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Convergence in Distribution of Row Sum Processes to Mixtures of Additive Processes

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Abstract. Consider the arrays $\{X_{nk}; k=1,2,...\}$ and $\{Y_{nk}; k=1,2,...\}$ adapted to the same array $\{\mathcal{A}_{nk}; k=0,1,2,...\}$ of σ-fields such that $A_{n,k-1} \subset A_{nk}$ for all $k \geq 1$, $n \geq 1$. Further, for each $n \geq 1$ and $t \in [0,1]$, let $k_n(t)$, $k_n(0)=0$, be integer valued, non-decreasing and right-continuous stopping times adapted to $\{\mathcal{A}_{nk}; k=0,1,2,...\}$. Conditions are given under which $\left(\sum_{k=1}^{k_n(t)} X_{nk}, \sum_{k=1}^{k_n(t)} g\left(\sum_{j=1}^{k-1} X_{nj}, Y_{nk}\right)\right)$, for a suitable function $g(u,v)$ (in particular for $g(u,v)=v$), converges in distribution to $(S(t), R(t))$ in the Skorokhod space $D_{\mathbb{R}^2}[0,1]$, such that $S(t)$ is an additive process and, conditionally on $(S(u), 0 \leq u \leq 1)$, the process $R(t)$ is (conditionally) additive, that is, $R(t)$ will be a suitable mixture of an additive process. The same convergence holds for $\left(\sum_{k=1}^{[nt]} X_{nk}, \sum_{k=1}^{k_n(t)} Y_{nk}\right)$ or $\left(\sum_{k=1}^{k_n(t)} X_{nk}, \sum_{k=1}^{[nt]} Y_{nk}\right)$ also. The results are motivated by several applications, some of which are indicated here, and some are presented elsewhere.

1 INTRODUCTION

Let $\{X_{nk}; k=1,2,...\}$ and $\{Y_{nk}; k=1,2,...\}$ be arrays of random variables and let $\{\mathcal{A}_{nk}; k=0,1,2,...\}$ be an array of σ-fields such that $\mathcal{A}_{n,k-1} \subset \mathcal{A}_{nk}$ for all $k \geq 1$, $n \geq 1$. Assume, throughout this paper, that both $\{X_{nk}; k=1,2,...\}$ and $\{Y_{nk}; k=1,2,...\}$ are adapted to $\{\mathcal{A}_{nk}; k=0,1,2,...\}$, that is, both X_{nk} and Y_{nk} are $\mathcal{A}_{nk}$ measurable for each $k \geq 1$, $n \geq 1$.

Further, for each $n \geq 1$ and $t \in [0,1]$, let $k_n(t)$ be stopping times adapted to $\{\mathcal{A}_{nk}; k=0,1,2,...\}$, that is, the event $\{k_n(t)=l\} \in \mathcal{A}_{nl}$, $l=0,1,2,...$ In addition, the sample paths of $k_n(t)$ are integer valued, non-decreasing and right-continuous with $k_n(0)=0$.

Let,

$$S_n(t) = \sum_{k=1}^{k_n(t)} X_{nk}, \quad t \in [0,1].$$

The conditions under which $S_n(t) \overset{fdd}{\Longrightarrow} S(t)$, in the sense of convergence in distribution of all finite dimensional distributions, or $S_n(t) \Longrightarrow S(t)$ in the Skorokhod space $D_{\mathbb{R}}[0,1]$, where $S(t)$ is an additive process, have been studied in several places, see for example Durrett and Resnick (1978) and Jeganathan (1983), Liptser and Shiryayev (1989, Chs 5 - 7) and Jacod and Shiryayev (1987, Ch. VIII). (See below for the notations $\overset{fdd}{\Longrightarrow}$ and $\Longrightarrow$ in $D_{\mathbb{R}}[0,1]$.) Here, an additive process is by definition a stochastically continuous process with independent increments, see for example Gikhman and Skorokhod (1969, Ch. 6).

Now let

$$R_n(t) = \sum_{k=1}^{k_n(t)} Y_{nk}, \qquad R_n^*(t) = \sum_{k=1}^{k_n(t)} g\left(\sum_{j=1}^{k-1} X_{nj}, Y_{nk}\right)$$

for a suitable function $g(u,v)$ (as specified in Theorem 2 in Section 2 below).

In this paper we study the conditions under which $(S_n(t), R_n(t)) \Longrightarrow (S(t), R(t))$ in $D_{\mathbb{R}^2}[0,1]$, such that $S(t)$ is an additive process as before and, conditionally on $(S(u), 0 \le u \le 1)$, the process $R(t)$ is (conditionally) additive, that is, $R(t)$ will be a suitable mixture of an additive process. Results of the same form are obtained for the sequence $(S_n(t), R_n^*(t))$ also. The same convergence holds for $\left(\sum_{k=1}^{[nt]} X_{nk}, \sum_{k=1}^{k_n(t)} Y_{nk}\right)$ or $\left(\sum_{k=1}^{k_n(t)} X_{nk}, \sum_{k=1}^{[nt]} Y_{nk}\right)$ also.

In the important special case in which one of limits $S(t)$ or $R(t)$ has only the jump component and the other has no jump component, the required conditions will involve the arrays $\{X_{nk}; k = 1, 2, ...\}$ and $\{Y_{nk}; k = 1, 2, ...\}$ only separately. In the general case we shall also require a condition to the effect that certain point processes associated with the limits $S(t)$ and $R(t)$ do not jump simultaneously.

It may be noted that the present results may not be confused with those in for example Jeganathan (1983) or Liptser and Shiryayev (1989, Chs. 5 - 7) or Shiryayev (1987, Ch. VIII, Section 5) where also mixtures of additive processes occur as limits but the conditions involved there are in terms of the convergence in probability, whereas in the present situations such conditions will not be satisfied.

In Section 2 we state the conditions and the main results. In Section 3 some illustrative examples are presented which partly motivated the present investigation, but the applications that will require the full force of the present results to certain network traffic models are presented separately (Jeganathan (2006b)), in view of the importance of the contexts of those models. (Another application is given in Jeganathan (2006a, Lemma 4).) Proofs are presented in Sections 4 and 5.

Notations and terminologies. $\mathbb{I}_{\{|x| \le \tau\}}$ stands for the indicator function of the interval $[-\tau, \tau]$ of the real line. The indicator function of an event such as $\{|X_{nj}| \le \tau\}$ will be denoted by $\mathbb{I}_{\{|X_{nj}| \le \tau\}}$. $a \wedge b$ and $a \vee b$ stand respectively for $\min\{a, b\}$ and

$\max\{a,b\}$. The real line will be denoted by $\mathbb{R}$, and $\mathbb{R}-\{0\}=(-\infty,0)\cup(0,\infty)$.

Convergence in distribution of a sequence of random vectors (of the same order) will be denoted by $\overset{d}{\Longrightarrow}$. Also, $\xi_{n,\epsilon}\overset{d}{\Longrightarrow}\xi$ as $n\to\infty$ first and then $\epsilon\to 0$ means that $\lim_{\epsilon\to 0}\lim_{n\to\infty}\rho(\xi_{n,\epsilon},\xi)=0$ where $\rho(\xi_{n,\epsilon},\xi)$ is the Lévy distance between the distribution functions of $\xi_{n,\epsilon}$ and ξ. (See for instance Loève (1963, page 215) for the Lévy distance.)

Convergence in probability will be denoted by $\overset{p}{\to}$. By $\xi_{n,\epsilon}\overset{p}{\to}0$ as $n\to\infty$ first and then $\epsilon\to 0$, we mean that $\lim_{\epsilon\to 0}\operatorname{limsup}_{n\to\infty}P[|\xi_{n,\epsilon}|>\eta]=0$ for every $\eta>0$.

The notation $\overset{fdd}{\Longrightarrow}$ stands for the convergence in distribution of a sequence of random processes in the sense of convergence in distribution of all finite dimensional distributions. Also, if the index set of the processes involved is not clear from the context it will be explicitly indicated; for example in the form $S_n(t)\overset{fdd}{\Longrightarrow}S(t)$, $t\in J$, instead of $S_n(t)\overset{fdd}{\Longrightarrow}S(t)$, where $J\subset[0,1]$.

The notation $\Longrightarrow$ in $D_{\mathbb{R}^q}[0,1]$, $q\geq 1$ means the convergence in distribution of a sequence of random processes in the Skorokhod space $D_{\mathbb{R}^q}[0,1]$. ($D_{\mathbb{R}^q}[0,1]$ is by definition the collection of all functions from $[0,1]$ to $\mathbb{R}^q$ that are right continuous and admit left-hand limits, equipped with the Skorokhod topology, see Billingsley (1968, Ch. 3) for $D_{\mathbb{R}}[0,1]$ and Jacod and Shiryayev (1987, Ch. VI) for $D_{\mathbb{R}^q}[0,1]$, $q\geq 1$.)

The process $S(t)$, $t\in[0,1]$, will be abbreviated to S, which will also stand for the trajectory of the process depending on the context.

We shall use the abbreviations

$$P_{k-1}\left[\ .\ \right]=P\left[\ .\ |\mathcal{A}_{n,k-1}\right],\qquad E_{k-1}\left[\ .\ \right]=E\left[\ .\ |\mathcal{A}_{n,k-1}\right]$$

for the conditional probability and the conditional expectation given $\mathcal{A}_{n,k-1}$.

In Section 5.2 below, we shall let $F_{k-1}^{Y}(x)=P_{k-1}[Y_{nk}\leq x]$, and in Sections 4 and 5, we shall let $\overline{F}_{k-1}^{X}(x)=P_{k-1}\left[X_{nk}^{(\tau)}\leq x\right]$ where $X_{nk}^{(\tau)}=X_{nk}-a_{X_{nk}}(\tau)$ with $a_{X_{nk}}(\tau)$ as defined in (2) below. Similar notations such as $\overline{F}_{k-1}^{Y}(y)$ and $\overline{F}_{k-1}^{X,Y}(x,y)$ will be defined analogously later when their use arises.

Throughout the paper, we let $k_n=k_n(1)$.

2 THE CONDITIONS AND THE MAIN RESULTS

We shall assume, throughout below, that the following *Conditional Uniform Asymptotic Negligibility* condition holds:

$$\max_{1\leq k\leq k_n}P_{k-1}[|X_{nk}|+|Y_{nk}|>\varepsilon]\overset{p}{\to}0\text{ for all }\varepsilon>0.\tag{1}$$

The conditions to be imposed for the array $\{Y_{nk};k=1,2,...\}$ will involve the limiting process $\{S(t);0\leq t\leq 1\}$ of the convergence $S_n(t)\overset{fdd}{\Longrightarrow}S(t)$. For this reason we first recall the conditions for this (known) convergence.

2.1 CONDITIONS ON $\{X_{nk}; k=1,2,...\}$ FOR $S_n(t) \Longrightarrow S(t)$.

We first introduce some notations. Let, for $\tau > 0$,

$$a_{X_{nk}}(\tau) = E_{k-1}\left[X_{nk}\mathbb{I}_{\{|X_{nk}|<\tau\}}\right] \tag{2}$$

(recall that $E_{k-1}\left[\;.\;\right] = E\left[\;.\;|\mathcal{A}_{n,k-1}\right]$) and

$$\sigma^2_{X_{nk}}(\tau) = E_{k-1}\left[X^2_{nk}\mathbb{I}_{\{|X_{nk}|<\tau\}}\right] - \left(a_{X_{nk}}(\tau)\right)^2$$

(which is the conditional variance of $X_{nk}\mathbb{I}_{\{|X_{nk}|<\tau\}}$ given $\mathcal{A}_{n,k-1}$.) Define

$$B_{n,t}(\tau) = \sum_{k=1}^{k_n(t)} \sigma^2_{X_{nk}}(\tau), \qquad A_{n,t}(\tau) = \sum_{k=1}^{k_n(t)} a_{X_{nk}}(\tau). \tag{3}$$

Also, for $x \neq 0$, let

$$L_{n,t}(x) = \begin{cases} \sum_{k=1}^{k_n(t)} P_{k-1}\left[X_{nk} \leq x\right] & \text{if } x < 0 \\ \sum_{k=1}^{k_n(t)} P_{k-1}\left[X_{nk} > x\right] & \text{if } x > 0. \end{cases} \tag{4}$$

Consider the following conditions.

(C1): There is a family $L_t(y), t \in [0,1]$, of *nonrandom,* real valued functions defined on $\mathbb{R} - \{0\}$ such that, for each t, $L_t(-\infty) = 0 = L_t(\infty)$ and

$$L_{n,t}(x) \xrightarrow{p} L_t(x) \text{ at all continuity point } x \neq 0 \text{ of } L_t.$$

(C2): There are nonrandom $B_t \geq 0, t \in [0,1]$, such that for each t,

$$\lim\nolimits_{\epsilon\to 0} \text{limsup}_{n\to\infty} P\left(|B_{n,t}(\epsilon) - B_t| \geq \eta\right) = 0 \text{ for all } \eta > 0.$$

(C3): For some $\tau > 0$ for which $\pm\tau$ are continuity points of L_t, there are nonrandom $A_t(\tau), t \in [0,1]$, such that

$$A_{n,t}(\tau) \xrightarrow{p} A_t(\tau).$$

(C4): $A_t(\tau)$ and B_t are continuous in t. In addition, for each t, $L_{t'}(x) \to L_t(x)$ as $t' \to t$ for all continuity point $x \neq 0$ of L_t.

Note that all the limits involved in (C1) - (C3), with the convergence in terms of $\xrightarrow{p}$, are non-random, and hence $\xrightarrow{p}$ is equivalent to $\xRightarrow{d}$.

Now, (C1) and (C2) entail that the limit $L_t(x)$ in (C1) induces, for each t, a Lévy measure L_t on $\mathbb{R}$ such that $L_t(\{0\}) = 0$ and $\int (1 \wedge x^2) L_t(dx) < \infty$. Then one can define

$$\phi_t(u) = iuA_t(\tau) - \frac{u^2}{2}B_t + \int \left(e^{iux} - 1 - iux\mathbb{I}_{\{|x|\leq\tau\}}\right) L_t(dx)$$

where B_t is as in (C2) and $A_t(\tau)$ is the limit in (C3).

Then, when (C1) - (C4) hold, $S_n(t) \overset{fdd}{\Longrightarrow} S(t)$, *where* $S(t)$ *is an additive process characterized by the family of triplets* (A_t, B_t, L_t) *in the sense that*

$$\log E\left[e^{iuS(t)}\right] = \phi_t(u) \quad \text{for all real } u.$$

(Note that $S(t)$ *being an additive process,* $S = (S(t); 0 \le t \le 1) \in D_{\mathbb{R}}[0,1]$ *a.s)*

If in addition (C3) is strengthened to

$$\sup_{0 \le t \le 1} |A_{n,t}(\tau) - A_t(\tau)| \overset{p}{\to} 0,$$

then $S_n(t) \Longrightarrow S(t)$ *in* $D_{\mathbb{R}}[0,1]$.

Remark 1 For the case $A_t(\tau) \equiv 0$, $B_t \equiv 0$ and L_t of the form that the process $S(t)$ has stationary increments, that is, $L_t(x,\infty) = t\nu(x,\infty), x > 0$, and $L_t(-\infty, x) = t\nu(-\infty, x), x < 0$, for a suitable non-random ν, the preceding result is due to Durrett and Resnick (1978, Theorem 4.1). For the statements in this generality, see for example Jeganathan (1983, Remark 2), Liptser and Shiryayev (1989, Chs. 5 - 7) and Jacod and Shiryayev (1987, Ch. VIII). (Note however that the proof of the preceding result will necessarily be implicit in the joint convergence of $S_n(t)$ and $R_n(t)$ to be established). ■

2.2 CONDITIONS ON $\{Y_{nk}; k = 1, 2, ...\}$

Let $a_{Y_{nk}}(\tau)$ and $\sigma^2_{Y_{nk}}(\tau)$ be as defined in (2) and (3) (with Y_{nk} involved in place of X_{nk}). We assume that the earlier (C1) - (C4) hold so that, as recalled above, the convergence $S_n(t) \overset{fdd}{\Longrightarrow} S(t)$ together with

$$S = (S(t); 0 \le t \le 1) \in D_{\mathbb{R}}[0,1] \ \ a.s.$$

is available. Below S will also stand for the trajectory of the process depending on the context.

In what follows, by a functional $\gamma(S)$ of S we mean $\gamma(S)$ is a random variable such that $\gamma(\omega)$ is a real number for each $\omega \in D_{\mathbb{R}}[0,1]$.

Also, analogous to (4) and (5), let

$$B^*_{n,t}(\tau) = \sum_{k=1}^{k_n(t)} \sigma^2_{Y_{nk}}(\tau), \qquad A^*_{n,t}(\tau) = \sum_{k=1}^{k_n(t)} a_{Y_{nk}}(\tau). \tag{5}$$

and, for $y \neq 0$,

$$L^*_{n,t}(x) = \begin{cases} \sum_{k=1}^{k_n(t)} P_{k-1}[Y_{nk} \le y] & \text{if } y < 0 \\ \sum_{k=1}^{k_n(t)} P_{k-1}[Y_{nk} > y] & \text{if } y > 0. \end{cases} \tag{6}$$

We now state the conditions, analogous to the earlier (C1) - (C4).

There is a family

$$\left\{A_t^*(\tau, S),\ B_t^*(S) \geq 0,\ L_t^*(y, S),\ t \in [0,1],\ y \in \mathbb{R}- \{0\}\right\}, \qquad \tau > 0,$$

of functionals of S, with, for each $t \in [0,1]$,

$$P\left[L_t(-\infty, S) = 0 = L_t(\infty, S)\right] = 1,$$

$$P\left[y \longmapsto L_t^*(y, S) \text{ is nonincreasing on } (0,\infty) \text{ and nondecreasing } (-\infty, 0)\right] = 1, \tag{7}$$

satisfying the following conditions:

(D1): There is a dense subset T of $[0,1]$ and a dense subset J of $\mathbb{R} - \{0\}$ such that

$$\left(S_n(t), L_{n,t}^*(y)\right) \overset{fdd}{\Longrightarrow} \left(S(t), L_t^*(y, S)\right), \quad (y,t) \in J \times T.$$

(D2): With $T \subset [0,1]$ as in (D1),

$$\left(S_n(t), B_{n,t}^*(\epsilon)\right) \overset{fdd}{\Longrightarrow} \left(S(t), B_t^*(S)\right),\ t \in T$$

as $n \to \infty$ first and then $\epsilon \to 0$ along the set J of (D1). (See Section 1 for the meaning of this convergence.)

(D3): For some $\tau > 0$ satisfying

$$P\left[L_t^*(y, S) \text{ is continuous at } y = \pm\tau\right] = 1 \text{ for all } t \in T, \tag{8}$$

$$\left(S_n(t), A_{n,t}^*(\tau)\right) \overset{fdd}{\Longrightarrow} \left(S(t), A_t^*(\tau, S)\right),\ t \in T.$$

(D4): $B_t^*(S)$, $A_t^*(\tau, S)$, as well as $L_t^*(y, S)$ for each $y \in J$, are continuous in t for almost all S.

(D5): All the $\overset{fdd}{\Longrightarrow}$ convergencies in (D1) - (D3) hold jointly.

Remark 2. Note that when T in (D1) is countable, the set of all $\tau > 0$ satisfying (8) is dense in $(0,\infty)$. To see this let $J_{t,m} \subset \mathbb{R} - \{0\}$ be the set of all $\tau \neq 0$ such that $E\left[m \wedge L_t^*(y, S)\right]$ is continuous at $y = \pm\tau$, where m is a positive integer. Then, because of (7), $\mathbb{R}- J_{t,m} \cup \{0\}$ is countable. Let $J_* = \cap_{m \geq 1} \cap_{t \in T} J_{t,m}$. Then J_* is dense in $\mathbb{R}- \{0\}$.

Thus for each $\tau \in J_*$, and for all $t \in T$ and $m \geq 1$, $E\left[m \wedge L_t^*(y, S)\right]$ is continuous at $y = \pm\tau$, that is, $P\left[m \wedge L_t^*(y, S) \text{ is continuous at } y = \pm\tau\right] = 1$ because (7) entails that, for each y, either $m \wedge L_t^*(y+, S) - m \wedge L_t^*(y-, S) \geq 0$ a.s or ≤ 0 a.s. Because this is true for every positive integer m, we thus have $P\left[L_t^*(y, S) \text{ is continuous at } y = \pm\tau\right] = 1$ for each $\tau \in J_*$ and for all $t \in T$. (Here $L_t^*(y+, S) = \lim_{y' \downarrow y} L_t^*(y', S)$.) ■

Remark 3. Note that, under (D4), (D1) will hold for $T = [0,1]$ itself. The same is the case for (D2). To see this for (D2), it is enough to note that, for each $n \geq 1$, $B^*_{n,t}(\epsilon)$ is nondecreasing in t and its limit $B^*_t(S)$ is such that $B^*_{t''}(S) - B^*_{t'}(S) \xrightarrow{p} 0$ as $t' - t'' \to 0$ by (D4). The same argument holds for (D1) because of (D4) and because $L^*_{n,t}(y)$ and $L^*_t(y)$ are nondecreasing in t. ■

Remark 4. The limits in (C1) - (C3) associated with $\{X_{nk}; k = 1, 2, ...\}$ are non-random. Hence when the marginal convergencies in (D1) - (D3) are restated to the corresponding situation of the array $\{X_{nk}; k = 1, 2, ...\}$, they are implied by the conditions (C1) - (C3). To see this, it is enough to lake $J = \cap_{t \in T} J_t$, where J_t is the set of all continuity points of $x \longmapsto L_t(x)$ and T is any countable dense subset of $[0,1]$. Note that J is dense because T is countable and the complement of each J_t is countable. ■

2.3 FURTHER CONDITIONS

We shall also impose two further conditions that involve both the arrays $\{X_{nk}; k = 1, 2, ...\}$ and $\{Y_{nk}; k = 1, 2, ...\}$.

(E1): For every bounded *closed* intervals I_1 and I_2 contained in $\mathbb{R} - \{0\}$,

$$\sum_{k=1}^{k_n} P_{k-1}\left[X_{nk} \in I_1, Y_{nk} \in I_2\right] \xrightarrow{p} 0.$$

This condition together with (C1) and (D1) will entail that certain two-dimensional point process generated by the arrays $\{X_{nk}, Y_{nk}; k = 1, 2, ...\}$ converges in a suitable sense to a two-dimensional point process such that its marginals do not jump simultaneously. In the case when the limiting Lévy measure L^*_t is non-random (note that L_t is already non-random), this would mean that the indicated marginals are mutually independent Poisson point processes, see, for example, Kasahara and Watanabe (1986, Section 8).

To introduce the second one, let

$$\sigma_{X_{nk}, Y_{nk}}(\tau) = E_{k-1}\left[X_{nk} Y_{nk} \mathbb{I}_{\{|X_{nk}| < \tau, |Y_{nk}| < \tau\}}\right] - a_{X_{nk}}(\tau)\, a_{Y_{nk}}(\tau),$$

which is just the conditional covariance between $X_{nk}\mathbb{I}_{\{|X_{nk}|<\tau\}}$ and $Y_{nk}\mathbb{I}_{\{|Y_{nk}|<\tau\}}$.

(E2):

$$\sum_{k=1}^{k_n} \sigma_{X_{nk}, Y_{nk}}(\epsilon) \xrightarrow{p} 0 \quad \text{as } n \to \infty \text{ first and then } \epsilon \to 0.$$

We note that this condition as well as the condition (E1) are automatically satisfied in many important situations, see the Remarks 7 and 8 below.

2.4 STATEMENTS OF THE RESULTS

In the statements below, a conditional additive process $R(t)$ given the process $S = (S(t); 0 \leq t \leq 1)$ is to be understood as explained in for example Jacod and Shiryayev

(1987, Ch. II, Section 6, page 124, Definition 6.2 and Lemma 6.10 with $\mathcal{F}_t = \sigma(S(v); 0 \le v < \infty) \wedge \sigma(R(u); 0 \le u \le t)$).

Theorem 1. *Let $S_n(t) = \sum_{k=1}^{k_n(t)} X_{nk}$ and $R_n(t) = \sum_{k=1}^{k_n(t)} Y_{nk}$. Assume that the conditions (C1) - (C4), (D1) - (D5), (E1) and (E2) hold. Further assume that $A_{n,t}(\tau)$ and $A_{n,t}^*(\tau)$ are tight in $D_{\mathbb{R}}[0,1]$.*

Then $(S_n(t), R_n(t)) \Longrightarrow (S(t), R(t))$ in $D_{\mathbb{R}^2}[0,1]$, where $S = (S(t); 0 \le t \le 1)$ is as before, and conditionally on S, the process $R(t)$ is additive such that

$$\log E\left[e^{iuR(t)} \middle| S\right] = iuA_t^*(\tau, S) - \frac{u^2}{2} B_t^*(S) + \int \left(e^{iux} - 1 - iux\mathbb{I}_{\{|x| \le \tau\}}\right) L_t^*(dx, S).$$

Note that in Theorem 1 if the characteristics (A_t^*, B_t^*, L_t^*) of $R(t)$ are nonrandom (that is, they do depend on S), then the processes $S(t)$ and $R(t)$ are independent.

Remark 5. Theorem 1 holds also (see Remark 11 in Section 4.4 below) when $(S_n(t), R_n(t)) = \left(\sum_{k=1}^{[nt]} X_{nk}, \sum_{k=1}^{k_n(t)} Y_{nk}\right)$, under the following modifications of the conditions. The conditions (C1) - (C4) are now assumed to hold with $[nt]$ involved in place of $k_n(t)$, as well as the present $S_n(t) = \sum_{k=1}^{[nt]} X_{nk}$ is now involved in the conditions (D1) and (D2), and in addition (1) and the conditions (E1) and (E2) hold with $k_n \vee n$ involved in place of k_n. It is (the modification of) the conditions (E1) and (E2) that make the $\overset{fdd}{\Longrightarrow}$ convergence of this extension feasible. The tightness in $D_{\mathbb{R}^2}[0,1]$ will be reduced to what is called Aldous criterion, which will be satisfied for $(S_n(t), R_n(t))$ whenever it is satisfied for each of $S_n(t)$ and $R_n(t)$ separately.

Similarly Theorem 1 holds also when $(S_n(t), R_n(t)) = \left(\sum_{k=1}^{k_n(t)} X_{nk}, \sum_{k=1}^{[nt]} Y_{nk}\right)$, under similar modifications. ■

Remark 6. It is important to note that the shifting quantity for the vector $(S_n(t), R_n(t))$ in Theorem 1 is the vector $\left(A_{n,t}(\tau), A_{n,t}^*(\tau)\right)$ composed of those of $S_n(t)$ and $R_n(t)$. This form is essential for the applications we have in mind, and we are able to obtain this form only by using the condition (E1) in a rather crucial manner. On the other hand, the usual shifting quantity suggested by the results for the case of sums of independent random vectors (see for instance Jacod and Shiryayev (1987)), would be of the form $\left(\sum_{k=1}^{k_n(t)} E_{k-1}\left[X_{nk}\mathbb{I}_{\{|X_{nk}|+|Y_{nk}|<\tau\}}\right], \sum_{k=1}^{k_n(t)} E_{k-1}\left[Y_{nk}\mathbb{I}_{\{|X_{nk}|+|Y_{nk}|<\tau\}}\right]\right)$. We do not know the relationship between these two forms in general under the present given conditions. ■

Remark 7. We note that (E2) is satisfied if either $B_t \equiv 0$ in (C2) (which means $\sum \sigma_{X_{nk}}^2(\epsilon) \xrightarrow{p} 0$) or $B_t^*(S) \equiv 0$ a.s. in (D2) (which means $\sum \sigma_{Y_{nk}}^2(\epsilon) \xrightarrow{p} 0$), because of the inequality $\left|\sum \sigma_{X_{nk},Y_{nk}}(\epsilon)\right|^2 \le \sum \sigma_{X_{nk}}^2(\epsilon) \sum \sigma_{Y_{nk}}^2(\epsilon)$. ■

Remark 8. If either $L_t^* \equiv 0$ a.s. together with $B_t \equiv 0$, or $L_t \equiv 0$ together with $B_t^*(S) = 0$ a.s, then both (E1) and (E2) are satisfied. This is because of the preceding

Remark 7 and the fact that (E1) is satisfied if either $L_t^* \equiv 0$ a.s. or $L_t \equiv 0$ holds.

We now give a situation (occurring in Jeganathan (2006a, Lemma 4)) where the modified (E1) and (E2) of Remark 5 hold. If, as $n \to \infty$ first and then $\epsilon \to 0$, $\sum_{k=1}^{k_n \vee n} \sigma_{X_{nk}}^2(\epsilon) \xrightarrow{p} 0$ and $\sum_{k=1}^{k_n \vee n} \sigma_{Y_{nk}}^2(\epsilon)$ is bounded in probability then the modified (E2) is satisfied. If $\sum_{k=1}^{k_n \vee n} P_{k-1}[|Y_{nk}| \geq \eta] \xrightarrow{p} 0$ for every $\eta > 0$, then the modified (E1) is satisfied. ∎

The next result is directly applicable in, and in fact possibly broadens the scope of, many applications, see Jeganathan (2006b). To state the result, let $g(u,v)$ be a function satisfying the following conditions.

(i) $(u,v) \longmapsto g(u,v)$ is continuous,

(i) $g(u,0) = 0$ for each u,

(ii) there are continuous functions $u \longmapsto g_u'$ and $u \longmapsto g_u''$ such that if

$$\rho_\eta(u) = \sup_{0<|v|\leq\eta} \frac{1}{v^2}\left| g(u,v) - v g_u' - \frac{v^2}{2} g_u'' \right|,$$

then

$$\sup_{|u|\leq a} \rho_\eta(u) \to 0 \quad \text{as } \eta \to 0 \text{ for each } a > 0.$$

In the next statement, it is convenient to let

$$A^*(t,\tau,S) = A_t^*(\tau,S),\, B^*(t,S) = B_t^*(S) \text{ and } L^*(x,t,S) = L_t^*(x,S).$$

Theorem 2. *Let $R_n^*(t) = \sum_{k=1}^{k_n(t)} g\left(\sum_{j=1}^{k-1} X_{nj}, Y_{nk}\right)$ with $g(u,v)$ and the corresponding g_u' and g_u'' as above. In addition to the conditions of Theorem 1, assume that*

$$\sum_{k=1}^{k_n} |a_{Y_{nk}}(\tau)|^2 \xrightarrow{p} 0. \tag{9}$$

Further assume that $P[t \longmapsto A^*(t,\tau,S)$ is of bounded variation$] = 1$.

Then $(S_n(t), R_n^*(t)) \Longrightarrow (S(t), R^*(t))$ in $D_{\mathbb{R}^2}[0,1]$ where conditionally on S the process $R^*(t)$ is additive such that

$$\begin{aligned} &\log E\left[e^{iuR^*(t)} \middle| S\right] \\ = \;& iuA_t^\#(\tau,S) - \frac{u^2}{2} B_t^\#(S) + \int_0^t \int \left(e^{iug(S(z),x)} - 1 - iux\mathbb{I}_{\{|x|\leq\tau\}} g_{S(z)}'\right) L^*(dx,dz,S) \end{aligned}$$

with $B_t^\#(S) = \int_0^t \left(g_{S(z)}'\right)^2 B^*(dz,S)$ and

$$A_t^\#(\tau,S) = \int_0^t g_{S(z)}' A^*(dz,\tau,S) + \frac{1}{2}\int_0^t g_{S(z)}'' B^*(dz,S).$$

Remark 9. In the particular case $g(u,v) = h(v)$, the conditions on $g(u,v)$ reduce to: (i) $v \longmapsto h(v)$ is continuous, (ii) $h(0) = 0$ and (iii) there are constants h' and h'' such that $\frac{1}{v^2}\left|h(v) - vh' - \frac{v^2}{2}h''\right| \to 0$ as $v \to 0$. For this particular case, the extension indicated in Remark 5 above holds also. For the case of independent summands, a form of this particular case is contained in LeCam (1986, Proposition 5, page 443).

Note that the condition $P\left[t \longmapsto A_t^*(\tau, S) \text{ is of bounded variation}\right] = 1$ is imposed in order that $\int_0^t g'_{S(s)} A^*(ds, \tau, S)$ is well-defined (and is not required for the preceding particular case because h' is a constant). Further, this condition, as well as (9), will be satisfied in many applications, see Section 3 below.

We further remark that for the particular case $g(u,v) = f(u)v$, one can also use Kurtz and Protter (1991, Theorem 2.2, page 1039) to obtain from Theorem 1 the conclusion of Theorem 2 with $R^*(t) = \int_0^t f(S(z))\,dR(z)$. This will be implicit in the proof of the general case of Theorem 2 given in Section 5 below, which proof also relies on some of the ideas contained in Kurtz and Protter (1991). Also the additional condition (9) will not needed for this case because $g''_u \equiv 0$. ■

Remark 10. For convenience we have taken $R_n(t)$ and $R_n^*(t)$ in Theorems 1 and 2 to be scalars. In applications one would often require these to be vectors. We now describe one such extension that is in particular needed in Jeganathan (2006b). Assume that the assumptions of Theorem 1 are satisfied. Consider an additional array $\{Z_{nk}; k = 1, 2, ...\}$ adapted to $\{\mathcal{A}_{nk}; k = 0, 1, 2, ...\}$, for which the conditions (D1) - (D5) hold (with limits that are functionals of the same $S = (S(t); 0 \le t \le 1)$) such that the $\overset{fdd}{\Longrightarrow}$ convergencies involved in these conditions hold jointly with corresponding ones for the array $\{Y_{nk}; k = 1, 2, ...\}$. Assume further that (E1) and (E2) hold in addition for the array $\{X_{nk}, Z_{nk}; k = 1, 2, ...\}$ as well as for the array $\{Y_{nk}, Z_{nk}; k = 1, 2, ...\}$ also.

Then Theorem 1 extend to the of vectors $\left(\sum_{k=1}^{k_n(t)} X_{nk}, \sum_{k=1}^{k_n(t)} Y_{nk}, \sum_{k=1}^{k_n(t)} Z_{nk}\right)$ in $D_{\mathbb{R}^3}[0,1]$ with the limit $(S(t), R(t), Z(t))$ such that *conditionally on S, the processes $R(t)$ and $Z(t)$ are independent additive processes.* Theorem 2 is similarly extended. ■

3 SOME EXAMPLES

As noted earlier, extensive applications that will require the full force of the preceding results to certain network traffic models are presented separately in Jeganathan (2006b). (See further Jeganathan (2006a, Lemma 4), where the extension indicated in Remark 5 above has been invoked.)

In the examples below will shall use the sequence ξ_j, $-\infty < j < \infty$, of iid random variables belonging to the domain of attraction of a stable law with index $0 < \alpha \le 2$. (Later on we shall indicate that the results of the Example 1 below extend to the situation of a more general sequence of chain dependent variables described in Durrett and Resnick

(1978, Example 4.1).) Details regarding the stable distributions and their domains of attraction can be found in many sources, for instance in Feller (1971, Chapter XVII).

Consider the case $0 < \alpha < 2$, where the preceding requirement amounts to

$$P(\xi_1 \geq x) \sim c_1 x^{-\alpha} H(x), \quad P(\xi_1 \leq -x) \sim c_2 x^{-\alpha} H(x), \quad x \to \infty$$

for some slowly varying function $H(x)$ and nonnegative constants c_1, c_2 with $c_1 + c_2 > 0$. With the constants a_n chosen such that

$$nP(|\xi_1| \geq a_n x) \to x^{-\alpha},$$

define

$$X_{nk} = \begin{cases} a_n^{-1}\xi_k & \text{if } 0 < \alpha < 1 \\ a_n^{-1}(\xi_k - E[\xi_1]) & \text{if } 1 < \alpha < 2 \\ a_n^{-1}\left(\xi_k - E\left[\xi_1 \mathbb{I}_{\{|\xi_1| \leq \tau a_n\}}\right]\right) & \text{if } \alpha = 1. \end{cases} \tag{10}$$

We next recall that (C1) - (C4) hold for X_{nk}. First, (C1) holds with $L_t(x, \infty) = tpx^{-\alpha}$, $x > 0$, and $L_t(-\infty, -y) = tqy^{-\alpha}$, $y > 0$, where $p = \frac{c_1}{c_1 + c_2}$ and $q = 1 - p$ with c_1 and c_2 are as above.

In addition (C3) holds, with

$$A_{n,t}(\tau) \to A_t(\tau) = \begin{cases} \frac{t\alpha}{\alpha - 1}(p - q)\tau^{1-\alpha} & \text{if } 0 < \alpha < 1 \\ -\frac{t\alpha}{\alpha - 1}(p - q)\tau^{1-\alpha} & \text{if } 1 < \alpha < 2 \\ 0 & \text{if } \alpha = 1. \end{cases} \tag{11}$$

The verification of (C1) with the limit as above, as well as the preceding convergence (11) for the cases $\alpha \neq 1$ are contained, for example, in Durrett and Resnick (1978, Example 4.1). For $\alpha = 1$, note that $E\left[X_{nk}\mathbb{I}_{\{|\xi_1| \leq \tau a_n\}}\right] = P[|\xi_1| > \tau a_n]\left(a_n^{-1}E\left[\xi_1 \mathbb{I}_{\{|\xi_1| \leq \tau a_n\}}\right]\right)$, so that

$$\sum \left| E\left[X_{nk}\mathbb{I}_{\{|\xi_1| \leq \tau a_n\}}\right]\right| \to 0$$

because $nP[|\xi_1| > \tau a_n]$ is bounded and $a_n^{-1}E\left[\xi_1 \mathbb{I}_{\{|\xi_1| \leq \tau a_n\}}\right] \to 0$ in view of $P[|\xi_1| \geq \epsilon a_n] \to 0$ for all $\epsilon > 0$. In addition, by (C1),

$$\sum \left| E\left[X_{nk}\left(\mathbb{I}_{\{|X_{nk}| \leq \tau\}} - \mathbb{I}_{\{|\xi_1| \leq \tau a_n\}}\right)\right]\right| \leq \sum E\left[|X_{nk}| \, \mathbb{I}_{\left\{\tau < |X_{nk}| \leq \tau + a_n^{-1} E\left[\xi_1 \mathbb{I}_{\{|\xi_1| \leq \tau a_n\}}\right]\right\}}\right] \to 0,$$

Thus (11) holds for $\alpha = 1$ by the preceding two displayed convergencies.

The condition (C2) also holds, with $B_t \equiv 0$, see Durrett and Resnick (1978, Example 4.1) mentioned above. Thus, according to Section 2.1 above,

$$\sum_{k=1}^{[nt]} X_{n,k} \Longrightarrow S_\alpha(t) \text{ in } D_{\mathbb{R}}[0,1],$$

where the limit $S_\alpha(t)$ is now a stable process with index α.

Let us also note that the condition (9) is satisfied because $\sup_k |a_{X_{nk}}(\tau)| \to 0$ and, similar to (11), $\sum_{k=1}^n |a_{X_{nk}}(\tau)| = \sum_{k=1}^n \left|E\left[X_{nk}\mathbb{I}_{\{|X_{nk}|\le\tau\}}\right]\right|$ converges to a finite quantity. (Note that in the examples below (9) will not be required, see the Remark 9 in Section 2 above)

In the case $\alpha = 2$, we shall assume for convenience that $E[\xi_1^2] < \infty$, in which case $X_{n,k} = n^{-1/2}(\xi_k - E[\xi_1])$. In this case $L_t \equiv 0$, $A_t(\tau) \equiv 0$ and $B_t = tE[\xi_1^2]$.

Example 1. First consider a simpler situation in which (ξ_j, ζ_j), $-\infty < j < \infty$, form an iid sequence of pairs of random variables, where ξ_j are as before belonging to the domain of attraction of a stable law with index $0 < \alpha < 2$. (The case $\alpha = 2$ will be discussed at the end of this example.) Here ξ_j and ζ_j need not be independent for each j. For example ζ_j can be a function of $(\xi_1, ..., \xi_j)$. Let $S_{n,k} = \sum_{j=1}^k X_{n,j}$, where $X_{n,k}$ is as in (10).

We shall assume that $E[\zeta_1] = 0$ and $\sigma^2 = E[\zeta_1^2] < \infty$ (though it will be enough to assume that ζ_1 is in the domain of attraction of a normal distribution). In the case (13) below we shall assume in addition that $E[\zeta_1 | \xi_1] = 0$.

In certain asymptotic inference problems, asymptotic behaviors of the pairs of the forms

$$\left(\frac{1}{n}\sum_{k=1}^{[nt]} h(S_{n,k-1}), \frac{1}{\sqrt{n}}\sum_{k=1}^{[nt]} g(S_{n,k-1})\zeta_k\right), \tag{12}$$

$$\left(\frac{1}{n}\sum_{k=1}^{[nt]} h(S_{n,k}), \frac{1}{\sqrt{n}}\sum_{k=1}^{[nt]} g(S_{n,k})\zeta_k\right) \tag{13}$$

are required, where g and h are continuous functions (for example $g(x) = x$ and $h(x) = x^2$). See Jeganathan (1997) where specific cases of (12) and (13) arose in connection with the asymptotic representation of the likelihood ratios or of the approximate maximum likelihood estimates (MLEs) in certain time series models. There the asymptotic behavior was studied as a consequence of direct asymptotic analysis of the likelihood ratios. However, that method itself is applicable only for the case of approximate MLEs. Thus the present results would be useful if estimates more general than the approximate MLEs are considered (such as M-estimates).

To deal with (12), take $\mathcal{A}_{nk} = \sigma((\xi_j, \zeta_j), j \le k)$. Recall that $E[\zeta_1] = 0$ and $\sigma^2 = E[\zeta_1^2] < \infty$. Then, taking $Y_{n,k} = \frac{\zeta_k}{\sqrt{n}}$, in view of the Remark 8 (the case $B_t \equiv 0$ with $L_t^* \equiv 0$), the assumptions of Theorem 1 are satisfied. Hence Theorems 1 and 2 give

$$\left(S_{n,[nt]}, \frac{1}{\sqrt{n}}\sum_{k=1}^{[nt]} g(S_{n,k-1})\zeta_k\right) \Longrightarrow \left(S_\alpha(t), B\left(\int_0^t g^2(S_\alpha(z))\,dz\right)\right) \text{ in } D_{\mathbb{R}^2}[0,1], \tag{14}$$

where $B(t)$ is a Brownian motion with variance $E[\zeta_1^2]$, independent of $S_\alpha(t)$. Note that, because of the independence of $B(t)$ and $S_\alpha(t)$, the distribution of the preceding limit is the same as that of $\left(S_\alpha(t), \int_0^t g(S_\alpha(z))\,dB(z)\right)$ or that of $\left(S_\alpha(t), W\sqrt{\int_0^t g^2(S_\alpha(z))\,dz}\right)$ with W standard normal independent of $S_\alpha(t)$.

Note that $\sup_{1\le k\le n}|S_{n,k}|$ is stochastically bounded, because $S_{n,[nt]} \Longrightarrow S_\alpha(t)$ in $D_{\mathbb{R}}[0,1]$. Hence one can assume that both the functions g and h are supported by a compact subset of the real line.

Then, because h is continuous and compactly supported, and hence uniformly continuous and uniformly bounded, it will follow from the convergence (14) that

$$\left(S_{n,[nt]}, \frac{1}{\sqrt{n}}\sum_{k=1}^{[nt]} g(S_{n,k-1})\zeta_k, \frac{1}{n}\sum_{k=1}^{[nt]} h(S_{n,k-1})\right)$$
$$\Longrightarrow \left(S_\alpha(t), W\sqrt{\int_0^t g^2(S_\alpha(z))\,dz}, \int_0^t h(S_\alpha(z))\,dz\right) \text{ in } D_{\mathbb{R}^3}[0,1]. \qquad (15)$$

To indicate briefly the relevant arguments for this convergence (for the details see Jeganathan (2006b, Proposition 2.2, Section 2)), first the $\overset{fdd}{\Longrightarrow}$ convergence in (15) follows for example using the arguments of the proof of Theorem 1, page 485 in Gikhman and Skorokhod (1969). If in addition $h(u)$ nonnegative, then the tightness in $D_{\mathbb{R}^3}[0,1]$ in (15) follows using (14) and Jacod and Shiryayev (1987, Theorem 3.37 (Statement (a)), page 318 and Corollary 3.33 (Statement (b)), page 317), in view of the fact that $\frac{1}{n}\sum_{k=1}^{[nt]} h(S_{n,k-1})$ is monotone in t and its limit $\int_0^t h(S_\alpha(z))\,dz$ is continuous in t. When $h(u)$ is not nonnegative, the positive part $h^+(u)$ and the negative part $h^-(u)$ are considered jointly, obtaining the tightness in $D_{\mathbb{R}^4}[0,1]$, from which the required tightness in $D_{\mathbb{R}^3}[0,1]$ in (15) will follow.

To deal with the case (13), we show that it reduces to the situation of (12). Recall that we assume in addition that $E[\zeta_1|\,\xi_1] = 0$. Then, assuming without loss of generality that $g(u)$ is, in addition to being continuous, compactly supported as is indicated above (and hence uniformly continuous and uniformly bounded), $(g(S_{n,k}) - g(S_{n,k-1}))\zeta_k$ form martingale differences with respect to the σ-fields $\sigma(\xi_j, j \le k+1, \zeta_l, l \le k)$. The sum of the conditional variances of this differences with respect to these σ-fields is given by $E[\zeta_1^2|\,\xi_1]\frac{1}{n}\sum(g(S_{n,k}) - g(S_{n,k-1}))^2$, where

$$E\left[\frac{1}{n}\sum_{k=1}^{n}(g(S_{n,k}) - g(S_{n,k-1}))^2\right] = \int_0^1 E\left[\left(g\left(S_{n,[nt]}\right) - g\left(S_{n,[nt]-1}\right)\right)^2\right]dt \to 0$$

because $\left(g\left(S_{n,[nt]}\right) - g\left(S_{n,[nt]-1}\right)\right)^2 \xrightarrow{p} 0$ for each t and $g(u)$ is uniformly bounded. This will imply $\sup_{0\leq t\leq 1}\left|\frac{1}{\sqrt{n}}\sum_{k=1}^{[nt]}\left(g\left(S_{n,k}\right)-g\left(S_{n,k-1}\right)\right)\zeta_k\right|\xrightarrow{p}0$, because $\left(g\left(S_{n,k}\right)-g\left(S_{n,k-1}\right)\right)\zeta_k$ are martingale differences. Thus the situation (13) reduces to that of (12).

We remark that if ξ_j are such that $E\left[\xi_1\right]=0$ and $E\left[\xi_1^2\right]<\infty$, or more generally if ξ_1 is in the domain of attraction of a normal distribution, then under the assumption $E\left[\zeta_1|\,\xi_1\right]=0$, the preceding results will remain true with $S_2\left(t\right)$ ($\alpha=2$) and $B\left(t\right)$ independent Browning motions. This is because, with $X_{n,k}=n^{-1/2}\left(\xi_k-E\left[\xi_1\right]\right)$ and $Y_{n,k}=\frac{\zeta_k}{\sqrt{n}}$ as before, one has $E_{k-1}\left[X_{n,k}Y_{n,k}\right]=E\left[X_{n,k}Y_{n,k}\right]=0$ in view of $E\left[\zeta_1|\,\xi_1\right]=0$, and hence (E2) holds. As already noted $L_t\equiv 0$, $A_t\left(\tau\right)\equiv 0$ and $B_t=tE\left[\xi_1^2\right]$, in addition to $L_t^*\equiv 0$, $A_t^*\left(\tau\right)\equiv 0$ and $B_t^*=tE\left[\zeta_1^2\right]$.

We also remark that because the framework of Theorems 1 and 2 are quite general, the iid structure of (ξ_j,ζ_j) can possibly be relaxed in many ways. One particularly interesting situation is the case of chain dependent variables described in Durrett and Resnick (1978, Example 4.1, and the references given there). In that work, the earlier details that preceed the Example 1 above for the iid sequence (ξ_j,ζ_j) of the present example are actually essentially carried out for such chain dependent variables. One can possibly use these details, as was done above for the present particular iid case, to obtain suitable extensions of the present example to such chain dependent variables. ■

Example 2. A situation related to this example arises in Jeganathan (2006a), where the results of the present paper are invoked. Because that situation is a bit complex, we describe a case which is simpler but still important enough to be applicable to many situations, as will become clear below. We also note that Theorem 2 was applicable in Example 1 above but not in this example.

Let the iid sequence (ξ_j,ζ_j), $-\infty<j<\infty$, be as in Example 1 above with $E\left[\zeta_1\right]=0$ and $\sigma^2=E\left[\zeta_1^2\right]<\infty$. We shall also require that the index α of the domain of attraction of ξ_j is such that $1<\alpha\leq 2$. (The reason for this restriction on α will become clear below.) In addition we shall assume that ξ_1 satisfies the Cramér's condition

$$\limsup_{|u|\to\infty}\left|E\left[e^{iu\xi_1}\right]\right|<1. \tag{16}$$

Let the functions $h\left(u\right)$ and $g\left(u\right)$ be *uniformly bounded and locally Riemann* integrable functions, such that

$$\int_{-\infty}^{\infty}\left|h\left(u\right)\right|du<\infty,\qquad \int_{-\infty}^{\infty}\left|g\left(u\right)\right|du<\infty.$$

Consider the pairs (with X_{nk} as in (10) and $S_{n,k}=\sum_{k=1}^{k}X_{nk}$)

$$\left(\frac{\beta_n}{n}\sum_{k=1}^{[nt]}h\left(\beta_nS_{n,k-1}\right),\sqrt{\frac{\beta_n}{n}}\sum_{k=1}^{[nt]}g\left(\beta_nS_{n,k-1}\right)\zeta_k\right), \tag{17}$$

where β_n is such that $\beta_n \to \infty$ but $\frac{\beta_n}{n} \to 0$. Pairs of this form arises in asymptotic inference problems in certain nonlinear time series regression models, see Park and Phillips (2001) for the particular case $\alpha = 2$, in a manner analogous to that described in Example 1 above.

Now, because $g(u)$ is uniformly bounded,

$$Y_{n,k} = \sqrt{\frac{\beta_n}{n}} g(\beta_n S_{n,k-1}) \zeta_k$$

form martingale differences, with $\sum E_{k-1}\left[Y_{n,k}^2\right] = E\left[\zeta_1^2\right] \frac{\beta_n}{n} \sum g^2(\beta_n S_{n,k-1})$. It follows from Borodin and Ibragimov (1995, Ch. III. Sections 2 and 3) or Jeganathan (2004, Theorem 2) for a more explicit statement, that (when $1 < \alpha \leq 2$ and when (16) and the stated conditions on $h(u)$ and $g(u)$ hold)

$$\left(S_{n,[nt]}, \; \frac{\beta_n}{n} \sum_{k=1}^{[nt]} g^2(\beta_n S_{n,k-1}) \right) \overset{fdd}{\Longrightarrow} \left(S_\alpha(t), \; \mathcal{L}_t \int g^2(u)\, du \right) \tag{18}$$

where $\mathcal{L}_t \geq 0$ a.s. is random, called the *local time* of the process $S_\alpha(t)$ at 0 up to the time t. (**Remark**: Actually Borodin and Ibragimov (1995) considers explicitly only the marginal convergence of $\frac{\beta_n}{n} \sum_{k=1}^{[nt]} g^2(\beta_n S_{n,k-1})$ but it is implicit in their proof that the preceding joint convergence holds whenever $S_{n,[nt]} \overset{fdd}{\Longrightarrow} S_\alpha(t)$, see Jeganathan (2004, Proposition 6 and Lemma 8).

The details regarding the local time $\mathcal{L}_t$ can be found for example in Borodin and Ibragimov (1995, Ch. 1, Section 4). In particular, the local time does not exist for $0 < \alpha \leq 1$, and when $1 < \alpha \leq 2$, it has the representation $\mathcal{L}_t = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_0^t e^{iuS_\alpha(z)} dz du$. Note that $\mathcal{L}_t$ is a functional of $S_\alpha(t)$.

Now,

$$\sum E_{k-1}\left[Y_{n,k}^2 \mathbb{I}_{\left\{|Y_{n,k}| \geq \varepsilon\right\}}\right] = \frac{\beta_n}{n} \sum g^2(\beta_n S_{n,k-1}) \int_{\left\{\sqrt{\frac{\beta_n}{n}} \left|g\left(\beta_n S_{n,k-1}\right)\right| |u| \geq \varepsilon\right\}} u^2 dF_{\zeta_1}(u)$$

where

$$\sup_k \int_{\left\{\sqrt{\frac{\beta_n}{n}} \left|g\left(\beta_n S_{n,k-1}\right)\right| |u| \geq \varepsilon\right\}} u^2 dF_{\zeta_1}(u) \leq \int_{\left\{\sup_k \sqrt{\frac{\beta_n}{n}} \left|g\left(\beta_n S_{n,k-1}\right)\right| |u| \geq \varepsilon\right\}} u^2 dF_{\zeta_1}(u) \xrightarrow{p} 0$$

because $\sup_k \sqrt{\frac{\beta_n}{n}} |g(\beta_n S_{n,k-1})| \xrightarrow{p} 0$ and $E\left[\zeta_1^2\right] < \infty$. Hence $\sum E_{k-1}\left[Y_{n,k}^2 \mathbb{I}_{\left\{|Y_{n,k}| \geq \varepsilon\right\}}\right] \xrightarrow{p} 0$, by (18).

Thus, (D1) - D4) hold with $L_t^* \equiv 0$, $B_t^* = \mathcal{L}_t \int g^2(u)\, du$ and $A_t^*(\tau) = 0$ for all $\tau > 0$. Then, when $1 < \alpha < 2$, it follows from Remark 8 that

$$\left(\sum_{k=1}^{[nt]} X_{nk}, \sqrt{\frac{\beta_n}{n}} \sum_{k=1}^{[nt]} g(\beta_n S_{n,k-1}) \zeta_k \right) \Longrightarrow \left(S_\alpha(t), W \sqrt{\mathcal{L}_t \int g^2(u)\, du} \right) \text{ in } D_{\mathbb{R}^2}[0,1], \tag{19}$$

where W is standard normal independent of $S_\alpha(t)$ (and $\mathcal{L}_t$).

Next suppose that $\alpha = 2$ and assume for convenience that $E[\xi_1] = 0$ and $E[\xi_1^2] < \infty$ (in which case $a_n = \sqrt{n}$.) Then

$$\sum E_{k-1}[X_{nk}Y_{nk}] = \frac{\sqrt{\beta_n}}{n}\sum g(\beta_n S_{n,k-1}) E[\xi_1\zeta_1] \xrightarrow{p} 0$$

because $\frac{\beta_n}{n}\sum g(\beta_n S_{n,k-1})$ is stochastically bounded (see (18)) and $\beta_n \to \infty$. Thus, by Remark 8, the statement (19) holds for the case $\alpha = 2$ also.

Now using the remark made immediately after (18), it follows from (19) (which holds for $1 < \alpha \leq 2$) that

$$(17) \stackrel{fdd}{\Longrightarrow} \left(\mathcal{L}_t \int h(u)\,du, W\sqrt{\mathcal{L}_t \int g^2(u)\,du}\right).$$

We note that the $\stackrel{fdd}{\Longrightarrow}$convergence here can be strengthened to $\Longrightarrow$ in $D_{\mathbb{R}^2}[0,1]$ using the earlier arguments given in the context of obtaining $\Longrightarrow$ in $D_{\mathbb{R}^2}[0,1]$ in (15) of Example 1 above, because it is known (Borodin and Ibragimov (1995, Ch. 1, Section 4)) that $t \longmapsto \mathcal{L}_t$ is continuous in t with probability one.

4 PROOF OF THEOREM 1

4.1 PRELIMINARY APPROXIMATIONS

We recall that the conditions (D1) - (D3) for $\{Y_{nk}; k = 1, ...\}$ are assumed to hold for a dense subset $T \subset [0,1]$ and for a dense subset $J \subset \mathbb{R} - \{0\}$. Because J is dense, it is easy to see using the familiar arguments (see for example Loève, M. (1963, Section 11.3, pages 180)) that (D1) is equivalent to the statement that for every bounded continuous functions $h_1, ..., h_q$ that *vanish in some neighborhood of* 0 and for every $t_1, ..., t_q$ in T

$$\left(S_n(t), \int h_j(y) L^*_{n,t_l}(dy), 1 \leq j,l \leq q\right) \stackrel{fdd}{\Longrightarrow} \left(S(t), \int h_j(y) L^*_{t_l}(dy, S), 1 \leq j,l \leq q\right)$$

where the limit $L^*_t(y,S)$ is as in (D1). In addition, for every bounded continuous functions $g_1, ..., g_q$

$$\begin{aligned}&\left(S_n(t), \int g_j(y)\,\mathbb{I}_{\{|y|\geq a\}} L^*_{n,t_l}(dy), 1 \leq j,l \leq q\right)\\ &\stackrel{fdd}{\Longrightarrow} \left(S(t), \int g_j(y)\,\mathbb{I}_{\{|y|\geq a\}} L^*_{t_l}(dy,S), 1 \leq j,l \leq q\right)\end{aligned} \tag{20}$$

for any $a > 0$ such that, with probability one, $y \longmapsto L^*_{t_l}(y,S)$ is continuous at $y = \pm a$ for all $t_1, ..., t_q$. The preceding two convergence statements are equivalent because the set

$$I_{t_1,...,t_q} = \left\{ a > 0 : \begin{array}{l} L_{t_1}(x), ..., L_{t_q}(x) \text{ are continuous at } x = \pm a \text{ and} \\ P\left[L^*_{t_1}(y,S), ..., L^*_{t_q}(y,S) \text{ are continuous at } y = \pm a\right] = 1 \end{array} \right\} \tag{21}$$

is dense in $(0,\infty)$. ($I_{t_1,\ldots,t_q}$ is the complement of a countable set, see Remark 2 in Section 2.2.).

Now, below in this subsection,

$$\sum \text{ stands for } \sum_{k=k_n(\mu)+1}^{k_n(\lambda)} \text{ for some fixed } \mu<\lambda \text{ in } T.$$

In this section we shall use the notations

$$X_{nk}^{(\tau)} = X_{nk} - a_{X_{nk}}(\tau), \quad Y_{nk}^{(\tau)} = Y_{nk} - a_{Y_{nk}}(\tau),$$
$$\overline{F}_{k-1}^{X}(x) = P_{k-1}\left[X_{nk}^{(\tau)} \le x\right], \quad \overline{F}_{k-1}^{Y}(y) = P_{k-1}\left[Y_{nk}^{(\tau)} \le y\right],$$
$$\overline{F}_{k-1}^{X,Y}(x,y) = P_{k-1}\left[X_{nk}^{(\tau)} \le x, Y_{nk}^{(\tau)} \le y\right].$$

(Note that τ in (C3) and τ in (D4) are different but we use the same notation for convenience.) Now, for any reals u and v, we have

$$\begin{aligned} & \sum E_{k-1}\left[e^{iuX_{nk}^{(\tau)}+ivY_{nk}^{(\tau)}}-1\right] \\ = & \sum \int \left(e^{iux+ivy}-1\right) d\overline{F}_{k-1}^{X,Y}(x,y) \\ = & \; iu\sum a_{X_{nk}^{(\tau)}}(\tau) + iv\sum a_{Y_{nk}^{(\tau)}}(\tau) \\ & +\sum\int \left(e^{iux+ivy}-1-iux\mathbb{I}_{\{|x|<\tau\}}-ivy\mathbb{I}_{\{|y|<\tau\}}\right) d\overline{F}_{k-1}^{X,Y}(x,y) \end{aligned} \tag{22}$$

where the last term itself can be split into the sum of three parts

$$\sum\int \left(e^{iux+ivy}-1-iux\mathbb{I}_{\{|x|<\tau\}}-ivy\mathbb{I}_{\{|y|<\tau\}}\right) \mathbb{I}_{\{|x|<\epsilon,|y|<\epsilon\}} d\overline{F}_{k-1}^{X,Y}(x,y), \tag{23}$$

$$\sum\int \left(e^{iux+ivy}-1-iux\mathbb{I}_{\{|x|<\tau\}}-ivy\mathbb{I}_{\{|y|<\tau\}}\right) \mathbb{I}_{\{|x|\ge\epsilon\}} d\overline{F}_{k-1}^{X,Y}(x,y) \tag{24}$$

and

$$\sum\int \left(e^{iux+ivy}-1-iux\mathbb{I}_{\{|x|<\tau\}}-ivy\mathbb{I}_{\{|y|<\tau\}}\right) \mathbb{I}_{\{|x|<\epsilon,|y|\ge\epsilon\}} d\overline{F}_{k-1}^{X,Y}(x,y) \tag{25}$$

The first step consists of the following important

Proposition 3. *Assume that all the conditions of Theorem 1, except (C3) and (D3), are satisfied. Then the following approximations hold.*

(i).

$$\sum_{k=1}^{k_n} \left| a_{X_{nk}^{(\tau)}}(\tau) \right| \xrightarrow{p} 0, \quad \sum_{k=1}^{k_n} \left| a_{Y_{nk}^{(\tau)}}(\tau) \right| \xrightarrow{p} 0.$$

(ii). The difference between (24) and

$$\sum \int \left(e^{iux} - 1 - iux\mathbb{I}_{\{|x|<\tau\}}\right) \mathbb{I}_{\{|x|\geq\epsilon\}} d\overline{F}_{k-1}^{X}(x), \tag{26}$$

as well as the difference between (25) and

$$\sum \int \left(e^{ivy} - 1 - ivy\mathbb{I}_{\{|y|<\tau\}}\right) \mathbb{I}_{\{|y|\geq\epsilon\}} d\overline{F}_{k-1}^{Y}(y), \tag{27}$$

converge to 0 *in probability as* $n \to \infty$ *first, and then* $\epsilon \to 0$ *along the set* $I_{\mu,\lambda}$ *in (21).*

(iii). The difference between (23) and

$$-\frac{u^2}{2} \sum \sigma_{X_{nk}}^2(\epsilon) - \frac{v^2}{2} \sum \sigma_{Y_{nk}}^2(\epsilon) - uv \sum \sigma_{X_{nk},Y_{nk}}(\epsilon) \tag{28}$$

converges to 0 *in probability as* $n \to \infty$ *first, and then* $\epsilon \to 0$ *along the set* $I_{\mu,\lambda}$ *in (21).*

Proof of the statement (i) of Proposition 3. Recall that $X_{nk}^{(\tau)} = X_{nk} - a_{X_{nk}}(\tau)$ and

$$\sum \left| a_{X_{nk}^{(\tau)}}(\tau) \right| = \sum \left| E_{k-1}\left[X_{nk}^{(\tau)} \mathbb{I}_{\left\{\left|X_{nk}^{(\tau)}\right|<\tau\right\}} \right] \right|. \tag{29}$$

Also, $\sum \left| E_{k-1}\left[X_{nk}^{(\tau)} \left(\mathbb{I}_{\left\{\left|X_{nk}^{(\tau)}\right|<\tau\right\}} - \mathbb{I}_{\{|X_{nk}|<\tau\}} \right) \right] \right|$ coincides (note that $a_{X_{nk}}(\tau)$ is $\mathcal{A}_{n,k-1}$ measurable) on the event $\{\max_k |a_{X_{nk}}(\tau)| < \eta\}$, $\eta < \tau$, with

$$\begin{aligned} & \sum \mathbb{I}_{\left\{\left|a_{X_{nk}}(\tau)\right|<\eta\right\}} \left| E_{k-1}\left[X_{nk}^{(\tau)} \left(\mathbb{I}_{\left\{\left|X_{nk}^{(\tau)}\right|<\tau\right\}} - \mathbb{I}_{\{|X_{nk}|<\tau\}} \right) \right] \right| \\ \leq\ & 2(\tau+\eta) \sum_{k=1}^{k_n} P_{k-1}\left[\tau - \eta \leq |X_{nk}| < \tau + \eta\right] \xrightarrow{p} 0 \end{aligned}$$

as $n \to \infty$ first and then $\eta \to 0$ because $\pm\tau$ are continuity points of the Lévy measure L. Further, $\max_k |a_{X_{nk}}(\tau)| \leq \delta + \tau \max P_{k-1}[\delta \leq |X_{nk}| < \tau]$ for every $\delta > 0$, and hence by (1),

$$\max_k |a_{X_{nk}}(\tau)| \xrightarrow{p} 0. \tag{30}$$

Thus

$$\sum \left| E_{k-1}\left[X_{nk}^{(\tau)} \left(\mathbb{I}_{\left\{\left|X_{nk}^{(\tau)}\right|<\tau\right\}} - \mathbb{I}_{\{|X_{nk}|<\tau\}} \right) \right] \right| \xrightarrow{p} 0.$$

In addition

$$\begin{aligned} & \sum \left| E_{k-1}\left[X_{nk}^{(\tau)} \mathbb{I}_{\{|X_{nk}|<\tau\}} \right] \right| = \sum |a_{X_{nk}}(\tau)|\, P_{k-1}\left[|X_{nk}| \geq \tau\right] \\ \leq\ & \max_k |a_{X_{nk}}(\tau)| \sum_{k=1}^{k_n} P_{k-1}\left[|X_{nk}| \geq \tau\right] \xrightarrow{p} 0 \text{ by (30) and (C1).} \end{aligned}$$

Thus (29) $\xrightarrow{p} 0$. Similarly $\sum \left| a_{Y_{nk}^{(\tau)}}(\tau) \right| \xrightarrow{p} 0$, completing the proof of Statement (i). ■

For the proofs of the remaining statements, we need

Lemma 4.

$$\epsilon^2 \sum_{k=1}^{k_n} \left\{ P_{k-1} \left[\left| X_{nk}^{(\tau)} \right| \geq \epsilon \right] + P_{k-1} \left[\left| Y_{nk}^{(\tau)} \right| \geq \epsilon \right] \right\} \xrightarrow{p} 0$$

as $n \to \infty$ first and then $\epsilon \to 0$.

Proof. For each $\epsilon \in I_{\mu,\lambda}$, where $I_{\mu,\lambda}$ is as in (21), we have

$$\epsilon^2 \sum_{k=1}^{k_n} P_{k-1} \left[\left| Y_{nk}^{(\tau)} \right| \geq \epsilon \right] \stackrel{d}{\Longrightarrow} \epsilon^2 \int_{\{|y| \geq \epsilon\}} L^* (dy) \text{ as } n \to \infty.$$

Now, in view of the fact $\int (1 \wedge y^2) L^* (dy) < \infty$ a.s., we have, with $0 < \epsilon < 1$ and as $\epsilon \to 0$,

$$\epsilon^2 \int_{\{|y|>1\}} L^* (dy) \xrightarrow{p} 0,$$

$$\epsilon^2 \int_{\{1 \geq |y| > \sqrt{\epsilon}\}} L^* (dy) \leq \epsilon \int_{\{1 \geq |y| > 0\}} y^2 L^* (dy) \xrightarrow{p} 0$$

and

$$\epsilon^2 \int_{\{\sqrt{\epsilon} \geq |y| > \epsilon\}} L^* (dy) \leq \int_{\{\sqrt{\epsilon} \geq |y| > \epsilon\}} y^2 L^* (dy) \xrightarrow{p} 0.$$

Hence $\epsilon^2 \int_{\{|y| \geq \epsilon\}} L^* (dy) \xrightarrow{p} 0$. Similarly $\epsilon^2 \int_{\{|y| \geq \epsilon\}} L (dy) \xrightarrow{p} 0$. The proof follows. ■

Proof of the statement (ii) of Proposition 3. The proof will depend on the use the condition (E1), in addition to the preceding Lemma 4. For notational convenience, we restrict to $u = 1 = v$. Then note that

$$\begin{aligned} & \left(e^{ix+iy} - 1 - ix\mathbb{I}_{\{|x|<\tau\}} - iy\mathbb{I}_{\{|y|<\tau\}}\right) - \left(e^{ix} - 1 - ix\mathbb{I}_{\{|x|<\tau\}}\right) \\ = \; & e^{ix} \left(e^{iy} - 1 - iy\mathbb{I}_{\{|y|<\tau\}}\right) + i \left(e^{ix} - 1\right) y\mathbb{I}_{\{|y|<\tau\}} \end{aligned}$$

where $\left| \left(e^{ix} - 1\right) y\mathbb{I}_{\{|y|<\tau\}} \right| \leq |x| \, |y| \wedge 2\tau$ and

$$\left| e^{iy} - 1 - iy\mathbb{I}_{\{|y|<\tau\}} \right| \leq \begin{cases} y^2 = y^2 \wedge \tau^2 & \text{if } |y| < \tau \\ 2 = \frac{2}{\tau^2} \left(y^2 \wedge \tau^2\right) & \text{if } |y| \geq \tau. \end{cases}$$

Thus the difference between the summand in (24) and that in (26) is bounded in absolute value by

$$C \int \left(|y|^2 \wedge \tau^2 + |x| \, |y| \wedge 2\tau \right) \mathbb{I}_{\{|x| \geq \epsilon\}} d\overline{F}_{k-1}^{X,Y} (x, y) \tag{31}$$

for some $C > 0$ (C depends on τ). Now note that

$$\begin{aligned} & \int \left(|y|^2 \wedge \tau^2 + |x|\,|y| \wedge 2\tau\right)\left(\mathbb{I}_{\{|x|\geq\epsilon\}} - \mathbb{I}_{\{M\geq|x|\geq\epsilon,|y|\leq\epsilon^2\}}\right) d\overline{F}^{X,Y}_{k-1}(x,y) \\ \leq\ & \left(\tau^2 + 2\tau\right)\int \left(\mathbb{I}_{\{|x|\geq M\}} + \mathbb{I}_{\{|y|\geq M\}} + \mathbb{I}_{\{M\geq|x|\geq\epsilon,M\geq|y|\geq\epsilon^2\}}\right) d\overline{F}^{X,Y}_{k-1}(x,y) \\ =\ & I_{1n}(M) + I_{2n}(M) + I_{3n}(M,\epsilon)\text{, say.} \end{aligned}$$

In addition, if $\epsilon \leq \tau$,

$$\int \left(|y|^2 \wedge \tau^2\right)\mathbb{I}_{\{M\geq|x|\geq\epsilon,|y|\leq\epsilon^2\}} d\overline{F}^{X,Y}_{k-1}(x,y) \leq \epsilon^4 \int \mathbb{I}_{\{|x|\geq\epsilon\}} d\overline{F}^{X}_{k-1}(x) = I_{4n}(\epsilon)\text{, say,}$$

and in the same way,

$$\int \left(|x|\,|y| \wedge 2\tau\right)\mathbb{I}_{\{M\geq|x|\geq\epsilon,|y|\leq\epsilon^2\}} d\overline{F}^{X,Y}_{k-1}(x,y) \leq M\epsilon^2 \int \mathbb{I}_{\{M\geq|x|\geq\epsilon\}} d\overline{F}^{X}_{k-1}(x) = I_{5n}(M,\epsilon)\text{, say.}$$

Now, for each $\epsilon > 0$ and $M > 0$, $\sum I_{3n}(M,\epsilon) \xrightarrow{p} 0$ as $n \to \infty$ by the condition (E1).

In addition, for each $M > 0$, $\sum \left(I_{4n}(\epsilon) + I_{5n}(M,\epsilon)\right) \xrightarrow{p} 0$ as $n \to \infty$ and then $\epsilon \to 0$, by Lemma 4.

Furthermore, $\sum \left(I_{1n}(M) + I_{2n}(M)\right) \xrightarrow{p} 0$ as $n \to \infty$ and then $M \to \infty$.

We have thus shown that the sum $\sum$ of (31) converges to 0 in probability as $n \to \infty$ and then $\epsilon \to 0$, completing the proof of the first part of the statement (ii). In the same way the second part also follows. ■

Proof of the statement (iii) of Proposition 3. Note that, for some constant $C > 0$ (depending on u and v),

$$\begin{aligned} & \left| e^{iux+ivy} - 1 - iux - ivy + \frac{1}{2}(ux+vy)^2 \right| \\ \leq\ & \frac{1}{6}|ux+vy|^3 \leq C\epsilon\left((ux)^2 + (vy)^2\right) \quad \text{if } |x| < \epsilon, |y| < \epsilon. \end{aligned}$$

Hence the difference between (23) and

$$\begin{aligned} & -\frac{1}{2}\sum \int (ux+vy)^2\, \mathbb{I}_{\{|x|<\epsilon,|y|<\epsilon\}} d\overline{F}^{X,Y}_{k-1}(x,y) \\ =\ & -\frac{1}{2}\sum E_{k-1}\left[\left(uX^{(\tau)}_{nk} + vY^{(\tau)}_{nk}\right)^2 \mathbb{I}_{\left\{\left|X^{(\tau)}_{nk}\right|<\epsilon,\left|Y^{(\tau)}_{nk}\right|<\epsilon\right\}}\right] \end{aligned} \tag{32}$$

is bounded in absolute value by $C\epsilon\sum\int\left((ux)^2 + (vy)^2\right)\mathbb{I}_{\{|x|<\epsilon,|y|<\epsilon\}} d\overline{F}^{X,Y}_{k-1}(x,y)$. Therefore it is enough to show that the difference between (28) and (32) converges to 0 in probability, as $n \to \infty$ first and then $\epsilon \to 0$.

We have,

$$\left| E_{k-1}\left[\left|X^{(\tau)}_{nk}\right|^2 \left(\mathbb{I}_{\left\{\left|X^{(\tau)}_{nk}\right|<\epsilon,\left|Y^{(\tau)}_{nk}\right|<\epsilon\right\}} - \mathbb{I}_{\left\{\left|X^{(\tau)}_{nk}\right|<\epsilon\right\}}\right)\right]\right| \leq \epsilon^2 P_{k-1}\left[\left|Y^{(\tau)}_{nk}\right| \geq \epsilon\right]. \tag{33}$$

Further, noting $X_{nk}^{(\tau)} = X_{nk} - a_{X_{nk}}(\tau)$,

$$\begin{aligned}
&\left| E_{k-1}\left[\left| X_{nk}^{(\tau)} \right|^2 \left| \mathbb{I}_{\left\{ \left| X_{nk}^{(\tau)} \right| < \epsilon \right\}} - \mathbb{I}_{\{|X_{nk}|<\epsilon\}} \right| \right] \right| \\
\leq\ & E_{k-1}\left[\left| X_{nk}^{(\tau)} \right|^2 \left(\mathbb{I}_{\left\{ \left| X_{nk}^{(\tau)} \right| < \epsilon, |X_{nk}| \geq \epsilon \right\}} + \mathbb{I}_{\left\{ \left| X_{nk}^{(\tau)} \right| \geq \epsilon, |X_{nk}| < \epsilon \right\}} \right) \right] \\
\leq\ & \epsilon^2 P_{k-1}\left[|X_{nk}| \geq \epsilon \right] + 2\left(\epsilon^2 + \max_k \left| a_{X_{nk}}(\tau) \right|^2 \right) P_{k-1}\left[\left| X_{nk}^{(\tau)} \right| \geq \epsilon \right]. \qquad (34)
\end{aligned}$$

In addition,

$$\begin{aligned}
& E_{k-1}\left[\left\{ X_{nk}^{(\tau)} \mathbb{I}_{\{|X_{nk}|<\epsilon\}} - \left(X_{nk}\mathbb{I}_{\{|X_{nk}|<\epsilon\}} - a_{X_{nk}}(\epsilon) \right) \right\}^2 \right] \\
=\ & E_{k-1}\left[\left\{ a_{X_{nk}}(\tau)\, \mathbb{I}_{\{|X_{nk}|<\epsilon\}} - a_{X_{nk}}(\epsilon) \right\}^2 \right] \\
\leq\ & 2\left(E_{k-1}\left[X_{nk}\mathbb{I}_{\{\epsilon \leq |X_{nk}| < \tau\}} \right] \right)^2 + 2\left(a_{X_{nk}}(\tau) \right)^2 P_{k-1}\left[|X_{nk}| \geq \epsilon \right] \\
\leq\ & \left\{ \tau^2 \max_k P_{k-1}\left[|X_{nk}| \geq \epsilon \right] + 2 \max_k \left| a_{X_{nk}}(\tau) \right|^2 \right\} P_{k-1}\left[|X_{nk}| \geq \epsilon \right]. \qquad (35)
\end{aligned}$$

In view of (1), (30) and Lemma 4, it follows from the above bounds (33) - (35) that, as $n \to \infty$ first and then $\epsilon \to 0$,

$$\sum E_{k-1}\left[\left\{ X_{nk}^{(\tau)} \mathbb{I}_{\left\{ \left| X_{nk}^{(\tau)} \right| < \epsilon, \left| Y_{nk}^{(\tau)} \right| < \epsilon \right\}} - \left(X_{nk}\mathbb{I}_{\{|X_{nk}|<\epsilon\}} - a_{X_{nk}}(\varepsilon) \right) \right\}^2 \right] \xrightarrow{p} 0. \qquad (36)$$

Using the inequality, $2\left|c^2 - d^2\right| = 2\left|c - d\right|\left|c + d\right| \leq M\left|c - d\right|^2 + M^{-1}\left|c + d\right|^2$, we have

$$\left| c^2 - d^2 \right| \leq (M+1)\left| c - d \right|^2 + 4M^{-1}\left| d \right|^2 \qquad \text{for all } M > 0. \qquad (37)$$

Using this inequality, and in view of (36) and because $E_{k-1}\left[\left(\left(X_{nk}\mathbb{I}_{\{|X_{nk}|<\epsilon\}} - a_{X_{nk}}(\varepsilon) \right) \right)^2 \right] = \sigma_{X_{nk}}^2(\epsilon)$, we see that $\sum \left| E_{k-1}\left[\left| X_{nk}^{(\tau)} \right|^2 \mathbb{I}_{\left\{ \left| X_{nk}^{(\tau)} \right| < \epsilon, \left| Y_{nk}^{(\tau)} \right| < \epsilon \right\}} \right] - \sigma_{X_{nk}}^2(\epsilon) \right|$ is bounded by a quantity that $\xrightarrow{p} 0$ as $n \to \infty$ first, then $\epsilon \to 0$ and then $M \to \infty$. ($\sum \sigma_{X_{nk}}^2(\epsilon)$ is stochastically bounded.) Thus

$$\sum E_{k-1}\left[\left| X_{nk}^{(\tau)} \right|^2 \mathbb{I}_{\left\{ \left| X_{nk}^{(\tau)} \right| < \epsilon, \left| Y_{nk}^{(\tau)} \right| < \epsilon \right\}} \right] - \sum \sigma_{X_{nk}}^2(\epsilon) \xrightarrow{p} 0 \qquad (38)$$

as $n \to \infty$ first and then $\epsilon \to 0$.

In exactly the same way,

$$\sum E_{k-1}\left[\left| Y_{nk}^{(\tau)} \right|^2 \mathbb{I}_{\left\{ \left| X_{nk}^{(\tau)} \right| < \epsilon, \left| Y_{nk}^{(\tau)} \right| < \epsilon \right\}} \right] - \sum \sigma_{Y_{nk}}^2(\epsilon) \xrightarrow{p} 0. \qquad (39)$$

Next consider $\sum E_{k-1}\left[X_{nk}^{(\tau)}Y_{nk}^{(\tau)}\mathbb{I}_{\left\{\left|X_{nk}^{(\tau)}\right|<\epsilon,\left|Y_{nk}^{(\tau)}\right|<\epsilon\right\}}\right] = E_{k-1}[\varphi_k\phi_k]$, where we let $\varphi_k = X_{nk}^{(\tau)}\mathbb{I}_{\left\{\left|X_{nk}^{(\tau)}\right|<\epsilon,\left|Y_{nk}^{(\tau)}\right|<\epsilon\right\}}$, $\phi_k = Y_{nk}^{(\tau)}\mathbb{I}_{\left\{\left|X_{nk}^{(\tau)}\right|<\epsilon,\left|Y_{nk}^{(\tau)}\right|<\epsilon\right\}}$. Then, with $U_k = X_{nk}\mathbb{I}_{\{|X_{nk}|<\epsilon\}} - a_{X_{nk}}(\epsilon)$ and $V_k = Y_{nk}\mathbb{I}_{\{|Y_{nk}|<\epsilon\}} - a_{Y_{nk}}(\epsilon)$,

$$E_{k-1}[|\varphi_k\phi_k - U_kV_k|] \leq E_{k-1}[|\varphi_k||\phi_k - V_k|] + E_{k-1}[|V_k||\varphi_k - U_k|],$$

where

$$\begin{aligned}\sum E_{k-1}[|V_k||\varphi_k - U_k|] &\leq \sum\sqrt{E_{k-1}\left[|V_k|^2\right]E_{k-1}\left[|\varphi_k - U_k|^2\right]} \\ &\leq \left(\sum E_{k-1}\left[|V_k|^2\right]\right)\sum E_{k-1}\left[|\varphi_k - U_k|^2\right] \xrightarrow{p} 0\end{aligned}$$

by (36) and because $E_{k-1}\left[|V_k|^2\right] = \sigma^2_{Y_{nk}}(\epsilon)$. In the same way $\sum E_{k-1}[|\varphi_k||\phi_k - V_k|] \xrightarrow{p} 0$. Hence, noting that $\sigma_{X_{nk},Y_{nk}}(\epsilon) = E_{k-1}[U_kV_k]$, we have

$$\sum E_{k-1}\left[X_{nk}^{(\tau)}Y_{nk}^{(\tau)}\mathbb{I}_{\left\{\left|X_{nk}^{(\tau)}\right|<\epsilon,\left|Y_{nk}^{(\tau)}\right|<\epsilon\right\}}\right] - \sum\sigma_{X_{nk},Y_{nk}}(\epsilon) \xrightarrow{p} 0.$$

This together with (38) and (39) implies that the difference between (28) and (32) converges to 0 in probability, as $n\to\infty$ first and then $\epsilon\to 0$, completing the proof of the statement (iii) of the proposition 3. ■

We shall also need the following bound for (22), already implicit in the proof of Proposition 3.

Lemma 5.

$$\begin{aligned}&\left|\int\left(e^{iux+ivy} - 1 - iux\mathbb{I}_{\{|x|<\tau\}} - ivy\mathbb{I}_{\{|y|<\tau\}}\right)d\overline{F}_{k-1}^{X,Y}(x,y)\right| \\ \leq\ & C\left(\int\left(x^2\wedge\tau^2\right)d\overline{F}_{k-1}^{X}(x) + \int\left(y^2\wedge\tau^2\right)d\overline{F}_{k-1}^{Y}(y)\right)\end{aligned}$$

for some constant $C>0$ depending on τ, u and v.

Proof. First, the summand in (23) with $\epsilon=\tau$ is bounded in absolute value by

$$C\int\left(x^2+y^2\right)\mathbb{I}_{\{|x|<\tau,|y|<\tau\}}d\overline{F}_{k-1}^{X,Y}(x,y) \tag{40}$$

for some $C>0$ depending on u and v. Also

$$\begin{aligned}&\left|\int x^2\left(\mathbb{I}_{\{|x|<\tau,|y|<\tau\}} - \mathbb{I}_{\{|x|<\tau\}}\right)d\overline{F}_{k-1}^{X,Y}(x,y)\right| \\ =\ &\left|\int x^2\mathbb{I}_{\{|x|<\tau,|y|\geq\tau\}}d\overline{F}_{k-1}^{X,Y}(x,y)\right| \leq \tau^2\int\mathbb{I}_{\{|y|\geq\tau\}}d\overline{F}_{k-1}^{Y}(y).\end{aligned}$$

In the same way $\left|\int y^2\left(\mathbb{I}_{\{|x|<\tau,|y|<\tau\}}-\mathbb{I}_{\{|y|<\tau\}}\right)d\overline{F}_{k-1}^{X,Y}(x,y)\right|\leq\tau^2\int\mathbb{I}_{\{|x|\geq\tau\}}d\overline{F}_{k-1}^{X}(x)$. Thus, (40) is bounded by

$$\begin{aligned}
& C\int\left(x^2\mathbb{I}_{\{|x|<\tau\}}+\mathbb{I}_{\{|x|\geq\tau\}}\right)d\overline{F}_{k-1}^{X}(x)+C\int\left(y^2\mathbb{I}_{\{|y|<\tau\}}+\mathbb{I}_{\{|y|\geq\tau\}}\right)d\overline{F}_{k-1}^{Y}(y) \\
= \; & C\int\left(x^2\wedge\tau^2\right)d\overline{F}_{k-1}^{X}(x)+C\int\left(y^2\wedge\tau^2\right)d\overline{F}_{k-1}^{Y}(y)\text{ for some }C>0. \qquad (41)
\end{aligned}$$

Next, the difference between the summands in (24) and (26), both with $\epsilon=\tau$, is bounded in absolute value by (31), which in turn is bounded by $(\tau^2+2\tau)\int\mathbb{I}_{\{|x|\geq\tau\}}d\overline{F}_{k-1}^{X}(x)$. In addition, the summand in (26) itself, with $\epsilon=\tau$ is bounded in absolute value by $2\int\mathbb{I}_{\{|x|\geq\tau\}}d\overline{F}_{k-1}^{X}(x)$.

Thus the summand in (24) is bounded in absolute value by $C\int\mathbb{I}_{\{|x|\geq\tau\}}d\overline{F}_{k-1}^{X}(x)$ for some $C>0$ depending on τ, u and v. In the same way, the summand in (25), with $\epsilon=\tau$, is bounded in absolute value by $C\int\mathbb{I}_{\{|y|\geq\tau\}}d\overline{F}_{k-1}^{Y}(y)$. This gives the required bound (because the preceding two bounds are already factored into (41)). ■

4.2 PROOF OF THEOREM 1 UNDER THE RESTRICTION (46)

We need to show that $(S_n(t),R_n(t))$ is tight in $D_{\mathbb{R}^2}[0,1]$, and that the finite dimensional distributions converge in distribution to those of the limit. The tightness is essentially known (for instance Jacod and Shiryayev (1987, Ch VI, Sections 4 and 5)), but for convenience we shall consider it separately in Section 4.4 below. Then, *assuming tightness*, we next consider the convergence of finite dimensional distributions. For this purpose it will be enough to restrict the index set to $T\subset[0,1]$ with T as in (D1), because T is dense in $[0,1]$.

Thus we consider the convergence of the sum

$$\sum_{l=1}^{r}u_l\left(S_n(\lambda_l)-S_n(\lambda_{l-1})\right)+\sum_{j=1}^{q}v_j\left(R_n(t_j)-R_n(t_{j-1})\right)$$

for finite values $0=\lambda_0<\lambda_1<...<\lambda_r\leq 1$ and $0=t_0<t_1<...<t_q\leq 1$ (contained in T) and for reals $u_1,...,u_r$ and $v_1,...,v_q$.

We **remark** that without loss of generality we can take $q=r$ and $(\lambda_1,...,\lambda_r)=(t_1,....t_q)$. Define (recall $X_{nk}^{(\tau)}=X_{nk}-a_{X_{nk}}(\tau)$, $Y_{nk}^{(\tau)}=Y_{nk}-a_{Y_{nk}}(\tau)$)

$$U_{nk}=u_lX_{nk}^{(\tau)},\quad V_{nk}=v_lY_{nk}^{(\tau)},\quad\text{ for }k_n(t_{l-1})<k\leq k_n(t_l),\ l=1,...,q.$$

(For notational convenience, we suppress the dependence of U_{nk} on u_i's and of V_{nk} on

v_i's). Then we need to consider, for each finite $0 = t_0 < t_1 < ... < t_q = 1$, the sum

$$\begin{aligned} & \sum_{l=1}^{q} \left(u_l \left(S_n \left(t_l \right) - S_n \left(t_{l-1} \right) \right) + v_l \left(R_n \left(t_l \right) - R_n \left(t_{l-1} \right) \right) \right) \\ = & \sum_{k=1}^{k_n(t_q)} \left(U_{nk} + V_{nk} \right) + \sum_{l=1}^{q} \sum_{k=k_n(t_{i-1})+1}^{k_n(t_i)} \left(u_l a_{X_{nk}} \left(\tau \right) + v_l a_{Y_{nk}} \left(\tau \right) \right). \end{aligned}$$

Now it is clear that the conditions (C1) - (C4) hold with appropriate limits when the array $\{U_{nk}, k_n(t_{l-1}) < k \le k_n(t_l),\ l = 1, ..., q\}$ is involved in place of the array $\{X_{nk},\ k = 1, ...\}$. Similarly, (D1) - (D5) hold for the array$\{V_{nk}, k_n(t_{l-1}) < k \le k_n(t_l),\ l = 1, ..., q\}$. (These arrays are now adapted to the array $\left\{\mathcal{A}_{n,k_n(t_{l-1})}, ..., \mathcal{A}_{n,k_n(t_l)},\ l = 1, ..., q\right\}$.) Thus the Proposition 3 also holds for these arrays.

Further note that these arrays depend on $a_{X_{nk}}(\tau)$ and $a_{Y_{nk}}(\tau)$, but because of the statement (i) of Proposition 3 for these arrays, the corresponding $L^*_{n,t}(y)$ of (C1) and (D1) will be approximately the same as the ones obtained with the same arrays but with $a_{X_{nk}}(\tau) = 0 = a_{Y_{nk}}(\tau)$.

For convenience we next state this fact and some of its consequences in the form we shall need.

Corollary 6. *Under the assumptions of Proposition 3, the difference between*

$$\sum_{k=1}^{k_n(t_q)} E_{k-1} \left[e^{iU_{nk} + iV_{nk}} - 1 \right]$$

and the sum

$$\begin{aligned} & \sum_{l=1}^{q} \left\{ \int \left(e^{iu_l x} - 1 - iu_l x \mathbb{I}_{\{|x| < \tau\}} \right) \mathbb{I}_{\{|x| \ge \epsilon\}} d \left(L_{n,t_i}(x) - L_{n,t_{i-1}}(x) \right) \right. \\ & + \int \left(e^{iv_l y} - 1 - iv_l y \mathbb{I}_{\{|y| < \tau\}} \right) \mathbb{I}_{\{|y| \ge \epsilon\}} d \left(L^*_{n,t_i}(y) - L^*_{n,t_{i-1}}(y) \right) \\ & \left. + \sum_{k=k_n(t_{i-1})+1}^{k_n(t_i)} \left(-\frac{u_l^2}{2} \sigma^2_{X_{nk}}(\epsilon) - \frac{v_l^2}{2} \sigma^2_{Y_{nk}}(\epsilon) \right) \right\} \end{aligned}$$

converges in probability to 0 *as* $n \to \infty$ *first, and then* $\epsilon \to 0$ *along the set* $I_{t_0, t_1, ..., t_q}$ *defined in (21). In particular*

$$\begin{aligned} & \left(S_n(t), A_{n,t}, A^*_{n,t}, \sum_{k=1}^{k_n(t_q)} E_{k-1} \left[e^{iU_{nk} + ivV_{nk}} - 1 \right] \right) \stackrel{fdd}{\Longrightarrow} \\ & \left(S(t), A_t, A^*_t, \sum_{l=1}^{q} \left(\psi_{t_l}(u_l) - \psi_{t_{l-1}}(u_l) + \psi^*_{t_l}(u_l) - \psi^*_{t_{l-1}}(u_l) \right) \right), \qquad (42) \end{aligned}$$

where

$$\psi_t(u) = -\frac{1}{2}u^2 B_t + \int \left(e^{iux} - 1 - iux\mathbb{I}_{\{|x|<\tau\}}\right) dL_t(x),$$

$$\psi_t^*(v) = -\frac{1}{2}u^2 B_t^* + \int \left(e^{ivy} - 1 - ivy\mathbb{I}_{\{|y|<\tau\}}\right) dL_t^*(y).$$

■

In the same way, in view of (22) and Lemma 5, we also in particular have the bound

$$\left|\sum_{k=l_1\wedge k_n(t_q)+1}^{l_2\wedge k_n(t_q)} E_{k-1}\left[e^{iU_{nk}+iV_{nk}} - 1\right]\right| \leq \Gamma_{nl_2} - \Gamma_{nl_1} + \Gamma_{nl_2}^* - \Gamma_{nl_1}^* \tag{43}$$

(in particular $\left|\sum_{k=1}^{l\wedge k_n(t_q)} E_{k-1}\left[e^{iU_{nk}+iV_{nk}} - 1\right]\right| \leq \Gamma_{nl} + \Gamma_{nl}^*$), where we let, for a suitable constant $C > 0$,

$$\begin{aligned}
\Gamma_{nl} &= C\sum_{k=1}^{l\wedge k_n(t_q)} \left|a_{X_{nk}^{(\tau)}}(\tau)\right| + C\sum_{k=1}^{l\wedge k_n(t_q)} \int \left(x^2\wedge\tau^2\right) d\overline{F}_{k-1}^X(x),\\
\Gamma_{nl}^* &= C\sum_{k=1}^{l\wedge k_n(t_q)} \left|a_{Y_{nk}^{(\tau)}}(\tau)\right| + C\sum_{k=1}^{l\wedge k_n(t_q)} \int \left(y^2\wedge\tau^2\right) d\overline{F}_{k-1}^Y(y)
\end{aligned} \tag{44}$$

Note further that, in view of (1), we always have

$$\sup_{1\leq k\leq k_n} \left|E_{k-1}\left[e^{iU_{nk}+iV_{nk}} - 1\right]\right| \xrightarrow{p} 0. \tag{45}$$

As a first step we complete the proof of Theorem 1 under the restriction

$$\Gamma_{n,k_n} + \Gamma_{n,k_n}^* \leq K \text{ a.s} \tag{46}$$

for some constant $K > 0$, which restriction will then be removed in the next step (Section 4.3 below). We then have the following approximation result.

Proposition 7.

$$E\left[e^{\sum_{k=1}^{k_n(t_q)}(U_{nk}+V_{nk}) - \sum_{k=1}^{k_n(t_q)} E_{k-1}\left[e^{iU_{nk}+iV_{nk}} - 1\right]}\right] - 1 \quad \to 0.$$

Proof. In view of (43), (45) and (46), the proof is identical with Lemma 6 in Jeganathan (1982) (which proof itself is based on Brown and Eagleson (1971)). ■

Now to complete the proof of the $\overset{fdd}{\Longrightarrow}$, we shall use the preceding Proposition 7 together with the convergence (42) of Corollary 6. Note that the tightness in $D_{\mathbb{R}}[0,1]$ of each of the first three components follows from the assumptions. Recall also that we have assumed that $(S_n(t), R_n(t))$ is tight in $D_{\mathbb{R}^2}[0,1]$ (see Section 4.4 below), and hence is tight in $(D_{\mathbb{R}}[0,1])^2$ also.

Thus the process $\left(S_n(t), R_n(t), A_{n,t}, A^*_{n,t}, \sum_{k=1}^{k_n(t_q)} E_{k-1}\left[e^{iU_{nk}+ivV_{nk}}-1\right]\right)$ is relatively compact in the product space $(D_{\mathbb{R}}[0,1])^4 \times \mathbb{R}$. Assume for convenience that it converges in $(D_{\mathbb{R}}[0,1])^4 \times \mathbb{R}$ for the sequence $\{n\}$ itself to the limit

$$\left(S(t), R^{\#}(t), A_t, A^*_t, \sum_{l=1}^{q}\left(\psi_{t_l}(u_l)-\psi_{t_{l-1}}(u_l)+\psi^*_{t_l}(u_l)-\psi^*_{t_{l-1}}(u_l)\right)\right), \text{ say,}$$

for some limiting process $R^{\#}(t)$. Note that this convergence in particular entails the $\stackrel{fdd}{\Longrightarrow}$. Then, in view of Proposition 7 and because of (43),

$$\begin{aligned}
& E\left[e^{\sum(U_{nk}+V_{nk})-\sum E_{k-1}\left[e^{iU_{nk}+iV_{nk}}-1\right]}\right] \\
\to\ & E\left[e^{\sum_{l=1}^{q}\left\{u_l(S(t_l)-S(t_{l-1}))+v_l\left(R^{\#}(t_l)-R^{\#}(t_{l-1})-\left(A^*_{t_l}-A^*_{t_{l-1}}\right)\right)-\left(\psi^*_{t_l}(v_l)-\psi^*_{t_{l-1}}(v_l)\right)\right\}}\right] \\
& \times e^{-\sum_{l=1}^{q}\left\{u_l\left(A_{t_l}-A_{t_{l-1}}\right)+\left(\psi_{t_l}(u_l)-\psi_{t_{l-1}}(u_l)\right)\right\}} \\
\equiv\ & 1. \qquad (47)
\end{aligned}$$

(Recall that A_t and $\psi_t(u)$ are nonrandom and A^*_t and $\psi^*_t(v)$ are functionals of $S = (S(u), 0 \le u \le 1)$.) Taking $v_l = 0$, this in particular gives

$$E\left[e^{\sum_{l=1}^{q} u_l(S(t_l)-S(t_{l-1}))}\right] = e^{\sum_{l=1}^{q}\left\{u_l\left(A_{t_l}-A_{t_{l-1}}\right)+\left(\psi_{t_l}(u_l)-\psi_{t_{l-1}}(u_l)\right)\right\}}.$$

Substituting this in the identity (47), and taking into account the remark made at the beginning of this subsection, the identity (47) gives

$$E\left[e^{\sum_{j=1}^{r} u_j S(\lambda_j)+\sum_{l=1}^{q}\left\{v_l\left(R^{\#}(t_l)-R^{\#}(t_{l-1})-\left(A^*_{t_l}-A^*_{t_{l-1}}\right)\right)-\left(\psi^*_{t_l}(v_l)-\psi^*_{t_{l-1}}(v_l)\right)\right\}}\right] = E\left[e^{\sum_{j=1}^{r} u_j S(\lambda_j)}\right]$$

for every $0 = \lambda_0 < \lambda_1 < \ldots < \lambda_r \le 1$ and $0 = t_0 < t_1 < \ldots < t_q \le 1$ and for reals $u_1, \ldots, u_r$ and $v_1, \ldots, v_q$. Because the preceding identity holds for every $0 < \lambda_1 < \ldots < \lambda_r \le 1$, we have

$$E\left[e^{\sum_{l=1}^{q}\left\{v_l\left(R^{\#}(t_l)-R^{\#}(t_{l-1})-\left(A^*_{t_l}-A^*_{t_{l-1}}\right)\right)-\left(\psi^*_{t_l}(v_l)-\psi^*_{t_{l-1}}(v_l)\right)\right\}}\,\middle|\, S\right] \equiv 1.$$

Because $A^*_{t_l}$ and $\psi^*_{t_l}(v_l)$ are functionals of S, this is equivalent to

$$\begin{aligned}
E\left[e^{\sum_{l=1}^{q} v_l\left(R^{\#}(t_l)-R^{\#}(t_{l-1})\right)}\,\middle|\, S\right] &= e^{\sum_{l=1}^{q}\left\{v_l\left(A^*_{t_l}-A^*_{t_{l-1}}\right)+\left(\psi^*_{t_l}(v_l)-\psi^*_{t_{l-1}}(v_l)\right)\right\}} \\
&= E\left[e^{\sum_{l=1}^{q} v_l(R(t_l)-R(t_{l-1}))}\,\middle|\, S\right].
\end{aligned}$$

(Note that the process $R(t)$ as defined in the statement of Theorem 1 is well defined.) This gives the required finite dimensional convergence (under the restriction (46)). ■

4.3 RELAXATION OF THE RESTRICTION (46).

We have

$$\left(\Gamma_{n,k_n(t)},\Gamma^*_{n,k_n(t)}\right)\overset{fdd}{\Longrightarrow}\left(\Gamma_t,\Gamma^*_t\right)\tag{48}$$

where (with C as in the definitions of $\Gamma_{n,k_n(t)}$ and $\Gamma^*_{n,k_n(t)}$ in (44))

$$\Gamma_t=CB_t+C\int x^2\wedge\tau^2dL_t\left(x\right),\quad \Gamma^*_t=CB^*_t+C\int y^2\wedge\tau^2dL^*_t\left(y\right).$$

We note that Γ_{nk} and Γ^*_{nk} are $\mathcal{A}_{n,k-1}$ measurable. Now let, for some $\eta>0$,

$$\kappa=\Gamma_1+\eta.\tag{49}$$

(Recall that Γ_t are nonrandom and $\Gamma_t\le\Gamma_1$.) Define

$$X^*_{nk}=\mathbb{I}_{\{\Gamma_{nk}\le\kappa\}}X_{nk},\qquad Y^*_{nk}=\mathbb{I}_{\left\{\Gamma^*_{nk}\le\alpha\right\}}Y_{nk}$$

for a given $\alpha>0$. We then have

$$a_{X^*_{nk}}\left(\tau\right)=\mathbb{I}_{\{\Gamma_{nk}\le\kappa\}}a_{X_{nk}}\left(\tau\right),\tag{50}$$

$$\delta^2_{X^*_{nk}}\left(\epsilon\right)=E_{k-1}\left[X^{*2}_{nk}\mathbb{I}_{\{|X_{nk}|<\epsilon\}}\right]-\left(E_{k-1}\left[X^*_{nk}\mathbb{I}_{\{|X_{nk}|<\epsilon\}}\right]\right)^2=\mathbb{I}_{\{\Gamma_{nk}\le\kappa\}}\sigma^2_{X_{nk}}\left(\epsilon\right)\tag{51}$$

and

$$\begin{aligned}&E\left[\left(e^{iuX^*_{nk}}-1-iuX^*_{nk}\mathbb{I}_{\{|X_{nk}|<\tau\}}\right)\mathbb{I}_{\{|X_{nk}|\ge\epsilon\}}\right]\\=\;&\mathbb{I}_{\{\Gamma_{nk}\le\kappa\}}E\left[\left(e^{iuX_{nk}}-1-iuX_{nk}\mathbb{I}_{\{|X_{nk}|<\tau\}}\right)\mathbb{I}_{\{|X_{nk}|\ge\epsilon\}}\right].\end{aligned}\tag{52}$$

The same relations (50) - (52) hold with the X^*_{nk}, X_{nk}, Γ_{nk} and κ replaced respectively by Y^*_{nk}, Y_{nk}, Γ^*_{nk} and α.

Lemma 8.*(i).*

$$P\left[\begin{array}{c}E\left[\left(e^{iuX^*_{nk}}-1-iuX^*_{nk}\mathbb{I}_{\{|X_{nk}|<\tau\}}\right)\mathbb{I}_{\{|X_{nk}|\ge\epsilon\}}\right]\\=E\left[\left(e^{iuX_{nk}}-1-iuX_{nk}\mathbb{I}_{\{|X_{nk}|<\tau\}}\right)\mathbb{I}_{\{|X_{nk}|\ge\epsilon\}}\right],\\a_{X^*_{nk}}\left(\tau\right)=a_{X_{nk}}\left(\tau\right),\;\;\delta^2_{X^*_{nk}}\left(\epsilon\right)=\sigma^2_{X_{nk}}\left(\epsilon\right)\text{ for all }1\le k\le k_n\text{ and }\epsilon>0\end{array}\right]\to 1.$$

(ii).

$$\sum_{k=1}^{k_n(t)}\delta^2_{Y^*_{nk}}\left(\epsilon\right)=\sum_{k=1}^{k_n(t)}\mathbb{I}_{\left\{\Gamma^*_{nk}\le\alpha\right\}}\sigma^2_{Y_{nk}}\left(\epsilon\right)\overset{fdd}{\Longrightarrow}\int_0^t\mathbb{I}_{\left\{\Gamma^*_{\lambda}\le\alpha\right\}}dB^*_{\lambda}=B^*_{t,\alpha},\text{ say,}\tag{53}$$

and, when $y \longmapsto L^*_{t_l}(y,S)$ is continuous at $y = \pm\epsilon$,

$$\sum_{k=1}^{k_n(t)} E\left[\left(e^{ivY^*_{nk}} - 1 - ivY^*_{nk}\mathbb{I}_{\{|Y_{nk}|<\tau\}}\right)\mathbb{I}_{\{|Y_{nk}|\geq\epsilon\}}\right]$$
$$\overset{fdd}{\Longrightarrow} \int \left(e^{ivy} - 1 - ivy\mathbb{I}_{\{|y|<\tau\}}\right)\mathbb{I}_{\{|y|\geq\epsilon\}} dL^*_{t,\alpha}(y) \tag{54}$$

where (with $L^*(y,t) = L^*_t(y)$)

$$L^*_{t,\alpha}(y) = \int_0^t \mathbb{I}_{\left\{\Gamma^*_\lambda \leq \alpha\right\}} dL^*(y, d\lambda). \tag{55}$$

Proof. The proof of the statement (i) is clear because the event $\{\Gamma_{n,k_n} \leq \kappa\}$ implies the event in question by (50) - (52), and because $P[\Gamma_{n,k_n} > \kappa] \to 0$ in view of (49).

For the statement (ii), we establish (53), and the proof of (54) is similar in view of the fact that (52) holds for Y_{nk} also (with $\mathbb{I}_{\left\{\Gamma^*_{nk}\leq\alpha\right\}}$ in place of $\mathbb{I}_{\{\Gamma_{nk}\leq\kappa\}}$). First, for a given $\eta > 0$, one can find continuous, nonnegative, nonincreasing functions $h_1(y)$ and $h_2(y)$ such that

$$0 \leq h_1(y) \leq \mathbb{I}_{\{y\leq\alpha\}} \leq h_2(y) \quad \text{with} \quad h_2(y) - h_1(y) \leq \mathbb{I}_{\{\alpha-\eta<y\leq\alpha+\eta\}} \quad \text{for all } y.$$

Also, because $\Gamma^*_{n,k_n(t)}$ is nondecreasing in t, with its limit Γ^*_t continuous in t, and because $-h_1(y)$ is nondecreasing and continuous in y, we have

$$\sup_{|\lambda-\mu|\leq 1/q} \left|h_1\left(\Gamma^*_{n,k_n(\lambda)}\right) - h_1\left(\Gamma^*_{n,k_n(\mu)}\right)\right| \xrightarrow{p} 0 \tag{56}$$

as $n \to \infty$ first and then $q \to \infty$. The same holds for $h_2(y)$.

We have $\sum_{k=1}^{k_n(t)} h_1(\Gamma^*_{nk})\,\sigma^2_{Y_{nk}}(\epsilon) = \sum_{l=1}^{q} \sum_{k=k_n\left(t\wedge\frac{l-1}{q}\right)+1}^{k_n\left(t\wedge\frac{l}{q}\right)} h_1(\Gamma^*_{nk})\,\sigma^2_{Y_{nk}}(\epsilon)$. The difference between this and

$$\sum_{l=1}^{q} h_1\left(\Gamma^*_{n,k_n\left(t\wedge\frac{l-1}{q}\right)}\right) \sum_{k=k_n\left(t\wedge\frac{l-1}{q}\right)+1}^{k_n\left(t\wedge\frac{l}{q}\right)} \sigma^2_{Y_{nk}}(\epsilon) \tag{57}$$

is bounded in absolute value by

$$\sup_{|\lambda-\mu|\leq 1/q} \left|h_1\left(\Gamma^*_{n,k_n(\lambda)}\right) - h_1\left(\Gamma^*_{n,k_n(\mu)}\right)\right| \sum_{k=1}^{k_n} \sigma^2_{Y_{nk}}(\epsilon) \xrightarrow{p} 0$$

as $n \to \infty$ and then $q \to \infty$, in view of (56). Further,

$$(57) \overset{fdd}{\Longrightarrow} \sum_{l=1}^{q} h_1\left(\Gamma^*_{t\wedge\frac{l-1}{q}}\right)\left(B^*_{t\wedge\frac{l}{q}} - B^*_{t\wedge\frac{l-1}{q}}\right)$$

which in turn converges in probability to $\int_0^t h_1(\Gamma_\lambda^*)\, dB_\lambda^*$. Thus we have shown that

$$\sum_{k=1}^{k_n(t)} h_1(\Gamma_{nk}^*)\, \sigma_{Y_{nk}}^2(\epsilon) \overset{fdd}{\Longrightarrow} \int_0^t h_1(\Gamma_\lambda^*)\, dB_\lambda^*.$$

In the same way $\sum_{k=1}^{k_n(t)} h_2(\Gamma_{nk}^*)\, \sigma_{Y_{nk}}^2(\epsilon) \overset{fdd}{\Longrightarrow} \int_0^t h_2(\Gamma_\lambda^*)\, dB_\lambda^*$.

Now $\sum_{k=1}^{k_n(t)} (h_2(\Gamma_{nk}^*) - h_1(\Gamma_{nk}^*))\, \sigma_{Y_{nk}}^2(\epsilon) \overset{fdd}{\Longrightarrow} \int_0^t (h_2(\Gamma_\lambda^*) - h_1(\Gamma_\lambda^*))\, dB_\lambda^*$, which limit is bounded by

$$\int_0^1 \mathbb{I}_{\{\alpha-\eta<\Gamma_\lambda^*\le\alpha+\eta\}} dB_\lambda^* \le \int_0^1 \mathbb{I}_{\{\alpha-\eta<\Gamma_\lambda^*\le\alpha+\eta\}} d\Gamma_\lambda^* = \Gamma_{\rho_1}^* - \Gamma_{\rho_2}^* \le 2\eta$$

where ρ_1 is such that $\Gamma_{\rho_1}^* = \alpha+\eta$ if $\alpha+\eta < \Gamma_1^*$ and $\rho_1 = 1$ if $\alpha+\eta \ge \Gamma_1^*$. Here we have used the continuity of Γ_t^* in t. Similarly ρ_2 is defined with $\alpha-\eta$. Hence (53) follows. ■

Now define U_{nk}^* and V_{nk}^* in terms of X_{nk}^* and Y_{nk}^* in exactly the same way as U_{nk} and V_{nk} were defined in terms of X_{nk} and Y_{nk} in the preceding Section 4.2. Then Proposition 3 and Corollary 6 hold for the variables U_{nk}^* and V_{nk}^* also, with the following modifications. The bound (43) will now take the form (with the constant C as in (44))

$$C \sum_{k=l_1\wedge k_n(t_q)+1}^{l_2\wedge k_n(t_q)} \Big\{ \mathbb{I}_{\{\Gamma_{nk}\le\kappa\}} \Big(\Big| a_{X_{nk}^{(\tau)}}(\tau) \Big| + \int x^2 \wedge \tau^2 d\overline{F}_{k-1}^X(x) \Big) + \mathbb{I}_{\{\Gamma_{nk}^*\le\alpha\}} \Big(\Big| a_{Y_{nk}^{(\tau)}}(\tau) \Big| + \int y^2 \wedge \tau^2 d\overline{F}_{k-1}^Y(y) \Big) \Big\} \le \alpha + \kappa$$

where the bound $\alpha+\kappa$ is obtained in view of the definitions of Γ_{nk} and Γ_{nk}^* in (44).

Hence, Proposition 7 holds when $U_{nk}^* + V_{nk}^*$ are involved in place of $U_{nk} + V_{nk}$. Therefore, if now $Q_\alpha(t)$ is the limit of $\sum_{j=1}^{k_n(t)} \left(Y_{nj}^* - a_{Y_{nj}^*}(\tau) \right)$, then in view of Lemma 8,

$$E\left[e^{\sum_{l=1}^q v_l (Q_\alpha(t_l) - Q_\alpha(t_{l-1}))} \Big| S \right] = e^{\sum_{l=1}^q \left(\psi_{t_l,\alpha}^*(v_l) - \psi_{t_{l-1},\alpha}^*(v_l) \right)} \tag{58}$$

where

$$\psi_{t,\alpha}^*(v) = -\frac{1}{2} v^2 B_{t,\alpha}^* + \int \left(e^{ivy} - 1 - ivy\mathbb{I}_{\{|y|<\tau\}} \right) dL_{t,\alpha}^*(y)$$

with $B_{t,\alpha}^*$ and $L_{t,\alpha}^*(y)$ as defined in (53) and (55). Note that

$$P\left[\sup_t \left| \psi_{t,\alpha}^*(v) - \psi_t^*(v) \right| \ne 0 \right] \le P[\Gamma_1^* > \alpha] \to 0 \text{ as } \alpha \to \infty$$

and in the same way

$$P\left[\sum_{j=1}^{k_n(t)} \left(Y_{nj}^* - a_{Y_{nj}^*}(\tau) \right) \ne \sum_{j=1}^{k_n(t)} \left(Y_{nj} - a_{Y_{nj}}(\tau) \right) \text{ for some } t \in [0,1] \right] \le P[\Gamma_{nk}^* > \alpha]$$

where $\lim_{\alpha\to\infty} \operatorname{limsup}_{n\to\infty} P\left[\Gamma_{nk}^* > \alpha\right] = 0$. Thus taking the limit as $\alpha \to \infty$ on both sides of (58)

$$E\left[e^{\sum_{l=1}^{q} v_l \left(R(t_l) - R(t_{l-1}) - \left(A_{t_l}^* - A_{t_{l-1}}^* \right) \right)} \middle| S \right] = e^{\sum_{l=1}^{q} \left(\psi_{t_l}^*(v_l) - \psi_{t_{l-1}}^*(v_l) \right)}.$$

Noting that A_t^* are functionals of S, this proves the convergence of finite dimensional distributions. ■

4.4 TIGHTNESS OF $(S_n(t), R_n(t))$

The tightness of $(S_n(t), R_n(t))$ in $D_{\mathbb{R}^2}[0,1]$ is essentially contained in Jacod and Shiryayev (1987, Ch VI, Section 4b, page 322), based on a certain criterion due to Aldous (1978). Because we shall need this criterion in Section 5 below, we shall present it. Actually, in view of the specific nature of our centering (see the earlier Remark 6 in Section 2.4), this criterion will be satisfied for $(S_n(t), R_n(t))$ whenever it is satisfied for each of $S_n(t)$ and $R_n(t)$ separately.

To state the criteria, it is convenient to extend $(S_n(t), R_n(t))$ to $t \in [0,\infty)$, by taking X_{nj} and Y_{nj} as before for $1 \le j \le k_n$ and letting $X_{nj} = 0 = Y_{nj}$ if $j > k_n$. (Recall that $k_n = k_n(1)$ is a stopping time). Similarly, take $\mathcal{A}_{n,j} = \mathcal{A}_{n,k_n}$ if $j > k_n$. Then note that $(S_n(t), R_n(t)) = (S_n(1), R_n(1))$ if $1 < t < \infty$ (as well as $a_{Y_{nj}}(\tau) = 0 = a_{X_{nj}}(\tau)$ if $j > k_n$).

As in Aldous (1978), the preceding extension is just to avoid the inconvenience of dealing with the stopping times T that will take their values in $[0,1]$ but $T+\delta$ may lie outside $[0,1]$. Then, the criteria of Aldous (1978) (as modified in Jacod and Shiryayev (1987, Ch VI, Section 4a) states that $(S_n(t), R_n(t))$ will be tight in $D_{\mathbb{R}^2}[0,1]$ if

$$\lim_{\alpha\to\infty} \operatorname{limsup}_{n\to\infty} P\left[\sup_{0\le t\le 1} \{|S_n(t)| + |R_n(t)|\} > \alpha \right] = 0 \tag{59}$$

and for every $\eta > 0$,

$$\lim_{\delta\to 0} \operatorname{limsup}_{n\to\infty} \sup_{T^*\le T\le T^*+\delta} P\left[|S_n(T) - S_n(T^*)| + |R_n(T) - R_n(T^*)| > \eta \right] \to 0 \tag{60}$$

where the supremum is with respect to all stopping times T and T^* satisfying $T^* \le T \le T^* + \delta$ and adapted to $\left\{\mathcal{A}_{nk_n(t)}; t \in [0,1]\right\}$.

Now when (59) and (60) involves either $S_n(t)$ only or $R_n(t)$ only, the verification of them based on the present assumptions (C1) - (C4) or (D1) - (D4) follow from Jacod and Shiryayev (1987, Ch VI, Theorem 4.18, page 323), where note that the "C-tightness" in part (iii) of Theorem 4.18 follows because of the assumptions (C4) and (D4). (Note however that the stopping times are with respect to the common filtration $\left\{\mathcal{A}_{nk_n(t)}; t \in [0,\infty)\right\}$) Hence (59) and (60) follow. ■

Remark 11. We now indicate the modifications required for the validity of Theorem 1 for the situation of Remark 5. We consider the case $(S_n(t), R_n(t)) = \left(\sum_{k=1}^{[nt]} X_{nk}, \sum_{k=1}^{k_n(t)} Y_{nk}\right)$. Regarding the modifications for Section 4.2, define now (with $X_{nk}^{(\tau)}$ and $Y_{nk}^{(\tau)}$ defined in Section 4.2), letting $k_n^*(t_q) = [nt_q] \vee k_n(t_q)$,

$$U_{nk} = u_l X_{nk}^{(\tau)},\ [nt_{l-1}] < k \leq [nt_l],\ l = 1, ..., q,\ U'_{nk} = 0,\ [nt_q] < k \leq k_n^*(t_q),$$

$$U'_{nk} = U_{nk},\ k_n(t_{l-1}) < k \leq k_n(t_l),\ l = 1, ..., q,\ k_n(t_q) < k \leq k_n^*(t_q),$$

and

$$V_{nk} = v_l Y_{nk}^{(\tau)},\ k_n(t_{l-1}) < k \leq k_n(t_l),\ l = 1, ..., q,\ V_{nk} = 0,\ k_n(t_q) < k \leq k_n^*(t_q).$$

This gives the arrays $\{U'_{nk}; k = 1, ..., k_n^*(t_q)\}$ and $\{V_{nk}; k = 1, ..., k_n^*(t_q)\}$, *both adapted to the same array*

$$\left\{\mathcal{A}_{n,k_n(t_{l-1})}, ..., \mathcal{A}_{n,k_n(t_l)},\ l = 1, ..., q,\ \mathcal{A}_{n,k_n(t_q)}, ..., \mathcal{A}_{n,k_n^*(t_q)}\right\}.$$

(We note that this statement breaks down in general if $S_n(t) = \sum_{k=1}^{k'_n(t)} X_{nk}$ where the stopping time $k'_n(t)$ is different from $k_n(t)$ or from $[nt]$.)

Now Proposition 3 holds for these arrays when the sums involved in its statements are taken to be $\sum_{k=1}^{k_n(t_q)} + \sum_{k=k_n(t_q)+1}^{k_n^*(t_q)}$. This is because all the bounds involved in the proof of Proposition 3 will be in terms of either the sum $\sum_{k=1}^{n}$ of the quantities associated with the array $\{X_{nk}; k = 1, ..., n\}$ or the sum $\sum_{k=1}^{k_n}$ of those associated with the array $\{Y_{nk}; k = 1, ..., k_n\}$. In addition note that the sum (26) in terms of U'_{nk} can be rewritten in terms of U_{nk}, in which case the sum takes the form $\sum_{l=1}^{q} \sum_{k=[nt_{i-1}]+1}^{[nt_i]}$.

The remaining modifications in Sections 4.2 and 4.3 are clear, obtaining the $\overset{fdd}{\Longrightarrow}$. The tightness also follows because as noted above (59) and (60) involve $S_n(t)$ and $R_n(t)$ only separately. ■

5 PROOF OF THEOREM 2

5.1 PROOF FOR THE SPECIAL CASE $g(u, v) = h(v)$

We first consider the particular case $g(u, v) = h(v)$, because much of the arguments of this case essentially become applicable to the more delicate general situation of Theorem 2. The arguments below are based on LeCam (1986, Proposition 5, page 443). We shall verify the conditions of Theorem 1 (which dealt with the array $\{X_{nk}, Y_{nk}, k = 1, ...\}$) for the array $\{X_{nk}, h(Y_{nk}), k = 1, ...\}$.

First consider (D1). Note that if $g_j(y)$ vanishes in some neighborhood of 0, then $y \longmapsto g_j(h(y))$ also vanishes in some neighborhood of 0 in view of the continuity of $h(y)$. Also if $L_{n,t}^{**}(dy)$ is as defined in (6) with Y_{nk} replaced by $h(Y_{nk})$, then $\int g_j(y) L_{n,t}^{**}(dy) =$

$\int g_j\left(h\left(y\right)\right) L^*_{n,t}\left(dy\right)$. Hence it follows from (20) that (D1) holds for the array $\{h\left(Y_{nk}\right), k=1,...\}$, with the limit $L^{**}_t\left(y,S\right)$ defined by

$$\int g\left(y\right) L^*_t\left(dy,S\right) = \int g\left(h\left(y\right)\right) L^*_t\left(dy,S\right)$$

for every $g\left(y\right)$ vanishing outside a neighborhood of 0.

Next, note that (E1) holds because $h\left(v\right)$ is continuous. Regarding the conditions (D2) and (E2), they follow from the following Lemma 9, where we define

$$\delta^2_{h(Y_{nk})}\left(\epsilon\right) = E_{k-1}\left[h^2\left(Y_{nk}\right)\mathbb{I}_{\{|Y_{nk}|<\epsilon\}}\right] - \left(E_{k-1}\left[h\left(Y_{nk}\right)\mathbb{I}_{\{|Y_{nk}|<\epsilon\}}\right]\right)^2.$$

Recall that the conditions on $g\left(u,v\right)$ reduce to (see Remark 9): (i) $v \longmapsto h\left(v\right)$ is continuous, (ii) $h\left(0\right)=0$ and (iii) there are constants h' and h'' such that $\frac{1}{v^2}\left|h\left(v\right) - vh' - \frac{v^2}{2}h''\right| \to 0$ as $v \to 0$.

Lemma 9 *As $n \to \infty$ first and then $\epsilon \to 0$, we have*

$$\sum\left(\sigma^2_{h(Y_{nk})}\left(\epsilon\right) - \delta^2_{h(Y_{nk})}\left(\epsilon\right)\right) \xrightarrow{p} 0, \tag{61}$$

$$\sum \sigma_{X_{nk},h(Y_{nk})}\left(\epsilon\right) - h'\sum \sigma_{X_{nk},Y_{nk}}\left(\epsilon\right) \xrightarrow{p} 0 \tag{62}$$

and

$$\sum \delta^2_{h(Y_{nk})}\left(\epsilon\right) - h'^2\sum E_{k-1}\left[Y^2_{nk}\mathbb{I}_{\{|Y_{nk}|<\epsilon\}}\right] \xrightarrow{p} 0. \tag{63}$$

Proof. We first verify (63). Note that by (1) and (D1), for each $\epsilon > 0$,

$$\sum\left|E_{k-1}\left[Y_{nk}\mathbb{I}_{\{\epsilon\le|Y_{nk}|<\tau\}}\right]\right|^2 \le \tau^2 \sup_j P_{j-1}\left[|Y_{nj}|\ge\epsilon\right]\sum P_{k-1}\left[|Y_{nk}|\ge\epsilon\right] \xrightarrow{p} 0$$

for any $0<\epsilon<\tau$. Hence, in view of the assumption (9),

$$\sum\left|E_{k-1}\left[Y_{nk}\mathbb{I}_{\{|Y_{nk}|<\epsilon\}}\right]\right|^2 \xrightarrow{p} 0. \tag{64}$$

In particular

$$\sum \sigma^2_{Y_{nk}}\left(\epsilon\right) - \sum E_{k-1}\left[Y^2_{nk}\mathbb{I}_{\{|Y_{nk}|<\epsilon\}}\right] \xrightarrow{p} 0 \tag{65}$$

Now, by assumption, $\left(|h\left(Y_{nk}\right)| + |Y_{nk}|\right)\mathbb{I}_{\{|Y_{nk}|<\epsilon\}} \le K\epsilon$ and

$$\left|h\left(Y_{nk}\right) - h'Y_{nk}\right|\mathbb{I}_{\{|Y_{nk}|<\epsilon\}} \le KY^2_{nk}\mathbb{I}_{\{|Y_{nk}|<\epsilon\}} \tag{66}$$

for some constant K, so that (using $|a^2-b^2| \le |a-b|\left(|a|+|b|\right)$),

$$\left|\left|E_{k-1}\left[h\left(Y_{nk}\right)\mathbb{I}_{\{|Y_{nk}|<\epsilon\}}\right]\right|^2 - \left|h'E_{k-1}\left[Y_{nk}\mathbb{I}_{\{|Y_{nk}|<\epsilon\}}\right]\right|^2\right| \le K^2\epsilon E_{k-1}\left[Y^2_{nk}\mathbb{I}_{\{|Y_{nk}|<\epsilon\}}\right].$$

Hence, by (64) and (65),

$$\sum \left| E_{k-1} \left[h\left(Y_{nk}\right) \mathbb{I}_{\{|Y_{nk}|<\epsilon\}} \right] \right|^2 \xrightarrow{p} 0 \tag{67}$$

as $n \to \infty$ first and then $\epsilon \to 0$. Similarly, by (66),

$$\sum \left| E_{k-1} \left[\left| h\left(Y_{nk}\right) - h' Y_{nk} \right|^2 \mathbb{I}_{\{|Y_{nk}|<\epsilon\}} \right] \right| \leq K\epsilon^2 \sum E_{k-1} \left[Y_{nk}^2 \mathbb{I}_{\{|Y_{nk}|<\epsilon\}} \right], \tag{68}$$

and hence using (65) and the inequality (37)

$$\sum E_{k-1} \left[h^2\left(Y_{nk}\right) \mathbb{I}_{\{|Y_{nk}|<\epsilon\}} \right] - h'^2 \sum E_{k-1} \left[Y_{nk}^2 \mathbb{I}_{\{|Y_{nk}|<\epsilon\}} \right] \xrightarrow{p} 0$$

as $n \to \infty$ first and then $\epsilon \to 0$. Hence (63) follows from (67).

We next show (61). We have, for some positive constants K_1 and K_2,

$$\begin{aligned} & \sum E_{k-1} \left[h^2\left(Y_{nk}\right) \left| \mathbb{I}_{\{|Y_{nk}|<\epsilon\}} - \mathbb{I}_{\{|h(Y_{nk})|<\epsilon\}} \right| \right] \\ \leq \;& E_{k-1} \left[h^2\left(Y_{nk}\right) \left(\mathbb{I}_{\{|Y_{nk}|<\epsilon, |h(Y_{nk})| \geq \epsilon\}} + \mathbb{I}_{\{|h(Y_{nk})|<\epsilon, |Y_{nk}| \geq \epsilon\}} \right) \right] \\ \leq \;& K_1 \epsilon^2 P_{k-1} \left[|Y_{nk}| \geq \epsilon K_2 \right] \xrightarrow{p} 0 \end{aligned} \tag{69}$$

as $n \to \infty$ first and then $\epsilon \to 0$. Here $\xrightarrow{p} 0$ is obtained by Lemma 4, and the last inequality is obtained because the assumption on $h(v)$ entails that there is a η such that $|h(v) - h'v| \leq Kv^2$ for all $0 < |v| \leq \eta$ for some $K > 0$, so that for all sufficiently small $\epsilon > 0$, $\mathbb{I}_{\{|Y_{nk}|<\epsilon, |h(Y_{nk})| \geq \epsilon\}} \leq \mathbb{I}_{\{|Y_{nk}|<\epsilon, |Y_{nk}| \geq c\epsilon\}}$ for some $c > 0$. The same arguments also give

$$\sum E_{k-1} \left[h^2\left(Y_{nk}\right) \left| \mathbb{I}_{\{|Y_{nk}|<\epsilon\}} - \mathbb{I}_{\{|h(Y_{nk})|<\epsilon\}} \right|^2 \right] \xrightarrow{p} 0 \tag{70}$$

as $n \to \infty$ first and then $\epsilon \to 0$, from which, using (67) and the inequality (37), we get

$$\left| \sum \left| E_{k-1} \left[h\left(Y_{nk}\right) \mathbb{I}_{\{|h(Y_{nk})|<\epsilon\}} \right] \right|^2 - \sum \left| E_{k-1} \left[h\left(Y_{nk}\right) \mathbb{I}_{\{|Y_{nk}|<\epsilon\}} \right] \right|^2 \right| \xrightarrow{p} 0. \tag{71}$$

This together with (67) and (68) gives (61).

To prove (62), recall that

$$\sigma_{X_{nk}, Y_{nk}}(\epsilon) = \sum E_{k-1} \left[\left(X_{nk} \mathbb{I}_{\{|X_{nk}|<\epsilon\}} - a_{X_{nk}}(\epsilon) \right) \left(Y_{nk} \mathbb{I}_{\{|Y_{nk}|<\epsilon\}} - a_{Y_{nk}}(\epsilon) \right) \right].$$

Hence

$$\begin{aligned} & \left| \sum \sigma_{X_{nk}, Y_{nk}}(\epsilon) - \sum E_{k-1} \left[\left(X_{nk} \mathbb{I}_{\{|X_{nk}|<\epsilon\}} - a_{X_{nk}}(\epsilon) \right) Y_{nk} \mathbb{I}_{\{|Y_{nk}|<\epsilon\}} \right] \right| \\ \leq \;& \sum \left| E_{k-1} \left[\left(X_{nk} \mathbb{I}_{\{|X_{nk}|<\epsilon\}} - a_{X_{nk}}(\epsilon) \right) a_{Y_{nk}}(\epsilon) \right] \right| \\ \leq \;& \sum \sqrt{\sigma_{X_{nk}}^2(\epsilon) \left(a_{Y_{nk}}(\epsilon) \right)^2} \leq \sum \sigma_{X_{nk}}^2(\epsilon) \sum \left(a_{Y_{nk}}(\epsilon) \right)^2 \xrightarrow{p} 0 \end{aligned}$$

as $n\to\infty$ first and then $\epsilon\to 0$, in view of (9) and (C3). In the same way, because of (67) and (71),

$$\sum \sigma_{X_{nk},h(Y_{nk})}(\epsilon)-\sum E_{k-1}\left[\left(X_{nk}\mathbb{I}_{\{|X_{nk}|<\epsilon\}}-a_{X_{nk}}(\epsilon)\right)h(Y_{nk})\,\mathbb{I}_{\{|h(Y_{nk})|<\epsilon\}}\right]\xrightarrow{p}0.$$

Further,

$$\begin{aligned}&\sum\left|E_{k-1}\left[\left(X_{nk}\mathbb{I}_{\{|X_{nk}|<\epsilon\}}-a_{X_{nk}}(\epsilon)\right)\left(h(Y_{nk})\,\mathbb{I}_{\{|h(Y_{nk})|<\epsilon\}}-h'Y_{nk}\mathbb{I}_{\{|Y_{nk}|<\epsilon\}}\right)\right]\right|\\ \leq\ &\sum\sigma^2_{X_{nk}}(\epsilon)\sum E_{k-1}\left[\left(h(Y_{nk})\,\mathbb{I}_{\{|h(Y_{nk})|<\epsilon\}}-h'Y_{nk}\mathbb{I}_{\{|Y_{nk}|<\epsilon\}}\right)^2\right]\xrightarrow{p}0\end{aligned}$$

because of (68) and (69). Thus (62) follows from the three displayed convergencies, completing the proof of the Lemma. ■

Thus, Proposition 3 together with Lemma 5 and Corollary 6 hold for the array $\{X_{nk},h(Y_{nk}),k=1,...\}$. In particular, the difference between

$$\sum E_{k-1}\left[e^{iu\left(X_{nk}-a_{X_{nk}}(\tau)\right)+iv\left(h(Y_{nk})-a_{h(Y_{nk})}(\tau)\right)}-1\right]$$

and

$$\sum\int\left(e^{iux}-1-iux\mathbb{I}_{\{|x|<\tau\}}\right)d\overline{F}^X_{k-1}(x)+\sum\int\left(e^{ivh(y)}-1-ivh(y)\,\mathbb{I}_{\{|h(y)|<\tau\}}\right)d\overline{F}^Y_{k-1}(y)$$

converges to 0 in probability. In view of (61), the second term of the last sum is approximated by

$$-\frac{v^2}{2}\sum\delta^2_{h(Y_{nk})}(\epsilon)+\int\left(e^{ivh(y)}-1-ivh(y)\,\mathbb{I}_{\{|h(y)|<\tau\}}\right)\mathbb{I}_{\{|y|\geq\epsilon\}}dL^*_{n,t}(dx)$$

as $n\to\infty$ first and then $\epsilon\to 0$. Further

$$iv\sum a_{h(Y_{nk})}(\tau)+\int\left(e^{ivh(y)}-1-ivh(y)\,\mathbb{I}_{\{|h(y)|<\tau\}}\right)\mathbb{I}_{\{|y|\geq\epsilon\}}dL^*_{n,t}(y)$$

can be rewritten in the form

$$\begin{aligned}&iv\sum\left\{E_{k-1}\left[h(Y_{nk})\,\mathbb{I}_{\{|h(Y_{nk})|<\tau,|Y_{nk}|<\epsilon\}}\right]+h'E_{k-1}\left[Y_{nk}\mathbb{I}_{\{\epsilon\leq Y_{nk}<\tau\}}\right]\right\}\\&+\int\left(e^{ivh(y)}-1-ivh'y\mathbb{I}_{\{|y|<\tau\}}\right)\mathbb{I}_{\{|y|\geq\epsilon\}}L^*_{n,t}(dy).\end{aligned}\tag{72}$$

In view of $\mathbb{I}_{\{|h(Y_{nk})|<\tau,|Y_{nk}|<\epsilon\}}=\mathbb{I}_{\{|Y_{nk}|<\epsilon\}}$ and

$$\sum E_{k-1}\left[\left(h(Y_{nk})-h'Y_{nk}-\frac{h''}{2}Y^2_{nk}\right)\mathbb{I}_{\{|Y_{nk}|<\epsilon\}}\right]\xrightarrow{p}0$$

as $n\to\infty$ first and then $\epsilon\to 0$, the sum in (72) is approximated by

$$h'\sum E_{k-1}\left[Y_{nk}\mathbb{I}_{\{||Y_{nk}||<\tau\}}\right]+\frac{h''}{2}\sum E_{k-1}\left[Y^2_{nk}\mathbb{I}_{\{|Y_{nk}|<\epsilon\}}\right]$$

as $n \to \infty$ first and then $\epsilon \to 0$. In view of (65), this gives the required forms for the limit of $\sum_{k=1}^{k_n(t)} h\left(Y_{nk}\right)$.

Also note that it is implicit in the above arguments that $\sum_{k=1}^{k_n(t)} a_{h(Y_{nk})}(\tau)$ is approximated by a quantity that converges in $D_{\mathbb{R}}[0,1]$ with limit in $C_{\mathbb{R}}[0,1]$. This completes the arguments for the particular case.

5.2 PROOF OF THE GENERAL CASE

As in the particular case above, we shall verify the conditions of Theorem 1 for the array $\{X_{nk}, W_{nk}, k = 1, ...\}$, where

$$W_{nk} = g\left(\sum_{j=1}^{k-1} X_{nj}, Y_{nk}\right). \tag{73}$$

As in Section 4.4, we assume for convenience that the process $\left(S_n(t), R_n(t)\right)$ is extended to $[0, \infty)$.

We first obtain a certain discrete approximation, in the form of a step function, to $S_n(t)$ in the space $D_{\mathbb{R}}[0, \infty)$. It is taken directly from Kurtz and Protter (1991, page 1067). In this approximation the locations of the jumps of the approximating step function are stopping times.

To describe the approximation, let $\theta_l, l \geq 1$, be a sequence of i.i.d. uniform random variables on $\left[\frac{1}{2}, 1\right]$, independent of $\sigma\left(\cup_n \cup_k \mathcal{A}_{nk}\right)$. Fix $\varepsilon > 0$. For each $z \in D_{\mathbb{R}^2}[0, \infty)$, define recursively $T_l^\varepsilon = T_l^\varepsilon(z), l \geq 0$, such that

$$T_0^\varepsilon = 0, \quad T_{l+1}^\varepsilon = \inf\left\{t > T_l^\varepsilon : \left|z_t - z_{T_l^\varepsilon}\right| \vee \left|z_{t-} - z_{T_l^\varepsilon}\right| \geq \varepsilon\theta_l\right\}.$$

Letting $y_l^\varepsilon(z) = z_{T_l^\varepsilon}$, define $I^\varepsilon(z) \in D_{\mathbb{R}^2}[0, \infty)$ such that

$$I^\varepsilon(z)_t = y_l^\varepsilon(z) \quad \text{if } T_l^\varepsilon(z) \leq t < T_{l+1}^\varepsilon(z).$$

Then note that

$$\left|z_t - I^\varepsilon(z)_t\right| \leq \varepsilon \quad \text{for all } t \in [0, \infty). \tag{74}$$

In addition, it is shown in Kurtz and Protter (1991, Lemma 6.1, page 1067) that if $z_n \to z$ in $D_{\mathbb{R}^2}[0, \infty)$, then, for every integer $q > 0$,

$$\begin{aligned} & \left(z_n, I^\varepsilon(z_n), y_l^\varepsilon(z_n), T_l^\varepsilon(z_n), l = 1, ..., q\right) \\ \to \ & \left(z, I^\varepsilon(z), y_l^\varepsilon(z), T_l^\varepsilon(z), l = 1, ..., q\right) \text{ in } D_{\mathbb{R}^4}[0, \infty) \times \mathbb{R}^{3q} \text{ a.s.} \end{aligned} \tag{75}$$

(Note that I^ε depends on the random variables θ_l.) This leads to the next lemma, where we define

$$S_n^\varepsilon(t) = I^\varepsilon(S_n)(t), \qquad S^\varepsilon(t) = I^\varepsilon(S^\varepsilon)(t),$$

$$T^{\varepsilon}_{n,l} = T^{\varepsilon}_{l}(z) \text{ when } z = (S_n, R_n), \qquad T^{\varepsilon}_{l} = T^{\varepsilon}_{l}(z) \text{ when } z = (S, R).$$

Note that $S^{\varepsilon}_n(t), S_n(t)$ and $R_n(t)$ are all adapted to $\left\{\mathcal{A}^*_{nk_n(t)}; t \in [0,\infty)\right\}$, where $\mathcal{A}^*_{nk_n(t)} = \mathcal{A}_{nk_n(t)} \vee \sigma(\theta_l, l \geq 1)$. (The σ-field $\sigma(\theta_l, l \geq 1)$ is independent of $\sigma(\cup_n \cup_k \mathcal{A}_{nk})$.)

Lemma 10 *For each $q > 0$,*

$$\begin{aligned} &\left(S^{\varepsilon}_n(t), S_n(t), R_n(t), S^{\varepsilon}_n\left(T^{\varepsilon}_{n,l}\right), T^{\varepsilon}_{n,l}, l = 1, ..., q\right) \\ \Longrightarrow\ &\left(S^{\varepsilon}(t), S(t), R(t), S^{\varepsilon}(T^{\varepsilon}_l), T^{\varepsilon}_l, l = 1, ..., q\right) \text{ in } D_{\mathbb{R}^3}[0,\infty) \times \mathbb{R}^{2q}. \end{aligned}$$

In addition, for every $\rho > 0$ and $\eta > 0$, there are integers $q > 0$ and $n_0 > 0$ such that

$$P\left[1 - \rho \leq T^{\varepsilon}_{n,q} \leq 1\right] \geq 1 - \eta \quad \text{for all } n \geq n_0.$$

Proof. The first statement is a consequence of (75) and Theorem 1 (already established). According to the arguments in Aldous (1978, pages 339 and 340), the second statement is a consequence of (60) and the fact that $X_{nj} = 0 = Y_{nj}$ for $j > k_n = k_n(1)$. ■

With the help of the preceding approximations we next verify the condition (D1) for the array $\{W_{nk},\ k = 1, ...\}$ defined in (73). For this purpose let

$$S_{n,k-1} = \sum_{j=1}^{k-1} X_{nj},$$

so that $W_{nk} = g(S_{n,k-1}, Y_{nk})$. Also, below we let

$$F^{Y}_{k-1}(y) = P_{k-1}[Y_{nk} \leq y], \qquad F^{W}_{k-1}(y) = P_{k-1}[W_{nk} \leq y].$$

Lemma 11. *For every continuous function $h(y)$ that vanishes in a neighborhood of 0,*

$$\sum_{k=1}^{k_n(t)} \int h(y)\, dF^{W}_{k-1}(y) - \sum_{l=1}^{\infty} \sum_{k=k_n\left(t \wedge T^{\varepsilon}_{n,l-1}\right)+1}^{k_n\left(t \wedge T^{\varepsilon}_{n,l}\right)} \int h\left(g\left(S^{\varepsilon}_n\left(t \wedge T^{\varepsilon}_{n,l-1}\right), y\right)\right) dF^{Y}_{k-1}(y) \tag{76}$$

converges to 0 in probability, uniformly over $t \in [0,1]$, as $n \to \infty$ first and then $\varepsilon \to 0$.

Proof. Suppose that h vanishes outside the neighborhood $[-\eta, \eta]$, $\eta > 0$. Also note that continuity of $(u, y) \longmapsto g(u, y)$ entails that $y \longmapsto \sup_{|u| \leq \alpha} |g(u, y)|$ is continuous (at $y = 0$ in particular) for every $\alpha > 0$, and hence $\sup_{|u| \leq \alpha} |g(u, y)| \geq \eta$ entails $|y| \geq \gamma$ for some $\gamma > 0$.

Hence if

$$\sup_{0 \leq t \leq 1} |S_n(t)| \leq \alpha \text{ and } \sup_{0 \leq t \leq 1} |S^{\varepsilon}_n(t)| \leq \alpha \tag{77}$$

are true, then the left hand side of (76) takes the form

$$\begin{aligned}\sum\int h(y)\,dF^W_{k-1}(y) &= \sum\int_{|y|>\gamma} h\left(g\left(S_{n,k-1},y\right)\right)dF^Y_{k-1}(y)\\ &= \sum_{l=1}^{\infty}\sum_{k=k_n\left(t\wedge T^{\varepsilon}_{n,l-1}\right)+1}^{k_n\left(t\wedge T^{\varepsilon}_{n,l}\right)}\int_{|y|>\gamma} h\left(g\left(S_{n,k-1},y\right)\right)dF^Y_{k-1}(y).\end{aligned}$$

Similarly, the integral on the right hand side of (76) can be restricted to $|y|>\gamma$. Hence, when (77) is true and in view of (74), (76) is bounded in absolute value by

$$\begin{aligned}&\sum_{k=1}^{k_n}\int_{|y|\geq\gamma}\left(\sup_{|u-v|\leq\varepsilon,0\leq|u|,|v|\leq\alpha}\left|h\left(g\left(u,y\right)\right)-h\left(g\left(v,y\right)\right)\right|\right)dF^Y_{k-1}(y)\\ &= \int_{|y|\geq\gamma}\left(\sup_{|u-v|\leq\varepsilon,0\leq|u|,|v|\leq\alpha}\left|h\left(g\left(u,y\right)\right)-h\left(g\left(v,y\right)\right)\right|\right)dL^*_{n,1}(y)\\ &\leq K_1(\gamma,\lambda)\int_{|y|\geq\gamma}dL^*_{n,1}(y)+K_2\int_{|y|>\lambda}dL^*_{n,1}(y)\qquad\text{for every }\lambda>0,\end{aligned}$$

where $K_1(\gamma,\lambda)=\sup_{|u-v|\leq\varepsilon,0\leq|u|,|v|\leq\alpha,\gamma\leq|y|\leq\lambda}\left|h\left(g\left(u,y\right)\right)-h\left(g\left(v,y\right)\right)\right|$ and $K_2=2\sup_y|h(y)|$.

We have $K_1(\gamma,\lambda)\to 0$ as $\varepsilon\to 0$ for every $\lambda>0$. Also $\int_{|y|\geq\gamma}dL^*_{n,1}(y)$ is stochastically bounded for each $\gamma>0$. Further,

$$\lim_{\lambda\to\infty}\ \text{limsup}_{n\to\infty}P\left[\int_{|y|>\lambda}dL^*_{n,1}(y)>\delta\right]=0\text{ for all }\delta>0.$$

In addition, $\lim_{\alpha\to\infty}\ \text{limsup}_{n\to\infty}P\left[\sup_{0\leq t\leq 1}|S_n(t)|>\alpha\right]=0$ and the same holds true for $S^{\varepsilon}_n(t)$. Hence the proof of the lemma follows. ■

To complete the verification of (D1), note that we can write

$$\begin{aligned}&\sum_{k=k_n\left(t\wedge T^{\varepsilon}_{n,l-1}\right)+1}^{k_n\left(t\wedge T^{\varepsilon}_{n,l}\right)}\int h\left(g\left(S^{\varepsilon}_n\left(t\wedge T^{\varepsilon}_{n,l-1}\right),y\right)\right)dF^Y_{k-1}(y)\\ &= \int h\left(g\left(S^{\varepsilon}_n\left(t\wedge T^{\varepsilon}_{n,l-1}\right),y\right)\right)d\left(L^*_{n,t\wedge T^{\varepsilon}_{n,l}}-L^*_{n,t\wedge T^{\varepsilon}_{n,l-1}}\right)(y).\end{aligned}$$

Also, in view of the second statement of Lemma 10, together with

$$\int_{1-\rho}^{1}\int_{|y|\geq\gamma}dL^*_{n,t}(y)\leq L^*_{n,1}(\gamma)-L^*_{n,1-\rho}(\gamma)+L^*_{n,1}(-\gamma)-L^*_{n,1-\rho}(-\gamma)\xrightarrow{p}0$$

as $n\to\infty$ first and then $\rho\to 0$, and the fact $\left(S_n(t),R_n(t)\right)=\left(S_n(1),R_n(1)\right)$ if $1<t<\infty$, we have, for each $\varepsilon>0$,

$$\sup_{0\leq t\leq 1}\left|\sum_{l=q+1}^{\infty}\int_{|y|\geq\gamma}h\left(g\left(S^{\varepsilon}_n\left(t\wedge T^{\varepsilon}_{n,l-1}\right),y\right)\right)d\left(L^*_{n,t\wedge T^{\varepsilon}_{n,l}}-L^*_{n,t\wedge T^{\varepsilon}_{n,l-1}}\right)(y)\right|\xrightarrow{p}0 \tag{78}$$

as $n \to \infty$ first and then $q \to \infty$.

Now suppose that $h_1, ..., h_l$ are functions that vanish in some neighborhood of the origin. Using the arguments similar to those employed in the above proof of Lemma 11, one can assume without loss of generality that the functions $y \longmapsto h_j(g(z,y))$ also vanish in some neighborhood (independent of z) of the origin. Therefore,

$$\left(S_n(t), \sum_{l=1}^{q} \int h_j\left(g\left(S_n^\varepsilon\left(t \wedge T_{n,l-1}^\varepsilon\right), y\right)\right) d\left(L_{n,t\wedge T_{n,l}^\varepsilon}^* - L_{n,t\wedge t\wedge T_{n,l-1}^\varepsilon}^*\right)(y),\ 1 \le j \le l\right)$$
$$\overset{fdd}{\Longrightarrow} \left(S(t), \sum_{l=1}^{q} \int h_j\left(g\left(S^\varepsilon\left(t \wedge T_{l-1}^\varepsilon\right), y\right)\right) d\left(L_{t\wedge T_l^\varepsilon}^* - L_{t\wedge t\wedge T_{l-1}^\varepsilon}^*\right)(y),\ 1 \le j \le l\right)$$

As in (78), we can approximate the preceding limit, as $n \to \infty$ first and then $q \to \infty$, by

$$\left(S(t), \sum_{l=1}^{\infty} \int h_j\left(g\left(S^\varepsilon\left(t \wedge T_{l-1}^\varepsilon\right), y\right)\right) d\left(L_{t\wedge T_l^\varepsilon}^* - L_{t\wedge t\wedge T_{l-1}^\varepsilon}^*\right)(y)\right),$$

which in turn is approximated, as in Lemma 11, by $\int h_j(g(S(t), y))\, dL_t^*(y)$ as $n \to \infty$ first and then $\varepsilon \to 0$.

We have thus established (D1) for the array $\{W_{nk},\ k = 1, ...\}$, as required.

We now complete the proof of Theorem 2 by proceeding analogous to the arguments of the particular case of Section 5.1. Recall the condition $\sup_{|u|\le\alpha} \rho_\eta(u) \to 0$ as $\eta \to 0$ for each $\alpha > 0$ where $\rho_\eta(u) = \sup_{0<|v|\le\eta} \frac{1}{v^2}\left|g(u,v) - vg_u' - \frac{v^2}{2} g_u''\right|$. This, together with the fact that $\sup_{0\le t\le 1} |S_n(t)|$ is stochastically bounded, entails that $\sup_k \rho_\eta(S_{n,k-1}) \xrightarrow{p} 0$ as $n \to \infty$ first and then $\eta \to 0$. In addition, both $\sup_k \left|g_{S_{n,k-1}}'\right|$ and $\sup_k \left|g_{S_{n,k-1}}''\right|$ are stochastically bounded. Therefore all the arguments that lead to Lemma 9 in connection with the particular case of Section 5.1 become applicable here, so that we have the following lemma, where (recall $W_{nk} = g(S_{n,k-1}, Y_{nk})$)

$$\delta_{W_{nk}}^2(\epsilon) = E_{k-1}\left[W_{nk}^2 \mathbb{I}_{\{|Y_{nk}|<\epsilon\}}\right] - \left(E_{k-1}\left[W_{nk}\mathbb{I}_{\{|Y_{nk}|<\epsilon\}}\right]\right)^2,$$

which replaces the $\delta_{h(Y_{nk})}^2(\epsilon)$ of Lemma 9.

Lemma 12. *As $n \to \infty$ first and then $\epsilon \to 0$, we have*

$$\sum \left(\sigma_{W_{nk}}^2(\epsilon) - \delta_{W_{nk}}^2(\epsilon)\right) \xrightarrow{p} 0,$$

$$\sum \sigma_{X_{nk},W_{nk}}(\epsilon) - \sum g_{S_{n,k-1}}' \sigma_{X_{nk},Y_{nk}}(\epsilon) \xrightarrow{p} 0$$

and

$$\sum \delta_{W_{nk}}^2(\epsilon) - \sum g_{S_{n,k-1}}'^2 E_{k-1}\left[Y_{nk}^2 \mathbb{I}_{\{|Y_{nk}|<\epsilon\}}\right] \xrightarrow{p} 0.$$

Now, using (65) and the approximation arguments that lead to (53), we have

$$\sum_{k=1}^{k_n(t)} g'^2_{S_{n,k-1}} E_{k-1}\left[Y_{nk}^2 \mathbb{I}_{\{|Y_{nk}|<\epsilon\}}\right] \stackrel{fdd}{\Longrightarrow} \int_0^t g'^2_{S(z)} dB^*(z).$$

Similarly, the remaining steps to complete the proof of Theorem 2 are essentially the same as those for the particular case of Section 5.1.

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