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February 15th,2006
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# Limits of the Number of Level Crossings and Related Functionals of Sums of Linear Processes

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First version: January, 2004. This version: February 2006.

**Abstract**. Consider a sequence $X_k = \sum_{j=0}^{\infty} c_j \xi_{k-j}$, $k \geq 1$, where $c_j$, $j \geq 0$, are constants and $\xi_j$, $-\infty < j < \infty$, are iid random variables belonging to the domain of attraction of a strictly stable law with index $0 < \alpha \leq 2$. Let $S_k = \sum_{j=1}^{k} X_j$. Under certain conditions on the constants $c_j$ and on the distribution of $\xi_1$, it has been established elsewhere that for a suitable slowly varying function $\kappa_1(n)$ and for a suitable constant $0 < H < 1$, the sequence $n^{-(1-H)} \kappa_1(n) \sum_{k=1}^{n} f(S_k, S_{k+1}, ..., S_{k+r})$ converges in distribution to $L_1^0 \int_{-\infty}^{\infty} f_*(x)\, dy$ where $f_*(x) = E[f(x, x+S_1, ..., x+S_r)]$ and $L_1^0$ is the local time at 0 of the Linear Fractional Stable Motion. The assumptions imposed on the function $f(x_0, ..., x_r)$ included the assumption that $f(x_0, ..., x_r)$ itself is Lebesgue integrable. This assumption does not hold in important situations such as when $f(x_0, x_1)$ is such that $\sum_{l=1}^{n} f_1(S_l, S_{l+1})$ gives the number of times $S_l$, $1 \leq l \leq n$, crosses the level 0. In this paper the above convergence result is obtained in such situations but the result is restricted either to the case $\alpha = 2$ (with $0 < H < 1$) or to the case $H = 1/\alpha$, $1 < \alpha \leq 2$ (in the later situation the Fractional Stable Motion reduces to the $\alpha$-stable motion ). The reasons for these restrictions are indicated, and they appear to be largely intrinsic to the situations under consideration. The results have motivation in large sample theory for certain nonlinear time series models where functions of the form $f(S_k, S_{k+1}, ..., S_{k+r})$ occur as regressions.

# 1 Introduction

Consider the linear process

$$X_k = \sum_{j=0}^{\infty} c_j \xi_{k-j}, \;\; k \geq 1, \tag{1}$$

where $c_j$, $j \geq 0$, are constants and $\xi_j$, $-\infty < j < \infty$, is a sequence of iid random variables belonging to the domain of attraction of a strictly stable law with index $0 < \alpha \leq 2$. This last statement means that, for a suitable slowly varying function $\kappa(n)$, the finite dimensional distributions of the process $t \longmapsto \left(n^{\frac{1}{\alpha}} \kappa(n)\right)^{-1} \sum_{j=1}^{[nt]} \xi_j$, $-\infty < t < \infty$, converge in distribution to those of the $\alpha$-stable Levy motion $Z_\alpha(t)$, that is,

$$t \longmapsto \left(n^{\frac{1}{\alpha}} \kappa(n)\right)^{-1} \sum_{j=1}^{[nt]} \xi_j \stackrel{fdd}{\Longrightarrow} Z_\alpha(t), \tag{2}$$

see for example Samorodnitsky and Taqqu (1994) for the definition of $\alpha$-stable Levy motion. Without further mentioning we shall assume that $E[\xi_1] = 0$ when $1 < \alpha \leq 2$.

Let

$$S_k = \sum_{j=1}^{k} X_j. \tag{3}$$

Under suitable conditions (see Section 2 below) on the constants $c_j$ it is known that for a suitable $H$, $0 < H < 1$, and for a suitable normalizing constant $\gamma_n = n^H \kappa_1(n)$ with $\kappa_1(n)$ slowly varying,

$$\gamma_n^{-1} S_{[nt]} \stackrel{fdd}{\Longrightarrow} \Lambda_{\alpha,H}(t), \tag{4}$$

where the limiting process $\{\Lambda_{\alpha,H}(t), t \geq 0\}$ is a Linear Fractional Stable Motion (LFSM). Here, by definition,

$$\Lambda_{\alpha,H}(t) = a \int_{-\infty}^{0} \left\{(t-u)^{H-\frac{1}{\alpha}} - (-u)^{H-\frac{1}{\alpha}}\right\} Z_\alpha(du) + a \int_0^t (t-u)^{H-\frac{1}{\alpha}} Z_\alpha(du),$$

where $Z_\alpha(t)$ is the $\alpha$-stable Levy motion introduced in (2), $a$ is a non-zero constant. By *convention*, when $H = \frac{1}{\alpha}$, $1 < \alpha \leq 2$, $\Lambda_{\alpha,H}(t) = \Lambda_{\alpha,\frac{1}{\alpha}}(t)$ is taken to be $Z_\alpha(t)$. When $\alpha = 2$, the LFSM reduces to the Fractional Brownian Motion. See Samorodnitsky and Taqqu (1994) and Maejima (1989) for the details of LFSM.

In J(2004, Statement (ii), Theorem 3) (where J(2004) = Jeganathan (2004)), it is shown that if

$$\int \left|E\left[e^{i\lambda\xi_1}\right]\right|^p d\lambda < \infty \quad \text{for some } p > 0, \tag{5}$$

then for any $f(x)$ such that $\int (|f(x)| + f^2(x))\, dx < \infty$,

$$n^{-1}\gamma_n \sum_{k=1}^{n} f(S_k) \Longrightarrow L_1^0 \int_{-\infty}^{\infty} f(x)\, dx, \tag{6}$$

where $L_1^0$ is the local time of the LFSM $\Lambda_{\alpha,H}(t)$ at 0 upto the time $t = 1$. (See J(2004) for the details of the local time of the LFSM.) Further, under certain mild additional restriction on $f(x)$ (see J(2004, Theorem 2)), the same convergence (6) holds if instead of (5), the weaker Cramér's condition

$$\lim \sup_{|\lambda|\to\infty} \left| E\left[e^{i\lambda\xi_1}\right] \right| < 1 \tag{7}$$

is assumed. (Note that (5) entails (7).)

Furthermore, it is shown in Jeganathan (2006, Theorem 3) that, corresponding to a function $f(x_0, ..., x_r)$,

$$n^{-1}\gamma_n \sum_{k=1}^{n} f(S_k, ..., S_{k+r}) \Longrightarrow L_1^0 \int_{-\infty}^{\infty} f_*(x)\, dx, \tag{8}$$

for all $0 < H < 1$, where

$$f_*(x) = E\left[f(x, x + S_1 ..., x + S_r)\right]$$

with $f_*(x)$ Lebesgue integrable. The conditions imposed there on $f(x_0, ..., x_r)$ are

$$\int |f(x_0, ..., x_r)|^i\, dx_0 ... dx_r < \infty, \;\; i = 1, 2, \;\;\; \int \left( \int |f(x_0, ..., x_r)|^2\, dx_r \right)^{\frac{1}{2}} dx_0 ... dx_{r-1} < \infty. \tag{9}$$

The convergence result (8) is well known for the random walk case $S_k = \sum_{j=1}^{k} \xi_j$, where $\xi_j$ are as earlier but with $1 < \alpha \leq 2$, see Borodin and Ibragimov (1995, Section 6 of Chapter III). (When $0 < \alpha \leq 1$, the local time of $Z_\alpha(t)$ does not exist.) The later work also contains references to the earlier important works, especially Skorokhod (1961). In this case the requirements in (9) are stronger than actually needed. Specifically, the requirements

$$\int |f_*(x)|\, dx < \infty, \qquad \int E\left[f^2(x, x + S_1 ..., x + S_r)\right] dx < \infty \tag{10}$$

are sufficient. (Note that $\int E\left[f^2(x, x + S_1 ..., x + S_r)\right] dx < \infty$ implies $\int |f_*(x)|^2\, dx < \infty$.) We shall briefly recall in Section 2 below the method of this random walk case, which consists of reducing the convergence (8) to that of (6) with $f(x) = f_*(x)$, in order to indicate that they are not directly applicable, and hence the relevance of the conditions in (9) in the situation (3), though the method of the present paper itself will also in some form consist of reducing the convergence (8) to (6).

Unfortunately the integrability conditions in (9) are not satisfied in situations such as $f(x_0, x_1) = \mathbb{I}_{(-\infty,0)}(x_0 x_1)$ or $f(x_0, x_1) = |x_0 - x_1| \mathbb{I}_{(-\infty,0)}(x_0 x_1)$, where in addition suitable moment restrictions are intrinsically involved (see below) when the conditions in (10) are required to be satisfied. In addition the method employed in Jeganathan (2006, Theorem 3) under (9) is not applicable for these situations. In the present paper we obtain the convergence (8) that will include the above situations. However, we shall restrict either to the case $H = \frac{1}{\alpha}$, $1 < \alpha \leq 2$ or to the case $\alpha = 2$ $(0 < H < 1)$ with additional moment restrictions. (This is in contract to the situation under (9) where no further restrictions are involved.) These restrictions appear to be largely intrinsic to the situations dealt with, as we shall explain below in Section 2.

In Section 2, we state the assumptions and the main results, together with the motivations for the stated assumptions. Section 3 contains the proof, which mainly consist of reducing the present situation to that treated in J(2004).

The results have motivation in some current research in large sample theory for certain nonlinear time series models where functions of the form $f(S_k, S_{k+1}, ..., S_{k+r})$ will occur as regressions, see for instance Park and Phillips (2001) for a specific situation where $f(S_k)$ is used as regressions.

*Notations*: In addition to the notation $\overset{fdd}{\Longrightarrow}$ used earlier with regard to the convergence in distribution of ransom processes (see (2), $\Longrightarrow$ stands as usual the convergence in distribution of a sequence of random vectors, $\overset{\mathcal{D}}{=}$ denotes the equivalence in distribution (in law). Throughout below we let $\psi(\lambda) = E\left[e^{i\lambda\xi_1}\right]$. Both $\mathbb{I}_B(X)$ and $\mathbb{I}(X \in B)$ will stand for the indicator function of $\{X \in B\}$. *The notation $C$ stands for generic constants that may differ even within the same expression.*

## 2 Assumptions and the main result

To begin with, let us indicate that in the situations of the present particular interest, appropriate moment restrictions are intrinsically involved. First consider the situation

$$f(x, y) = \mathbb{I}_{(-\infty,0)}(xy) = \mathbb{I}_{(-\infty,0)}(x)\, \mathbb{I}_{(0,\infty)}(y) + \mathbb{I}_{(0,\infty)}(x)\, \mathbb{I}_{(-\infty,0)}(y),$$

where the corresponding $\sum_{l=1}^{n} f(S_l, S_{l+1})$ gives the number of times $S_l$, $1 \leq l \leq n$, changes its sign. If we take $f_1(x, y) = \mathbb{I}_{(-\infty,0]}(x)\, \mathbb{I}_{(0,\infty)}(y) + \mathbb{I}_{[0,\infty)}(x)\, \mathbb{I}_{(-\infty,0)}(y)$, then the corresponding $\sum_{l=1}^{n} f_1(S_l, S_{l+1})$ gives the number of times $S_l$, $1 \leq l \leq n$, crosses the level 0, and the treatment of which is the same as that for $f(x, y)$ in addition to the fact that the limiting distributions of both cases will coincide.

In this case, we have

$$f_*(x) = E[f(x, x + X_1)] = \mathbb{I}_{(-\infty,0)}(x)\, P(X_1 > -x) + \mathbb{I}_{(0,\infty)}(x)\, P(X_1 < -x)$$

and hence, assuming that $E[|X_1|] < \infty$,

$$\int_{-\infty}^{\infty} f_*(x)\,dx = \int_0^{\infty} P(|X_1| > x)\,dx = E[|X_1|] < \infty.$$

Further, noting that $E[f^2(x, x+X_1)] \le f_*(x)$, we also have $\int_{-\infty}^{\infty} E[f^2(x, x+X_1)]\,dx \le E[|X_1|]$.

The requirement $E[|X_1|] < \infty$ is implied by the requirements $E[|\xi_1|] < \infty$ and $\sum_{j=0}^{\infty} |c_j| < \infty$, where $E[|\xi_1|] < \infty$ holds when $\alpha > 1$.

Next, consider the situation

$$f(x,y) = |x-y|\,\mathbb{I}_{(-\infty,0)}(xy).$$

It can be checked as in the first example that when $E\left[|X_1|^3\right] < \infty$,

$$\int_{-\infty}^{\infty} f_*(x)\,dx = E\left[|X_1|^2\right], \qquad \int_{-\infty}^{\infty} E\left[f^2(x, x+X_1)\right]dx = E\left[|X_1|^3\right].$$

Thus in this example the requirement $E\left[|\xi_1|^3\right] < \infty$ is required.

Another example of related interest is $f(x,y) = |y|\,\mathbb{I}_{(-\infty,0)}(xy)$, where $\int_{-\infty}^{\infty} f_*(x)\,dx = \frac{3}{2}E\left[|X_1|^2\right]$ and $\int_{-\infty}^{\infty} E[f^2(x, x+X_1)]\,dx = CE\left[|X_1|^3\right]$.

Some of the steps of the method of the present paper will in part consist of reducing the convergence (8) to essentially a form of the convergence (6). This is the method employed in the random walk case $S_k = \sum_{j=1}^{k} \xi_j$. We next recall it in order to indicate the further conditions, especially on $f(x_0, \dots, x_r)$, that will be needed in the present situation (3) in order to effect this reduction. For convenience consider the case $f(x,y)$, in which case we have $f_*(x) = E[f(x, x+\xi_1)]$. Then noting that

$$f_*(S_k) = E[f(S_k, S_{k+1}) | S_1, \dots, S_k] \tag{11}$$

in the present random walk case $S_k = \sum_{j=1}^{k} \xi_j$, the differences $n^{\frac{1}{\alpha}-1}\kappa(n)(f(S_k, S_{k+1}) - f_*(S_k))$, $1 \le k \le n$, form a martingale difference array, and the sum of their conditional variances is given by

$$\sum_{k=1}^{n} E\left[\left(n^{\frac{1}{\alpha}-1}\kappa(n)(f(S_k, S_{k+1}) - f_*(S_k))\right)^2 \middle| S_1, \dots, S_k\right] \le \left(n^{\frac{1}{\alpha}-1}\kappa(n)\right)^2 \sum_{k=1}^{n} f_{**}(S_k) \tag{12}$$

where $f_{**}(S_k) = E[f^2(S_k, S_{k+1}) | S_1, \dots, S_k]$. Note that $f_{**}(x) = E[f^2(x, x+\xi_1)]$. When $\int f_{**}(x)\,dx < \infty$, the expected value of $n^{\frac{1}{\alpha}-1}\kappa(n)\sum_{k=1}^{n} f_{**}(S_k)$ is bounded (see J(2004) or the proof of the Lemma 4 below). Therefore (12) is of order $O_p\left(n^{\frac{1}{\alpha}-1}\kappa(n)\right)$, in particular it converges to 0 in probability, because $1 < \alpha \le 2$. This implies

$$n^{\frac{1}{\alpha}-1}\kappa(n)\sum_{k=1}^{n}(f(S_k, S_{k+1}) - f_*(S_k)) \xrightarrow{p} 0.$$

Because $n^{\frac{1}{\alpha}-1}\kappa(n)\sum_{k=1}^{n} f_*(S_k) \Longrightarrow L_1^0\int_{-\infty}^{\infty} f_*(x)\,dx$ (see (6)), this gives the required convergence for $n^{\frac{1}{\alpha}-1}\kappa(n)\sum_{k=1}^{n} f(S_k, S_{k+1})$.

It is clear that the preceding arguments use (11) in an essential manner, which is a consequence of the Markov structure for the sequence $S_l = \sum_{j=1}^{l}\xi_j$, $l \geq 1$. Under the integrability conditions in (9), some approximate form of such a property for the case (3) may be considered to be implicit in the arguments in Jeganathan (2006, Theorem 3), though the arguments used there are different from the above. Because now we do not have the conditions in (9), we need to obtain the property in a different manner, which will require further appropriate conditions.

With these preliminaries, we now state the conditions. We first recall the conditions for the convergence (4). Assume that the coefficients $c_j$, where $c_0 = 1$, of the process (1) satisfy (without further mentioning) one of the following two mutually exclusive requirements.

(A1): (**The case** $H = \frac{1}{\alpha}$, $0 < H < 1$).

$$\sum_{j=0}^{\infty}|c_j| < \infty \text{ with } \sum_{j=0}^{\infty}c_j \neq 0.$$

(A2): (**The case** $H \neq \frac{1}{\alpha}$, $0 < H < 1$). $c_j = j^{H-1-\frac{1}{\alpha}}u(j)$, with $H \neq \frac{1}{\alpha}$, $0 < H < 1$, where $u(j)$ is slowly varying at infinity, satisfying

$$\sum_{j=0}^{\infty}c_j = 0 \text{ when } H - \frac{1}{\alpha} < 0. \tag{13}$$

Now let

$$\gamma_n = \begin{cases} n^H u(n)\kappa(n) & \text{if (A1) is satisfied} \\ \left(\sum_{j=0}^{\infty}c_j\right) n^{\frac{1}{\alpha}}\kappa(n) & \text{if (A2) is satisfied,} \end{cases} \tag{14}$$

where $\kappa(n)$ is as in (2) and $u(n)$ as in (A2). Then under (A1) or (A2), $\gamma_n^{-1}S_{[nt]} \overset{fdd}{\Longrightarrow} \Lambda_{\alpha,H}(t)$. See, for example, Kasahara and Maejima (1988, Theorems 5.1, 5.2 and 5.3.). Here it is to be understood that under (A1) the limit is $Z_\alpha(t)$ with $1 < \alpha \leq 2$. Recall our convention that $Z_\alpha(t) = \Lambda_{\alpha,\frac{1}{\alpha}}(t)$ when $H = \frac{1}{\alpha}$, $1 < \alpha \leq 2$.

Next, in order to introduce the restrictions on the $f(x_0, ..., x_r)$, define,

$$f_{+,\varepsilon}(x_0, ..., x_r) = \sup\left\{f(x_0, u_1..., u_r) : |x_i - u_i| \leq \varepsilon, 1 \leq i \leq r\right\}, \quad \varepsilon > 0,$$

$$f_{-,\varepsilon}(x_0, ..., x_r) = \inf\left\{f(x_0, u_1..., u_r) : |x_i - u_i| \leq \varepsilon, 1 \leq i \leq r\right\}, \quad \varepsilon > 0.$$

(Here note that the first argument $x_0$ is left unaffected when the sup is taken.) Further, for a function $h(x_0,...,x_r)$, define, letting

$$c_j = 0 \quad \text{if } j < 0,$$

$$M_{h,n}(x) = E\left[h\left(x, x+\sum_{j=0}^{\nu_n} c_j\xi_j, ..., x+\sum_{j=0}^{\nu_n}(c_j+...+c_{j-r+1})\xi_j\right)\right].$$

Here the integers $\nu_n$ will be taken to be as defined in (15) below. Note that, because $\nu_n \uparrow \infty$,

$$\left(\sum_{j=0}^{\nu_n} c_j\xi_j, ..., \sum_{j=0}^{\nu_n}(c_j+...+c_{j-r+1})\xi_j\right) \xrightarrow{p} \left(\sum_{j=0}^{\infty} c_j\xi_j, ..., \sum_{j=0}^{\infty}(c_j+...+c_{j-r+1})\xi_j\right) = (S_1,...,S_r).$$

We impose the following restrictions on the function $f(x_0,...,x_r)$. (These restrictions are verified below for the motivating examples indicated earlier.)

**(B0)**: There is a $\eta > 0$ and a $0 < \theta < 1 - H$, such that

$$\max_{1\le l\le [n^\eta]} E\left[|f(S_l, S_{l+1}, ..., S_{l+r})|\right] \le Cn^\theta.$$

(Here note that the max is restricted only to $1 \le l \le [n^\eta]$ where $\eta > 0$ can be chosen arbitrarily small.)

**(B1)**: For some $\varepsilon_0 > 0$, $\sup_{n,0<\varepsilon<\varepsilon_0} M_{|f_{\pm,\varepsilon}|^2,n}(x) < \infty$ for each $x$.

**(B2)**: For some $\varepsilon_0 > 0$, $\sup_{n,0<\varepsilon<\varepsilon_0} \int_{-\infty}^{\infty}\left(M_{|f_{\pm,\varepsilon}|,n}(x) + M_{|f_{\pm,\varepsilon}|^2,n}(x)\right)dx \le C$.

**(B3)**: $\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\int_{-\infty}^{\infty} M_{f_{\pm,\varepsilon},n}(x)\,dx = \int_{-\infty}^{\infty} E\left[f(x, x+S_1, ..., x+S_r)\right]dx$.

**(B4)**: There is an $\varepsilon_0 > 0$ and functions $h_{\pm,\varepsilon}(x_0,...,x_r)$ such that, for all $0 < \varepsilon < \varepsilon_0$,

$$|f_{\pm,\varepsilon}| \le h_{\pm,\varepsilon} \quad \text{and} \quad M_{h_{\pm,\varepsilon},n}(x) \to E\left[h_{\pm,\varepsilon}(x, x+S_1, ..., x+S_r)\right]$$

for almost all $x$ and

$$\int_{-\infty}^{\infty} M_{h_{\pm,\varepsilon},n}(x)\,dx \to \int_{-\infty}^{\infty} E\left[h_{\pm,\varepsilon}(x, x+S_1,...,x+S_r)\right]dx < \infty.$$

Throughout below, the sequences $\kappa_n$ and $\nu_n$ stand for integers such that

$$\kappa_n = \left[n^{\frac{(1-H)}{2}}\right], \qquad \nu_n = \left[\frac{n^{\frac{(1-H)}{2}}}{\log n}\right]. \tag{15}$$

Note that $\frac{\nu_n}{\kappa_n} \to 0$.

We now state the main result of this paper in which the first statement deals with the case $|f(x_0,...,x_r)| \le C$ and the second one deals with the case where $f(x_0,...,x_r)$ is not necessarily bounded. Third statement relaxes the restriction (5) to (7) under some further mild restrictions on $M_{|f_{\pm,\varepsilon}|,n}(x)$ defined earlier.

**Theorem 1**. *Assume that $f(x_0,...,x_r)$ is such that the conditions (B0) - (B4) are satisfied. In addition, in the Statements (I) and (II) below assume that (5) holds.*

*(I): Suppose that $|f(x_0,...,x_r)| \le C$. Suppose in addition that* either one *of the following two requirements hold, for a $\nu_n$ of the form (15).*

*(a): $1 < \alpha \le 2$, $H = \frac{1}{\alpha}$, and there is a $1 < \tau < \alpha$ such that $\sum_{j=\nu_n}^{\infty} |c_j|^{\tau} = o\left(n^{-H}\right)$.*

*(b): $0 < H < 1$ and $E\left[|\xi_1|^p\right] < \infty$ for $p \ge 3$. When $H \ne \frac{1}{2}$ assume in addition that $p > \frac{2H}{(1-H)^2}$, and when $H = \frac{1}{2}$ assume in addition that $\sum_{j=\nu_n}^{\infty} c_j^2 = o\left(n^{-\frac{2H}{p}}\right)$.*

*Then the convergence (8) holds.*

*(II): (Here $f(x_0,...,x_r)$ is not necessarily bounded.) Suppose that*

*(c):$0 < H < 1$ and $E\left[|\xi_1|^p\right] < \infty$ for $p \ge 3$. When $H \ne \frac{1}{2}$ assume in addition that $p > \max\left\{\frac{2}{(1-H)^2}, \frac{2(3-H)}{1+H+2H^2}\right\}$, and when $H = \frac{1}{2}$ assume in addition that $\sum_{j=\nu_n}^{\infty} c_j^2 = o\left(n^{-\frac{2}{p}}\right)$.*

*Then the conclusion of the statement (I) holds, that is, (8) holds.*

*(III). The preceding Statements (II) and (III) hold also if instead of (5) the weaker condition (7) is assumed, provided that in addition $M_{|f_{\pm,\varepsilon}|,n}(x)$ is locally Riemann integrable and there is are $\delta > 0$ and $\varepsilon_0 > 0$ such that $\sup_{n,0<\varepsilon<\varepsilon_0} \int_{-\infty}^{\infty} \left(\sup_{|u-x|\le\delta} M_{|f_{\pm,\varepsilon}|,n}(u)\right) dx < \infty$.*

**Remark.** The proof of Theorem 1 will also give

$$n^{-1}\gamma_n \sum_{k=1}^{[nt]} f(S_k,...,S_{k+r}) \overset{fdd}{\Longrightarrow} L_t^0 \int_{-\infty}^{\infty} f_*(x)\, dx. \tag{16}$$

This can be strengthened to the convergence in the Skorokhod space $D_{\mathbf{R}}[0,1]$ as follows. First suppose that $f(x_0,...,x_r) \ge 0$. Then the left hand side in (16) is nondecreasing in $t$. In such a case the convergence in $D_{\mathbf{R}}[0,1]$ is a consequence (see Jacod and Shiryayev (1987, Theorem 3.37 (Statement (a)), page 318 and Corollary 3.33 (Statement (b)), page 317)) of the fact that the limit is continuous in $t$, that is, almost all trajectories $t \mapsto L_t^0$ are continuous. The continuity of $t \mapsto L_t^0$ is well known in the case $H = \frac{1}{\alpha}$ or in the case $H=2$, which are the cases involved in Theorem 1. (In the general case $0<\alpha\leq 2$, $0<H<1$, the continuity of $t\mapsto L_t^0$ is a consequence of the inequality (see J(2004, Remark 8 at the end of the paper)).) $E\left[\left(L_t^0-L_s^0\right)^{2l}\right]\leq C\left|t-s\right|^{l(1-H)}$ for all $l\geq 1$.

To remove the above restriction $f(x_0,...,x_r)\geq 0$, the proof below also gives

$$\left(n^{-1}\gamma_n\sum_{k=1}^{[nt]}f^{+}(S_k,...,S_{k+r}),\ n^{-1}\gamma_n\sum_{k=1}^{[nt]}f^{-}(S_k,...,S_{k+r})\right)$$
$$\overset{fdd}{\Longrightarrow}\left(L_t^0\int_{-\infty}^{\infty}f_*^{+}(x)\,dx,\ L_t^0\int_{-\infty}^{\infty}f_*^{-}(x)\,dx\right),$$

where $f^{+}$ and $f^{-}$ are respectively the positive part and the negative part of $f$. This convergence can be strengthened to that in $D_{\mathbf{R}^2}[0,1]$ using the same result in Jacod and Shiryayev (1987) mentioned above. This will strengthen the convergence (16) to that in $D_{\mathbf{R}}[0,1]$. ■

Some further comments on the statements of Theorem 1 are in order. As noted earlier, when (9) holds the conclusion of Theorem 1 holds without any of the further restrictions (a) - (c). Regarding the restrictions (a) - (c), first note that the case $H\neq\frac{1}{\alpha}$, $\alpha\neq 2$, is excluded in (a) of the Statement (I), whereas the case $H\neq\frac{1}{2}$, $\alpha=2$ is included in the Statement (II). The reason is that for the former case we do not know of any estimate analogous to the sharp bound (21) below invoked in the later case.

Also note that the condition on $c_j$'s for the case $H=\frac{1}{2}$ in the requirement (b) is much weaker than that in (a) for the case $\alpha=2$, but in (a) no extra moment condition is assumed. Further note that in each of the requirements (b) and (c), the rate of decay of the constants $c_j$ for the case $H=\frac{1}{2}$ is tied to an appropriate extra moment condition. In applications this rate of decay can often be identified. A related context in which a similar restriction is required for a similar reason is Davydov (1970), where the functional CLT for the process $\gamma_n^{-1}S_{[nt]}$ ($\alpha=2$, $H\neq\frac{1}{2}$) is dealt with and where the severity of the moment restriction is tied to the closeness of $H$ to 0. (The form $E\left[\left|\xi_1\right|^p\right]<\infty$ with $p>\max\left\{2,\frac{1}{H}\right\}$ appears to a best possible one available presently; note $\gamma_n=n^H u(n)$.) In the present context (where note that $n^{-1}\gamma_n=n^{-(1-H)}u(n)\kappa(n)$ is the normalizing factor) the severity of the moment restrictions are tied to the closeness of $1-H$ to 0 ■

We now verify (B0) - (B4) for the case $f(x,y)=\mathbb{I}_{(-\infty,0)}(xy)$. (Essentially the same arguments apply also for the other examples.) The condition (B0) is obvious. For (B4), we choose

$$h_{\pm,\varepsilon}(x,y)=\left|f_{\pm,\varepsilon}(x,y)\right|=\mathbb{I}_{(-\infty,0)}(x)\,\mathbb{I}_{(0,\infty)}(y\pm\varepsilon)+\mathbb{I}_{(0,\infty)}(x)\,\mathbb{I}_{(-\infty,0)}(y\mp\varepsilon).$$

Letting

$$\chi_{\nu_n}=\sum_{j=0}^{\nu_n}c_j\xi_j,$$

$$M_{|f_{\pm,\varepsilon}|,n}(x) = E\left[\left|f_{\pm,\varepsilon}(x, x+\chi_{\nu_n})\right|\right] \to \mathbb{I}_{(-\infty,0)}(x)\, P(S_1 > -x \mp \varepsilon) + \mathbb{I}_{(0,\infty)}(x)\, P(S_1 < -x \pm \varepsilon)$$

for all $x$ (because $\chi_{\nu_n} \Longrightarrow S_1$ and $y \mapsto P(S_1 \le y)$ is continuous.) Further

$$\int M_{|f_{\pm,\varepsilon}|,n}(x)\, dx = E\left[(\chi_{\nu_n} \pm \varepsilon)^+\right] + E\left[(\chi_{\nu_n} \mp \varepsilon)^-\right] \to E\left[(S_1 \pm \varepsilon)^+\right] + E\left[(S_1 \mp \varepsilon)^-\right].$$

This verifies (B4), and implicitly (B2) is also verified. Because $|f_{\pm,\varepsilon}| = f_{\pm,\varepsilon}$ and

$$E\left[(S_1 \pm \varepsilon)^+\right] + E\left[(S_1 \mp \varepsilon)^-\right] \to E\left[|S_1|\right] = \int E\left[f(x, x+S_1)\right] dx \quad \text{as } \varepsilon \to 0,$$

(B3) is also verified.

## 3 Proof of Theorem 1

For simplicity we present the proof for the case $r = 1$, that is, to the case $f(x_0, x_1)$, and the required modifications needed for the general case $f(x_0, ..., x_r)$ are mostly essentially notational. Also, for convenience, we shall restrict below to the situation where $\kappa(n)$ in (1) and $u(n)$ in (A2) are such that $\kappa(n) \equiv 1 \equiv u(n)$, so that we can take $\gamma_n = n^H$.

We have

$$S_l = \sum_{j=-\infty}^{0} (g(l-j) - g(1-j))\xi_j + \sum_{j=1}^{l} g(l-j)\xi_j,$$

where we let

$$g(k) = \sum_{s=0}^{k} c_s.$$

Define for $\nu_n < l$ and $l \ge \kappa_n$,

$$S_{nl}^* = \sum_{j=-\infty}^{0} (g(l-j) - g(1-j))\xi_j + \sum_{j=1}^{l-\nu_n} g(l-j)\xi_j,$$

$$S_{nl,1}^* = \sum_{j=-\infty}^{0} (g(l+1-j) - g(1-j))\xi_j + \sum_{j=1}^{l-\nu_n} g(l+1-j)\xi_j.$$

Note that $S_{nl,1}^*$ and $S_{nl}^*$ depend on $\nu_n$.

In the Lemmas 2 and 3 below we shall use the following four inequalities. First, if $E[\xi_j] = 0$ and $E[|\xi_1|^p] < \infty$, $p \ge 2$, we have (Whittle (1960))

$$E\left[\left|\sum_{j=0}^{\infty} d_j \xi_j\right|^p\right] \le C E\left[|\xi_1|^p\right] \left(\sum_{j=0}^{\infty} d_j^2\right)^{p/2} \qquad (17)$$

for constants $d_j$ such that $\sum_{j=0}^{\infty} d_j^2 < \infty$.

Second, if $0 < \alpha \leq 2$, then (see for instance Avram and Taqqu (1986, Lemma 1, Section 3, page 408))

$$E\left[\left|\sum_{j=\nu}^{\infty} d_j \xi_j\right|^{\tau}\right] \leq CE\left[|\xi_1|^{\tau}\right] \sum_{j=\nu}^{\infty} |d_j|^{\tau} \quad \text{for all } 0 < \tau < \alpha \text{ and } \nu \geq 0 \tag{18}$$

for constants $d_j$ such that $\sum_{j=\nu}^{\infty} |d_j|^{\tau} < \infty$.

To state the next two inequalities, note that, with $g(k)$ as before (and when $u(n) \equiv 1$ in (A2))

$$g(k) = \sum_{s=0}^{k} c_s \sim C k^{H-\frac{1}{2}} \quad \text{under (A2)}. \tag{19}$$

Then, as the third inequality, we have for any $0 < \tau < \alpha$, $0 < \alpha \leq 2$,

$$E\left[\left|\nu^{-H} \sum_{j=0}^{\nu-1} g(j)\xi_j\right|^{\tau}\right] \leq C. \tag{20}$$

This inequality follows using the arguments in for example Borodin and Ibragimov (1995, page 9).

Fourth, suppose that $E[\xi_j] = 0$ and $E[|\xi_1|^s] < \infty$ for $s \geq 3$ . Then, noting (19), we have (using Bhattacharya and Ranga Rao (1976, Corollary 17.13, 179)),

$$P\left[\left|\nu^{-H} \sum_{j=0}^{\nu-1} g(j)\xi_j\right| > a_n\right] = P\left[\left|\nu^{-\frac{1}{2}} \sum_{j=0}^{\nu-1} \frac{g(j)}{\nu^{H-\frac{1}{2}}}\xi_j\right| > a_\nu\right] \leq C\nu^{-\frac{s-2}{2}} a_\nu^{-s} \tag{21}$$

for every sequence $a_\nu$ of real numbers such that $a_\nu \geq C(s-2+\delta)\sqrt{\log \nu}$ for any $\delta > 0$.

Recall that the constants $\nu_n$ and $\kappa_n$ are as in (15). Also, in all the statements below, appropriate conditions of Theorem 1 are assumed to hold.

**Lemma 2.** *Assume that either one of the requirements (a) and (b) of Theorem 1 (with $S^*_{nl,1}$ and $S^*_{nl}$ corresponding to $\nu_n$). Then when $k_n^{-1}\nu_n \to 0$ suitably slowly,*

$$\frac{1}{n^{1-H}} \sum_{l=\kappa_n}^{n} \left\{P\left(\left|S^*_{nl,1} - S^*_{nl}\right| > \varepsilon\right) + P\left(\left|n^{-H}\left(S_l - S^*_{nl}\right)\right| > \varepsilon\right)\right\} \to 0.$$

*Similarly under the requirement (c) of Theorem 1,*

$$\sum_{l=\kappa_n}^{n} \left\{P\left(\left|S^*_{nl,1} - S^*_{nl}\right| > \varepsilon\right) + P\left(\left|n^{-H}\left(S_l - S^*_{nl}\right)\right| > \varepsilon\right)\right\} \to 0.$$

**Proof.** First assume that (a) holds, in which case recall that $H = \frac{1}{\alpha}$, $1 < \alpha \leq 2$. Note that

$$S^*_{nl,1} - S^*_{nl} = \sum_{j=-\infty}^{l-\nu_n} c_{l-j}\xi_j \stackrel{\mathcal{D}}{=} \sum_{j=\nu_n}^{\infty} c_j \xi_j.$$

Hence, using (18), for any $\tau$ for which $1 < \tau < \alpha$,

$$P\left(\left|S^*_{nl,1} - S^*_{nl}\right| > \varepsilon\right) \leq C \sum_{j=\nu_n}^{\infty} |c_j|^{\tau},$$

and hence $\frac{1}{n^{1-H}} \sum_{l=\kappa_n}^{n} P\left(\left|S^*_{nl,1} - S^*_{nl}\right| > \varepsilon\right) \leq C n^H \sum_{j=\nu_n}^{\infty} |c_j|^{\tau} \to 0$ when $\tau$ is such that (a) of Theorem 1 holds.

Next, note that

$$(S_l - S^*_{nl}) \stackrel{\mathcal{D}}{=} \sum_{j=0}^{\nu_n - 1} g(j)\xi_j.$$

Hence, using (20), for any $1 < \tau < \alpha$,

$$n^H E\left[\left|n^{-H} \sum_{j=0}^{\nu_n-1} g(j)\xi_j\right|^{\tau}\right] \leq C n^H n^{-\tau H} \nu_n^{\tau H} \leq C n^{H\left(1-\tau+\frac{\tau(1-H)}{2}\right)} \to 0$$

because $1 - \tau + \frac{\tau(1-H)}{2} = -\frac{\tau}{2}(1+H) + 1 < 0$, by choosing $\alpha > \tau > \frac{2}{1+H} = \frac{2\alpha}{1+\alpha}$. (Note that because $\alpha > 1$, $1 < \frac{2\alpha}{1+\alpha} < \alpha$.) Hence $\frac{1}{n^{1-H}} \sum_{l=\kappa_n}^{n} P\left(\left|n^{-H}\left(S_l - S^*_{nl}\right)\right| > \varepsilon\right) \to 0$. This proves the first part of the lemma under the requirement (a).

Now suppose that the requirement (b) of Theorem 1 holds. We have, using (17),

$$P\left(\left|S^*_{nl,1} - S^*_{nl}\right| > \varepsilon\right) \leq \frac{C}{\varepsilon^p}\left(\sum_{j=\nu_n}^{\infty} c_j^2\right)^{p/2}.$$

If $H \neq 1/2$, then $\sum_{j=\nu_n}^{\infty} c_j^2 \sim C \sum_{j=\nu_n}^{\infty} j^{2(H-1)-1} \sim C\nu_n^{2(H-1)}$. Hence in view of (1) (recall $n^{-\frac{1-H}{2}}\nu_n \sim k_n^{-1}\nu_n \sim \log n$ ),

$$\frac{1}{n^{1-H}} \sum_{l=\kappa_n}^{n} P\left(\left|S^*_{nl,1} - S^*_{nl}\right| > \varepsilon\right) \leq C n^H \nu_n^{p(H-1)} = C\left(n^{-\frac{1-H}{2}}\nu_n\right)^{p(H-1)} n^{-\frac{p}{2}(H-1)^2+H} \to 0,$$

because $H - \frac{p}{2}(1-H)^2 < 0$ is equivalent to $p > \frac{2H}{(1-H)^2}$, which is an assumption. This convergence holds when $H = 1/2$ also because $\left(\sum_{j=\nu_n}^{\infty} c_j^2\right)^{p/2} = o\left(n^{-H}\right)$ by assumption.

Next, we have using (21),

$$\begin{aligned}
P\left(\left|n^{-H}\left(S_l - S^*_{nl}\right)\right| > \varepsilon\right) &= P\left(\left|\nu_n^{-1/2} \sum_{j=0}^{\nu_n-1} \frac{g(j)}{\nu_n^{H-1/2}}\xi_j\right| > \varepsilon\left(\frac{n}{\nu_n}\right)^H\right) \\
&\leq C\nu_n^{-\frac{p-2}{2}}\left(\frac{n}{\nu_n}\right)^{-pH} \\
&\sim C\left(n^{-\frac{1-H}{2}}\nu_n\right)^{pH-\frac{p-2}{2}} n^{-\frac{(1+H)pH}{2}-\frac{(1-H)(p-2)}{4}}. \qquad (22)
\end{aligned}$$

Here $\frac{(1+H)pH}{2}+\frac{(1-H)(p-2)}{4} > H$ is equivalent to $p > \frac{2(1+H)}{1+H+2H^2}$, which is always true because $\frac{2(1+H)}{1+H+2H^2} < 2$ and $p \geq 3$. Thus $\frac{1}{n^{1-H}} \sum_{l=\kappa_n}^{n} P\left(\left|n^{-H}\left(S_l - S_{nl}^*\right)\right| > \varepsilon\right) \to 0$. This completes the proof of the first part of the lemma.

The proof of the second part under (c) is essentially the same as the preceding proof under the requirement (b). Note that in using (22), we now require $\frac{(1+H)pH}{2}+\frac{(1-H)(p-2)}{4} > 1$, which is equivalent to $p > \frac{2(3-H)}{1+H+2H^2}$, which is the assumption in the case $H \neq 1/2$. In the case $H = 1/2$, noting that $p \geq 3$, we always have $p \geq 3 > \frac{2(3-H)}{1+H+2H^2}$. This completes the proof of the lemma. ■

In view of Lemma 2, we obtain

**Lemma 3.** *The difference between* $\frac{1}{n^{1-H}} \sum_{l=\kappa_n}^{n} f(S_l, S_{l+1})$ *and*

$$\frac{1}{n^{1-H}} \sum_{l=\kappa_n}^{n} f(S_l, S_{l+1}) \mathbb{I}\left(\left|S_{nl,1}^* - S_{nl}^*\right| \leq \varepsilon, \left|n^{-H}\left(S_l - S_{nl}^*\right)\right| \leq \varepsilon\right)$$

*converges to* 0 *in probability.*

**Proof.** This is clear in the case of the second statement of Lemma 2, because its conclusion implies $\sup_{\kappa_n \leq l \leq n}\left(\left|S_{nl,1}^* - S_{nl}^*\right| + \left|n^{-H}\left(S_l - S_{nl}^*\right)\right|\right) \to 0$. In the case of the first statement of Lemma 2, the expected value of the difference in the statement is bounded in absolute value by $\frac{C}{n^{1-H}} \sum_{l=\kappa_n}^{n} \left\{P\left(\left|S_{nl,1}^* - S_{nl}^*\right| > \varepsilon\right) + P\left(\left|n^{-H}\left(S_l - S_{nl}^*\right)\right| > \varepsilon\right)\right\}$, because $|f(x,y)| \leq C$. ■

Now, using

$$S_{l+1} = S_{nl}^* + S_{l+1} - S_{nl,1}^* + S_{nl,1}^* - S_{nl}^*, \tag{23}$$

we have

$$f(S_l, S_{l+1}) \mathbb{I}\left(\left|S_{nl,1}^* - S_{nl}^*\right| \leq \varepsilon, \left|n^{-H}\left(S_l - S_{nl}^*\right)\right| \leq \varepsilon\right)$$
$$\begin{cases} \leq f_{+,\varepsilon}\left(S_l, S_{nl}^* + S_{l+1} - S_{nl,1}^*\right) \mathbb{I}\left(\left|n^{-H}\left(S_l - S_{nl}^*\right)\right| \leq \varepsilon\right) \\ \geq f_{-,\varepsilon}\left(S_l, S_{nl}^* + S_{l+1} - S_{nl,1}^*\right) \mathbb{I}\left(\left|S_{nl,1}^* - S_{nl}^*\right| \leq \varepsilon, \left|n^{-H}\left(S_l - S_{nl}^*\right)\right| \leq \varepsilon\right). \end{cases}$$

Hence, by invoking Lemma 2 again,

$$P\left[\begin{array}{l} n^{-(1-H)} \sum_{l=\kappa_n}^{n} f(S_l, S_{l+1}) \\ \begin{cases} \leq n^{-(1-H)} \sum_{l=\kappa_n}^{n} f_{+,\varepsilon}\left(S_l, S_{nl}^* + S_{l+1} - S_{nl,1}^*\right) \mathbb{I}\left(\left|n^{-H}\left(S_l - S_{nl}^*\right)\right| \leq \varepsilon\right) \\ \geq n^{-(1-H)} \sum_{l=\kappa_n}^{n} f_{-,\varepsilon}\left(S_l, S_{nl}^* + S_{l+1} - S_{nl,1}^*\right) \mathbb{I}\left(\left|n^{-H}\left(S_l - S_{nl}^*\right)\right| \leq \varepsilon\right) \end{cases} \end{array}\right] \to 1.$$

Then the idea of the proof of Theorem 1 will consist of completing the steps: (i) the bounds in the preceding statement converge in distribution, as $n \to \infty$ first and then $\varepsilon \to 0$, to one and the same limit, and (ii) the remaining part $n^{-(1-H)} \sum_{l=1}^{\kappa_n} f(S_l, S_{l+1}) \xrightarrow{p} 0$.

**Lemma 4.** *There is a* $\varepsilon_0 > 0$ *such that for all* $0 < \varepsilon < \varepsilon_0$,

$$n^{-(1-H)} \sum_{l=\kappa_n}^{n} E\left[h_\varepsilon\left(S_l, S_{nl}^* + S_{l+1} - S_{nl,1}^*\right)\right] \le C \quad \text{when } h_\varepsilon(x,y) = \left|f_{\pm,\varepsilon}(x,y)\right| \text{ or } \left|f_{\pm,\varepsilon}(x,y)\right|^2.$$

**Proof**. $S_{nl}^*$ and $\left(S_l - S_{nl}^*, S_{l+1} - S_{nl,1}^*\right)$ are independent, so that we can write

$$E\left[h_\varepsilon\left(S_l, S_{nl}^* + S_{l+1} - S_{nl,1}^*\right)\right] = E\left[\varphi\left(S_{nl}^*\right)\right]$$

with

$$\varphi_n(x) = E\left[h_\varepsilon\left(x + S_l - S_{nl}^*, x + S_{l+1} - S_{nl,1}^*\right)\right].$$

Now

$$\left(S_l - S_{nl}^*, S_{l+1} - S_{nl,1}^*\right) \overset{\mathcal{D}}{=} \left(\sum_{j=0}^{\nu_n - 1} g(\nu_n - 1 - j)\xi_j, \sum_{j=0}^{\nu_n} g(\nu_n - j)\xi_j\right)$$

and $\sum_{j=0}^{\nu_n} g(\nu_n - j)\xi_j - \sum_{j=0}^{\nu_n - 1} g(\nu_n - 1 - j)\xi_j = \sum_{j=0}^{\nu_n} c_{\nu_n - j}\xi_j$. Hence (see (23))

$$\varphi_n(x) = E\left[h_\varepsilon\left(x + T_{\nu_n}, x + T_{\nu_n} + \chi_{\nu_n}\right)\right]$$

where

$$T_{\nu_n} = \sum_{j=0}^{\nu_n - 1} g(\nu_n - 1 - j)\xi_j, \qquad \chi_{\nu_n} = \sum_{j=0}^{\nu_n} c_{\nu_n - j}\xi_j. \tag{24}$$

First suppose that (5) holds. Then, noting that $\varphi_n(x) = \frac{1}{2\pi}\int e^{-i\lambda x}\widehat{\varphi_n}(\lambda)\, d\lambda$, where $\widehat{\varphi_n}(\lambda)$ is the Fourier transform of $\varphi_n(x)$, and taking $x = S_{nl}^*$, we have

$$2\pi\left|E\left[\varphi\left(S_{nl}^*\right)\right]\right| \le \frac{1}{l^H}\int \left|E\left[e^{-i\frac{\lambda}{l^H}S_{nl}^*}\right]\right| \left|\widehat{\varphi_n}\left(\frac{\lambda}{l^H}\right)\right| d\lambda$$

where

$$\sup_\lambda \left|\widehat{\varphi_n}(\lambda)\right| \le \int \varphi_n(x)\, dx = \int E\left[h_\varepsilon\left(x, x + \chi_{\nu_n}\right)\right] dx \le C, \quad \text{by (B2)},$$

and, noting that $S_{nl}^* \overset{\mathcal{D}}{=} \sum_{j=\nu_n}^{l-1} g(j)\xi_j$ and $\nu_n < \left[\frac{l}{2}\right]$ when $l \ge \kappa_n$,

$$\int \left|E\left[e^{-i\frac{\lambda}{l^H}S_{nl}^*}\right]\right| d\lambda = \int \prod_{j=\nu_n}^{l} \left|\psi\left(\frac{\lambda}{l^H}g(j)\right)\right| d\lambda \le \int \prod_{j=\left[\frac{l}{2}\right]}^{l} \left|\psi\left(\frac{\lambda}{l^H}g(j)\right)\right| d\lambda \le C,$$

see for instance J(2004, Lemma 17). Thus $\left|E\left[\varphi\left(S_{nl}^*\right)\right]\right| \le \frac{C}{l^H}$. Hence, noting $0 < H < 1$,

$$\frac{1}{n^{1-H}} \sum_{l=\nu_n}^{n} E\left[\varphi\left(S_{nl}^*\right)\right] \le \frac{C}{n^{1-H}} \sum_{l=1}^{n} l^{-H} \le C.$$

The same holds under (7) and under the further restrictions of the statement (III) of Theorem 1, see J(2004, Section 6). This completes the proof of the lemma. ■

**Lemma 5.** *Let*

$$q_{n\varepsilon}(y) = E\left[f_{\pm,\varepsilon}\left(y + T_{\nu_n}, y + T_{\nu_n} + \chi_{\nu_n}\right)\mathbb{I}\left(\left|n^{-H}T_{\nu_n}\right| \le \varepsilon\right)\right]$$

*with* $(T_{\nu_n}, \chi_{\nu_n})$ *as defined in (24). Then the difference*

$$\frac{1}{n^{1-H}}\sum_{l=\kappa_n}^{n} f_{\pm,\varepsilon}\left(S_l, S_{nl}^* + S_{l+1} - S_{nl,1}^*\right)\mathbb{I}\left(\left|n^{-H}\left(S_l - S_{nl}^*\right)\right| \le \varepsilon\right) - \frac{1}{n^{1-H}}\sum_{l=\kappa_n}^{n} q_{n\varepsilon}\left(S_{nl}^*\right) \xrightarrow{p} 0.$$

**Proof.** Let $E_j$ denote the conditional expectation given $\{\xi_i : i \le j\}$. Further let

$$G_l = f_{\pm,\varepsilon}\left(S_l, S_{nl}^* + S_{l+1} - S_{nl,1}^*\right) I\left(\left|n^{-H}\left(S_l - S_{nl}^*\right)\right| \le \varepsilon\right).$$

Then note that (similar to $\varphi_n(x)$ in the proof of Lemma 4)

$$E_{l-\nu_n}\left[G_l\right] = q_{n\varepsilon}\left(S_{nl}^*\right).$$

Now for $k < \kappa_n$ and $l \ge \kappa_n$,

$$\begin{aligned} a_k &= E\left[\left(\sum_{l=\kappa_n}^{n}\left(G_l - E_{l-k}\left[G_l\right]\right)\right)^2\right] \\ &= E\left[\left(\sum_{l=\kappa_n}^{n}\left(G_l - E_{l-k+1}\left[G_l\right]\right)\right)^2\right] + E\left[\left(\sum_{l=\kappa_n}^{n}\left(E_{l-k+1}\left[G_l\right] - E_{l-k}\left[G_l\right]\right)\right)^2\right] \\ &\quad + 2E\left[\left(\sum_{l=\kappa_n}^{n}\left(G_l - E_{l-k+1}\left[G_l\right]\right)\right)\left(\sum_{j=\kappa_n}^{n}\left(E_{j-k+1}\left[G_j\right] - E_{j-k}\left[G_j\right]\right)\right)\right]. \end{aligned}$$

The first term on the right hand side is $a_{k-1}$. Because $E_{l-k+1}[G_l] - E_{l-k}[G_l]$ are martingale differences, the second term is, by Lemma 4, bounded by

$$\sum_{l=\kappa_n}^{n} E\left[G_l^2\right] \le Cn^{(1-H)} = b_n, \text{ say.}$$

Then the third term is (by Cauchy-Schwarz inequality) bounded by $2\sqrt{a_{k-1}b_n}$. Hence $a_k \le a_{k-1} + b_n + 2\sqrt{a_{k-1}b_n} = \left(\sqrt{a_{k-1}} + \sqrt{b_n}\right)^2$, that is, $\sqrt{a_k} \le \sqrt{a_{k-1}} + \sqrt{b_n}$ and hence $\sqrt{a_k} \le k\sqrt{b_n}$ because $a_0 = 0$. Hence $a_{\nu_n} \le \nu_n^2 b_n$ so that $n^{-2(1-H)}a_{\nu_n} \le C\nu_n^2 n^{-(1-H)} \to 0$, by assumption (15). Thus

$$a_{\nu_n} = E\left[\left(\sum_{l=\kappa_n}^{n}\left(G_l - E_{l-\nu_n}\left[G_l\right]\right)\right)^2\right] \to 0.$$

Because $E_{l-\nu_n}[G_l] = q_{n\varepsilon}(S^*_{nl})$, the proof is complete. ■

**Lemma 6.**

$$n^{-(1-H)} \sum_{l=1}^{\kappa_n} f(S_l, S_{l+1}) \xrightarrow{p} 0.$$

**Proof.** Recall that $\kappa_n = \left[n^{\frac{1-H}{2}}\right]$. Letting $\rho_{0n} = n$ and $\rho_{1n} = \left[n^{1-H}\right]$,

$$\rho_{0n}^{-(1-H)} \sum_{l=\rho_{1n}}^{\rho_{0n}} f(S_l, S_{l+1}) = O_p(1)$$

by Lemma 4 and by the remarks immediately prior to that lemma. Because this is true also when the pair $(\rho_{0n}, \rho_{1n})$ is replaced by $(\rho_{1n}, \rho_{2n})$ with $\rho_{2n} = \left[\rho_{1n}^{1-H}\right]$,

$$\rho_{1n}^{-(1-H)} \sum_{l=\rho_{2n}}^{\rho_{1n}} f(S_l, S_{l+1}) = O_p(1).$$

Hence

$$n^{-(1-H)} \sum_{l=\rho_{2n}}^{\rho_{1n}} f(S_l, S_{l+1}) \xrightarrow{p} 0.$$

Continuing this a finite number of times

$$n^{-(1-H)} \sum_{l=\left[n^{\beta}\right]}^{\rho_{1n}} f(S_l, S_{l+1}) \xrightarrow{p} 0$$

for any $\beta$ of the form $\beta = (1-H)^m$ for any integer $m > 1$. Now choose $m$ such that $\beta < \min(\eta, 1-H-\theta)$, where $\eta$ and $\theta$ are as in (B0). Then

$$n^{-(1-H)} \sum_{l=1}^{\left[n^{\beta}\right]} E\left[|f(S_l, S_{l+1})|\right] \le C\left[n^{\beta}\right] n^{\theta-(1-H)} \to 0.$$

This completes the proof of the lemma. ■

**Lemma 7.** *With $q_{n\varepsilon}(y)$ as in Lemma 5,*

$$n^{-(1-H)} \sum_{l=\kappa_n}^{n} q_{n\varepsilon}(S^*_{nl}) \Longrightarrow L_1^0 \int f_*(y)\, dy.$$

*as $n \to \infty$ first and then $\varepsilon \to 0$ where $f_*(x) = E[f(x, x+X_1)]$.*

**Proof.** Assume that (5) holds. According to J(2004, Statement (ii) of Theorem 3 and Remark 4 of Section 3), the convergence in the statement of the lemma holds for $n^{-(1-H)} \sum_{l=\kappa_n}^{n} q_{n\varepsilon}(S_l)$ (that is, when $q_{n\varepsilon}(S_l)$ is involved in place of $q_{n\varepsilon}(S^*_{nl})$ ) if the following conditions hold (note $n^{-1}\kappa_n \to 0$).

(i): $\sup_{n,0<\varepsilon<\varepsilon_0} \int \left(|q_{n\varepsilon}(x)| + |q_{n\varepsilon}(x)|^2\right) dx < \infty$ for some $\varepsilon_0 > 0$,

(ii): $\lim_{\varepsilon\to 0} \sup_n \int_{\{|x|>an^H\}} |q_{n\varepsilon}(x)|\, dx = 0$ for every $a > 0$, and

(iii): $\lim_{\varepsilon\to 0} \sup_n \int q_{n\varepsilon}(x)\, dx = \int f_*(x)\, dx$.

Now, because $S_l - S^*_{nl} = \sum_{j=l-\nu_n+1}^{l} g(l-j)\xi_j$ with $\frac{\nu_n}{\kappa_n} \to 0$, it is easy to see that the same arguments in J(2004) will give the same convergence for $n^{-(1-H)} \sum_{l=\kappa_n}^{n} q_{n\varepsilon}(S^*_{nl})$ also. (Specifically, the arguments of the proof of the approximation in Proposition 6 of J(2004) under (5) hold for $n^{-(1-H)} \sum_{l=\kappa_n}^{n} q_{n\varepsilon}(S^*_{nl})$ also.)

We now show that (i) - (iii) above follow from (B1) - (B4) (with $r = 1$). Recall that $\chi_{\nu_n} \stackrel{\mathcal{D}}{=} \sum_{j=0}^{\nu_n} c_j \xi_j$, see (24). We have

$$\begin{aligned}
\int_{\{|x|>an^H\}} |q_{n\varepsilon}(x)|\, dx &= E\left[\int_{\{|x|>an^H\}} |f_{\pm,\varepsilon}(x + T_{\nu_n}, x + T_{\nu_n} + \chi_{\nu_n})|\, \mathbb{I}\left(\left|n^{-H}T_{\nu_n}\right| \le \varepsilon\right)\right] dx \\
&\le \int_{\{|x|>(a-\varepsilon_0)n^H\}} E\left[|f_{\pm,\varepsilon}(x, x + \chi_{\nu_n})|\right] dx
\end{aligned}$$

when $0 < \varepsilon < \varepsilon_0 < a$, so that (ii) follows from (B4) by an extended version (Young) of Lebesgue dominated convergence theorem (see Loève (1963, page 162)). In the same way (i) follows from (B2). Now note that

$$\int q_{n\varepsilon}(x)\, dx = \int E\left[f_{\pm,\varepsilon}(x, x + \chi_{\nu_n})\, \mathbb{I}\left(\left|n^{-H}T_{\nu_n}\right| \le \varepsilon\right)\right] dx,$$

and

$$\int E\left[|f_{\pm,\varepsilon}(x, x + \chi_{\nu_n})|\, \mathbb{I}\left(\left|n^{-H}T_{\nu_n}\right| > \varepsilon\right)\right] dx \to 0$$

by (B4) using the extended dominated convergence theorem as above because, in view of $P\left(\left|n^{-H}T_{\nu_n}\right| > \varepsilon\right) \to 0$ and by (B1), $E\left[|f_{\pm,\varepsilon}(x, x + \chi_{\nu_n})|\, \mathbb{I}\left(\left|n^{-H}T_{\nu_n}\right| > \varepsilon\right)\right] \to 0$ for each $x$. Hence (iii) follows from (B3). This completes the proof of the lemma when (5) holds.

In the same way, now invoking an appropriate modification of Theorem 2 in J(2004) (in the same way the modification of Theorem 3 in J(2004) is used above), the proof of the lemma is obtained under (7) and other restrictions in the statement (III) of Theorem 1.

This completes the proof and hence the proof of Theorem 1 is completed. ■

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