alpha \in (0, 1)$ was resolved.

The equation $\Delta u = u^p$ in D with $u = \phi$ on ∂D has also been widely studied. For $1 \leq p \leq 2$, it has been studied probabilistically using the exit measure of super-Brownian motion (a measure valued

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branching process), by Dynkin, Le Gall, Kuznetsov, and others [20, 13, 14]. Properties of solutions when $f(u) = u^p, p \geq 1$, with both finite and singular boundary conditions have also been studied by a number of authors using analytic techniques. We briefly review a sample. Bandle and Marcus [2] give results on asymptotic behavior and uniqueness of the "blow-up solution" u which includes the case of the non-linear problem $\Delta u = u^p$ for any $p > 1$. Loewner and Nirenberg [21] had studied the special case of $p = (n+2)/(n-2)$, which has connection with problem of conformal deformation of metrics in Riemannian geometry. Fabbris and Veron [15] have studied the problem of removable singularities. Related work on boundary singularities can be found in [17].

Choi-Mckenna [8] and Lazer-Mckenna [19] have studied a variety of singular boundary value problems of the type $\Delta u = -g(x)u^{-\alpha}$ in D with $u = 0$ on ∂D, where $\alpha > 0$ and g is a non-negative function. From their work, existence of solutions bounded below by a positive harmonic function can be established.

In this paper we consider the equation $\Delta u = u^p$ in D with $u = \phi$ on ∂D. We assume that ϕ vanishes on an open subset Γ of ∂D. Then we study the existence and behavior of positive solutions for various $p \in \mathbb{R}$ and for bounded $C^{1,1}$ and Lipschitz domains. We are concerned with whether such positive solutions enjoy the boundary Harnack principle; that is whether any two positive solutions approach to zero at Γ at the same rate. In the next subsection we state our results precisely.

In the sequel, for two positive functions f and g, the notation $f(x) \approx g(x)$ for $x \in U$ means that there is a positive constant $c \geq 1$ such that $c^{-1}g(x) \leq f(x) \leq c\,g(x)$ for $x \in U$. We use $C(\overline{D})$ and $C_\infty(D)$ to denote the space of continuous functions on $\overline{D}$ and the space of continuous functions in D that vanishes on ∂D, respectively. For two positive constants a and b, $a \wedge b := \min\{a, b\}$. We will use $B(x, r)$ to denote the open ball in $\mathbb{R}^n$ centered at x with radius r.

1.1 Main Results

Let $D \subset \mathbb{R}^n$ be an open connected set, where $n \geq 2$. Let D_1 be an open subset of $\mathbb{R}^n$ such that $\Gamma := D_1 \cap \partial D \neq \emptyset$. Throughout the paper, we denote the Green function of $\frac{1}{2}\Delta$ in D with Dirichlet boundary condition by G_D.

Consider the equation

$$\tfrac{1}{2}\Delta u = u^p \qquad \text{in } D, \tag{1.2}$$

where $p \in \mathbb{R}$.

Definition 1.1. *We say that $u \in C(\overline{D})$ is a mild solution to (1.2) if*

$$u(x) = h(x) - \int_D G_D(x, y)u^p(y)dy, \quad x \in D,$$

where $h \in C(\overline{D})$ is a harmonic function in D satisfying $h|_{\partial D} = u|_{\partial D}$.

We consider the following classes of functions.

- $\mathcal{H}_+ = \mathcal{H}_+(D, \Gamma)$ denotes the class of functions $h \in C(\overline{D})$, that are positive and harmonic in D and vanish on Γ.
- $\mathcal{S}^p_+ = \mathcal{S}^p_+(D, \Gamma)$ denotes the class of mild solutions $u \in C(\overline{D})$ to (1.2) that are positive in D and vanish on Γ.

- $\mathcal{S}^p_H = \mathcal{S}^p_H(D, \Gamma)$ denotes the class of $u \in \mathcal{S}^p_+$ for which $u \approx h$ in D for some $h \in \mathcal{H}_+$.

The reason of introducing the subclass $\mathcal{S}^p_H$ is due to the following BHP for positive harmonic function in bounded Lipschitz domains due to Ancona, B. Dahlberg and J. M. Wu (see [3, p.176] for a proof).

Theorem 1.2. *Suppose that D is a bounded Lipschitz domain in $\mathbb{R}^n$ with $n \geq 2$. Then there is a constant $c \geq 1$ such that for every $z \in \partial D$, $r > 0$ and two positive harmonic functions u and v in $B(z, 2r) \cap D$ that vanish continuously on $\partial D \cap B(z, 2r)$, we have*

$$\frac{u(x)}{v(x)} \leq c\,\frac{u(y)}{v(y)} \qquad \text{for every } x, y \in D \cap B(z, r). \tag{1.3}$$

In view of the above theorem, we see that when D is a bounded Lipschitz domain in $\mathbb{R}^n$, functions in $\mathcal{S}^p_H$ enjoy the boundary Harnack principle (1.3). So the purpose of this paper is to investigate how large the class $\mathcal{S}^p_H$ is, in particular, when is $\mathcal{S}^p_H = \mathcal{S}^p_+$ or $\mathcal{S}^p_H = \emptyset$.

We are now ready to state the main results of this paper. We begin with a result when D is a bounded $C^{1,1}$-domain. Recall that a bounded domain $D \subset \mathbb{R}^n$ is said to be $C^{1,1}$-smooth if for every point $z \in \partial D$, there is $r > 0$ such that $D \cap B(z, r)$ is the region in $B(z, r)$, under some local coordinate system centered at z, that lies above the graph of a function whose first derivatives are Lipschitz continuous.

Theorem 1.3. *Assume that $n \geq 2$ and D is a bounded $C^{1,1}$-domain in $\mathbb{R}^n$.*

(i) *For $p \geq 1$, $\emptyset \neq \mathcal{S}^p_H = \mathcal{S}^p_+$;*

(ii) *For $-1 < p < 1$, $\emptyset \neq \mathcal{S}^p_H \subset \mathcal{S}^p_+$;*

(iii) *For $p \leq -1$, $\mathcal{S}^p_H = \mathcal{S}^p_+ = \emptyset$.*

Remark 1.4. (i) The proof of Theorem 1.3 is based on s two-sided Green function estimate for Brownian motion in D; see Proposition 2.5 below. Theorem 1.3 in fact holds not only for Laplacian but also for a large class of uniformly elliptic operators in a bounded C^2-domain D. Let $\mathcal{L} = \frac{1}{2}\sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right)$, where a_{ij} has continuous derivatives on $\overline{D}$ (i.e. it is $C^1(\overline{D})$), and $A(x) = (a_{ij}(x))$ is a symmetric matrix-valued function that is uniformly bounded and elliptic. Then by Theorem 3.3 of Grüter and Widman [18], the Green function $G^{\mathcal{L}}_D(x, y)$ of $\mathcal{L}$ in D satisfies the following estimate

$$G^{\mathcal{L}}_D(x, y) \leq c\delta_D(x)\,|x-y|^{1-n}, \quad x, y \in D,$$

where $c > 0$ and $\delta_D(x)$ is the Euclidean distance between x and D^c. On the other hand, we know from Lemma 4.6.1 and Theorem 4.6.11 of Davies [10] that $G^{\mathcal{L}}_D(x, y) \geq c\,\delta_D(x)\delta_D(y)$. For $r > 0$, define $D_r = \{x \in D : \delta_D(x) < r\}$. Thus for fixed $y_0 \in D$ and $r < \delta_D(y_0)$,

$$G^{\mathcal{L}}_D(x, y_0) \approx \delta_D(x) \quad \text{for } x \in D_r. \tag{1.4}$$

It is well known that Harnack and boundary Harnack principles hold for $\mathcal{L}$ and the Green function

$$G^{\mathcal{L}}_D(x, y) \approx \begin{cases} |x-y|^{2-n} & \text{when } n \geq 3, \\ \log(1 + |x-y|^{-2}) & \text{when } n = 2. \end{cases}$$

for $x, y \in D \setminus D_r$ with $r > 0$. Hence by a similar argument as that in Bogdan [6], we conclude that the estimate (2.6)-(2.7) hold for the Green function $G^{\mathcal{L}}_D$ of $\mathcal{L}$ in D . Then by imitating the argument as in the proof of Theorem 1.3 we can obtain the result for $\mathcal{L}$ as well.

(ii) Now suppose $D = [0,1]^n$ and $-1 < p < 1$. Let $u(x) = c_p(x_1)^{\frac{2}{1-p}}$, where $c_p = \left(\frac{2(1+p)}{(1-p)^2}\right)^{\frac{2}{1-p}}$ and x_1 is the first coordinate of $x = (x_1, \ldots, x_n)$. Now $u \in \mathcal{S}_+^p$ clearly and due to the one dimensional nature of this example one can establish $u \notin \mathcal{S}_H^p$. This suggests that Theorem 1.3(ii) could be replaced by: $-1 < p < 1$, $\emptyset \neq \mathcal{S}_H^p \subsetneq \mathcal{S}_+^p$. However, we were not able to generalize the above example to general bounded $C^{1,1}$- domains D. $\square$

We now consider the case when D is a bounded Lipschitz domain in $\mathbb{R}^n$. Let X be Brownian motion in $\mathbb{R}^n$ and $\tau_D := \inf\{t > 0 : X_t \notin D\}$ the first exit time of X from D. It is well-known that the Green function $G_D(x,y)$ for the part process X^D of X killed upon leaving domain D (or equivalently, for $\frac{1}{2}\Delta$ in D with Dirichlet boundary condition) exists and is continuous on $D \times D$ except along the diagonal. Fix some $x_0 \in D$ and set $\varphi(x) := G_D(x, x_0) \wedge 1$.

Theorem 1.5. *Let D be a bounded Lipschitz domain satisfying*

$$\mathbb{E}_x[\tau_D] \leq c\,\varphi(x) \qquad \text{for every } x \in D. \tag{1.5}$$

Then there exists $p_0 \in (-\infty, 0]$ such that the following holds.

(i) *If $p \geq 1$ then $\mathcal{S}_H^p \neq \emptyset$ and $\mathcal{S}_H^p = \mathcal{S}_+^p$,*

(ii) *$\mathcal{S}_H^p \neq \emptyset$ for $p > p_0$,*

(iii) *$\mathcal{S}_H^p = \mathcal{S}_+^p = \emptyset$ for $p < p_0$.*

We conjecture that for $p \in (p_0, 1)$, $\emptyset \neq \mathcal{S}_H^p \subsetneq \mathcal{S}_+^p$. Next, we study when condition (1.5) holds. For $\theta \in (0, \pi)$, let $\Gamma(\theta)$ be the truncated circular cone in $\mathbb{R}^n$ with angle θ defined by

$$\Gamma(\theta) := \{x \in \mathbb{R}^n : |x| < 1 \text{ and } x \cdot e_1 > |x| \cos\theta\}, \tag{1.6}$$

where $e_1 := (1, 0, \cdots, 0) \in \mathbb{R}^n$. We say that a bounded Lipschitz domain D satisfies the interior cone condition with common angle θ, if there is some $a > 0$ such that for every point $x \in \partial D$, there is a cone $\Gamma \subset D$ with vertex at x that is conjugate to $a\Gamma(\theta)$.

The result below shows that condition (1.5) holds for bounded Lipschitz domains in $\mathbb{R}^n$ satisfying the interior cone condition with common angle strictly larger than $\cos^{-1}(1/\sqrt{n})$. In particular, it is satisfied for bounded Lipschitz domains in $\mathbb{R}^n$ whose Lipschitz constant is strictly less than $1/\sqrt{n-1}$.

Theorem 1.6. *Let D be a bounded Lipschitz domain in $\mathbb{R}^n$ with $n \geq 2$ satisfying an interior cone condition with common angle $\theta \in \left(\cos^{-1}(1/\sqrt{n}), \pi\right)$. Then there is a constant $c \geq 1$ such that*

$$c^{-1}\varphi(x) \leq \mathbb{E}_x[\tau_D] \leq c\,\varphi(x) \qquad \text{for every } x \in D.$$

While the lower bound for $\mathbb{E}_x[\tau_D]$ in fact holds for every bounded Lipschitz domain, we show in Theorem 3.3 that the condition on θ is sharp for the upper bound (hence for (1.5)). Namely, for domains that satisfy the interior cone condition with common angle less than $\cos^{-1}(1/\sqrt{n})$, condition (1.5) does not, in general, hold.

Remark 1.7. (i) Theorem 1.6 has also been obtained independently by M. Bieniek and K. Burdzy [5] using a different method.

(ii) We have stated all our results for solutions of the equation (1.2). However if we assume that $f(u) \approx u^p$ then the proofs of our main results can be suitably modified to yield the same quantitative behavior for solutions of the equation (1.1). $\square$

The rest of the paper is organised as follows. In the next section we present some technical lemmas and estimates on the Green function which are required for the proof of Theorem 1.3 and Theorem 1.5. In Section 3, we prove Theorem 1.6 and show that the condition on the common angle, so that the upper bound in this statement holds, is sharp (Theorem 3.3). Finally in Section 4 we prove Theorem 1.5 and in Section 5 we prove Theorem 1.3.

2 Preliminaries

2.1 Auxiliary estimates

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}_x, x \in \mathbb{R}^n)$ be a complete filtered probability space on which X is a Brownian motion on $\mathbb{R}^n$, starting from x. For h harmonic in D, we denote by $\mathbb{P}_x^h$ the h-transform of $\mathbb{P}_x$ under h. $\mathbb{E}_x$ ($\mathbb{E}_x^h$) denotes expectation with respect to $\mathbb{P}_x$ (respectively, $\mathbb{P}_x^h$). For any set $A \subset \mathbb{R}^n$ we denote

$$\tau_A = \inf\{t : X_t \notin A\}.$$

The following is a well known result. We provide a proof here for reader's convenience. This result in fact holds for more general potential $q \geq 0$, for example, q is in some Kato class.

Lemma 2.1. *Assume that every point of ∂D is regular with respect to D^c. Let $h \in \mathcal{H}_+$ and $q \geq 0$. The function v given by*

$$v(x) = \mathbb{E}_x\left[h(X_{\tau_D})e^{-\int_0^{\tau_D} q(X_s)ds}\right], \quad x \in D \tag{2.1}$$

satisfies

$$v(x) = h(x) - \int_D G_D(x,y)q(y)v(y)dy \quad x \in D. \tag{2.2}$$

The converse is true if q is bounded.

Proof. The proof is along the lines of [7]. Suppose that v is given by (2.1). Then for $x \in D$, by the Markov property of X,

$$\begin{aligned}
v(x) &= \mathbb{E}_x\left[h(X_{\tau_D})e^{-\int_0^{\tau_D} q(X_s)ds}\right] \\
&= h(x) + \mathbb{E}_x\left[h(X_{\tau_D})\left(e^{-\int_0^{\tau_D} q(X_s)ds} - 1\right)\right] \\
&= h(x) - \mathbb{E}_x\left[h(X_{\tau_D})\int_0^{\tau_D} q(X_t)e^{-\int_t^{\tau_D} q(X_s)ds}dt\right] \\
&= h(x) - \mathbb{E}_x\left[\int_0^{\tau_D} q(X_t)\,\mathbb{E}_{X_t}\left[h(X_{\tau_D})e^{-\int_0^{\tau_D} q(X_s)ds}\right]dt\right] \\
&= h(x) - \mathbb{E}_x\left[\int_0^{\tau_D} q(X_t)v(X_t)dt\right] \\
&= h(x) - \int_D G_D(x,y)q(y)v(y)dy
\end{aligned}$$

For the converse, assume $q \geq 0$ is bounded. Suppose now that v satisfies (2.1), then v solves the equation:

$$\frac{1}{2}\Delta v - qv = 0 \quad \text{in } D \quad \text{with} \quad v|_{\partial D} = h|_{\partial D}. \tag{2.3}$$

As $q \geq 0$ is bounded, it is well known that solutions to equation (2.3) are continuous on $\overline{D}$ and C^1 in D (see, e.g., [16]). Furthermore solution of (2.3) enjoys the maximum principle and therefore is unique. This proves the Lemma. □

Lemma 2.2. *There exists a constant $\gamma = \gamma(D) > 0$ such that for every $h \in \mathcal{H}_+$ and $p > 1 - 2\gamma$, we have*

$$\sup_{x\in D} \mathbb{E}_x^h \left[\int_0^{\tau_D} h^{p-1}(X_s)ds\right] \leq C_1 < \infty,$$

where C_1 depends only on h, p and D.

Proof. We will be mainly using the notation in [3], page 200-201. Let $l_k = \{x : h(x) = 2^k\}$ for any $k \in \mathbb{Z}$. Note that there exists k_0 such that $l_k = \emptyset$ for $k \geq k_0$. Define $S_{-1} = 0$ and let $S_0 = \tau_D \wedge \inf\{t : X_t \in \cup l_k\}$. For $i \geq 1$, let $S_i = \tau_D \wedge \inf\{t > S_{i-1} : X_t \in \cup l_k \backslash l_{W_{i-1}}\}$, where W_{i-1} is the k such that $X_{S_{i-1}} \in l_k$. Let $v_k = \sup_{x\in l_k} \mathbb{E}_x^h[S_1]$. From [3], one has that:
(a) there exists a constant $\gamma(D) > 0$ such that for any k, $v_k \leq c_0 2^{2k\gamma(D)}$
(b) there exists a constant c_1 such that $\sum_{i=0}^{\infty} \mathbb{P}_x^h(W_i = k) \leq c_1$ for all $x \in D$. Hence

$$\begin{aligned}
\mathbb{E}_x^h \left[\int_0^{\tau_D} h^{p-1}(X_s)ds\right] &= \sum_{i=0}^{\infty} \mathbb{E}_x^h \left[\int_{S_{i-1}}^{S_i} h^{p-1}(X_s)ds\right] \\
&\leq c_2 \sum_{i=0}^{\infty} \mathbb{E}_x^h \left[2^{W_{i-1}(p-1)}(S_i - S_{i-1})\right] \\
&= c \sum_{i=0}^{\infty} \mathbb{E}_x^h \left[2^{W_{i-1}(p-1)} \mathbb{E}_{X_{S_{i-1}}}^h (S_1)\right] \\
&\leq c \sum_{i=0}^{\infty} \mathbb{E}_x^h \left[2^{W_{i-1}(p-1)} v_{W_{i-1}}\right] \\
&= c \sum_{i=0}^{\infty} \sum_{k=-\infty}^{k_0} \mathbb{E}_x^h \left[2^{W_{i-1}(p-1)} v_{W_{i-1}} 1(W_{i-1} = k)\right] \\
&= c \sum_{k=-\infty}^{k_0} v_k 2^{k(p-1)} \mathbb{E}_x^h \left[\sum_{i=0}^{\infty} 1(W_{i-1} = k)\right] \\
&\leq c \sum_{k=-\infty}^{k_0} 2^{2k\gamma(D)} 2^{k(p-1)}
\end{aligned} \tag{2.4}$$

Hence if $p > 1 - 2\gamma(D)$ then $\mathbb{E}_x^h \left[\int_0^{\tau_D} h^{p-1}(X_s)\right] < C_1$ for all $x \in D$. □

Lemma 2.3. *Let $h \in \mathcal{H}_+$. Suppose that p is a real number such that*

$$\sup_{x\in D} \int_D G_D(x,y) h(y)^{p(1+\varepsilon)} dy < \infty$$

for some $\varepsilon > 0$. Then

(i) *The family of functions $\{G_D(x,\cdot)h^p(\cdot) : x \in D\}$ is uniformly integrable over D.*

(ii) *Let* $B_{h,p} = \{g : D \to \mathbb{R} : g$ *is Borel measurable and* $|g(x)| \le h^p(x)$ *for all* $x \in D\}$. *The family of functions* $\{\int G_D(\cdot, y)g(y)dy : g \in B_{h,p}\}$ *is uniformly bounded and equicontinuous in* $C_\infty(D)$, *and, consequently, it is relatively compact in* $C_\infty(D)$.

The above assertions hold especially when $p > 1 - 2\gamma(D)$, *where* $\gamma(D)$ *is the constant in Lemma 2.2*

Proof. Let $q > 1$ be such that $q^{-1} + (1+\varepsilon)^{-1} = 1$. For any Borel measurable set A, by Hölder inequality,

$$\int_A G(x,y)h(y)^p\,dy \le \left(\int_D G(x,y)^{p(1+\varepsilon)}\,dy\right)^{1/(1+\varepsilon)} \left(\int_A G(x,y)\,dy\right)^{1/q}.$$

Since D is a bounded, it follows that $\sup_{x\in D}\int_D G(x,y)h(y)^p\,dy < \infty$ and

$$\lim_{\delta\to 0}\sup_{A:m(A)<\delta}\sup_{x\in D}\int_A G(x,y)h(y)^p\,dy = 0,$$

where m denotes the Lebesgue on $\mathbb{R}^d$. Therefore the family of functions $\{G(x,\cdot)h(\cdot)^p,\ x \in D\}$ is uniformly integrable over D. This in particularly implies that, due to the continuity of $x \to G(x,y)$ on $\overline{D}\setminus\{y\}$, function $x \to \int_D G_D(x,y)h(y)^p\,dy$ is continuous on $\overline{D}$ and vanishes on ∂D. On the other hand, by using triangle inequality, the family of functions $\{|G_D(x,\cdot) - G_D(y,\cdot)|\,h(\cdot)^p : x, y \in D\}$ is uniformly integrable on D. Therefore function $(x,y) \to \int_D |G_D(x,z) - G_D(y,z)|\,h(z)^p\,dz$ is continuous on $\overline{D}\times\overline{D}$.

For each $g \in B_{h,p}$, as $|g| \le h^p$, the family of functions $B_{h,p}$ is continuous in D, uniformly bounded, and converge uniformly to zero as $x \to \partial D$. For any x, y in D and $g \in B_{h,p}$,

$$\left|\int_D G_D(x,z)g(z)dy - \int_D G_D(y,z)g(z)dz\right| \le \int_D |\ G_D(x,z) - G_D(y,z)\ |\ \ h(z)^p dz. \tag{2.5}$$

Therefore the family of functions in the statement of the lemma is equi-continuous in D.

When $p > 1 - 2\gamma(D)$, one can always find an $\varepsilon > 0$ such that $p(1+\varepsilon) > 1 - 2\gamma$. Thus by Lemma 2.2

$$\sup_{x\in D}\int_D G(x,y)\,h(y)^{p(1+\varepsilon)}\,dy = \sup_{x\in D} h(x)\,\mathbb{E}^h_x\left[\int_0^{\tau_D} h(X_s)^{p(1+\varepsilon)-1}ds\right] < \infty$$

and so the hypothesis of the Lemma is satisfied and the result holds. □

2.2 Green function estimates

We begin with an estimate for Green function in $C^{1,1}$ domains.

Lemma 2.4. *Suppose that* D *is a bounded* $C^{1,1}$ *domain. Then*

$$G_D(x,y) \approx \min\left\{\frac{1}{|x-y|^{n-2}}, \frac{\delta_D(x)\delta_D(y)}{|x-y|^n}\right\} \qquad \text{when } n \ge 3 \tag{2.6}$$

$$G_D(x,y) \approx \log\left(1 + \frac{\delta_D(x)\delta_D(y)}{|x-y|^2}\right) \qquad \text{when } n = 2. \tag{2.7}$$

Proof. Estimate (2.6) is due to K.-O. Widman and Z. Zhao (see [22]). Estimate (2.7) is established as Theorem 6.23 in [9] for bounded C^2-smooth domain D. However the proof carries over to bounded $C^{1,1}$-domains. □

For the rest of this subsection we will assume that D is a bounded Lipschitz domain in $\mathbb{R}^n$ with $n \geq 2$. Recall that we defined $\varphi(x) := G_D(x, x_0) \wedge 1$, where $x_0 \in D$ is fixed.

Let $r(x,y) := \delta_D(x) \vee \delta_D(y) \vee |x-y|$ and (r_0, λ) be the Lipschitz characteristics of D. For $x, y \in D$, we let $A_{x,y} = x_0$ if $r(x,y) \geq r_0/32$ and when $r := r(x,y) < r_0/32$, $A_{x,y}$ is any point in D such that

$$B(A_{x,y}, \kappa r) \subset D \cap B(x, 3r) \cap B(y, 3r),$$

with $\kappa := \frac{1}{2\sqrt{1+\lambda^2}}$.

Proposition 2.5. *Let $D \subset \mathbb{R}^n$ be a bounded Lipschitz domain with $n \geq 2$. Then there is a constant $c > 1$ such that on $D \times D \setminus d$,*

$$G_D(x,y) \approx \frac{\varphi(x)\varphi(y)}{\varphi(A_{x,y})^2} \frac{1}{|x-y|^{n-2}} \qquad \textit{when } n \geq 3, \tag{2.8}$$

$$c^{-1}\frac{\varphi(x)\varphi(y)}{\varphi(A_{x,y})^2} \leq G_D(x,y) \leq c\frac{\varphi(x)\varphi(y)}{\varphi(A_{x,y})^2} \log\left(1 + \frac{1}{|x-y|^2}\right) \qquad \textit{when } n = 2. \tag{2.9}$$

Proof. When $n = 3$, (2.8) is proved in [6] as Theorem 2. So it remains to show (2.9) when $n = 2$.

Since D is bounded, there is a ball $B \supset D$. It follows from [9, Lemma 6.19] that for $x, y \in D$,

$$G_D(x,y) \leq G_B(x,y) \leq \frac{1}{2\pi} \ln\left(1 + 4\frac{\delta_B(x)\delta_B(y)}{|x-y|^2}\right) \leq c \ln\left(1 + |x-y|^{-2}\right).$$

On the other hand, by [9, Lemma 6.7], for every c_1, there is a constant $c_2 > 0$ such that

$$G_D(x,y) \geq c_2 \qquad \text{for } x, y \in D \text{ with } |x-y| \leq c_1 \min\{\delta_D(x), \delta_D(y)\}.$$

From these, inequality (2.9) can be proved in the same way as the proofs for [6, Proposition 6 and Theorem 2]. □

3 Boundary decay rate for Lipschitz Domains

In this section we prove Theorem 1.6, and the sharpness of the requirement on the common angle in its statement. To prove Theorem 1.6 we will need the following result.

Proposition 3.1. *Let $D \subset \mathbb{R}^n$ be a bounded Lipschitz domain with $n \geq 2$ and φ_1 be the first positive eigenfunction for the Dirichlet Laplacian in D normalized to have $\int_D \varphi_1(x)^2 dx = 1$. If D satisfies interior cone condition with common angle $\theta \in \left(\cos^{-1}(1/\sqrt{n}), \pi\right)$, there are positive constants $\varepsilon > 0$ and $a > 0$ such that*

$$\varphi_1(x) \geq a\, \delta_D(x)^{2-\varepsilon} \qquad \textit{for every } x \in D,$$

where $\delta_D(x)$ denotes the Euclidean distance between x and D^c.

Proof. By Theorem 4.6.8 of [10] and its proof, there is some constant $a > 0$ such that

$$\varphi_1(x) \geq a\, \delta_D(x)^{\alpha} \qquad \text{for every } x \in D,$$

where $\alpha > 0$ is the constant determined by

$$\alpha(\alpha + n - 2) = \lambda_1(\theta). \tag{3.1}$$

Here $\lambda_1(\theta)$ is the first eigenvalue for the Dirichlet Beltrami-Laplace operator in the unit spherical cap determined by $\overline{\Gamma(\theta)} \cap \{x \in \mathbb{R}^n : |x| = 1\}$. The first eigenvalue $\lambda_1(\theta)$ can be determined in terms of hypergeometric function and so does α. Recall the hypergeometric function

$$F(\alpha, \beta, \gamma, z) := 1 + \frac{\alpha\beta}{\gamma}\frac{z}{1!} + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)}\frac{z^2}{2!} + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{\gamma(\gamma+1)(\gamma+2)}\frac{z^3}{3!} + \cdots$$

Let $\theta(p,n)$ be the smallest positive zero of $F\left(-p, p+n-2, \frac{n-1}{2}, \frac{1-\cos\theta}{2}\right)$. It is known that $p \mapsto \theta(p,n)$ is continuous and strictly decreasing with $\theta(1,n) = \pi/2$ (cf. p.62 of [12]). Let $\theta \mapsto p(\theta, n)$ be the inverse function of $p \mapsto \theta(p,n)$. We know from [12, p.59 and p.63] that α in (3.1) is equal to

$$\alpha = p(\theta, n). \tag{3.2}$$

Note that

$$F\left(-2, n, \frac{n-1}{2}, z\right) = 1 - \frac{4n}{n-1}z + \frac{4n}{n-1}z^2,$$

which has roots $\frac{n-\sqrt{n}}{2n}$ and $\frac{n+\sqrt{n}}{2n}$. Set $z = \frac{1-\cos\theta}{2}$. The corresponding smallest positive root for θ is $\cos\theta_0 = \frac{1}{\sqrt{n}}$ or $\theta_0 = \cos^{-1}(1/\sqrt{n})$. In other words, we have for $n \geq 2$,

$$\theta(2,n) = \cos^{-1}(1/\sqrt{n}), \qquad \text{or equivalently,} \qquad p(\cos^{-1}(1/\sqrt{n}),\, n) = 2. \tag{3.3}$$

As $\theta \mapsto p(\theta, n)$ is strictly decreasing, we have $p(\theta, n) < 2$ for every $\theta > \cos^{-1}(1/\sqrt{n})$. This proves the proposition. $\square$

Recall that $\varphi(x) := G_D(x, x_0) \wedge 1$. The following lemma, which is known to the experts, gives the relation between φ and φ_1.

Lemma 3.2. *Suppose that D is a bounded Lipschitz domain in $\mathbb{R}^n$ with $n \geq 2$. There is a constant $c \geq 1$ such that*

$$c^{-1}\varphi_1(x) \leq \varphi(x) \leq c\,\varphi_1(x) \qquad \text{for } x \in D.$$

Proof. It is well-known that D is intrinsic ultracontactive (cf. [10]) and so for every $t > 0$, there is a constant $c_t \geq 1$ such that

$$c_t^{-1}\,\varphi_1(x)\varphi_1(y) \leq p^D(t,x,y) \leq c_t\,\varphi_1(x)\varphi_1(y) \qquad \text{for every } x, y \in D. \tag{3.4}$$

For the definition of intrinsic ultracontractivity and its equivalent characterizations, see Davies and Simon [11]. By (3.4), we have

$$\varphi(x) \geq \int_D p^D(1,x,y)\varphi(y)dy \geq c\left(\int_D \varphi(y)\varphi_1(y)dy\right)\varphi_1(x).$$

Thus there is a constant $c_1 > 0$ such that

$$\varphi(x) \geq c_1\varphi_1(x) \qquad \text{for every } x \in D. \tag{3.5}$$

On the other hand, let $K := \{x \in D : G_D(x, x_0) \geq 1\}$, which is a compact subset of D. Observe that both φ and φ_1 are continuous and strictly positive in D. So $a := \sup_{x\in K}\frac{\varphi(x)}{\varphi_1(x)}$ is a positive and finite number. Since φ is harmonic in $D \setminus K$ and $\Delta\varphi_1(x) = -\lambda_1\varphi_1(x)$ with $\lambda_1 > 0$, we have $\Delta(\varphi - a^{-1}\varphi_1) \geq 0$

on $D \setminus K$. As both φ and φ_1 vanish continuously on ∂D and $\varphi(x) - a\varphi_1(x) \le 0$ on K, we have by the maximal principle for harmonic functions that

$$\varphi(x) \le a\, \varphi_1(x) \qquad \text{for every } x \in D \setminus K. \tag{3.6}$$

This proves the Lemma. □

Proof of Theorem 1.6 Let $K = \{x \in D : G_D(x, x_0) \ge 1\}$, which is a compact subset of D. By taking $r_0 > 0$, we may and do assume that the Euclidean distance between K and D^c is at least r_0. Since φ is a positive harmonic function in $D \setminus K$ that vanishes on ∂D, by Carleson's estimate (see, e.g., Theorem III.1.8 of [3]), there is a universal constant $c_1 = c_1(D, K) > 0$ such that $\varphi(y) \le c_1\, \varphi(A_{x,y})$ whenever $r(x, y) < r_0/32$. Note also that φ is bounded on D and that, by Proposition 3.1 and (3.5), $\varphi(x) \ge c\, \delta_D(x)^{2-\varepsilon}$.

When $n \ge 3$, we have by (2.8),

$$G_D(x, y) \le c\, \frac{\varphi(x)\varphi(y)}{\varphi(A_{x,y})^2} \frac{1}{|x - y|^{n-2}}, \qquad x, y \in D,$$

Thus we have

$$\begin{aligned}
\frac{G_D 1(x)}{\varphi(x)} &\le c \int_D \frac{\varphi(y)}{\varphi(A_{x,y})} \frac{1}{\varphi(A_{x,y})|x - y|^{n-2}} dy \\
&= c \int_{\{y \in D : r(x,y) \ge r_0/32\}} \frac{\varphi(y)}{\varphi(A_{x,y})^2} \frac{1}{|x - y|^{n-2}} dy \\
&\quad + c \int_{\{y \in D : r(x,y) < r_0/32\}} \frac{\varphi(y)}{\varphi(A_{x,y})} \frac{1}{\varphi(A_{x,y})\, |x - y|^{n-2}} dy \\
&\le c \int_D \frac{1}{|x - y|^{n-2}} dy + c \int_{\{y \in D : r(x,y) < r_0/32\}} \frac{1}{(r(x, y)^{2-\varepsilon}\, |x - y|^{n-2}} dy \\
&\le c + c \int_{\{y \in D : r(x,y) < r_0/32\}} \frac{1}{|x - y|^{n-\varepsilon}} dy \\
&\le c < \infty.
\end{aligned}$$

When $n = 2$, we have by (2.9),

$$\begin{aligned}
\frac{G_D 1(x)}{\varphi(x)} &\le c \int_D \frac{\varphi(y)}{\varphi(A_{x,y})} \frac{1}{\varphi(A_{x,y})} \log(1 + |x - y|^{-2}) dy \\
&= c \int_{\{y \in D : r(x,y) \ge r_0/32\}} \frac{\varphi(y)}{\varphi(A_{x,y})^2} \frac{1}{|x - y|^{\varepsilon/2}} dy \\
&\quad + c \int_{\{y \in D : r(x,y) < r_0/32\}} \frac{\varphi(y)}{\varphi(A_{x,y})} \frac{1}{\varphi(A_{x,y})\, |x - y|^{\varepsilon/2}} dy \\
&\le c \int_D \frac{1}{|x - y|^{\varepsilon/2}} dy + c \int_{\{y \in D : r(x,y) < r_0/32\}} \frac{1}{(r(x, y)^{2-\varepsilon}\, |x - y|^{\varepsilon/2}} dy \\
&\le c + c \int_{\{y \in D : r(x,y) < r_0/32\}} \frac{1}{|x - y|^{2-(\varepsilon/2)}} dy \\
&\le c < \infty.
\end{aligned}$$

For the lower bound, note that both $\varphi(x)$ and $x \mapsto \int_K G_D(x, y) dy$ are positive harmonic functions in $D \setminus K$ vanishing continuously on ∂D, and that $G_D 1$ is bounded on D. By the boundary Harnack inequality

(Theorem 1.2), there is a constant $c_0 \in (0,1)$ so that

$$\mathbb{E}_x[\tau_{DD}] \geq \int_K G_D(x,y)dy \geq c_0\varphi(x) \qquad \text{for every } x \in D \setminus K.$$

By taking $c_0 > 0$ smaller if necessary, we conclude that $\mathbb{E}_x[\tau_{DD}] \geq c_0\varphi(x)$ for every $x \in D$. The theorem is now proved. □

The next result says that Theorem 1.6 is sharp.

Theorem 3.3. *Let $D = \Gamma(\theta)$ be a truncated circular cone in $\mathbb{R}^n$ with common angle $\theta < \cos^{-1}(1/\sqrt{n})$ and $n \geq 2$, defined by (1.6). Then there are constants $c > 0$ and $\alpha > 2$ such that*

$$G_D 1(x) \geq c\,\delta_D(x)^{2-\alpha}\varphi(x) \qquad \text{for every } x = (x_1, 0, \cdots, 0) \text{ with } 0 < x_1 < 1/2. \tag{3.7}$$

Proof It is known (see, e.g., two lines above (4.6.6) on page 129 of [10]) that $\varphi_1(x)$ decays at rate $\delta_D(x)^\alpha$ as $x \to 0$ along the axis of the cone $\Gamma(\theta)$, where α is given by (3.1). We see from (3.2)-(3.3) that $\alpha > 2$ when $\theta < \cos^{-1}(1/\sqrt{n})$. Clearly there is $\varepsilon \in (0, 1/2)$ such that $B(x, \delta_D(x)) \subset D \setminus K$ for every $x = (x_1, 0, \cdots, 0)$ with $0 < x_1 < \varepsilon$. This together with (3.6) implies in particular that there is a constant $c > 0$ such that

$$\varphi(x) \leq a\,\varphi_1(x) \leq c\,\delta_D(x)^\alpha \qquad \text{for } x = (x_1, 0, \cdots, 0) \text{ with } 0 < x_1 < \varepsilon.$$

By Harnack inequality,

$$\varphi(y) \leq c\,\varphi(x) \leq c\delta_D(x)^\alpha \leq c\delta_D(y)^\alpha \tag{3.8}$$

for every $y \in B(x, \delta_D(x)/2)$ and every $x = (x_1, 0, \cdots, 0)$ with $0 < x_1 < \varepsilon$. By Proposition 2.5,

$$G_D(x,y) \geq c\,\frac{\varphi(x)\varphi(y)}{\varphi(A_{x,y})^2}\frac{1}{|x-y|^{n-2}}, \qquad x, y \in D,$$

where $A_{x,y}$ is as given in the proof of Theorem 1.6. Here for the case of $n = 2$, we use the convention that $0^0 = 1$. Let $x = (x_1, 0, \cdots, 0)$ with $0 < x_1 < \varepsilon$. For $y \in B(x, \delta_D(x)/4) \setminus B(x, \delta_D(x)/6)$, we can take $A_{x,y} = y$. Note that in this case, $\delta_D(y) \leq 5\delta_D(x)/4 \leq 15|x-y|$. We therefore have

$$\begin{aligned} G_D 1(x) &\geq \int_{B(x,\,\delta_D(x)/2)\setminus B(x,\,\delta_D(x)/3)} G_D(x,y)dy \\ &\geq c\,\varphi(x) \int_{B(x,\,\delta_D(x)/2)\setminus B(x,\,\delta_D(x)/3)} \frac{1}{\varphi(y)|x-y|^{n-2}}dy \\ &\geq c\,\varphi(x) \int_{B(x,\,\delta_D(x)/2)\setminus B(x,\,\delta_D(x)/3)} \frac{1}{|x-y|^\alpha\,|x-y|^{n-2}}dy \\ &\geq c\,\varphi(x)\delta_D(x)^{2-\alpha}. \end{aligned}$$

This establishes the theorem. □

Remark 3.4. Note that the circular cone $\Gamma(\theta)$ with angle $\theta = \cos^{-1}(1/\sqrt{n})$ has Lipschitz constant $1/\sqrt{n-1}$ at its vertex. So if D is a bounded Lipschitz domain in $\mathbb{R}^n$ with Lipschitz constant strictly less than $1/\sqrt{n-1}$, then D satisfies interior cone condition with common angle $\theta \in \left(\cos^{-1}(1/\sqrt{n}), \pi\right)$. We point out that this is only a sufficient condition. The aforementioned interior cone condition can be satisfied in some bounded Lipschitz domains with Lipschitz constant larger than $1/\sqrt{n-1}$. A smooth domain with an inward sharp cone is such an example. □

4 Proof of Theorem 1.5

(i) Fix $p \geq 1$ and $h \in \mathcal{H}_+$. Let $\gamma = \gamma(D)$ be the constant obtained in Lemma 2.2 for this h. Define $\Lambda = \left\{u \in C(\overline{D}) : e^{-\gamma} h \leq u \leq h\right\}$. Clearly, Λ is a closed non-empty convex sub-set of $C(\overline{D})$. Let $G_D(\cdot,\cdot)$ be the Green function of the domain D. Define $T : \Lambda \to C(\overline{D})$ as

$$T(u)(x) = \mathbb{E}_x \left[h(X_{\tau_D}) \exp\left(- \int_0^{\tau_D} u^{p-1}(X_s) ds \right) \right]$$

¿From Lemma 2.1 one can conclude that $T(u)(x) = h(x) - \int G_D(x,y) u^{p-1}(y) T(u)(y) dy$. Now for $u \in \Lambda$, $T(u) \leq h$ and hence $u^{p-1} T(u) \in B_{h,p}$. By Lemma 2.3, we conclude that

$$\left\{ \int G_D(\cdot, y) u^{p-1}(y) T(u)(y) dy : \ u \in \Lambda \right\}$$

is relatively compact in $C_\infty(D)$. Therefore, as $h \in \mathcal{H}_+$ we have

$$T(u) \in C(\overline{D}) \qquad \text{for every } u \in \Lambda.$$

Moreover

$$T(\Lambda) \ \text{ is relatively compact in } (C(\overline{D}), \|\cdot\|_\infty). \tag{4.1}$$

If $u_n \in \Lambda$ is such that $\|u_n - u\|_\infty \to 0$, then $u_n^{p-1}(x) \to u^{p-1}(x)$ for all $x \in D$. Now for $u \in \Lambda$, $u^{p-1}(x) \leq h^{p-1}(x)$ for all $x \in D$. An application of the Dominated Convergence Theorem implies that $T(u_n)(x) \to T(u)(x)$ for all $x \in D$ and by (4.1), the convergence holds in the uniform norm. We have shown that

$$T : \Lambda \to \Lambda \ \text{ is continuous.} \tag{4.2}$$

For any $u \in \Lambda$, $x \in D$

$$\begin{aligned}
\frac{T(u)(x)}{h(x)} &= \frac{\mathbb{E}_x \left[h(X_{\tau_D}) \exp(-\int_0^{\tau_D} u^{p-1}(X_s) ds) \right]}{h(x)} \\
&= \mathbb{E}_x^h \left[\exp\left(- \int_0^{\tau_D} h^{p-1}(X_s) ds \right) \right] \\
&\geq \exp\left(-\mathbb{E}_x^h \left[\int_0^{\tau_D} h^{p-1}(X_s) ds \right] \right) \\
&\geq \exp(-C_1)
\end{aligned}$$

where the last inequality follows from Lemma 2.2. By continuity we have $T(u)(x) \geq \exp(-C_1) h(x)$, $x \in \overline{D}$. Hence we have shown that

$$T(\Lambda) \subset \Lambda. \tag{4.3}$$

Therefore from (4.1), (4.2), (4.3) and Schauder's fixed point theorem [16, Theorem 11.1], T has a fixed point in Λ. Therefore there exists a u such that $u(x) = \mathbb{E}_x \left[h(X_{\tau_D}) \exp\left(- \int_0^{\tau_D} u^{p-1}(X_s) ds \right) \right]$. From Lemma 2.1 we can conclude that $u \in \mathcal{S}_H^p$. Therefore $\mathcal{S}_H^p \neq \emptyset$ for $p \geq 1$.

By definition, $\mathcal{S}_H^p \subset \mathcal{S}_+^p$. Let $u \in \mathcal{S}_+^p$. Since $p \geq 1$, it follows from Lemma 2.1 that

$$u(x) = \mathbb{E}_x \left[h(X_{\tau_D}) \exp\left(- \int_0^{\tau_D} u^{p-1}(X_s) ds \right) \right], \qquad x \in D$$

for some $h \in \mathcal{H}_+$. Clearly, $u(x) \le h(x)$ for $x \in D$. By Jensen's inequality

$$\begin{aligned}
\frac{u(x)}{h(x)} &\ge \frac{\mathbb{E}_x\left[h(X_{\tau_D})\exp(-\int_0^{\tau_D} h^{p-1}(X_s)ds)\right]}{h(x)} \\
&\ge \mathbb{E}_x^h\left[\exp\left(-\int_0^{\tau_D} h^{p-1}(X_s)ds\right)\right] \\
&\ge \exp\left(-\mathbb{E}_x^h\left[\int_0^{\tau_D} h^{p-1}(X_s)ds\right]\right) \\
&\ge \exp(-C_1)
\end{aligned}$$

Hence $u(x) \ge h(x)\exp(-\gamma)$ for any $x \in D$. This implies that $u \in \mathcal{S}_H^p$, and therefore $\mathcal{S}_H^p = \mathcal{S}_+^p$.

(ii) For any $h \in \mathcal{H}_+$, by (3.4),

$$h(x) \ge \int_D p^D(t,x,y)h(y)dy \ge c\,\varphi_1(x).$$

Hence for $p \ge 0$, by assumption (1.5),

$$\mathbb{E}_x^h\left[\int_0^{\tau_D} h^{p-1}(X_s)ds\right] = \frac{G_D h^p(x)}{h(x)} \le \frac{\|h\|_\infty^p}{c}\frac{G_D 1(x)}{\varphi(x)} \le c_1 < \infty.$$

Using a similar fixed point argument as in (i), we have $\mathcal{S}_H^p \ne \emptyset$ for $p \ge 0$.

For $h \in \mathcal{H}_+$, define

$$\alpha(h) = \inf\left\{p : \sup_{x\in D}\frac{G_D h^p(x)}{h(x)} < \infty\right\} \tag{4.4}$$

and define

$$p_0 = \inf_{h\in\mathcal{H}_+}\alpha(h). \tag{4.5}$$

We now show that $p_0 > -\infty$. By (4.4) and (4.5), it suffices to show that

$$\text{there exists } q \in \mathbb{R} \text{ such that } \sup_{x\in D}\frac{G_D h^{-q}(x)}{h(x)} = \infty. \tag{4.6}$$

By [9, Lemma 6.7], for every c_1, there is a constant $c_2 > 0$ such that

$$G_D(x,y) \ge c_2 \qquad \text{for } x,y \in D \text{ with } |x-y| \le c_1\min\{\delta_D(x),\,\delta_D(y)\}.$$

Fix $c_1 > 0$ and a corresponding $c_2 > 0$. Note that, for a suitable constant $c_3 > 0$, which depends only on c_1, $|y-x| \le c_3\delta_D(x)$ implies $|x-y| \le c_1\min\{\delta_D(x),\,\delta_D(y)\}$. Hence

$$\begin{aligned}
\frac{G_D h^{-q}(x)}{h(x)} &= \frac{\int_D G_D(x,y)h^{-q}(y)dy}{h(x)} \\
&\ge \frac{c_2\int_{\{|x-y|<c\min\{\delta_D(x),\delta_D(y)\}} h^{-q}(y)dy}{h(x)} \\
&\ge \frac{c_2\int_{B(x,c_3\delta_D(x))} h^{-q}(y)dy}{h(x)}
\end{aligned}$$

By [3, Lemma 1.9, page 185, equation (1.22)], there exist constants $c_4 > 0$ and $\beta > 0$, such that $h(y) \le c_4\delta_D(y)^\beta$ for all $y \in B(x,c_3\delta_D(x))$. Hence, for constants $c_5, c_6 > 0$,

$$\frac{G_D h^{-q}(x)}{h(x)} \ge c_5\frac{\int_{B(x,c_3\delta_D(x))}\delta_D(y)^{-q\beta}dy}{\delta_D(x)^\beta}$$

$$\geq c_6 \delta(x)^{-q\beta+d-\beta}.$$

If q is chosen sufficiently large, the last expression above is unbounded over D. This proves (4.6) and thus $p_0 > -\infty$.

We see from above that $p_0 \leq 0$ and that for every $p > p_0$, using the fixed point argument as in (i), one has $\mathcal{S}_H^p \neq \emptyset$.

(iii) Now we show that for every $p < p_0$, $\mathcal{S}_+^p = \emptyset$. Suppose $\mathcal{S}_+^p \neq \emptyset$. Then

$$u(x) = h(x) - \int_D G_D(x,y) u^p(y)\, dy$$

for some $h \in \mathcal{H}_+$. As $u \leq h$ and $p < p_0 \leq 0$,

$$\mathbb{E}_x^h \left[\int_0^{\tau_D} h^{p-1}(X_s) ds \right] = \frac{G_D h^p\,(x)}{h(x)} \leq \frac{G_D u^p\,(x)}{h(x)} = \frac{h(x) - u(x)}{h(x)} = 1 - \frac{u(x)}{h(x)} \leq 1.$$

This contradicts the definition of p_0 in (4.5). Hence $\mathcal{S}_+^p = \emptyset$ for every $p < p_0$. □

5 Proof of Theorem 1.3

(i) As any bounded $C^{1,1}$ domain satisfies the hypothesis of Theorem 1.5, the results follows directly from Theorem 1.5(1).

(ii). It is well known that for bounded $C^{1,1}$ domain D, the Euclidean boundary ∂D is the same as the minimal Martin boundary for Δ in D. So for any $h \in \mathcal{H}_+$, there is a finite positive measure μ on ∂D such that

$$h(x) = \int_{\partial D} K_D(x,z)\, \mu(dz),$$

where $K_D(x,z)$ is the Martin kernel for Δ in D. It is a direct consequence of (2.6) and (2.9) that

$$K_D(x,z) \approx \frac{\delta_D(x)}{|x-z|^n} \qquad \text{for } x \in D \text{ and } z \in \partial D. \tag{5.1}$$

Hence

$$h(x) \geq c\delta_D(x) \quad \text{for } x \in D. \tag{5.2}$$

Note that for each fixed $z \in \partial D$, $x \mapsto K_D(x,z)$ is a positive harmonic function in D.

We first assume that $n \geq 3$. It follows from Zhao [22]

$$G_D(x,y) \leq c \min\left\{ \delta_D(x)|x-y|^{1-n},\, \delta_D(x)\delta_D(y)|x-y|^{-n} \right\}. \tag{5.3}$$

If $-1 < p < 0$, then by (5.3) and (5.2)

$$\begin{aligned}
\sup_{x\in D} \mathbb{E}_x^h \left[\int_0^{\tau_D} h^{p-1}(X_s) ds \right] &= \sup_{x \in D} \frac{\int_D G_D(x,y) h^p(y) dy}{h(x)} \\
&\leq c \sup_{x\in D} \left(\delta_D(x)^{-1} \int_D \frac{\delta_D(x)\delta_D(y)^{-p}}{|x-y|^{n-1-p}} \delta_D(y)^p\, dy \right) \\
&= c \sup_{x \in D} \int_D \frac{1}{|x-y|^{n-1-p}} < \infty.
\end{aligned}$$

If $0 \le p < 1$, since h is bounded, by (5.3) and (5.2) we have

$$\begin{aligned}
\sup_{x\in D} \mathbb{E}_x^h \left[\int_0^{\tau_D} h^{p-1}(X_s)ds \right] &= \sup_{x\in D} \frac{\int_D G_D(x,y)h^p(y)dy}{h(x)} \\
&\le c \sup_{x\in D} \left(\delta_D(x)^{-1} \int_D \frac{\delta_D(x)}{|x-y|^{n-1}}\, dy \right) \\
&= c \sup_{x\in D} \int_D \frac{1}{|x-y|^{n-1}} < \infty.
\end{aligned}$$

Now we can imitate the arguments presented in the proof of Theorem 1.5(ii), to conclude that $\mathcal{S}_H^p \neq \emptyset$ when $-1 < p < 1$ and $n \ge 3$.

We now assume $n = 2$. If $-1 < p < 0$, then by (2.7) and (5.2)

$$\begin{aligned}
\sup_{x\in D} \mathbb{E}_x^h \left[\int_0^{\tau_D} h^{p-1}(X_s)ds \right] &= \sup_{x\in D} \frac{\int_D G_D(x,y)h^p(y)dy}{h(x)} \\
&\le c \sup_{x\in D} \left(\delta_D(x)^{-1} \int_D \log\left(1 + \frac{\delta_D(x)\delta_D(y)}{|x-y|^2}\right) \delta_D(y)^p\, dy \right).
\end{aligned}$$

Observe that $\log(1+ab) \le ab \le ab^{-p}$ for $a > 0$ and $0 < b \le 1$ and $\log(1+ab) \le (-1/p)b^{-p}b^{-p} \le (-1/p)ab^{-p}$ for $a \ge 1$ and $b > 0$. Thus

$$\log\left(1 + \frac{\delta_D(x)\delta_D(y)}{|x-y|^2}\right) \le c\, \frac{\delta_D(x)}{|x-y|} \frac{\delta(y)^{-p}}{|x-y|^{-p}} \tag{5.4}$$

when either $\delta_D(x) \ge |x-y|$ or $\delta_D(y) \le |x-y|$. It follows that

$$\begin{aligned}
&\sup_{x\in D} \mathbb{E}_x^h \left[\int_0^{\tau_D} h^{p-1}(X_s)ds \right] \\
\le\ & c \sup_{x\in D} \left(\int_D \frac{1}{|x-y|^{1-p}} dy + \delta_D(x)^{-1} \int_{\{y\in D: \delta_D(x) < |x-y| < \delta_D(y)\}} \log\left(1 + \frac{\delta_D(x)\delta_D(y)}{|x-y|^2}\right) \delta_D(y)^p\, dy \right) \\
\le\ & c \sup_{x\in D} \left(1 + \delta_D(x)^{-1} \int_{\{y\in D: \delta_D(x) < |x-y| < \delta_D(y)\}} \frac{\delta_D(x)^{-p}\delta_D(y)^{-p}}{|x-y|^{-2p}} \delta_D(y)^p\, dy \right) \\
\le\ & c \sup_{x\in D} \left(1 + \delta_D(x)^{-1-p} \int_{\{y\in D: |x-y| > \delta_D(x)\}} \frac{1}{|x-y|^{-2p}}\, dy \right) \\
\le\ & c + c \sup_{x\in D} \delta_D(x)^{1+p} < \infty.
\end{aligned}$$

Consider $0 \le p < 1$. Since h is bounded, by (2.7), (5.2) and as any $C^{1,1}$-domain satisfies the hypothesis of Theorem 1.5, we have

$$\begin{aligned}
\sup_{x\in D} \mathbb{E}_x^h \left[\int_0^{\tau_D} h^{p-1}(X_s)ds \right] &= \sup_{x\in D} \frac{\int_D G_D(x,y)h^p(y)dy}{h(x)} \\
&\le c \sup_{x\in D} \left(\delta_D(x)^{-1} G_D 1(x) \right) \\
&= c \sup_{x\in D} \left(\delta_D(x)^{-1} \varphi(x) \right) < \infty.
\end{aligned}$$

The last inequality is due to the fact that $\varphi(\cdot) \approx \delta_D(\cdot)$ in D, which is a consequence of (5.1) and the boundary Harnack principle (Theorem 1.2).

Now we can imitate the arguments presented in the proof of Theorem 1.5(ii), to conclude that $\mathcal{S}_H^p \neq \emptyset$ when $-1 < p < 0$ and $n = 2$. We thus obtain (ii).

(iii). Suppose that there exists a mild solution u to (1.2) which is positive in D and vanishing on Λ. Then, by definition, there is a positive harmonic function h that vanishes on Λ such that $u = h - G_D u^p$. Hence

$$u(x) \le h(x) \quad \text{and} \quad G_D u^p(x) \le h(x) \quad \text{for every } x \in D. \tag{5.5}$$

On the other hand there is a finite positive measure μ on ∂D such that $\mu(\Lambda) = 0$ and

$$h(x) = \int_{\partial D} K_D(x,z)\mu(dz) = \int_{\partial D\setminus\Lambda} K_D(x,z)\mu(dz), \quad x \in D.$$

Take $z_0 \in \Lambda$ and $r_0 > 0$ such that $B(z_0, 2r_0) \subset D_1$. Then by (5.1),

$$h(x) \approx \delta_D(x) \quad \text{for } x \in D \cap B(z_0, r_0). \tag{5.6}$$

Since $p \le -1$ and $u(y) \le h(y)$, we have

$$u(y)^p \ge c^p\, \delta_D(y)^p \ge c^p\, \delta_D(y)^{-1} \quad \text{for } y \in D \cap B(z_0, r_0).$$

Now take a sequence of points $\{x_k\}$ in $D \cap B(z_0, r_0)$ that converges to z_0. Then for $n \ge 3$, by Fatou's lemma,

$$\begin{aligned}
\liminf_{k\to\infty} \frac{G_D u^p(x_k)}{h(x_k)} &\ge c \liminf_{k\to\infty} \frac{\int_{D\cap B(z_0,r_0)} G_D(x_k,y)\delta_D(y)^{-1}\,dy}{\delta_D(x_k)} \\
&\approx c \liminf_{k\to\infty} \int_{D\cap B(z_0,r_0)} \delta_D(x_k)^{-1} \min\left\{\frac{1}{|x_k-y|^{n-2}}, \frac{\delta_D(x_k)\delta_D(y)}{|x_k-y|^n}\right\} \delta_D(y)^{-1}\,dy \\
&\ge c \int_{D\cap B(z_0,r_0)} \liminf_{k\to\infty} \delta_D(x_k)^{-1} \min\left\{\frac{1}{|x_k-y|^{n-2}}, \frac{\delta_D(x_k)\delta_D(y)}{|x_k-y|^n}\right\} \delta_D(y)^{-1}\,dy \\
&= c \int_{D\cap B(z_0,r_0)} |z_0-y|^{-n} dy \\
&= \infty.
\end{aligned}$$

This contradicts inequality (5.5). Therefore $\mathcal{S}_+^p = \emptyset$ when $n \ge 3$.

Similarly, when $n = 2$, by Fatou's lemma,

$$\begin{aligned}
\liminf_{k\to\infty} \frac{G_D u^p(x_k)}{h(x_k)} &\ge c \liminf_{k\to\infty} \frac{\int_{D\cap B(z_0,r_0)} G_D(x_k,y)\delta_D(y)^{-1}\,dy}{\delta_D(x_k)} \\
&\approx c \liminf_{k\to\infty} \int_{D\cap B(z_0,r_0)} \delta_D(x_k)^{-1} \log\left(1 + \frac{\delta_D(x_k)\delta_D(y)}{|x_k-y|^2}\right) \delta_D(y)^{-1}\,dy \\
&\ge c \int_{D\cap B(z_0,r_0)} \liminf_{k\to\infty} \delta_D(x_k)^{-1} \log\left(1 + \frac{\delta_D(x_k)\delta_D(y)}{|x_k-y|^2}\right) \delta_D(y)^{-1}\,dy \\
&= c \int_{D\cap B(z_0,r_0)} |z_0-y|^{-2} dy \\
&= \infty.
\end{aligned}$$

This again contradicts inequality (5.5). Therefore $\mathcal{S}_+^p = \emptyset$ when $n = 2$. $\square$

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