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On Jantzen's and Andersen's sum formulas for algebraic groups

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Abstract

Andersen's sum formula for tilting modules for reductive algebraic groups is derived as a consequence of the older Jantzen sum formula for Weyl modules. The connection between the two sum formulas is obtained by interpreting each formula as the calculation of a suitable Euler characteristic. Some observations about the weights involved in these sum formulas are also included.

0. *Introduction*

Consider the category of rational representations of a reductive algebraic group over an algebraically closed field of positive characteristic. H. H. Andersen defined a filtration of the space of equivariant homomorphisms from a Weyl module into a tilting module. He also proved in most cases a sum formula for this filtration. The filtration and the sum formula are reminiscent of the well-known Jantzen filtration of Weyl modules and the corresponding Jantzen sum formula. In section 2.2 below we will see a different and complete proof of Andersen's formula via the known Jantzen sum formula. For the proof we will need to independently interpret each sum formula as the calculation of the same Euler characteristic. For Andersen's formula this is observed in 1.3, based on some reasoning from [Andersen2] summarized in 1.1-1.2. For Jantzen's formula the interpretation was observed in [Kulkarni] and is recalled in 2.1. A slightly different description of the weights involved in both sum formulas appears in 2.3.

Let us fix notation and recall some standard facts. $\mathbf{Z}$, k and R will always denote respectively the ring of integers, a field of prime characteristic p and the ring $\mathbf{Z}_p$ of the corresponding p-adic integers. Let $G_{\mathbf{Z}}$ be a split and connected reductive algebraic group scheme over $\mathbf{Z}$. We will use standard machinery from [Jantzen] such as the dominant weights obtained by choosing a set of positive roots R^+ for a split maximal torus, the Weyl group W, etc.

For any commutative ring A by base change we get a corresponding group scheme G_A. The dominant integral weights λ index the Weyl modules $\Delta_A(\lambda)$ and the dual Weyl modules $\nabla_A(\lambda)$. These modules are characteristic-free, i.e., $\Delta_{\mathbf{Z}}(\lambda)$ is $\mathbf{Z}$-free, $\Delta_A(\lambda) = \Delta_{\mathbf{Z}}(\lambda) \otimes A$ and likewise for dual Weyl modules. A G_A-module with a (dual) Weyl filtration is one that has a finite filtration whose successive factors are (dual) Weyl modules. A tilting G_A-module is a module that has a Weyl filtration as well as a dual Weyl filtration. The multiplicity of a (dual) Weyl module corresponding to a given dominant weight λ in any (dual) Weyl filtration of a tilting G_A-module Q is the same (e.g., by considering the formal character of Q) and will be denoted by $[Q : \Delta_A(\lambda)]$. This can be extended to an arbitrary integral weight ν by considering the corresponding Weyl character $\chi(\nu)$, see [Jantzen II.5.7(1), II.5.9(1)]. So if there is an element $w \in W$ of length $\ell(w)$ with $\lambda = w \cdot \nu$ dominant, we let $[Q : \chi(\nu)] = (-1)^{\ell(w)}[Q : \Delta_A(\lambda)]$. Otherwise let $[Q : \chi(\nu)] = 0$.

For a dominant weight λ there is a unique indecomposable tilting G_k-module $T_k(\lambda)$ with highest weight λ and any tilting G_k-module is uniquely a direct sum of these. Finding the formal character of $T_k(\lambda)$ is an important open problem. Andersen's sum formula enables one to do this in some cases, just as Jantzen's sum formula permits calculation of formal characters of simple modules in some cases. See [Andersen2, Introduction and 2.13]. To define his filtration, Andersen works over G_R (see below). By [Andersen1], every tilting G_R-module is uniquely a direct sum of tilting G_R-modules $T_R(\lambda)$ with $T_R(\lambda) \otimes k = T_k(\lambda)$. In particular any tilting G_k-module lifts uniquely to a tilting G_R-module.

We will need some homological machinery. Define (see [Kulkarni])

$$E(M,N) = div|Hom_{G_{\mathbf{Z}}}(M,N)_{tor}| + \sum_{i>0} (-1)^i div|Ext^i_{G_{\mathbf{Z}}}(M,N)|$$

for finitely generated G_Z-modules M and N, where div stands for taking the divisor of a rational number. Below we will see how the calculation of E for pairs of Weyl modules is equivalent to Jantzen's as well as Andersen's sum formula. For the latter we will need E_R, the analogue of E defined by working over G_R. Note that if M and N are $\mathbf{Z}$-free $G_{\mathbf{Z}}$-modules of finite rank, then by the Universal Coefficient Theorem [Jantzen I.4.18] $Ext^i_{G_R}(M\otimes R, N\otimes R) = Ext^i_{G_{\mathbf{Z}}}(M,N)\otimes R$ as $\mathbf{Z}_p$ is flat over $\mathbf{Z}$. So for such M and N, $E_R(M\otimes R, N\otimes R) =$ (the coefficient of $[p]$ in $E(M,N))[p]$. Finally let us record the following fundamental homological facts, see [Jantzen, II.B].

Theorem 1. Let the base ring $A = \mathbf{Z}$ or R. Then for dominant weights λ and μ,

(i) $Ext^i_{G_A}(\Delta_A(\mu), \nabla_A(\lambda)) = 0$ unless ($\mu = \lambda$ and $i = 0$). $Hom_{G_A}(\Delta_A(\lambda), \nabla_A(\lambda)) = A$.

(ii) $Ext^i_{G_A}(\Delta_A(\mu), \Delta_A(\lambda)) = 0$ unless $\mu < \lambda$ or ($\mu = \lambda$ and $i = 0$), where $<$ is the usual dominance partial order on weights. $Hom_{G_A}(\Delta_A(\lambda), \Delta_A(\lambda)) = A$.

1. *Andersen's sum formula and its connection with E*

1.1. *Andersen's filtration.* In this section fix a prime p and a dominant weight λ. In 1.1 and 1.2 we will reproduce the setup and the reasoning from [Andersen2, 1.3-1.5]. For a tilting G_R-module Q, Andersen gives a descending filtration $F_\lambda(Q)^j$ of $F_\lambda(Q) = Hom_{G_R}(\Delta_R(\lambda)), Q) = R^{[Q:\Delta_R(\lambda)]}$ (by Theorem 1.(i)). To define this filtration, fix an enumeration of dominant weights such that $\lambda_i < \lambda_j$ implies $i < j$. Let $[Q : \Delta_R(\lambda_j)] = n_j$. By Theorem 1.(ii) Q has a finite filtration $Q = Q_0 \supset Q_1 \supset Q_2 \ldots$ with $Q_{i-1}/Q_i = \Delta(\lambda_i)^{n_i}$ such that $\lambda_i < \lambda_j$ implies $i < j$. Let i be the index such that the chosen $\lambda = \lambda_i$. Consider the two short exact sequences

$$(1) \quad 0 \to Q_{i-1} \to Q \to Q/Q_{i-1} \to 0 \qquad \text{and} \qquad (2) \quad 0 \to Q_i \to Q_{i-1} \to \Delta(\lambda)^{n_i} \to 0.$$

Since $[Q/Q_{i-1} : \Delta_R(\lambda)] = 0$ we have $Hom_{G_R}(\Delta_R(\lambda), Q/Q_{i-1}) = 0$. Therefore any map φ in $F_\lambda(Q)$ factors through Q_{i-1} and hence can be composed with the surjection in (2) to get $\Phi(\varphi)$ in $End_{G_R}(\Delta_R(\lambda))^{n_i}$. Since $[Q_i : \Delta_R(\lambda)] = 0$ we have $Hom_{G_R}(\Delta_R(\lambda), Q_i) = 0$ and hence Φ is an injection between modules isomorphic to R^{n_i}. Define

$$F_\lambda(Q)^j = \{\varphi \in F_\lambda(Q) | \Phi(\varphi) \in p^j End_{G_R}(\Delta_R(\lambda))^{n_i}\}.$$

(Earlier [Andersen1] used the pairing $Hom_{G_R}(\Delta_R(\lambda)), Q) \times Hom_{G_R}(Q, \nabla_R(\lambda)) \to R$, (see Theorem 1.(i)) to define the same filtration. The equivalence with the above description is proved in [Andersen2, Proposition 1.6].) Continuing the analogy with Jantzen's filtration, Andersen then defines $\bar{F}_\lambda(Q \otimes_R k)^j =$ the image of $F_\lambda(Q)^j$ in $F_\lambda(Q) \otimes_R k$, which is a filtration of $F_\lambda(Q) \otimes_R k = Hom_{G_k}(\Delta_k(\lambda), Q \otimes_R k)$. Henceforth we will write $\bar{Q}$ for $Q \otimes_R k$.

1.2. *Andersen's sum formula.* Continue with the setting of 1.1. As in the case of Jantzen's sum formula, to calculate $\sum_{j>0}\dim\bar{F}_\lambda(\bar{Q})^j$ amounts to calculating the p-adic valuation of the cokernel of Φ, see [Jantzen II.8.18]. To identify $coker(\Phi)$ consider the long exact sequences obtained by applying $Hom_{G_R}(\Delta_R(\lambda_i), -)$ to (1) and (2). The Hom terms in both sequences have been analyzed above. In the first sequence for $t>0$, $Ext^t_{G_R}(\Delta_R(\lambda), Q/Q_{i-1}) = 0$ (since $[Q/Q_{i-1} : \Delta_R(\lambda_j)] = 0$ for any $\lambda_j > \lambda$) and $Ext^t_{G_R}(\Delta_R(\lambda), Q) = 0$ (since Q has a dual Weyl filtration). Hence $Ext^t_{G_R}(\Delta_R(\lambda), Q_{i-1}) = 0$ for $t>0$. Using this for $t=1$ in the second long exact sequence one gets the short exact sequence

$$0 \to F_\lambda(Q) \xrightarrow{\Phi} End_{G_R}(\Delta_R(\lambda))^{n_i} \to Ext^1_{G_R}(\Delta_R(\lambda), Q_i) \to 0. \tag{3}$$

(3) is enough for Andersen's proof (see below), but we will need to make the remaining step explicit for use in 1.3. Since $Ext^t_{G_R}(\Delta_R(\lambda), \Delta_R(\lambda)) = 0$ for $t>0$, the entire long exact sequence reduces to (3). In particular $Ext^t_{G_R}(\Delta_R(\lambda), Q_i) = 0$ for $t \neq 1$.

Theorem 2. (Andersen's sum formula) $\sum_{j>0}\dim\bar{F}_\lambda(\bar{Q})^j = -$ the coefficient of $[p]$ in

$$\sum_{\alpha\in R^+}\ \sum_{m<0 \text{ or } m>\langle\lambda+\rho,\check{\alpha}\rangle} div(m)[\bar{Q} : \chi(\lambda - m\alpha)].$$

Andersen proves his sum formula as follows. Using the linkage principle, it is enough to work with one block at a time. Using translation functors, he reduces the problem to a regular block. Andersen shows by careful analysis that the claimed formula and the Ext^1 in (3) behave in the same fashion under wall crossing functors and then finishes the proof via an induction. See [Andersen2, Section 2]. The use of regular weights restricts the validity of this proof to $p \geq$ the Coxeter number for G, even though the formula was expected to be valid without this restriction. We will take a slightly different tack in 1.3 which will lead to another proof in Section 2 that works for all p.

1.3. *Andersen's sum formula and E.* Continue with the setting in 1.1 and 1.2. By the third sentence following (3), we have $\sum_{j>0}\dim\bar{F}_\lambda(\bar{Q})^j = -$ the coefficient of $[p]$ in $E_R(\Delta_R(\lambda), Q_i)$. Now by additivity of E_R (see [Kulkarni, 1.2]),

$$E_R(\Delta_R(\lambda), Q_i) = \sum_{j>i} n_j E_R(\Delta_R(\lambda), \Delta_R(\lambda_j)) = \sum_j n_j E_R(\Delta_R(\lambda), \Delta_R(\lambda_j)),$$

where we have used the fact that for $j \leq i$, $E_R(\Delta_R(\lambda), \Delta_R(\lambda_j)) = 0$ by Theorem 1.(ii). Combining this with the comparison between E and E_R from the introduction, we have $\sum_{j>0}\dim\bar{F}_\lambda(\bar{Q})^j = -$ the coefficient of $[p]$ in $\sum_j n_j E(\Delta_{\mathbf{Z}}(\lambda), \Delta_{\mathbf{Z}}(\lambda_j))$. To compare this with the statement of Theorem 2, define for a dominant weight μ

$$U(\lambda,\mu) = \{(\alpha, m) | \alpha \in R^+, m<0 \text{ or } m > \langle\lambda+\rho,\check{\alpha}\rangle, \lambda - m\alpha = w\cdot\mu \text{ for some } w \in W\}. \tag{4}$$

(Such w must be unique.) Then Theorem 2 is easily seen to be equivalent to the following statement.

$$E(\Delta_{\mathbf{Z}}(\lambda), \Delta_{\mathbf{Z}}(\mu)) = \sum_{U(\lambda,\mu)} (-1)^{\ell(w)} div(m). \tag{5}$$

2. *Connection between Jantzen's and Andersen's sum formulas via E*

2.1. *Jantzen's sum formula and E.* The Jantzen filtration $\Delta_k^i(\lambda)$ of the Weyl module $\Delta_k(\lambda)$ is obtained from a generator of $Hom_{G_{\mathbf{Z}}}(\Delta_{\mathbf{Z}}(\lambda), \nabla_{\mathbf{Z}}(\lambda))$ (see Theorem 1.(i)) exactly as Andersen's filtration was obtained from the map Φ in 1.1. Letting ch denote the formal character, Jantzen's sum formula [Jantzen II.8.19(1)] (see also [Jantzen II.8.16(1)]) states: $\sum_{i>0} ch(\Delta_k^i(\lambda)) = -$ the coefficient of $[p]$ in

$$\sum_{\beta\in R^+} \sum_{0<n<\langle\lambda+\rho,\beta\check{}\rangle} div(n)\chi(\lambda - n\beta).$$

[Kulkarni, Corollary 1.4] observes the following connection with E. For a dominant weight μ, the coefficient of $ch(\Delta_k(\mu))$ in $\sum_{i>0} ch(\Delta_k^i(\lambda)) = -$ the coefficient of $[p]$ in $E(\Delta_{\mathbf{Z}}(\mu), \Delta_{\mathbf{Z}}(\lambda))$. Thus in view of 1.3 either of the sum formulas must imply the other. Let us derive Andersen's formula using this connection. Define for a dominant weight μ

$$V(\lambda,\mu) = \{(\beta,n)|\beta \in R^+, 0 < n < \langle\lambda+\rho,\beta\check{}\rangle, \lambda - n\beta = x\cdot\mu \text{ for some } x \in W\}. \tag{6}$$

(Such x must be unique.) Then Jantzen's sum formula is easily seen to be equivalent to the following statement.

$$E(\Delta_{\mathbf{Z}}(\mu), \Delta_{\mathbf{Z}}(\lambda)) = \sum_{V(\lambda,\mu)} (-1)^{\ell(x)} div(n). \tag{7}$$

2.2. *A proof of Andersen's sum formula.* We will prove Theorem 2 by showing the equivalence of (5) and (7). Clearly it is enough to produce a bijection between $U(\lambda,\mu)$ and $V(\mu,\lambda)$ for which $n = \pm m$ and $x = w^{-1}$. This is an easy check as follows.

First suppose $(\alpha, m) \in U(\lambda,\mu)$ is given with $\lambda - m\alpha = w\cdot\mu$. So $w^{-1}\cdot\lambda = \mu + m(w^{-1}\alpha)$.

Case 1a. If $w^{-1}\alpha \in R^+$ then let $\beta = w^{-1}\alpha$ and $n = -m$. We have

$$\langle\mu+\rho, w^{-1}\alpha\check{}\rangle = \langle w^{-1}(\lambda+\rho) - m(w^{-1}\alpha), w^{-1}\alpha\check{}\rangle = \langle\lambda+\rho,\alpha\check{}\rangle - 2m. \tag{8}$$

Since $\langle\mu+\rho,\beta\check{}\rangle$ and $\langle\lambda+\rho,\alpha\check{}\rangle$ are both positive, the possibility $m > \langle\lambda+\rho,\alpha\check{}\rangle$ in (4) cannot be true. So $m < 0$ and hence $n = -m > 0$. Also $\langle\mu+\rho,\beta\check{}\rangle = \langle\lambda+\rho,\alpha\check{}\rangle - 2m > -2m = 2n$. So $0 < n < \frac{1}{2}\langle\mu+\rho,\beta\check{}\rangle$, in particular $(\beta, n) \in V(\mu,\lambda)$.

Case 1b. If $w^{-1}\alpha \in -R^+$ then let $\beta = -w^{-1}\alpha$ and $n = m$. By (8) we have $\langle\mu+\rho,\beta\check{}\rangle = 2m - \langle\lambda+\rho,\alpha\check{}\rangle$. Since $\langle\mu+\rho,\beta\check{}\rangle$ and $\langle\lambda+\rho,\alpha\check{}\rangle$ are both positive, the possibility $m < 0$ in (4) cannot be true. So $m > \langle\lambda+\rho,\alpha\check{}\rangle$. Thus $n = m > 0$. Also $\langle\mu+\rho,\beta\check{}\rangle = 2m - \langle\lambda+\rho,\alpha\check{}\rangle > m = n$, as desired. (Since $0 < \langle\lambda+\rho,\alpha\check{}\rangle = 2m - \langle\mu+\rho,\beta\check{}\rangle$, we actually have $\frac{1}{2}\langle\mu+\rho,\beta\check{}\rangle < m = n < \langle\mu+\rho,\beta\check{}\rangle$.)

To give the inverse map, suppose $(\beta, n) \in V(\mu,\lambda)$ is given with $\mu - n\beta = x\cdot\lambda$. So $x^{-1}\cdot\mu = \lambda + n(x^{-1}\beta)$.

Case 2a. If $x^{-1}\beta \in R^+$ then let $\alpha = x^{-1}\beta$ and $m = -n$. We have $m < 0$ since $n > 0$ by (6), so $(\alpha, m) \in U(\lambda, \mu)$. Clearly this case is inverse to Case 1a. (n must satisfy the bounds in the last sentence of Case 1a by the calculation there.)

Case 2b. If $x^{-1}\beta \in -R^+$ then let $\alpha = -x^{-1}\beta$ and $m = n$. Now via a calculation similar to (8) we have $\langle \lambda + \rho, \check{\alpha} \rangle = 2n - \langle \mu + \rho, \check{\beta} \rangle < 2n - n = m$, so $(\alpha, m) \in U(\lambda, \mu)$. Clearly this case is inverse to Case 1b (and again the bounds obtained there on n must hold in this case). This completes the proof of (5) and hence that of Theorem 2.

2.3. *Some observations regarding the sets $U(\lambda, \mu)$ and $V(\lambda, \mu)$.* Consider the following key condition involved in the definition of these sets. "The line through λ in the direction of some root intersects the orbit of μ (under the dot action of W)." Since $\lambda - m\alpha = w \cdot \mu \Leftrightarrow w^{-1} \cdot \lambda = \mu + m(w^{-1}\alpha)$, the stated condition is symmetric in λ and μ. The asymmetry in the definitions of comes from the numerical inequalities in (4) and (6) stipulating where on the line the intersection must occur. Some natural questions arise in this regard. For Andersen's (respectively Jantzen's) sum formula corresponding to a dominant weight λ, only those dominant weights μ for which $\mu < \lambda$ (respectively $\lambda < \mu$) matter. (This follows, e.g, by (5) and (7) respectively since nontriviality of $E(\Delta_{\mathbf{Z}}(\lambda), \Delta_{\mathbf{Z}}(\mu))$ requires $\lambda < \mu$ by Theorem 1.(ii).) So one may wonder how the asymmetry resulting from these conditions compares with the respective numerical inequalities. One may also seek a simple explanation for the apparent symmetry in 2.2 between Case 1a and Case 1b as well as that between the cases 2a and 2b. These questions are addressed in the next proposition.

Proposition. Let λ and μ be distinct dominant weights. Then the set $U(\lambda, \mu)$ (respectively $V(\lambda, \mu)$) is empty unless $\lambda < \mu$ (respectively $\mu < \lambda$), in which case it equals $\{(\alpha, m) | \alpha \in R^+, \lambda - m\alpha = w \cdot \mu \text{ for some } w \in W\}$. Moreover there is a fixed point free involution on each set given by $(\alpha, m) \leftrightarrow (\alpha, \langle \lambda + \rho, \check{\alpha} \rangle - m)$. In particular the cardinality of each of these sets is even.

Proof. Throughout the proof suppose that $w \cdot \mu = \lambda - m\alpha$ with $\alpha \in R^+$. One gets the needed involution by noting that

$$(s_\alpha w) \cdot \mu = s_\alpha \cdot (\lambda - m\alpha) = s_\alpha \cdot \lambda + m\alpha = \lambda - (\langle \lambda + \rho, \check{\alpha} \rangle - m)\alpha. \tag{9}$$

Since $(s_\alpha w)^{-1}\alpha = -w^{-1}\alpha$, (9) sets up a pairing between the two parts into which the set $U(\lambda, \mu)$ (respectively $V(\lambda, \mu)$) is divided by the cases 1a and 1b (respectively 2a and 2b–suitably modified by interchanging λ and μ since we used $V(\mu, \lambda)$ in 2.2). It also explains the bisection of the total numerical range allowed in each set around the midpoint $\frac{1}{2}\langle \lambda + \rho, \check{\alpha} \rangle$ by the respective pair of cases.

Since λ and μ are dominant, $w \cdot \mu \leq \mu$ and $w^{-1} \cdot \lambda \leq \lambda$ with equalities iff w is the identity. Recall that by (8) we have $\langle \mu + \rho, w^{-1}\check{\alpha} \rangle = \langle \lambda + \rho, \check{\alpha} \rangle - 2m$. It suffices to show that one must have $\lambda < \mu$ or $\mu < \lambda$ appropriately depending on the value of m. First note that $m = \frac{1}{2}\langle \lambda + \rho, \check{\alpha} \rangle$ is impossible, e.g., because that would mean $\langle \mu + \rho, w^{-1}\check{\alpha} \rangle = 0$.

Case I. If $m < 0$, then $\lambda < \lambda - m\alpha = w \cdot \mu \leq \mu$.

Case II. If $m = 0$, then $w =$ identity and $\mu = \lambda$, contrary to our assumption.

Case III. If $0 < m < \frac{1}{2}\langle \lambda + \rho, \check{\alpha} \rangle$, then $\langle \mu + \rho, w^{-1}\check{\alpha} \rangle > 0$. So $w^{-1}\alpha \in R^+$ and hence $\mu < \mu + m(w^{-1}\alpha) = w^{-1} \cdot \lambda \leq \lambda$.

Remaining three cases. If $\frac{1}{2}\langle \lambda+\rho, \check{\alpha} \rangle < m < \langle \lambda+\rho, \check{\alpha} \rangle$ (respectively, if $m = \langle \lambda+\rho, \check{\alpha} \rangle$, if $m > \langle \lambda + \rho, \check{\alpha} \rangle$), then using (9) one reduces to Case III (respectively, case II, Case I) to conclude that $\mu < \lambda$ (respectively, $\mu = \lambda$, $\lambda < \mu$). This finishes the proof of the proposition.

Notes. 1) Suppose for now that Case I holds, i.e., $w \cdot \mu = \lambda - m\alpha$ with $\alpha \in R^+$ and $m < 0$. Let $\beta = w^{-1}\alpha$ and $m' = \langle \lambda+\rho, \check{\alpha} \rangle - m$. One can write down all the order relations forced on the weights involved in the given situation and its "mirror image" under (9). These weights are μ, its translates by w and $s_\alpha w$, λ and its translates by w^{-1} and $w^{-1}s_\alpha$. One easily sees using available information that $w^{-1}s_\alpha \cdot \lambda = \mu - m'\beta < \mu + m\beta = w^{-1} \cdot \lambda \leq \lambda < \lambda - m\alpha = w \cdot \mu \leq \mu$. (So for instance $-m\alpha \leq \mu - \lambda \leq -m\beta$ and hence $\alpha \leq \beta$ with equality iff $w =$ identity.) For the remaining weight one has $s_\alpha w \cdot \mu = \lambda - m'\alpha < \lambda$ and examples show that in general $s_\alpha w \cdot \mu$ is not comparable to either of $w^{-1}s_\alpha \cdot \lambda$ and $w^{-1} \cdot \lambda$. A parallel analysis applies to Case III (e.g., via the bijection in 2.2). Then one uses (9) to transfer these results to the "mirror image" cases.

2) Via (8) the cases above can be formulated entirely in terms of the weights λ and μ without reference to m. Cases I, II and III are respectively equivalent to $\langle \mu + \rho, w^{-1}\check{\alpha} \rangle > \langle \lambda + \rho, \check{\alpha} \rangle > 0$, $\langle \mu + \rho, w^{-1}\check{\alpha} \rangle = \langle \lambda + \rho, \check{\alpha} \rangle > 0$ and $\langle \lambda + \rho, \check{\alpha} \rangle > \langle \mu + \rho, w^{-1}\check{\alpha} \rangle > 0$. For the "mirror image" cases, $w^{-1}\alpha \in -R^+$ so one replaces $\langle \mu + \rho, w^{-1}\check{\alpha} \rangle$ by $-\langle \mu + \rho, w^{-1}\check{\alpha} \rangle$.

2.4. *Remarks and questions.* 1) Exactly how do the sets $U(\lambda, \mu)$ and $V(\lambda, \mu)$ look like? By 2.2 and 2.3 it suffices to consider just one of the sets, say $V(\lambda, \mu)$. Let us comment on the special case of the general linear group. Here it is not hard to see that this set, when nonempty, has cardinality exactly two, i.e., the root α is determined uniquely by the pair of dominant weights in question. So $V(\lambda, \mu)$ is empty iff $E(\Delta_{\mathbf{Z}}(\mu), \Delta_{\mathbf{Z}}(\lambda))$ is trivial. [Kulkarni, Theorem 2.3] identifies for the general linear group the pairs of dominant weights for which the sets $V(\lambda, \mu)$ are nonempty and describes $E(\Delta_{\mathbf{Z}}(\mu), \Delta_{\mathbf{Z}}(\lambda))$ in the language of partitions. It should be interesting to get such explicit descriptions for other reductive groups.

2) Andersen has defined tilting filtrations for quantum groups at a root of unity. By a reasoning parallel to his proof in the modular case he also obtained an analogous sum formula for these filtrations in presence of regular weights. The above reasoning with appropriate modifications should work without restriction for the quantum sum formula as well.

3) G. McNinch established a completely different connection between Jantzen's and Andersen's filtrations in the presence of "Howe duality" in positive characteristic. [McNinch, Theorem 2] shows that $F_\lambda(Q)$ for a self-dual full titling module Q naturally becomes a Weyl module for the Howe dual group and moreover, under this identification, Andersen's filtration is carried to Jantzen's filtration. It should be interesting if one can find a

link between the above reasoning (which is able to deal only with sum formulas, not with individual filtration terms) and McNinch's work. In a similar vein it would be extremely interesting to find a homological interpretation for the dimensions of individual terms in Andersen's filtration.

4) It seems plausible that one should be able to directly prove Andersen's formula by reduction to SL_2 case in a way similar to Andersen's proof of Jantzen's sum formula. In fact H. H. Andersen has directly proved (7) using this approach (private communication). Coupled with the results in this paper, this gives a uniform and self-contained proof of both sum formulas. Such a treatment will appear in [AK].

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