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February 1, 2006
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Limit Theorems for Functionals of Sums that Converge to Fractional Stable Motions

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January, 2006. (Revised and extended version of the earlier version, December 2003.)

Abstract. Consider $X_k = \sum_{j=0}^{\infty} c_j \xi_{k-j}$, $k \geq 1$, where c_j, $j \geq 0$, are constants and ξ_j, $-\infty < j < \infty$, are iid random variables belonging to the domain attraction of a strictly stable law with index $0 < \alpha \leq 2$. Let $S_k = \sum_{j=1}^{k} X_j$. Under certain conditions on c_j, it is known that for a suitable slowly varying function $\kappa_1(n)$ and for a suitable constant $0 < H < 1$, $\left(n^H \kappa_1(n)\right)^{-1} S_{[nt]}$ converges in distribution to a fractional stable motion (indexed by α and H). In addition, it is known that if $f(y)$ is such that $\int \left(|f(y)| + |f(y)|^2\right) dy < \infty$, then under certain further conditions on the distribution of ξ_1, $n^{-(1-H)} \kappa_1(n) \sum_{k=1}^{n} f(S_k) \Longrightarrow L_1^0 \int_{-\infty}^{\infty} f(y)\, dy$, where L_t^x is the local time of the fractional stable motion at x upto time t.

In this paper we obtain three further results, motivated by asymptotic inference for certain nonlinear time series models. First, we show that if in addition $\int_{-\infty}^{\infty} f(y)\, dy = 0$, then when $1/3 < H < 1$ (which probably cannot be relaxed), $\sqrt{n^{-(1-H)} \kappa_1(n)} \sum_{k=1}^{n} f(S_k) \Longrightarrow W\sqrt{bL_1^0}$, where W is standard normal, independent of L_1^0, and b is a constant having an explicit expression in terms of the distributions of S_k, $k \geq 1$.

Now let, for $\nu \geq 1$, $\omega_k = \sum_{j=k-\nu+1}^{k} d_{k-j} \eta_j$ where (ξ_j, η_j), $-\infty < j < \infty$, are iid with ξ_j as before and $E[\eta_1] = 0$, $E[\eta_1^2] < \infty$ and $E[|\eta_1 \xi_1|] < \infty$. Then if $1/3 < H < 1$ as above but possibly $\int_{-\infty}^{\infty} f(y)\, dy \neq 0$, we show that $\sqrt{n^{-(1-H)} \kappa_1(n)} \sum_{k=1}^{n} f(S_k) \omega_k \Longrightarrow W\sqrt{b^* L_1^0}$. The constant b^* in the limit will be similar to that of b in the first result.

It is further shown that $n^{-(1-H)} \kappa_1(n) \sum_{k=1}^{n} f(S_k, S_{k+1}, ..., S_{k+r}) \Longrightarrow L_1^0 \int_{-\infty}^{\infty} f_*(x)\, dx$ for all $0 < H < 1$ and for all suitable $f(x_0, ..., x_r)$, $r \geq 1$, where $f_*(x) = E[f(x, x+S_1, ..., x+S_r)]$.

These convergencies are also shown to hold jointly with certain other random quantities.

Acknowledgments. I am grateful to Peter C.B.Phillips and the Cowles Foundation for Research in Economics, Yale University, for the hospitality and support for several visits related to the research on the applications of the results of this paper to nonlinear time series models (Jeganathan and Phillips (2006)). This version was completed during my Fall 2005 visit to the Cowles Foundation. I am thankful to my Bangalore colleagues Vishwambhar Pati, Professor Alladi Sitaram and Thangavelu for the help in Fourier analysis.

1 INTRODUCTION

Consider a sequence $\xi_j, -\infty < j < \infty$, of iid random variables belonging to the domain of attraction of a strictly stable law with index $0 < \alpha \leq 2$. We recall that this is equivalent to the statement that for a suitable slowly varying function $\kappa(n)$,

$$t \longmapsto \left(n^{1/\alpha}\kappa(n)\right)^{-1} \sum_{j=1}^{[nt]} \xi_j \stackrel{fdd}{\Longrightarrow} Z_\alpha(t)\,,\ t > 0, \tag{1}$$

where $\{Z_\alpha(t), t > 0\}$ is an α-stable Levy motion, that is, has stationary independent increments such that, for each $0 < t < \infty$,

$$E[e^{iuZ_\alpha(t)}] = \begin{cases} e^{-t|u|^\alpha\left(1-i\beta\,\mathrm{sign}(u)\,\tan\left(\frac{\pi\alpha}{2}\right)\right)} & \text{if } \alpha \neq 1 \\ e^{-t|u|} & \text{if } \alpha = 1 \end{cases}$$

with $|\beta| \leq 1$. (Here and in the rest of the paper, the notation $\stackrel{fdd}{\Longrightarrow}$ signifies the convergence in distribution of random processes in the sense of convergence in distribution of all finite dimensional distributions.) For the details of the above statement, see for instance Ibragimov and Linnik (1965, Chapter 2, Section 6) or Bingham et al (1987, page 344.). Note that this definition of strict α-stability for the case $\alpha = 1$ differs from the usual one in that we take the skewness parameter β to be 0. When $\alpha = 2$, $Z_2(t)$ becomes the Brownian Motion with variance 2.

In addition, we shall also assume without further mentioning that

$$\text{When } \alpha = 2, \quad E\left[\xi_1\right] = 0 \text{ and } E\left[\xi_1^2\right] < \infty.$$

Now consider the linear process

$$X_k = \sum_{j=0}^{\infty} c_j \xi_{k-j},\ \ k \geq 1, \tag{2}$$

where ξ_j, $-\infty < j < \infty$, are as earlier with index $0 < \alpha \leq 2$, and c_j, $j \geq 0$, are constants. Let

$$S_k = \sum_{j=1}^{k} X_j.$$

Under suitable conditions (specified in Section 2 below) on the constants c_j it is known that for a suitable H, $0 < H < 1$, and for a slowly varying $\kappa_1(n)$ the process

$$\left(n^H \kappa_1(n)\right)^{-1} S_{[nt]} \stackrel{fdd}{\Longrightarrow} \Lambda_{\alpha,H}(t),$$

where the limit $\{\Lambda_{\alpha,H}(t), t \geq 0\}$ is a *Linear Fractional Stable Motion* (LFSM). It is defined by

$$\Lambda_{\alpha,H}(t) = a\int_{-\infty}^{0} \left\{(t-u)^{H-1/\alpha} - (-u)^{H-1/\alpha}\right\} Z_\alpha(du) + a\int_{0}^{t} (t-u)^{H-1/\alpha} Z_\alpha(du)$$

if $H \neq 1/\alpha$, and

$$\Lambda_{\alpha,H}(t) = Z_\alpha(t) \quad \text{if } H = 1/\alpha$$

where a is a non-zero constant and $\{Z_\alpha(t), t \in R\}$ is an α-stable Levy motion, taken to be $Z_\alpha(t)$ as defined earlier for $0 < t < \infty$, and for $-\infty < t < 0$, it is taken to be $Z_\alpha(t) = Z^*_\alpha(-t)$ with $\{Z^*_\alpha(u), 0 < u < \infty\}$ an independent copy of $\{Z_\alpha(u), 0 < u < \infty\}$. See Samorodnitsky and Taqqu (1994) for the details of LFSM.

Note that when $H = 1/\alpha$, *the restriction* $0 < H < 1$ *reduces to* $1 < \alpha \leq 2$. When $\alpha = 2$, the LFSM reduces to the *Fractional Brownian Motion.*

Now suppose that $\int \left|E\left[e^{i\lambda\xi_1}\right]\right|^p d\lambda < \infty$ for some $p > 0$, and let the function $f(y)$ be such that $\int \left(|f(y)| + |f(y)|^2\right) dy < \infty$. Then, it follows from Jeganathan (2004a, Statement (ii) of Theorem 3) that

$$n^{-(1-H)}\kappa_1(n) \sum_{k=1}^{n} f(S_k) \Longrightarrow L_1^0 \int_{-\infty}^{\infty} f(y)\, dy,$$

where L_t^x is the *local time* of the LFSM $\Lambda_{\alpha,H}(t)$ at x upto the time t. See Jeganathan (2004a) for the existence and other details of the local time of the LFSM.

In this paper the first main result (Theorem 1 in Section 2) consists of showing that if the restrictions

$$\int |f(y)|^i\, dy < \infty,\ i = 1, 2, 3, 4, \quad \int_{-\infty}^{\infty} |yf(y)|\, dy < \infty, \tag{3}$$

$$\int_{-\infty}^{\infty} f(y)\, dy = 0, \tag{4}$$

and

$$\frac{1}{3} < H < 1$$

hold, then

$$\sqrt{n^{-(1-H)}\kappa_1(n)} \sum_{k=1}^{n} f(S_k) \Longrightarrow W\sqrt{bL_1^0} \tag{5}$$

where W has the standard normal $(0, 1)$ distribution independent of L_1^0, and b is a nonnegative constant having an explicit expression in terms of the distributions of S_k, $k \geq 1$. (*We remark that the restriction* $\frac{1}{3} < H < 1$ *probably cannot be relaxed* because it cannot be relaxed in the continuous time situation, see the Remark below in this section.)

This result is known for the random walk case $S_k = \sum_{j=1}^{k} \xi_j$ (that is, the case $c_j = 0$ for all $j \geq 1$, $c_0 = 1$), see Borodin and Ibragimov (1995, Theorem 3.3 of Chapter IV). For the symmetric Bernoulli random walk case, it was originally discovered by Dobrushin

(1955). But note however that many of the structural simplifications available in the random walk case (for example the fact that $S_{l+k} - S_k$ is independent of S_k and has the same distribution as that of S_l) are not available for the present case, in addition to the fact that in the present case the convergence of $\left(n^H \kappa_1(n)\right)^{-1} S_{[nt]}$ to $\Lambda_{\alpha,H}(t)$ does not hold in the Skorokhod space $D[0,1]$ when $0 < H < 1/\alpha$ and $\alpha \neq 2$ (see for instance Astrauskas (1983)).

Next let, for some integer $\nu \geq 1$,

$$\omega_k = \sum_{j=k-\nu+1}^{k} d_{k-j}\eta_j = \eta_k + d_1\eta_{k-1} + \ldots + d_{\nu-1}\eta_{k-\nu+1}, \tag{6}$$

where $(\xi_j, \eta_j), -\infty < j < \infty$, are iid ($\xi_j$ are as before) with

$$E[\eta_1] = 0, \quad E\left[\eta_1^2\right] < \infty \quad \text{and} \quad E[|\eta_1\xi_1|] < \infty. \tag{7}$$

Then as the second main result (Theorem 2, Section 2) we obtain the convergence

$$\sqrt{n^{-(1-H)}\kappa_1(n)} \sum_{k=1}^{n} f(S_k)\omega_k \Longrightarrow W\sqrt{b^* L_1^0} \tag{8}$$

where $f(y)$ satisfies all the conditions in (3) but now (4) need not hold, that is, possibly

$$\int_{-\infty}^{\infty} f(y)\, dy \neq 0.$$

The constant b^* in the limit will have the form similar to that of b in (5).

As far as we can determine, Theorem 2 has not been known previously, even for the for the random walk situation $S_k = \sum_{j=1}^{k} \xi_j$ with $\omega_k = \eta_k$.

Note that the requirement $E[|\eta_1\xi_1|] < \infty$ in (7) implicitly requires certain moment condition on ξ_1. It is satisfied when $\alpha = 2$ because then $E[\xi_1^2] < \infty$ ($E[\eta_1^2] < \infty$ already by assumption). It is also satisfied, using Cauchy-Schwarz inequality, when

$$E\left[|\eta_1|^{\frac{\gamma}{\gamma-1}}\right] < \infty \text{ for some } 1 < \gamma < \alpha \text{ when } 1 < \alpha < 2.$$

The convergence (8), which is needed in obtaining the asymptotic behavior of least squares or similar estimators in certain nonlinear time series models (Jeganathan and Phillips (2006b)), is one of the primary motivations of the present investigation. We identify the close relationship between the convergence results (5) and (8). Though unfortunately (8) is not directly deducible from (5), we shall see that, once the relationship has been identified, its proof will use similar ideas involved in (5), and in fact some of the steps can be transported or deducible from those of (5).

As the third main result (which in some form will also be required in obtaining (5) and (8)) we show, when $0 < H < 1$, that $n^{-1}\gamma_n \sum_{l=1}^{n} f(S_l, S_{l+1}, ..., S_{l+r}) \Longrightarrow L_1^0 \int_{-\infty}^{\infty} f_*(x)\, dx$ where $f_*(x) = E[f(x, x+S_1, ..., x+S_r)]$. We note that the conditions imposed on $f(x_0, ..., x_r)$ exclude the limits of the functionals such as the number of level crossings of $\sum_{j=1}^{k} X_j$; for the treatment of such functionals see Jeganathan (2004b).

The plan of the paper is as follows. The required assumptions as well as the statements of the main results will be stated in Section 2, where it is also noted that the main result can be related to a form of a martingale CLT. (Such a relationship to a martingale CLT is implicit in Borodin and Ibragimov (1995) though the methods employed there are tied in many ways to the iid structure of the random walk case $S_k = \sum_{j=1}^{k} \xi_j$ treated there.) The proof of the main results will then consists of the verification of the conditions of this martingale CLT, which verification will be done in Sections 3 - 5.

Notations. In addition to the convergence $\stackrel{fdd}{\Longrightarrow}$ introduced earlier, the convergence in distribution of a sequence of random variables or random vectors will be signified as usual by $\Longrightarrow$. Throughout below we let

$$\psi(\lambda) = E\left[e^{i\lambda\xi_1}\right].$$

For any Borel measurable function $f(y)$ with $\int |f(y)|\, dy < \infty$, $\widehat{f}(\lambda)$ stands for its Fourier transform, that is,

$$\widehat{f}(\lambda) = \int e^{i\lambda y} f(y) dy.$$

We let

$$g(j) = \begin{cases} \sum_{i=0}^{j} c_i & \text{if } j \geq 0 \\ 0 & \text{if } j < 0, \end{cases}$$

where the constants c_i are as in (2) with $c_0 = 1$.

E_l stands for the conditional expectation given the σ-field $\sigma(\xi_j;\ j \leq l)$.

The normalizing constant $b_n = n^{1/\alpha}\kappa(n)$ (where $\kappa(n)$ is as in (1)) will be used exclusively in the sense of (45) below. Similarly γ_n will be used in the sense of (13) or (46) below.

Throughout the paper the notation C stands for a generic constant that may take different values at different places of even the same proof of the same expression.

Remark. We note that the continuous time analogues of Theorem 1, in the forms of generalizations of the appropriate results in for instance Yor (1983), do not follow directly from Theorem 1. The reason is that in the method employed in the present paper the central limit phenomenon is involved at two different levels. One at the familiar level of the partial sum S_k itself, but another at the level of the partial sum of $f(S_k)$ themselves. It is the later level that is central to, and distinguishes, the present situation, whose continuous time analogue needs to be worked out separately. Despite this one would tend to believe that suitable versions of continuous time analogues will hold, though we have not worked out all the details. In this situation, it may be noted that the restriction $1/3 < H < 1$ cannot be relaxed, as can be seen from the known regularity properties of L_1^x with respect to the space variable x when L_1^x is the local time of the fractional Brownian motion (see Geman and Horowitz (1980, Table 2)).

As noted earlier, Theorem 2 has not been known previously, even for the situation $S_k = \sum_{j=1}^k \xi_j$ with $\omega_k = \eta_k$. Its possible continuous time versions in some specific forms have also been unknown. ■

2 THE MAIN RESULTS AND THE RELATION TO A MARTINGALE CLT

One of the following mutually exclusive conditions will be imposed on the coefficients c_j of the process X_k, where recall that $c_0 = 1$.

(A1) (**The case** $H \neq 1/\alpha$, $0 < H < 1$). $c_j = j^{H-1-1/\alpha} u(j)$, with $H \neq 1/\alpha$, $0 < H < 1$, where $u(j)$ is slowly varying at infinity, satisfying

$$\sum_{j=0}^{\infty} c_j = 0 \text{ when } H - 1/\alpha < 0. \tag{9}$$

In addition, there is an integer $l_0 > 0$ and constants c_1 and c_2 such that

$$0 < c_1 \leq \frac{u(l+j_1)}{u(l-j_2)} \leq c_2 \text{ for all } 0 \leq j_1, j_2 \leq [l/2] \text{ and } l \geq l_0. \tag{10}$$

(A2) (**The case** $H = 1/\alpha$, $0 < H < 1$). $\sum_{j=0}^{\infty} |c_j| < \infty$ and $\sum_{j=0}^{\infty} c_j \neq 0$. In addition

$$\sup_{j \geq 1} |j c_j| < \infty. \tag{11}$$

We note that the restriction (10) is automatically satisfied if $u(j)$ is monotone in j, because of the assumption of $u(j)$ being slowly varying. For instance if $u(j)$ is nondecreasing, then $1 \leq \frac{u(l+j_1)}{u(l-j_2)} \leq \frac{u(2l)}{u(l/2)}$ when $0 \leq j_1, j_2 \leq [l/2]$, where $\frac{u(2l)}{u(l/2)} \to 4$ as $l \to \infty$. (We do not know if the monotonicity of $u(j)$ can be assumed without loss of generality, in which case the restriction (10) then holds automatically.)

Note that if (9) is violated, then the case $c_j = j^{H-1-1/\alpha} u(j)$ with $H - 1/\alpha < 0$ comes under (A2). Also it is implicit that $u(j) \neq 0$ for all sufficiently large j.

Remark. A motivation of the condition (A1) is what has been called a *Fractional ARIMA* model with stable innovations, a detailed discussion of which can be found for instance in Samorodnitsky and Taqqu (1994, Section 7.13, page 380). In a simplest case of this model, (2) takes the form

$$X_k = (1-B)^{-d} \xi_k = \sum_{j=0}^{\infty} c_j(-d) B^j \xi_k = \sum_{j=0}^{\infty} c_j(-d) \xi_{k-j} \tag{12}$$

where B is the back-shift operator $B\xi_i = \xi_{i-1}$. Here we have used the formal expansion $(1-B)^{-d} = \sum_{j=0}^{\infty} c_j(-d)\, B^j$, so that using Stirling's approximation,

$$c_j(-d) = \frac{\Gamma(j+d)}{\Gamma(d)\,\Gamma(j+1)} \sim \frac{1}{\Gamma(d)} j^{d-1} \text{ as } j \to \infty \text{ if } d \neq 0, -1, \ldots$$

where $\Gamma(.)$ stands for the gamma function, and $c_j(-d) = 0$ for $j \geq d$ if $d = 0, -1, \ldots$.

Hence if we take $H = d + \frac{1}{\alpha}$, the condition (A1) is satisfied, including (9) because $H - \frac{1}{\alpha} < 0$ is the same as $d < 0$ and hence

$$\sum_{j=0}^{\infty} c_j(-d) = (1-x)^{-d}\Big|_{x=1} = 0 \quad (d < 0).$$

In addition, when $0 < H < 1$, the series (12) converges with probability one (see Samorodnitsky and Taqqu (1994, Theorem 7.13.1, page 381)). ■

Now let

$$\gamma_n = \begin{cases} n^H u(n)\kappa(n) & \text{if (A1) is satisfied} \\ \left(\sum_{j=0}^{\infty} c_j\right) n^{1/\alpha}\kappa(n) & \text{if (A2) is satisfied,} \end{cases} \tag{13}$$

where $\kappa(n)$ is as in (1) and $u(n)$ as in (A1). Then it is known (see Kasahara and Maejima (1988, Theorems 5.1, 5.2 and 5.3)) that when (A1) is satisfied, the process $\gamma_n^{-1} S_{[nt]} \overset{fdd}{\Longrightarrow} \Lambda_{\alpha,H}(t)$, $H \neq 1/\alpha$, and similarly when $1 < \alpha \leq 2$ and (A2) is satisfied, $\gamma_n^{-1} S_{[nt]} \overset{fdd}{\Longrightarrow} Z_\alpha(t)$. In view of our convention that $Z_\alpha(t) = \Lambda_{\alpha,1/\alpha}(t)$ when $1 < \alpha \leq 2$, the preceding statements will be combined in the form

$$\gamma_n^{-1} S_{[nt]} \overset{fdd}{\Longrightarrow} \Lambda_{\alpha,H}(t),$$

with the understanding that when (A2) is satisfied the limit is $Z_\alpha(t)$ *with* $1 < \alpha \leq 2$.

(A3) We shall also require the following assumptions on $\psi(\lambda)$ (recall $\psi(\lambda) = E\left[e^{i\lambda\xi_1}\right]$).

$$\int |\psi(\lambda)|^2 \, d\lambda < \infty \tag{14}$$

and

$$\int |\lambda|^3 \, |\psi(\lambda)|^p \, d\lambda < \infty \text{ for some } p > 0. \tag{15}$$

Note that because $|\psi(\lambda)| \leq 1$, (15) entails

$$\int |\psi(\lambda)|^p \, d\lambda < \infty. \tag{16}$$

(This is also implied by (14) for $p \geq 2$.)

Remarks on the restrictions (14) and (15). The restriction (14) entails that the Lebesgue density of the distribution of ξ_1 exists (Kawata (1972, Theorem 11.6.1)). If we denote this density by $\varphi(x)$, then $\psi(\lambda) = \widehat{\varphi}(\lambda)$ and, by Plancherel's theorem, $\int |\psi(\lambda)|^2 d\lambda = 2\pi \int |\varphi(x)|^2 dx$.

Now suppose that the preceding density $\varphi(x)$ has a *distributional derivative* $\varphi'(x)$ such that $\int |\varphi'(x)| dx < \infty$. Then it can be shown that $\widehat{\varphi}(\lambda) = i\widehat{\varphi'}(\lambda)\lambda^{-1}$ where $\widehat{\varphi'}(\lambda)$ is the Fourier transform of $\varphi'(x)$. (This follows from standard facts about Fourier transforms and distributional derivatives of Lebesgue integrable functions, see for instance Rudin (1991).) In this case, in addition to (14), (15) holds for $p = 5$ and hence for all $p \geq 5$. This is the case for instance when $\varphi(x)$ is suitably piecewise differentiable. From the point of view of statistical applications indicated earlier, such conditions are not very restrictive. As a simple example suppose that $\varphi(x) = \frac{1}{2}\mathbb{I}_{\{|x|\leq 1\}}$, the density function of the random variable uniformly distributed over the interval $[-1,1]$. Then the corresponding distributional derivative $\varphi'(x) = -\frac{1}{2}(\delta_1(x) - \delta_{-1}(x))$, where δ_a is the Dirac delta function. ■

Theorem 1. *Assume that $1/3 < H < 1$. Assume further that (14) and (15) hold. Let $f(y)$ be Borel measurable such that (3) and (4) hold.*

Furthermore, let $h(y)$ be Borel measurable such that $\int \left(|h(y)| + |h(y)|^2\right) dy < \infty$. Then

$$\left(\frac{1}{\gamma_n} S_{[nt]}, \frac{\gamma_n}{n}\sum_{k=1}^{n} h(S_k), \sqrt{\frac{\gamma_n}{n}}\sum_{k=1}^{n} f(S_k)\right) \Longrightarrow \left(\Lambda_{\alpha,H}(t), L_1^0 \int h(y)\,dy, W\sqrt{bL_1^0}\right),$$

where L_1^0 is the local time of $\Lambda_{\alpha,H}(t)$ as before, W is standard normal independent of the process $\Lambda_{\alpha,H}(t)$ and

$$0 \leq b = \frac{1}{2\pi}\int \left|\widehat{f}(\mu)\right|^2 \left(1 + 2\sum_{r=1}^{\infty} E\left[e^{-i\mu S_r}\right]\right) d\mu < \infty.$$

■

As noted earlier, the restriction $1/3 < H < 1$ cannot probably be relaxed. To state the next statement (recall $g(j) = \sum_{i=0}^{j} c_i$), define (the integer ν is as in (6))

$$\Phi_r(\mu) = \begin{cases} E\left[\omega_\nu e^{-i\mu\sum_{j=0}^{\nu-1} g(j)\xi_{\nu-j}}\right] E\left[\omega_\nu e^{-i\mu\sum_{j=0}^{\nu-1}(g(j+r)-g(j))\xi_{\nu-j}}\right] & \text{if } r \geq \nu \\ E\left[\omega_\nu\omega_{\nu+r} e^{-i\mu\left(\sum_{j=0}^{\nu+r-1} g(\nu+r-j)\xi_{\nu+r-j} - \sum_{j=0}^{\nu-1} g(j)\xi_{\nu-j}\right)}\right] & \text{if } 1 \leq r < \nu, \end{cases}$$

and, letting $g(j) = 0$ for $j < 0$,

$$\Psi_r(\mu) = \prod_{j=1+\nu, j\neq r,\ldots,j\neq r+\nu}^{\infty} \psi\left(-\left(g(j) - g(j-r)\right)\mu\right).$$

Theorem 2. *Suppose that all the assumptions of Theorem 1 except (4) are satisfied, that is, now possibly*

$$\int f(y)\, dy \neq 0.$$

Let the sequence ω_k be as in (6) with η_k satisfying (7).

Then

$$\left(\frac{1}{\gamma_n} S_{[nt]}, \frac{\gamma_n}{n} \sum_{k=1}^{n} h(S_k), \sqrt{\frac{\gamma_n}{n}} \sum_{k=1}^{n} f(S_k)\, \omega_k\right) \Longrightarrow \left(\Lambda_{\alpha,H}(t), L_1^0 \int h(y)\, dy, W\sqrt{b^* L_1^0}\right),$$

where, with $\Phi_r(\mu)$ and $\Psi_r(\mu)$ as defined above,

$$0 \leq b^* = \frac{1}{2\pi} \int \left|\widehat{f}(\mu)\right|^2 \left(E\left[\omega_\nu^2\right] + 2\sum_{r=1}^{\infty} \Phi_r(\mu)\, \Psi_r(\mu)\right) d\mu < \infty.$$

■

Note that in the case $\nu = 1$ (and hence $\omega_k = \eta_k$), we have for $r \geq 1$,

$$\Phi_r(\mu)\, \Psi_r(\mu) = E\left[\eta_1 e^{-i\mu\xi_1}\right] E\left[\eta_1 e^{-i\mu(g(r)-g(0))\xi_1}\right] \prod_{j=1, j\neq r}^{\infty} \psi\left(-\left(g(j) - g(j-r)\right)\mu\right).$$

Also if we take $\eta_1 \equiv 1$, this reduces to $\Phi_r(\mu)\, \Psi_r(\mu) = E\left[e^{-i\mu S_r}\right]$, as is to be expected.

Theorem 3. *Assume that (14) holds. Let $f(x_0, ..., x_r)$, $r \geq 1$, be such that*

$$\int |f(x_0, ..., x_r)|^i\, dx_0 ... dx_r < \infty, \;\; i = 1, 2, \quad \int \left(\int |f(x_0, ..., x_r)|^2\, dx_r\right)^{\frac{1}{2}} dx_0 ... dx_{r-1} < \infty. \tag{17}$$

Then, for all $0 < H < 1$,

$$\frac{\gamma_n}{n} \sum_{l=1}^{n} f(S_l, S_{l+1}, ..., S_{l+r}) \Longrightarrow L_1^0 \int_{-\infty}^{\infty} f_*(x)\, dx$$

where

$$f_*(x) = E\left[f(x, x + S_1, ..., x + S_r)\right].$$

■

As noted earlier, the restriction (17) excludes the situation such as the number of level crossings of $\sum_{j=1}^{k} X_j$, see Jeganathan (2004b) for the treatment of such functionals.

Note that the restrictions (15) and $H > 1/3$ are not involved in Theorem 3. Also note that the limit in Theorem 3 involves $f(x)$ only in terms of $\int_{-\infty}^{\infty} f_*(x)\, dx$. For instance the limits of $\frac{\gamma_n}{n} \sum_{l=1}^{n} f(S_{l+r-1}, S_{l+r})$ and $\frac{\gamma_n}{n} \sum_{l=1}^{n} f(S_l, S_{l+1})$ must be identical, which is indeed the case, though their respective limits involve $f_*(x) = E[f(x+S_{r-1}, x+S_r)]$ and $f_*(x) = E[f(x, x+S_1)]$. This is because

$$\begin{aligned}\int E[f(x+S_{r-1}, x+S_r)]\,dx &= E\left[\int f(x+S_{r-1}, x+S_r)dx\right] \\ &= E\left[\int f(x, x+S_r-S_{r-1})dx\right] \\ &= E\left[\int f(x, x+S_1)dx\right] = \int E[f(x, x+S_1)]\,dx,\end{aligned}$$

where we have used the fact that $S_r - S_{r-1}$ and $S_1 - S_0 = S_1$ are identically distributed by stationarity.

Also note that in the case $f(x_0, ..., x_r) = f_0(x_0)...f_r(x_r)$, the conditions in (17) hold when $\int |f_l^i(x))|\,dx < \infty$, $l = 0, ..., r$, $i = 1, 2$.

RELATION TO A MARTINGALE CLT. We next relate Theorems 1 and 2 to a martingale CLT. For this purpose, fix an integer $l_0 \geq 2$ and corresponding to Theorem 1 define, for each positive integer m,

$$\zeta_{nmk} = \sqrt{\frac{\gamma_n}{n}} \sum_{l=\left[n\frac{k-1}{m}\right]+l_0}^{\left[n\frac{k}{m}\right]} f(S_l), \quad k \geq 1, \tag{18}$$

$$R_{nmk} = \sqrt{\frac{\gamma_n}{n}} \sum_{l=\left[n\frac{k-1}{m}\right]+1}^{\left[n\frac{k-1}{m}\right]+l_0-1} f(S_l), \quad k \geq 1. \tag{19}$$

Similarly, corresponding to Theorem 2 define (with ω_l as in (6))

$$\zeta^*_{nmk} = \sqrt{\frac{\gamma_n}{n}} \sum_{l=\left[n\frac{k-1}{m}\right]+l_0}^{\left[n\frac{k}{m}\right]} f(S_l)\,\omega_l, \quad k \geq 1, \tag{20}$$

$$R^*_{nmk} = \sqrt{\frac{\gamma_n}{n}} \sum_{l=\left[n\frac{k-1}{m}\right]+1}^{\left[n\frac{k-1}{m}\right]+l_0-1} f(S_l)\,\omega_l, \quad k \geq 1. \tag{21}$$

In these definitions we follow the usual convention that a sum is to be interpreted as 0 if it is with respect to an empty index set. Note that

$$\sqrt{\frac{\gamma_n}{n}} \sum_{l=1}^{n} f(S_l) = \sum_{k=1}^{m} (\zeta_{nmk} + R_{nmk}), \tag{22}$$

$$\sqrt{\frac{\gamma_n}{n}} \sum_{l=1}^{n} f(S_l)\,\omega_l = \sum_{k=1}^{m} \left(\zeta^*_{nmk} + R^*_{nmk}\right). \tag{23}$$

We shall show that below (Lemma 9), for each l_0,

$$\lim_{m\to\infty} \limsup_{n\to\infty} P\left[\left|\sum_{k=1}^{m} R_{nmk}\right| + \left|\sum_{k=1}^{m} R^*_{nmk}\right| > \epsilon\right] = 0 \text{ for all } \epsilon > 0, \tag{24}$$

and therefore, the respective limiting behaviors of (22) and (23) will be the same as those of $\sum_{k=1}^{m} \zeta_{nmk}$ and $\sum_{k=1}^{m} \zeta^*_{nmk}$ (For this same reason, and for notational convenience, the dependence on l_0 is not explicitly indicated.)

In Sections 3 - 5 below we establish that there is an integer $l_0 > 1$ such that the following facts hold (recall that E_l stands for the conditional expectation given $\sigma(\xi_j;\ j \leq l)$).

(R1) There is a nonrandom $\Delta(n,m)$ such that

$$\sum_{k=1}^{m} \left|E_{\left[n\frac{k-1}{m}\right]}\left[\zeta_{nmk}\right]\right| \leq \Delta(n,m) \to 0 \text{ as } n \to \infty, \text{ for each } m.$$

(R2)

$$\sum_{k=1}^{m} E_{\left[n\frac{k-1}{m}\right]}\left[\zeta^2_{nmk}\right] \Longrightarrow bL^0_1$$

as $n \to \infty$ first and then $m \to \infty$, where the constant b is as specified in Theorem 1.

Recall that the convergence in distribution of a sequence of distribution functions is metrizable, for example by the Lévy distance (see for instance Loève (1963, page 215)). Then the preceding convergence means that the distribution of $\sum_{k=1}^{m} E_{\left[n\frac{k-1}{m}\right]}\left[\zeta^2_{nmk}\right]$ converges in such a metric to that of bL^0_1 as $n \to \infty$ first and then $m \to \infty$.

(R3)

$$\lim_{m\to\infty} \limsup_{n\to\infty} \sum_{k=1}^{m} E\left[\zeta^4_{nmk}\right] = 0.$$

The next condition (R4) pertains only to the case $\alpha = 2$. To state it define

$$\chi_{nmk} = \frac{1}{\sqrt{n}} \sum_{l=\left[n\frac{k-1}{m}\right]+1}^{\left[n\frac{k}{m}\right]} \xi_l. \tag{25}$$

(R4) *When* $\alpha = 2$ (in which case we have $E[\xi_1] = 0$ and $E[\xi_1^2] < \infty$)

$$\limsup_{n\to\infty} P\left[\sum_{k=1}^{m}\left|E_{\left[n\frac{k-1}{m}\right]}\left[\zeta_{nmk}\chi_{nmk}\right]\right| > \varepsilon\right] = 0 \text{ for each } m \text{ and } \varepsilon > 0.$$

- **(R*1) - (R*2)**: *In the case of Theorem 2, we shall verify the preceding conditions with* ζ^*_{nmk} *in place of* ζ_{nmk}*, in which case the corresponding conditions will be referred to as (R*1), (R*2), (R*3) and (R*4).*

Note that the preceding conditions involve iterated limits in the sense that the limits are taken as $n \to \infty$ first and then $m \to \infty$. To proceed further it is convenient to note that they can be restated in an alternative form involving only the index n that goes to ∞. For this purpose recall that if $h(n, m)$ is a nonrandom function of n and m such that

$$\lim_{m\to\infty}\limsup_{n\to\infty}|h(n,m)| = 0$$

then one can find a sequence $m_n \uparrow \infty$ such that

$$h(n, m_n) \to 0.$$

If $G(n,m)$ is random, then note that $G(n,m) \xrightarrow{p} 0$ as $n \to \infty$ first and then $m \to \infty$, that is,

$$\lim_{m\to\infty}\limsup_{n\to\infty} P[|G(n,m)| \geq \eta] = 0 \text{ for all } \eta > 0,$$

is equivalent to $\lim_{m\to\infty}\limsup_{n\to\infty} E[\min(|G(n,m)|, 1)] = 0$, and therefore, taking $h(n,m) = E[\min(|G(n,m)|, 1)]$, there is a sequence $m_n \uparrow \infty$ such that $E[\min(|G(n,m_n)|, 1)] \to 0$, which is equivalent to

$$G(n, m_n) \xrightarrow{p} 0.$$

Thus (noting that the convergence in (R2) can be restated in terms of a suitable metric), (R1) - (R4) entail that there is a sequence $m_n \uparrow \infty$ such that

$$\sum_{k=1}^{m_n}\left|E_{\left[n\frac{k-1}{m_n}\right]}\left[\zeta_{nm_nk}\right]\right| + \sum_{k=1}^{m_n} E\left[\zeta^4_{nm_nk}\right] \xrightarrow{p} 0, \tag{26}$$

$$\sum_{k=1}^{m_n}\left|E_{\left[n\frac{k-1}{m_n}\right]}\left[\zeta_{nm_nk}\chi_{nm_nk}\right]\right| \xrightarrow{p} 0 \quad (\text{for } \alpha = 2) \tag{27}$$

and

$$\sum_{k=1}^{m_n} E_{\left[n\frac{k-1}{m_n}\right]}\left[\zeta^2_{nm_nk}\right] \Longrightarrow bL_1^0 \tag{28}$$

In the same way, the conditions (R*1) - (R*4) imply that (26) - (28) hold with ζ_{nmk} replaced by ζ^*_{nmk}.

We are now in a position to present the proof of Theorem 1, when (R1) - (R4) hold. First, for convenience, we let

$$\zeta_{m_n k} = \zeta_{n m_n k}, \quad \chi_{m_n k} = \chi_{n m_n k}, \quad k = 1, ..., m_n.$$

Next, for the purpose of the proof, we

- *extend the array* $\zeta_{m_n k}$*,* $1 \le k \le m_n$*, to all* $k \ge 1$*, by taking* $\{\zeta_{m_n k}; k = m_n + 1, ...\}$ *to be an array of iid Gaussian* $\left(0, \frac{1}{m_n}\right)$ *random variables, independent of* $\{\xi_j; -\infty < j < \infty\}$.

Further, we use the **notation** $E_{m_n,l}$ for the conditional expectation given the σ-field

$$\mathcal{F}_{m_n l} = \begin{cases} \sigma\left(\xi_j,\ j \le \left[n\frac{l}{m_n}\right]\right) & \text{if } 1 \le l \le m_n \\ \sigma\left(\xi_j,\ j \le \left[n\frac{l}{m_n}\right] \text{ and } \zeta_{m_n k}, m_n + 1 \le k \le l\right) & \text{if } l > m_n. \end{cases}$$

Explicitly,

$$E_{m_n,l}\left[\ .\ \right] = E\left[\ .\ \middle| \mathcal{F}_{m_n l}\right].$$

With this extension, (26) and (27) take the strengthened forms, for any $0 < \gamma < 1$,

$$\sum_{k=1}^{\left[m_n^{1+\gamma}\right]} \left|E_{m_n,k-1}\left[\zeta_{m_n k}\right]\right| \to 0, \tag{29}$$

$$\sum_{k=1}^{\left[m_n^{1+\gamma}\right]} E\left[\zeta^4_{m_n k}\right] \to 0, \tag{30}$$

and

$$\sum_{k=1}^{\left[m_n^{1+\gamma}\right]} \left|E_{m_n,k-1}\left[\zeta_{m_n k}\chi_{m_n k}\right]\right| \xrightarrow{p} 0 \quad (\text{for } \alpha = 2). \tag{31}$$

Now, define the martingale differences

$$\zeta'_{m_n k} = \zeta_{m_n k} - E_{m_n,k-1}\left[\zeta_{m_n k}\right], \quad k = 1, 2, ...$$

with respect to the σ-fields $\mathcal{F}_{m_n k}$, $k = 1, 2, ...$It is easily seen, in view of (29), that

$$\text{(30) and (31) hold with } \zeta_{m_n k} \text{ replaced by } \zeta'_{m_n k}. \tag{32}$$

In addition, if we define

$$T_{m_n}(q) = \sum_{k=1}^{q} E_{m_n,k-1}\left[\left|\zeta'_{m_nk}\right|^2\right] = \sum_{k=1}^{q}\left\{E_{m_n,k-1}\left[\zeta^2_{m_nk}\right] - \left(E_{m_n,k-1}\left[\zeta_{m_nk}\right]\right)^2\right\},$$

then, in view of (29) and because ζ_{m_nk}, $k = m_n + 1, \ldots$ are iid Gaussian $\left(0, \frac{1}{m_n}\right)$, for any $s \geq 1$,

$$T_{m_n}(sm_n) \Longrightarrow bL_1^0 + s - 1, \quad s \geq 1. \tag{33}$$

Now for each fixed $t > 0$, define

$$\tau_{m_n}(t) = \inf\{q \geq 1 : T_{m_n}(q) \geq t\}.$$

Note that

$$\tau_{m_n}(t) = m_n \quad \text{if } t = T_{m_n}(m_n). \tag{34}$$

We have

$$\{\tau_{m_n}(t) \leq l\} = \{T_{m_n}(l) \geq t\} \in \mathcal{F}_{m_n,l-1}, \quad l = 1, 2, \ldots,$$

so that for each n and $t > 0$,

$$\tau_{m_n}(t) \text{ is a stopping time with respect to the } \sigma\text{-fields } \mathcal{F}_{m_n,l-1}, l = 1, 2, \ldots.$$

Note that for any positive integer J, $P\left[\frac{\tau_{m_n}(t)}{m_n} > J\right] \leq P\left[T_{m_n}(Jm_n) \leq t\right]$ and hence, in view of (33),

$$P\left[\frac{\tau_{m_n}(t)}{m_n} > J\right] \to 0 \quad \text{if } J > t - 1. \tag{35}$$

We thus have shown, in view of (29) - (31), (35) and because $m_n \uparrow \infty$,

$$\sum_{k=1}^{\tau_{m_n}(t)} E_{m_n,k-1}\left[\zeta_{m_nk}\right] \xrightarrow{p} 0, \tag{36}$$

$$\sum_{k=1}^{\tau_{m_n}(t)} E_{m_n,k-1}\left[\left|\zeta'_{m_nk}\right|^4\right] \xrightarrow{p} 0 \tag{37}$$

and

$$\sum_{k=1}^{\tau_{m_n}(t)} E_{m_n,k-1}\left[\left|\zeta'_{m_nk}\chi_{m_nk}\right|\right] \xrightarrow{p} 0 \quad (\text{for } \alpha = 2) \tag{38}$$

Further, because of (32) and (35),

$$E_{m_n,\tau_{m_n}(t)-1}\left[\left|\zeta'_{m_n,\tau_{m_n}(t)}\right|^2\right] \xrightarrow{p} 0. \tag{39}$$

Hence, because

$$T_{m_n}\left(\tau_{m_n}(t)\right) \geq t \geq T_{m_n}\left(\tau_{m_n}(t)-1\right) = T_{m_n}\left(\tau_{m_n}(t)\right) - E_{m_n,\tau_{m_n}(t)-1}\left[\left|\zeta'_{m_n,\tau_{m_n}(t)}\right|^2\right],$$

$$T_{m_n}\left(\tau_{m_n}(t)\right) \xrightarrow{p} t. \tag{40}$$

Now let

$$W_n(t) = \sum_{k=1}^{\tau_{m_n}(t)} \zeta'_{m_n k}.$$

By making the convention that the sum $\sum_{k=1}^{\tau_{m_n}(t)}$ is empty when $t < 0$, we may assume for convenience that $W_n(t)$ is defined for all $-\infty < t < \infty$. Similarly, let $W(t)$ be the Brownian motion for $0 \leq t < \infty$ and $W(t) \equiv 0$ for $t < 0$. We then have

Lemma 4. *Let $W(t)$ be as above, and as before let $Z_\alpha(t)$ be the α-stable motion. Then, for $0 < \alpha \leq 2$ and for every $M > 0$ and every integer $l > 0$,*

$$t \longmapsto \left(n^{-1/\alpha}\sum_{j=-nl}^{[nt]} \xi_j, W_n(t)\right) \Longrightarrow \left(Z_\alpha(t) - Z_\alpha(-l), W(t)\right) \quad \text{in } D_{\mathbb{R}^2}[-l, M],$$

where the processes $W(t)$ and $Z_\alpha(t)$ are independent. Here " $\Longrightarrow$ in $D_{\mathbb{R}^2}[-l, M]$ " signifies the convergence in distribution in the Skorokhod space $D_{\mathbb{R}^2}[-l, M]$.

Proof. The proof consists of reducing the situation to that of Jeganathan (2006a, Theorem 1 and Remarks 5 and 8). **First suppose that $0 < \alpha < 2$.** Note that, with

$$\chi_{m_n k} = \frac{1}{n^{1/\alpha}\kappa(n)} \sum_{l=\left[n\frac{k-1}{m_n}\right]+1}^{\left[n\frac{k}{m_n}\right]} \xi_l, \qquad (\kappa(n) \text{ as in } (1)),$$

we have $\sum_{k=-m_n l+1}^{[m_n t]} \chi_{m_n k} = \frac{1}{n^{1/\alpha}\kappa(n)} \sum_{j=-nl}^{\left[n\frac{[m_n t]}{m_n}\right]} \xi_j$. Hence, using (1) and noting that l is an integer,

$$\sum_{k=-m_n l+1}^{[m_n t]} \chi_{m_n k} - \frac{1}{n^{1/\alpha}\kappa(n)} \sum_{j=-nl}^{[nt]} \xi_j \xrightarrow{p} 0. \tag{41}$$

Therefore, by (1),

$$\sum_{k=-m_n l+1}^{[m_n t]} \chi_{m_n k} \overset{fdd}{\Longrightarrow} Z_\alpha(t) - Z_\alpha(-l).$$

In addition, because $\frac{m_n}{n} \to 0$,

$$\sup_{-m_n l+1 \le k \le [m_n M]} P\left[|\chi_{m_n k}| > \varepsilon\right] \to 0.$$

The preceding two facts will imply that the conditions (C1) - (C4) of Jeganathan (2006a, Section 2.1) hold (with the stopping time $k_n(t)$ there taken to be $[m_n t]$) for the array $\{\chi_{m_n k}, k = -m_n(l+1), ...\}$ (of independent random variables), with the limit $B_t \equiv 0$ in Condition (C2), see Loève(1963, Section 22.4, Central Convergence Criterion, page 311). Here $B_t \equiv 0$ explicitly means $\sum_{k=-m_n l+1}^{[m_n L]} \sigma^2_{\chi_{m_n k}}(\tau) \xrightarrow{p} 0$ for all $L > 0$ where $\sigma^2_{\chi_{m_n k}}(\tau)$ is the truncated variance as defined in Jeganathan (2006a) or as in Loève(1963, Condition (ii) of the Central Convergence Criterion, page 311). It is clear that this implies, in view of (35), $\sum_{k=-m_n l+1}^{[m_n M]\vee\tau_{m_n}(M)} \sigma^2_{\chi_{m_n k}}(\tau) \xrightarrow{p} 0$.

It is also clear from (36), (37) and (40) that the conditions (D1) - (D5) of that paper (with the stopping time $k_n(t)$ there taken to be $\tau_{m_n}(t)$) hold for the the array $\left\{\zeta'_{m_n k}, k = 1, ...\right\}$, with the limiting triplets (A^*_t, B^*_t, L^*_t) such that $A^*_t \equiv 0 \equiv L^*_t$ and $\sum_{k=1}^{\tau_{m_n}(t)} \sigma^2_{\zeta'_{m_n k}}(\tau) \xrightarrow{p} B^*_t \equiv t$. In addition, using (29), (30) and the fact $\sum_{k=1}^{[m_n M]} E_{m_n,k-1}\left[\left|\zeta'_{m_n k}\right|^2\right] = \sum_{k=1}^{m_n} E_{m_n,k-1}\left[\left|\zeta'_{m_n k}\right|^2\right] + \frac{[m_n M]-m_n}{m_n}$ (recall that $\{\zeta_{m_n k}; k = m_n+1, ...\}$ are iid Gaussian $\left(0, \frac{1}{m_n}\right)$), it can be seen that $\sum_{k=1}^{[m_n M]} \sigma^2_{\zeta'_{m_n k}}(\tau)$ is bounded in probability.

Thus all the requirements specified in Jeganathan (2006a, Remarks 5 and 8) are satisfied. This proves the lemma when $0 < \alpha < 2$.

In the case $\alpha = 2$, (31) entails that the condition (E2) in Jeganathan (2006a, as modified in Remarks 5) holds. Hence, similar to the case $0 < \alpha < 2$ above, the proof for this case also follows. This completes the proof of the Lemma. ■

We now come back to the proof of Theorem 1. Because Lemma 4 is true for every $l > 0$, it entails (keeping in mind the conditions (A1) and (A2), see Kasahara and Maejima (1988))

$$\left(\gamma_n^{-1} S_{[nt]}, W_n(t)\right) \overset{fdd}{\Longrightarrow} (\Lambda_{\alpha,H}(t), W(t))$$

where the processes $W(t)$ and $\Lambda_{\alpha,H}(t)$ are independent. Further, in Section 5 below (see the Remark following (100) below) it is shown that (or see Jeganathan (2004a, Proposition 6 and Lemmas 7 and 8))

$$T_n = \sum_{k=1}^{m_n} E_{m_n,k-1}\left[\left|\zeta'_{m_n k}\right|^2\right]$$

is approximated by a functional of the process $\gamma_n^{-1} S_{[nt]}$ such that T_n converges in distribution if $\gamma_n^{-1} S_{[nt]} \overset{fdd}{\Longrightarrow} \Lambda_{\alpha,H}(t)$. We then have

$$\left(\gamma_n^{-1} S_{[nt]}, W_n(t), T_n\right) \overset{fdd}{\Longrightarrow} \left(\Lambda_{\alpha,H}(t), W(t), bL^0_1\right). \tag{42}$$

Now with q a positive integer and $J > 0$, let

$$0 = \tau_{q0} < \tau_{q1} < ...\tau_{q,q-1} < \tau_{qq} = J$$

be such that

$$\sup_{1 \leq i \leq q} |\tau_{qi} - \tau_{q,i-1}| \to 0 \text{ as } q \to \infty.$$

Define

$$T_{n,q,J} = \begin{cases} \tau_{qi} & \text{if } \tau_{qi} \leq T_n < \tau_{q,i+1},\ i = 0, 1, ...q-1, \\ J & \text{if } T_n \geq J. \end{cases}$$

Letting

$$T = bL_1^0,$$

define $T_{q,J}$ analogously. Now, taking $\tau_{q,q+1} = \infty$,

$$\{W_n(T_{n,q,J}) \leq v\} = \cup_{i=0}^q \{W_n(\tau_{qi}) \leq v, \tau_{qi} \leq T_n < \tau_{q,i+1}\}$$

where $\{W_n(\tau_{qi}) \leq v, \tau_{qi} \leq T_n < \tau_{q,i+1}\}$ are disjoint, and hence, for $0 \leq u_1 \leq ... \leq u_k < \infty$ and for any reals $d_j, j = 1, ..., k$,

$$\begin{aligned} & P\left(W_n(T_{n,q,J}) \leq v, \gamma_n^{-1} S_{[nu_j]} \leq d_j, j = 1, ..., k\right) \\ = \ & P\left(\cup_{i=0}^q \left\{W_n(\tau_{qi}) \leq v, \tau_{qi} \leq T_n < \tau_{q,i+1}, \gamma_n^{-1} S_{[nu_j]} \leq d_j, j = 1, ..., k\right\}\right) \\ = \ & \sum_{i=0}^q P\left(W_n(\tau_{qi}) \leq v, \tau_{qi} \leq T_n < \tau_{q,i+1}, \gamma_n^{-1} S_{[nu_j]} \leq d_j, j = 1, ..., k\right). \end{aligned}$$

One can assume without loss of generality that $\tau_{q1}, ...\tau_{qq}$ are continuity points of T. Then (42) together with the preceding identity entail that

$$\begin{aligned} & P\left(W_n(T_{n,q,J}) \leq v, \gamma_n^{-1} S_{[nu_j]} \leq d_j, j = 1, ..., k\right) \\ \to \ & \sum_{i=0}^q P\left(W(\tau_{qi}) \leq v, \tau_{qi} \leq T < \tau_{q,i+1}, \Lambda_{\alpha,H}(u_j) \leq d_j, j = 1, ..., k\right) \\ = \ & P\left(W(T_{q,J}) \leq v, \Lambda_{\alpha,H}(u_j) \leq d_j, j = 1, ..., k\right). \end{aligned}$$

In other words, we have

$$\left(W_n(T_{n,q,J}), \gamma_n^{-1} S_{[nt]}\right) \overset{fdd}{\Longrightarrow} \left(W(T_{q,J}), \Lambda_{\alpha,H}(t)\right).$$

(Note that $T_{q,J}$ is a function of L_1^0, which, being a functional of $\Lambda_{\alpha,H}(t)$, is independent of $W(t)$ by Lemma 4.) In addition, because $W_n(t) \Longrightarrow W(t)$ in the Skorkhod space $D[0, M]$ with $W(t) \in C[0, M]$ for every $M > 0$, we have

$$\lim_{h \to 0} \limsup_{n \to \infty} P\left[\sup_{|t-s| \leq h, t, s \in [0,M]} |W_n(t) - W_n(s)| > \varepsilon\right] = 0$$

for all $\varepsilon > 0$ and all $M > 0$. Hence

$$\lim_{J\to\infty}\lim_{q\to\infty}\limsup_{n\to\infty} P\left[\left|W_n\left(T_{n,q,J}\right) - W_n\left(T_n\right)\right| > \varepsilon\right] = 0.$$

Similarly

$$\lim_{J\to\infty}\lim_{q\to\infty} P\left[\left|W\left(T_{q,J}\right) - W\left(T\right)\right| > \varepsilon\right] = 0.$$

It follows that

$$\left(W_n\left(T_n\right), \gamma_n^{-1} S_{[nt]}\right) \Longrightarrow \left(W(T), \Lambda_{\alpha,H}(t)\right).$$

Noting that $\tau_{m_n}\left(T_n\right) = m_n$ (see (34)) so that $W_n\left(T_n\right) = \sum_{k=1}^{m_n} \zeta_{m_n k}$, and in view of the independence of the processes $W(t)$ and $\Lambda_{\alpha,H}(t)$ so that the distribution of $(W(T), \Lambda_{\alpha,H}(t))$ is the same as that of $\left(W\sqrt{bL_1^0}, \Lambda_{\alpha,H}(t)\right)$ where W is standard normal independent of the process $\Lambda_{\alpha,H}(t)$ (recall $T = bL_1^0$), the preceding convergence takes the form

$$\left(\sum_{k=1}^{m_n} \zeta_{m_n k}, \gamma_n^{-1} S_{[nt]}\right) \Longrightarrow \left(W\sqrt{bL_1^0}, \Lambda_{\alpha,H}(t)\right) \tag{43}$$

(Recall that $\sum_{k=1}^{m_n} \zeta_{m_n k} = \sqrt{\frac{\gamma_n}{n}} \sum_{k=1}^{n} f\left(S_k\right)$.) Now in Section 5 (see the Remark following (100) below (or in Jeganathan (2004a, Proposition 6 and Lemmas 7 and 8)) it is shown that $\frac{\gamma_n}{n}\sum_{k=1}^{n} h\left(S_k\right)$ occurring the statement of Theorem 1 is approximated by a functional of the process $\gamma_n^{-1} S_{[nt]}$ such that the former converges in distribution to $L_1^0 \int h(y)\,dy$ if $\gamma_n^{-1} S_{[nt]} \overset{fdd}{\Longrightarrow} \Lambda_{\alpha,H}(t)$. Thus the convergence (43) holds jointly with $n^{-1}\gamma_n \sum_{k=1}^{n} h\left(S_k\right)$. This being the conclusion of Theorem 1, the proof is completed. ■

3 SOME PRELIMINARIES

In this section we first present some preliminaries for the purpose of verification of the requirements (R1) - (R4) and (R*1) - (R*4). In this section itself we shall illustrate the intent of these preliminaries by verifying the conditions (R1) and (R*1).

To begin with recall the fact that ξ_1 belongs to the domain of attraction of a strictly stable law with index $0 < \alpha \le 2$, in the sense of Section 1 above, means in particular (see Ibragimov and Linnik (1965, Theorem 2.6.5, page 85)) that, for all u in some neighborhood of 0,

$$\psi(u) = E\left[e^{iu\xi_1}\right] = \begin{cases} e^{-|u|^\alpha G(|u|)\left(1 - i\beta\,\mathrm{sign}(u)\tan\left(\frac{\pi\alpha}{2}\right)\right)} & \text{if } \alpha \neq 1 \\ e^{-|u| G(|u|)} & \text{if } \alpha = 1 \end{cases}$$

with $|\beta| \le 1$, where $G(u)$ is slowly varying as $u \to 0$. In particular there are constants $\eta > 0$ and $d > 0$ such that

$$|\psi(u)| \le e^{-d|u|^\alpha G(|u|)} \qquad \text{for all } |u| \le \eta. \tag{44}$$

In addition, if one lets

$$b_n^{-1} = \inf\left\{u > 0 : u^\alpha G(u) = n^{-1}\right\},$$

then $b_n^\alpha \backsim nG(b_n^{-1})$ as $n \to \infty$, and in (1) one can take $\kappa(n) \backsim G^{\frac{1}{\alpha}}(b_n^{-1})$, so that we henceforth assume for convenience that $\kappa(n)$ in (1) and the above b_n are such that

$$b_n = n^{\frac{1}{\alpha}} G^{\frac{1}{\alpha}}(b_n^{-1}) = n^{\frac{1}{\alpha}} \kappa(n). \tag{45}$$

See for instance Bingham et al (1987, page 344) for the details of these facts. Then note that, (13) takes the form

$$\gamma_n = \begin{cases} n^{H-1/\alpha} u(n) b_n & \text{if the condition (A1) is satisfied} \\ (\sum_{j=0}^\infty c_j) b_n & \text{if the condition (A2) is satisfied.} \end{cases} \tag{46}$$

The following result is essentially well-known, and we supply its proof for completeness.

Lemma 5. *Let η be as in (44) and b_n be as in (45). Let κ_j be integers such that for some integer $j_0 > 0$ and a constant $C > 0$,*

$$\kappa_j \geq Cj \quad \text{for all } j \geq j_0. \tag{47}$$

Then for every $0 < c < \alpha$ there is a constant $a > 0$ such that

$$\left|\psi\left(\lambda b_j^{-1}\right)\right|^{\kappa_j} \leq Ce^{-a|\lambda|^c} \quad \text{for all } |\lambda| \leq \eta b_j,\ j \geq 1. \tag{48}$$

Further, for every $\delta > 0$ there is a $0 < \rho < 1$ such that

$$\sup_{|\lambda| \geq \delta b_j} \left|\psi\left(\lambda b_j^{-1}\right)\right|^{\kappa_j} = \sup_{|\mu| \geq \delta} |\psi(\mu)|^{\kappa_j} \leq C\rho^j \quad \text{for all } j \geq 1. \tag{49}$$

Proof. According to (44), $\left|\psi\left(\lambda b_j^{-1}\right)\right|^{\kappa_j} \leq e^{-d\kappa_j |\lambda|^\alpha b_j^{-\alpha} G\left(|\lambda| b_j^{-1}\right)}$ for all $|\lambda| \leq \eta b_j$. Therefore we first recall a bound for $b_j^{-\alpha} G\left(|\lambda| b_j^{-1}\right)$ for all sufficiently large j.

According to Potter's inequality (see Bingham et al (1987, Theorem 1.5.6, Statement (ii), page 25), for every $\delta > 0$ there is a $B > 0$ such that $|\frac{G(x)}{G(y)}| \leq B \max\{(x/y)^\delta, (x/y)^{-\delta}\}$ for all $x > 0, y > 0$. In particular $\left|\frac{G\left(b_j^{-1}\right)}{G\left(|\lambda| b_j^{-1}\right)}\right| \leq B \max\{|\lambda|^\delta, |\lambda|^{-\delta}\}$. Because $\max\{|\lambda|^\delta, |\lambda|^{-\delta}\} = |\lambda|^\delta$ if $|\lambda| \geq 1$, it then follows from (45) that there is a j_1 such that

$$b_j^{-\alpha} G\left(|\lambda| b_j^{-1}\right) \geq B^{-1} j^{-1} |\lambda|^{-\delta} \quad \text{for all } j \geq j_1 \text{ and } |\lambda| \geq 1.$$

Therefore, by (44), for every $0 < c < \alpha$ there is a $a > 0$ such that

$$\left|\psi\left(\lambda b_j^{-1}\right)\right|^{\kappa_j} \leq e^{-d\kappa_j |\lambda|^\alpha b_j^{-\alpha} G\left(|\lambda| b_j^{-1}\right)} \leq e^{-a|\lambda|^c} \quad \text{for all } 1 \leq |\lambda| \leq \eta b_j,\ j \geq j_2$$

where $j_2 = \max(j_0, j_1)$ (j_0 as in (47)). On the other hand, if $j \le j_2$,

$$\left|\psi\left(\lambda b_j^{-1}\right)\right|^{\kappa_j} \le 1 = e^{a|\eta b_j|^c} e^{-a|\eta b_j|^c} \le \left(\max_{j \le j_2} e^{a|\eta b_j|^c}\right) e^{-a|\lambda|^c} \quad \text{for all } |\lambda| \le \eta b_j,\ j \le j_2.$$

Further,

$$\left|\psi\left(\lambda b_j^{-1}\right)\right|^{\kappa_j} \le 1 = ee^{-1} \le ee^{-|\lambda|^c} \quad \text{if } |\lambda| \le 1,\ j \ge 1.$$

Hence the proof of the first part follows from the preceding three inequalities.

Regarding the second part note that, the condition (16) entails the Cramér's condition $\limsup_{|\lambda| \to \infty} |\psi(\lambda)| < 1$, which is equivalent to the statement that for every $\delta > 0$, there is a $0 < \tau = \tau(\delta) < 1$ such that

$$\sup_{|\lambda| \ge \delta} |\psi(\lambda)| \le \tau < 1.$$

Hence the second statement follows, completing the proof of the lemma. ■

The following consequences of Lemma 5 will be used below. First, for any $\kappa \ge 0$,

$$\int_{\{|\lambda| \le \eta b_l\}} |\lambda|^\kappa \left|\psi\left(\lambda b_l^{-1}\right)\right|^{[l/2]} d\lambda \le C \int |\lambda|^\kappa e^{-a|\lambda|^c} d\lambda \le C, \tag{50}$$

using the Statement (i) of Lemma 5. Next let l_0 be such that for some $0 < \gamma < 1$, $[l/2] - p \ge [l\gamma]$ for all $l \ge l_0$, where p is as in (15). Then, for any $\delta > 0$ and $0 \le \kappa \le 3$, using the Statement (ii) of Lemma 5,

$$\begin{aligned} \int_{\{|\lambda| > \delta b_l\}} |\lambda|^\kappa \left|\psi\left(\lambda b_l^{-1}\right)\right|^{[l/2]} d\lambda &\le C\rho^l \int_{\{|\lambda| > \delta b_l\}} |\lambda|^\kappa \left|\psi\left(\lambda b_l^{-1}\right)\right|^p d\lambda \\ &= C\rho^l b_l^{1+\kappa} \int |\lambda|^\kappa |\psi(\lambda)|^p d\lambda \le C\rho_*^l, \quad l \ge l_0, \end{aligned} \tag{51}$$

for some constant $0 < \rho_* < 1$, using (15).

We shall also need to use the next inequality, which is a direct consequence of Hölder's inequality, see for instance Hewitt and Stromberg (1965, page 200, Exercise (13.26)). For convenience of reference we state it as a lemma.

Lemma 6. *For any functions $\varphi_i(u) : R^k \to R, i = 1, ..., q,$*

$$\int \prod_{i=1}^q |\varphi_i(u)|\, du \le \prod_{i=1}^q \left(\int |\varphi_i(u)|^q du\right)^{\frac{1}{q}}, \ q \ge 1.$$

By replacing $|\varphi_i(u)|$ by $|\ell(u)|^{1/q} |\varphi_i(u)|$ in this inequality, we also have

$$\int |\ell(u)| \prod_{i=1}^q |\varphi_i(u)|\, du \le \prod_{i=1}^q \left(\int |\ell(u)|\, |\varphi_i(u)|^q du\right)^{1/q}, \quad q \ge 1. \tag{52}$$

We now state one consequence of this, which will be used later. For this purpose note that, when (A1) holds,

$$g(j) = \sum_{s=0}^{j} c_s \sim C j^{H-1/\alpha} u(j), \quad j \to \infty.$$

(Note that in the case $H - 1/\alpha < 0$, the requirement $\sum_{j=0}^{\infty} c_j = 0$ (see (9)) is invoked here.) Therefore the requirement (10) on $u(j)$ holds for $g(j)$ also, that is, there is an integer $l_0 > 0$ and constants c_1 and c_2 such that $g(l) \neq 0$ and

$$0 < c_1 \leq \frac{g(l+j_1)}{g(l-j_2)} \leq c_2 \quad \text{for all } 0 \leq j_1, j_2 \leq [l/2] \tag{53}$$

for all $l \geq l_0$. This also entails that, recalling that $\gamma_l = l^{H-1/\alpha} u(l) b_l$ so that $\frac{\gamma_l}{b_l |g(q)|} = \frac{l^{H-1/\alpha} u(l)}{|g(q)|} \sim \left|\frac{g(l)}{g(q)}\right|$, there is an l_0 such that for all $l \geq l_0$,

$$0 < D_1 \leq \frac{\gamma_l}{b_l |g(q)|} \leq D_2 \quad \text{for } [l/2] \leq q < l. \tag{54}$$

Also note that, for p as in (16), there is an l_0 such that for some $0 < \gamma < 1$,

$$l - [l/2] - p \geq [l\gamma] \quad \text{for all } l \geq l_0.$$

Then, for $\delta > 0$ such that $D_1^{-1}\delta = \eta$ with η as in the Statement (i) of Lemma 5, we have for $l \geq l_0$ and $\kappa \geq 0$,

$$\begin{aligned}
& \int_{\{|\lambda| \leq \delta b_l\}} |\lambda|^\kappa \prod_{j=0}^{l} \left|\psi\left(\lambda \frac{g(j)}{\gamma_l}\right)\right| d\lambda \\
\leq\; & \int_{\{|\lambda| \leq \delta b_l\}} |\lambda|^\kappa \prod_{j=[l/2]+1}^{l} \left|\psi\left(\lambda \frac{g(j)}{\gamma_l}\right)\right| d\lambda, \quad \text{using } |\psi(\lambda)| \leq 1 \\
\leq\; & \prod_{j=[l/2]+1}^{l} \left(\int_{\{|\lambda| \leq \delta b_l\}} |\lambda|^\kappa \left|\psi\left(\lambda \frac{g(j)}{\gamma_l}\right)\right|^{l-[l/2]} d\lambda\right)^{\frac{1}{l-[l/2]}} \quad \text{by (52)} \\
=\; & \prod_{j=[l/2]+1}^{l} \left(\left|\frac{\gamma_l}{g(j)\, b_l}\right|^{1+\kappa} \int_{\left\{\left|\frac{\gamma_l}{g(j) b_l}\lambda\right| \leq \delta b_l\right\}} |\lambda|^\kappa \left|\psi\left(\frac{\lambda}{b_l}\right)\right|^{l-[l/2]} d\lambda\right)^{\frac{1}{l-[l/2]}} \\
\leq\; & D_2^{1+\kappa} \int_{\{|\lambda| \leq D_1^{-1}\delta b_l\}} |\lambda|^\kappa \left|\psi\left(\frac{\lambda}{b_l}\right)\right|^{l-[l/2]} d\lambda \leq C, \quad \text{by (50) and (54).}
\end{aligned} \tag{55}$$

In the same way, for every $\delta > 0$, $0 \leq \kappa \leq 3$ and $l \geq l_0$,

$$\int_{\{|\lambda| > \delta b_l\}} |\lambda|^\kappa \prod_{j=0}^{l-1} \left|\psi\left(g(j)\frac{\lambda}{\gamma_l}\right)\right| d\lambda \leq C \int_{\{|\lambda| > D_2^{-1}\delta b_l\}} |\lambda|^\kappa \left|\psi\left(\frac{\lambda}{b_l}\right)\right|^{l-[l/2]} d\lambda \leq C\rho^l, \tag{56}$$

using (51), where $0 < \rho < 1$. In addition, noting that $g(0) = 1$ and $|\psi(\lambda)| \le 1$, for any constants u_l, v_l, h_l such that $\min_{0\le l\le l_0}|u_l| > 0$ and $\min_{0\le l\le l_0}|v_l| > 0$, we have for $0 \le l \le l_0$

$$\max_{l\le l_0}\left|\int \prod_{q=0}^{l}|\psi(u_l\lambda g(q))|\left|\widehat{f}(v_l\lambda - h_l)\right| d\lambda\right|^2$$
$$\le \max_{l\le l_0}\left|\int\left|\psi(u_l\lambda)\widehat{f}(v_l\lambda - h_l)\right| d\lambda\right|^2 \le \max_{l\le l_0}\frac{1}{|u_l v_l|}\int|\psi(\lambda)|^2 d\lambda\int\left|\widehat{f}(\lambda)\right|^2 d\lambda \le C, \quad (57)$$

where we have used (14) and the fact $\int\left|\widehat{f}(\lambda)\right|^2 d\lambda = 2\pi\int|f(x)|^2 dx < \infty$.

As a further preliminary, we next introduce a decomposition for S_k which will be repeatedly used throughout below. In this section itself we shall illustrate the intent of this decomposition, as well as the Lemmas 5 and 6, by verifying the conditions (R1) and (R*1). Recall that

$$S_k = \sum_{l=-\infty}^{0}(g(k-l) - g(-l))\xi_l + \sum_{l=1}^{k} g(k-l)\xi_l,$$

where recall that $g(j) = \sum_{s=0}^{j} c_s$. The indicated decomposition is

$$S_k = S_{k,j} + S_{k,j}^*, \quad 1 \le j \le k-1, \quad (58)$$

where

$$S_{k,j} = \sum_{l=-\infty}^{0}(g(k-l) - g(-l))\xi_l + \sum_{l=1}^{k-j} g(k-l)\xi_l \quad (59)$$

and

$$S_{k,j}^* = \sum_{l=k-j+1}^{k} g(k-l)\xi_l = \sum_{q=0}^{j-1} g(q)\,\xi_{k-q}.$$

Here it is important to note that

$$S_{k,j} \text{ and } S_{k,j}^* \text{ are independent.}$$

In addition note that the marginal distribution of $S_{k,j}^*$ is the same as that of $\sum_{i=0}^{j-1} g(i)\,\xi_i$.

Next, in order to deal with ζ_{nmk}^*, we have (recall that E_j stands for the conditional expectation given $\{\xi_k, k \le j\}$)

$$E_{k-\nu}[f(S_k)\,\omega_k] = f_*(S_{k,\nu}) \quad (60)$$

where $S_{k,\nu}$ is as in (59) and

$$\begin{aligned} f_*(x) &= E\left[f\left(x+\sum_{j=k-\nu+1}^{k} g(k-j)\xi_j\right)\sum_{j=k-\nu+1}^{k} d_{k-j}\eta_j\right] \\ &= E\left[f\left(x+\sum_{j=1}^{\nu} g(\nu-j)\xi_j\right)\sum_{j=1}^{\nu} d_{\nu-j}\eta_j\right]. \end{aligned}$$

In the verifications of the conditions (R*1) - (R*4) for the variables ζ^*_{nmk}, the function $f_*(x)$ will take the role of $f(x)$ of Theorem 1, and therefore we need to check that $f_*(x)$ satisfies the conditions (3) and (4) with $f_*(x)$ involved in place of $f(x)$. We state this fact separately.

Lemma 7. *For $f_*(x)$ as in (60), the requirements (3) and (4) hold with $f_*(x)$ involved in place of $f(x)$.*

Proof. First note that

$$\begin{aligned} \int |f_*(x)|\,dx &\le \int E\left[\left|f\left(x+\sum_{j=1}^{\nu} g(\nu-j)\xi_j\right)\right|\left|\sum_{j=1}^{\nu} d_{\nu-j}\eta_j\right|\right]dx \\ &= E\left[\left|\sum_{j=1}^{\nu} d_{\nu-j}\eta_j\right|\int\left|f\left(x+\sum_{j=1}^{\nu} g(\nu-j)\xi_j\right)\right|dx\right] \\ &= \left(\int |f(x)|\,dx\right)E\left[\left|\sum_{j=1}^{\nu} d_{\nu-j}\eta_j\right|\right] < \infty, \quad \text{by (7).} \end{aligned} \tag{61}$$

Next, by Cauchy-Schwarz inequality and using $E\left[\left|\sum_{j=1}^{\nu} d_{\nu-j}\eta_j\right|^2\right] \le C$ (see (7)),

$$f_*^2(x) \le E\left[\left|f\left(x+\sum_{j=1}^{\nu} g(\nu-j)\xi_j\right)\right|^2\right]E\left[\left|\sum_{j=1}^{\nu} d_{\nu-j}\eta_j\right|^2\right] \le CE\left[\left|f\left(x+\sum_{j=1}^{\nu} g(\nu-j)\xi_j\right)\right|^2\right].$$

This also implies, noting $|f_*(x)|^3 = |f_*^2(x)|^{\frac{3}{2}}$,

$$|f_*(x)|^3 \le C\left(E\left[\left|f\left(x+\sum_{j=1}^{\nu} g(\nu-j)\xi_j\right)\right|^2\right]\right)^{\frac{3}{2}} \le CE\left[\left|f\left(x+\sum_{j=1}^{\nu} g(\nu-j)\xi_j\right)\right|^3\right].$$

The same holds for $|f_*(x)|^4$. Hence, for $i = 2, 3, 4$,

$$\begin{aligned}\int |f_*(x)|^i\, dx &\le C\int E\left[\left|f\left(x + \sum_{j=1}^{\nu} g(\nu - j)\xi_j\right)\right|^i\right] dx \\ &= CE\left[\int \left|f\left(x + \sum_{j=1}^{\nu} g(\nu - j)\xi_j\right)\right|^i dx\right] \\ &= CE\left[\int |f(x)|^i\, dx\right] = C\int |f(x)|^i\, dx < \infty.\end{aligned}$$

The arguments in (61) also entails that, noting that $E\left[\sum_{j=1}^{\nu} d_{\nu-j}\eta_j\right] = 0$,

$$\int f_*(x)\, dx = \left(\int f(x)\, dx\right) E\left[\sum_{j=1}^{\nu} d_{\nu-j}\eta_j\right] = 0.$$

Next,

$$\int |x f_*(x)|\, dx \le E\left[\left|\sum_{j=1}^{\nu} d_{\nu-j}\eta_j\right| \int \left(|x| + \left|\sum_{j=1}^{\nu} g(\nu - j)\xi_j\right|\right) |f(x)|\, dx\right] < \infty$$

by (3) and (7). This completes the proof. ■

For the next result we note that, using the condition $\int_{-\infty}^{\infty} |y f(y)|\, dy < \infty$,

$$\left|\widehat{f}(\lambda_1) - \widehat{f}(\lambda_2)\right| \le C\,|\lambda_1 - \lambda_2|\,.$$

Now (4) entails that $\widehat{f}(0) = \int_{-\infty}^{\infty} f(y)\, dy = 0$. Thus $\left|\widehat{f}(\lambda)\right| \le C\,|\lambda|$. We also have $\left|\widehat{f}(\lambda)\right| \le C$ using $\int |f(y)|\, dy < \infty$. Thus, when (3) and (4) hold,

$$\left|\widehat{f}(\lambda)\right| \le C \min(|\lambda|, 1)\,. \tag{62}$$

Lemma 8. *There is a $0 < \rho < 1$ and a positive integer l_0 such that*

$$\sup_{j\ge 0} |E_j[f(S_{j+l})]| \le \frac{C}{\gamma_l^2} \text{ for all } l \ge l_0,$$

where recall that E_j stands for the conditional expectation given $\{\xi_k, k \le j\}$.

Under the conclusion of Lemma 7, the same bound holds for $\sup_{j\ge 0} |E_j[f(S_{j+l})\,\omega_{j+l}]|$ if $l_0 > \nu$, where ω_{j+l} and ν are as in (6).

Proof. We have $f(y) = \frac{1}{2\pi}\int e^{-i\lambda y}\widehat{f}(\lambda)\, d\lambda$. Hence, using (58),

$$f(S_{j+l}) = \frac{1}{2\pi}\int e^{-i\lambda\left(S_{j+l,l} + S^*_{j+l,l}\right)}\widehat{f}(\lambda)\, d\lambda.$$

Therefore, because $S_{j+l,l}$ and $S^*_{j+l,l}$ are independent,

$$\begin{aligned} |E_j[f(S_{j+l})]| &\leq \frac{C}{\gamma_l}\int \left|E\left[e^{-i\frac{\lambda}{\gamma_l}S^*_{j+l,l}}\right]\right|\left|\widehat{f}\left(\frac{\lambda}{\gamma_l}\right)\right| d\lambda \\ &= \frac{C}{\gamma_l}\int \left|\widehat{f}\left(\frac{\lambda}{\gamma_l}\right)\right|\prod_{q=0}^{l-1}\left|\psi\left(\frac{g(q)}{\gamma_l}\lambda\right)\right| d\lambda, \end{aligned}$$

where we have used $\left|E\left[e^{-i\frac{\lambda}{\gamma_l}S^*_{j+l,l}}\right]\right| = \left|\prod_{i=0}^{l-1}\psi\left(\frac{\lambda g(i)}{\gamma_l}\right)\right|$.

Now let l_0 be such that (55) and (56) hold. Then using (62), if $l \geq l_0$,

$$\int \left|\widehat{f}\left(\frac{\lambda}{\gamma_l}\right)\right|\prod_{q=0}^{l-1}\left|\psi\left(\frac{g(q)}{\gamma_l}\lambda\right)\right| d\lambda \leq \frac{C}{\gamma_l} + C\rho^l \leq \frac{C}{\gamma_l}, \quad l \geq l_0,$$

using $0 < \rho < 1$. Hence the first part of the lemma follows.

Regarding the second part, it is implied by the first part of the lemma and by the conclusion of Lemma 7, because we have $E_j[f(S_{j+l})\,\omega_{j+l}] = E_j[E_{j+l-\nu}[f(S_{j+l})\,\omega_{j+l}]]$, where $E_{j+l-\nu}[f(S_{j+l})\,\omega_{j+l}] = f_*(S_{j+l,\nu})$ with $f_*(x)$ as in Lemma 7. (Note that S_{j+l} and $S_{j+l,\nu}$ have the same structure and hence the conclusion of the first part of the lemma for S_{j+l} holds for $S_{j+l,\nu}$ also.) This completes the proof. ■

We next verify (24).

Lemma 9. *(24) holds for each positive integer l_0 (recall that R_{nmk} depends on l_0). More specifically,*

$$\max_{1\leq k\leq m}\left(|R_{nmk}| + |R^*_{nmk}|\right) = O_p\left(\sqrt{\frac{\gamma_n}{n}}\right),$$

where recall that $\frac{\gamma_n}{n} \to 0$.

Proof. First suppose that $k \geq 2$. We have (using the notation $|f|(x) = |f(x)|$)

$$E[|R_{nmk}|] \leq \sqrt{\frac{\gamma_n}{n}}\sum_{l=\left[n\frac{k-1}{m}\right]+1}^{\left[n\frac{k-1}{m}\right]+l_0-1} E[|f|(S_l)].$$

Now, according to the arguments of Lemma 8 with $j = 0$ and with $|f|(S_l)$ in place of $f(S_l)$ (note that $\left|\widehat{|f|}(\lambda)\right| \leq \int |f|(y)\,dy < \infty$), there is an n_0 and a constant $C > 0$ (both independent of $k \geq 2$) such that

$$\max_{\left[n\frac{k-1}{m}\right]<l\leq\left[n\frac{k-1}{m}\right]+l_0-1} E[|f|(S_l)] \leq C \quad \text{for all } n \geq n_0.$$

Thus $E[|R_{nmk}|] \leq C\sqrt{\frac{\gamma_n}{n}}$ for all $n \geq n_0$ and $k \geq 2$.

In the case $k = 1$, note that $\left(\sqrt{\frac{\gamma_n}{n}}\right)^{-1} R_{nm1} = \sum_{l=1}^{l_0-1} f(S_l)$, which is a fixed random variable and hence is of order $O_p(1)$.

In view of Lemma 7 and similar to Lemma 8, the preceding arguments for R_{nmk} apply for R^*_{nmk} also. Hence the lemma follows. ■

We next verify (R1) and (R*1) as a consequence of Lemma 8.

Verification of (R1) and (R*1): First consider (R1) corresponding to ζ_{nmk}, where ζ_{nmk} is as defined in (18). We have, by the first part of Lemma 8, there is an l_0 (independent of $k \geq 1$) such that

$$\left|E_{\left[n\frac{k-1}{m}\right]}\left[\zeta_{nmk}\right]\right| \leq \sqrt{\frac{\gamma_n}{n}} \sum_{l=l_0}^{\left[n\frac{k}{m}\right]-\left[n\frac{k-1}{m}\right]} \left|E_{\left[n\frac{k-1}{m}\right]}\left[f\left(S_{\left[n\frac{k-1}{m}\right]+l}\right)\right]\right| \leq C\sqrt{\frac{\gamma_n}{n}} \sum_{l=1}^{n} \frac{1}{\gamma_l^2}.$$

Here recall that $\gamma_n = n^{-H}u(n)$, where $u(n)$ is slowly varying.

Hence if $1/2 \leq H < 1$, it is clear that $\sqrt{\frac{\gamma_n}{n}} \sum_{l=1}^{n} \frac{1}{\gamma_l^2} \to 0$.

In the case $0 < H < 1/2$, we have $\sum_{l=1}^{n} \frac{1}{\gamma_l^2} \sim C\frac{n}{\gamma_n^2}$, so that

$$\sqrt{\frac{\gamma_n}{n}} \sum_{l=1}^{n} \frac{1}{\gamma_l^2} \sim C\sqrt{n}\gamma_n^{-\frac{3}{2}} = Cn^{-(3H-1)/2}\left(u(n)\right)^{-3/2}.$$

Because $1/3 < H < 1$, this converges to 0, and hence (R1) is verified. In the same way (R*1) is verified using the second part of Lemma 8. ■

4 VERIFICATION OF (R2), (R*2) AND (R4)

We first consider (R2) and then we shall indicate the modifications required for (R*2). We have

$$\begin{aligned} E_{\left[n\frac{k-1}{m}\right]}\left[\zeta_{nmk}^2\right] &= \frac{\gamma_n}{n}\sum_{l=l_0}^{n_{mk}} E_{\left[n\frac{k-1}{m}\right]}\left[f^2\left(S_{\left[n\frac{k-1}{m}\right]+l}\right)\right] \\ &+2\frac{\gamma_n}{n}\sum_{l=l_0}^{n_{mk}}\sum_{r=1}^{n_{mk}-l} E_{\left[n\frac{k-1}{m}\right]}\left[f\left(S_{\left[n\frac{k-1}{m}\right]+l}\right)f\left(S_{\left[n\frac{k-1}{m}\right]+l+r}\right)\right]\end{aligned}$$

where and throughout below

$$n_{mk} = \left[n\frac{k}{m}\right] - \left[n\frac{k-1}{m}\right].$$

Clearly, (R2) is a consequence of the next Lemmas 10 and 11 and Propositions 12 and 13.

Lemma 10. *For each $1 \leq k \leq m$,*

$$\max_{1\leq k\leq m} \frac{\gamma_n}{n}\sum_{l=l_0}^{n_{mk}}\sum_{r=q}^{n_{mk}} E\left[\left|E_{\left[n\frac{k-1}{m}\right]}\left[f\left(S_{\left[n\frac{k-1}{m}\right]+l}\right)f\left(S_{\left[n\frac{k-1}{m}\right]+l+r}\right)\right]\right|\right] \to 0$$

as $n \to \infty$ first and then $q \to \infty$.

Lemma 11.

$$\sum_{l=1}^{\infty} \int |\psi_{S_l}(\mu)| \left|\widehat{f}(\mu)\right|^2 d\mu < \infty,$$

where $\psi_{S_l}(\mu)$ is the characteristic function of S_l. In particular the quantity b defined in Theorem 1 is finite.

Noting that (R2) involves $\sum_{k=1}^{m} E_{[n\frac{k-1}{m}]}[\zeta_{nmk}^2]$, the preceding two lemmas allow us to concentrate, for each $q \geq 1$, on the limit of

$$\frac{\gamma_n}{n} \sum_{k=1}^{m} \sum_{l=l_0}^{n_{mk}} E_{[n\frac{k-1}{m}]} \left[f^2 \left(S_{[n\frac{k-1}{m}]+l} \right) \right]$$
$$+2\frac{\gamma_n}{n} \sum_{k=1}^{m} \sum_{l=l_0}^{n_{mk}} \sum_{r=1}^{q} E_{[n\frac{k-1}{m}]} \left[f\left(S_{[n\frac{k-1}{m}]+l} \right) f\left(S_{[n\frac{k-1}{m}]+l+r} \right) \right]$$

as $n \to \infty$ first and then $m \to \infty$. This purpose is served by the next two results.

Proposition 12.

$$\frac{\gamma_n}{n} \sum_{k=1}^{m} \sum_{l=l_0}^{n_{mk}} E_{[n\frac{k-1}{m}]} \left[f^2 \left(S_{[n\frac{k-1}{m}]+l} \right) \right] \Longrightarrow \widehat{f^2}(0)\, L_1^0$$

as $n \to \infty$ first and then $m \to \infty$.

Here note that

$$\widehat{f^2}(0) = \int f^2(x)\, dx = \frac{1}{2\pi} \int \left|\widehat{f}(\mu)\right|^2 d\mu,$$

where the last equality is obtained by Plancherel's theorem.

Proposition 13. *For each $r \geq 1$,*

$$\frac{\gamma_n}{n} \sum_{k=1}^{m} \sum_{l=1}^{n_{mk}} E_{[n\frac{k-1}{m}]} \left[f\left(S_{[n\frac{k-1}{m}]+l} \right) f\left(S_{[n\frac{k-1}{m}]+l+r} \right) \right] \Longrightarrow \left(\frac{1}{2\pi} \int \psi_{S_r}(\mu) \left|\widehat{f}(\mu)\right|^2 d\mu \right) L_1^0,$$

as $n \to \infty$ first and then $m \to \infty$.

In place of Proposition 13, we shall obtain the following more general result, from which Theorem 3 will also be obtained and which will also be required to verify (R*2).

Proposition 13*. *Let $w(u,v)$ be such that*

$$\int\int |w(x,y)|^i\, dxdy < \infty, \quad i = 1, 2, \quad \int \left(\int |w(x,y)|^2\, dy \right)^{\frac{1}{2}} dx < \infty.$$

Then for each $r \geq 1$,

$$\frac{\gamma_n}{n} \sum_{k=1}^{m} \sum_{l=1}^{n_{mk}} E_{[n\frac{k-1}{m}]} \left[w\left(S_{[n\frac{k-1}{m}]+l}, S_{[n\frac{k-1}{m}]+l+r} \right) \right] \Longrightarrow \left(\frac{1}{2\pi} \int \psi_{S_r}(\mu)\, \widehat{w}(-\mu,\mu)\, d\mu \right) L_1^0$$

as $n \to \infty$ first and then $m \to \infty$.

We start with

Proof of Lemma 11. In view of (58), $|\psi_{S_l}(\mu)| \le \left|\prod_{j=0}^{l-1} \psi(g(j)\mu)\right|$. Hence it is enough to show that

$$\sum_{l=l_0}^{\infty} \int \prod_{j=0}^{l} |\psi(g(j)\mu)| \left|\widehat{f}(\mu)\right|^2 d\mu = \sum_{l=l_0}^{\infty} \frac{1}{\gamma_l} \int \left|\prod_{j=0}^{l} \psi\left(g(j)\frac{\mu}{\gamma_l}\right)\right| \left|\widehat{f}\left(\frac{\mu}{\gamma_l}\right)\right|^2 d\mu \tag{63}$$

is finite, for a suitable l_0. Because $\left|\widehat{f}\left(\frac{\mu}{\gamma_l}\right)\right| \le C\left|\frac{\mu}{\gamma_l}\right|$ (see (62)), we have using (55) and (56),

$$\int \left|\prod_{j=0}^{l-1} \psi\left(g(j)\frac{\mu}{\gamma_l}\right)\right| \left|\widehat{f}\left(\frac{\mu}{\gamma_l}\right)\right|^2 d\mu \le \frac{C}{\gamma_l^2} + C\rho^l \le \frac{C}{\gamma_l^2}, \quad l \ge l_0,$$

for a suitable l_0 and for some $0 < \rho < 1$. Hence, (63) is bounded by $C\sum_{l=l_0}^{\infty} \frac{1}{\gamma_l^3}$, where note that $\sum_{l=l_0}^{\infty} \frac{1}{\gamma_l^3} < \infty$ when the assumed restriction $3H > 1$ holds. Hence the proof. ■

We now give some preliminaries. First recall from (58) that

$$S_{[n\frac{k-1}{m}]+l} = S_{[n\frac{k-1}{m}]+l,l} + \sum_{j=1}^{l} g(l-j)\,\xi_{[n\frac{k-1}{m}]+j}$$

where recall

$$S_{[n\frac{k-1}{m}]+l,l} = \sum_{j=-\infty}^{0} \left(g\left(\left[n\frac{k-1}{m}\right]+l-j\right) - g(1-j)\right)\xi_j + \sum_{j=1}^{[n\frac{k-1}{m}]} g\left(\left[n\frac{k-1}{m}\right]+l-j\right)\xi_j.$$

Here note that the r.h.s. involves the array $\left\{\xi_j : -\infty < j \le \left[n\frac{k-1}{m}\right]\right\}$ which does not depend on l. We observe that

- The vectors $\left\{S_{[n\frac{k-1}{m}]+l,l}; 1 < l \le n_{mk}\right\}$ and $\left\{\sum_{j=1}^{l} g(l-j)\,\xi_{[n\frac{k-1}{m}]+j}; 1 < l \le n_{mk}\right\}$ are independent. Further the distribution of $\left\{\sum_{j=1}^{l} g(l-j)\,\xi_{[n\frac{k-1}{m}]+j}; 1 < l \le n_{mk}\right\}$ is the same as that of $\{T_l; 1 < l \le n_{mk}\}$ where

$$T_l = \sum_{j=1}^{l} g(l-j)\,\xi_j.$$

Hence we can write

$$\begin{aligned} & E_{[n\frac{k-1}{m}]}\left[f\left(S_{[n\frac{k-1}{m}]+l}\right) f\left(S_{[n\frac{k-1}{m}]+l+r}\right)\right] \\ = \; & E\left[f(y_1+T_l)\, f(y_2+T_{l+r})\right]_{(y_1,y_2)=\left(S_{[n\frac{k-1}{m}]+l,l},\, S_{[n\frac{k-1}{m}]+l+r,l+r}\right)}. \end{aligned} \tag{64}$$

Letting, for any $0 \leq \nu_n < l$ (ν_n will also be allowed to tend to ∞ appropriately),

$$T_{nl}^* = \sum_{j=1}^{l-\nu_n} g(l-j)\xi_j, \qquad T_{nl,r}^* = \sum_{j=1}^{l-\nu_n} g(l+r-j)\xi_j,$$

we have

$$T_l = T_{nl}^* + \sum_{j=l-\nu_n+1}^{l} g(l-j)\xi_j, \qquad T_{l+r} = T_{nl,r}^* + \sum_{j=l-\nu_n+1}^{l+r} g(l+r-j)\xi_j.$$

(Note that T_{nl}^* and $T_{nl,r}^*$ depend on ν_n.) Hence, letting $\widehat{w}(\lambda,\mu)$ for the corresponding Fourier transform of $w(x_1,x_2)$, we have for any $0 \leq \nu_n < l$,

$$\begin{aligned}
& (2\pi)^2 E\left[w\left(y_1+T_l, y_2+T_{l+r}\right)\right] \\
= & \int e^{-i\lambda y_1 - i\mu y_2} E\left[e^{-i\lambda T_l - i\mu T_{l+r}}\right] \widehat{w}(\lambda,\mu)\, d\lambda d\mu \\
= & \int e^{-i\lambda y_1 - i\mu y_2} E\left[e^{-i(\lambda+\mu)T_{nl}^* - i\mu\left(T_{nl,r}^* - T_{nl}^*\right)}\right] E\left[e^{-i\lambda\left(T_l - T_{nl}^*\right) - i\mu\left(T_{l+r} - T_{nl,r}^*\right)}\right] \widehat{w}(\lambda,\mu)\, d\lambda d\mu \\
= & \frac{1}{\gamma_n} \int e^{-i\frac{\lambda}{\gamma_n} y_1 - i\mu(y_2-y_1)} E\left[e^{-i\lambda\frac{T_{nl}^*}{\gamma_n} - i\mu\left(T_{nl}^* - T_{nl,r}^*\right)}\right] E\left[e^{-i\frac{\lambda}{\gamma_n}\left(T_l - T_{nl}^*\right) - i\mu\left(T_{l+r} - T_{nl,r}^* - T_l + T_{nl}^*\right)}\right] \\
& \times \widehat{w}\left(\frac{\lambda}{\gamma_n} - \mu, \mu\right) d\lambda d\mu. \qquad (65)
\end{aligned}$$

Now (recall $g(j) = 0$ if $j < 0$)

$$\begin{aligned}
& E\left[e^{-i\frac{\lambda}{\gamma_n}\left(T_l - T_{nl}^*\right) - i\mu\left(T_{l+r} - T_{nl,r}^* - T_l + T_{nl}^*\right)}\right] \\
= & E\left[e^{-i\frac{\lambda}{\gamma_n}\left(T_l - T_{nl}^*\right) - i\mu\sum_{j=l-\nu_n+1}^{l+r}(g(l+r-j)-g(l-j))\xi_j}\right] \\
= & E\left[e^{-i\frac{\lambda}{\gamma_n}\left(T_l - T_{nl}^*\right) - i\mu\sum_{j=l-\nu_n+1}^{l}\left(c_{l+1-j}+\ldots+c_{l+r-j}\right)\xi_j}\right] \prod_{j=0}^{r-1} \psi\left(-g(j)\mu\right),
\end{aligned}$$

where and throughout below we let

$$c_j = 0 \text{ for } j < 0.$$

Similarly

$$E\left[e^{-i\lambda\frac{T_{nl}^*}{\gamma_n} - i\mu\left(T_{nl,r}^* - T_{nl}^*\right)}\right] = \prod_{j=\nu_n}^{l-1} \psi\left(\frac{\lambda g(j)}{\gamma_n} - \mu\left(c_{j+1}+\ldots+c_{j+r}\right)\right).$$

Hence

$$\begin{aligned}
& \left| E\left[e^{-i\lambda\frac{T_{nl}^*}{\gamma_n} - i\mu\left(T_{nl}^* - T_{nl,r}^*\right)}\right] E\left[e^{-i\frac{\lambda}{\gamma_n}\left(T_l - T_{nl}^*\right) - i\mu\left(T_{l+r} - T_{nl,r}^* - T_l + T_{nl}^*\right)}\right] \right| \\
\leq & \prod_{j=\nu_n}^{l-1} \left| \psi\left(\frac{\lambda g(j)}{\gamma_n} - \mu\left(c_{j+1}+\ldots+c_{j+r}\right)\right) \right| \prod_{j_1=0}^{r-1} \left|\psi\left(-g(j_1)\mu\right)\right|. \qquad (66)
\end{aligned}$$

With these preliminaries, we now consider the proof of Proposition 13* through a series of steps. (The proof of Lemma 10 will be given in the next section because it involves computations similar to those in the verification of (R3).) In order to state and prove the first step, we need the following result.

Lemma 14. *Let $f(x_0, ..., x_r)$, $r \geq 1$, be such that $\int \left(\int |f(x_0, ..., x_r)|^2 dx_r \right)^{\frac{1}{2}} dx_0 ... dx_{r-1} < \infty$. Then*

$$\sup_{\lambda_0, ... \lambda_{r-1}, c} \int \left| \widehat{f}(\lambda_0, ... \lambda_{r-2}, \lambda_{r-1} + c\mu, \mu) \right|^2 d\mu \leq \int \left(\int |f(x_0, ..., x_r)|^2 dx_r \right)^{\frac{1}{2}} dx_0 ... dx_{r-1} \leq C.$$

In particular for $w(x, y)$ as in Proposition 13,*

$$\sup_{c, \lambda} \int |\widehat{w}(\lambda + c\mu, \mu)|^2 d\mu \leq \int \left(\int |w(x, y)|^2 dy \right)^{\frac{1}{2}} dx \leq C.$$

Proof. We have by definition

$$\begin{aligned} & \widehat{f}(\lambda_0, ... \lambda_{r-2}, \lambda_{r-1} + c\mu, \mu) \\ = & \int e^{i\lambda_0 x_0 + ... + i\lambda_{r-2} x_{r-2} + i(\lambda_{r-1} + c\mu) x_{r-1} + i\mu x_r} f(x_0, ..., x_r) dx_0 ... dx_r \\ = & \int e^{i\mu x_r} \left\{ \int e^{i\lambda_0 x_0 + ... + i\lambda_{r-1} x_{r-1}} f(x_0, ..., x_{r-1}, x_r - c x_{r-1}) dx_0 ... dx_{r-1} \right\} dx_r. \end{aligned}$$

Then by Plancherel's theorem, for each $\lambda_0, ... \lambda_{r-1}, c$,

$$\begin{aligned} & \int \left| \widehat{f}(\lambda_0, ... \lambda_{r-2}, \lambda_{r-1} + c\mu, \mu) \right|^2 d\mu \\ = & \int \left| \int e^{i\lambda_0 x_0 + ... + i\lambda_{r-1} x_{r-1}} f(x_0, ..., x_{r-1}, x_r - c x_{r-1}) dx_0 ... dx_{r-1} \right|^2 dx_r \\ \leq & \int \left| \int |f(x_0, ..., x_{r-1}, x_r - c x_{r-1})|^2 dx_r \right|^{1/2} dx_0 ... dx_{r-1} \\ = & \int \left(\int |f(x_0, ..., x_{r-1}, x_r)|^2 dx_r \right)^{1/2} dx_0 ... dx_{r-1}, \end{aligned}$$

where in the second step we have used the generalized Minskowski inequality (see for instance Folland (1984, page 186)). This proves the result. ■

Lemma 15. *Let $w(x, y)$ be as in Proposition 13*, and let $T^*_{nl,r}$ and T^*_{nl} correspond to $2\nu_n < [n\delta]$, $0 < \delta < 1$*

Further, let $R_n(y_1, y_2, a, \delta)$ be the difference between

$$(2\pi)^2 \frac{\gamma_n}{n} \sum_{l=[n\delta]+1}^{n} E\left[w\left(y_1 + T_l, y_2 + T_{l+r} \right) \right] \tag{67}$$

and

$$\frac{1}{n}\sum_{l=[n\delta]+1}^{n}\int_{\{|\mu|\le a,|\lambda|\le a\}} U_n(\lambda,\mu,y_1,y_2)\, E\left[e^{-i\lambda\frac{T^*_{nl}}{\gamma_n}-i\mu\left(T^*_{nl,r}-T^*_{nl}\right)}\right]$$
$$\times E\left[e^{-i\frac{\lambda}{\gamma_n}\left(T_l-T^*_{nl}\right)-i\mu\left(T_{l+1}-T^*_{nl,r}-T_l+T^*_{nl}\right)}\right]\widehat{w}\left(\frac{\lambda}{\gamma_n}-\mu,\mu\right)d\lambda d\mu \tag{68}$$

where

$$U_n(\lambda,\mu,y_1,y_2)=e^{-i\lambda\gamma_n^{-1}y_1-i\mu(y_2-y_1)}.$$

Then

$$\lim_{a\to\infty}\limsup_{n\to\infty}\left(\sup_{y_1,y_2}|R_n(y_1,y_2,a,\delta)|\right)=0 \quad \text{for each } \delta>0.$$

Proof. Note that (67) involves the left hand side of the identity (65). Further when in (68) the $\int_{\{|\mu|\le a,|\lambda|\le a\}}$ is replaced by $\int_{R^2}$, it reduces to that involving the right hand side of (65). Therefore the difference $R_n(y_1,y_2,a,\delta)$ in the statement of the lemma is simply the same as (68) but with the integral $\int_{\{|\mu|\le a,|\lambda|\le a\}}$ replaced by the $\int_{\{|\mu|\le a,|\lambda|\le a\}^c}$, where $\{|\mu|\le a,|\lambda|\le a\}^c$ stands for the complement of $\{|\mu|\le a,|\lambda|\le a\}$. For notational simplification, we treat the case $r=1$. Then, using (66) and noting that $|\psi(\lambda)|\le 1$, $|U_n(\lambda,\mu,y_1,y_2)|\le C$, and $\{|\mu|\le a,|\lambda|\le a\}^c\subset\{|\mu|>a,|\lambda|<\infty\}\cup\{|\mu|\le a,|\lambda|>a\}$, we have

$$|R_n(y_1,y_2,a,\delta)|$$
$$\le \frac{1}{n}\sum_{l=[n\delta]+1}^{n}\int_{\{|\mu|>a,|\lambda|<\infty\}\cup\{|\mu|\le a,|\lambda|>a\}}\left|F\left(\frac{\lambda}{\gamma_n}-\mu,\mu\right)\right|\prod_{j=\nu_n}^{l-1}\left|\psi\left(\frac{\lambda g(j)}{\gamma_n}-\mu c_{j+1}\right)\right|d\lambda d\mu$$

where we have let

$$F\left(\frac{\lambda}{\gamma_n}-\mu,\mu\right)=\psi(-\mu)\,\widehat{w}\left(\frac{\lambda}{\gamma_n}-\mu,\mu\right).$$

Note that $\prod_{j=\nu_n}^{l-1}\left|\psi\left(\frac{\lambda g(j)}{\gamma_n}-\mu c_{j+1}\right)\right|\le\prod_{j=[l/2]}^{l-1}\left|\psi\left(\frac{\lambda g(j)}{\gamma_n}-\mu c_{j+1}\right)\right|$ because $\nu_n<[n\delta]/2\le[l/2]$.

Now using (52),

$$\int_{\{|\mu|>a,|\lambda|<\infty\}}\left|F\left(\frac{\lambda}{\gamma_n}-\mu,\mu\right)\prod_{j=[l/2]}^{l-1}\psi\left(\frac{\lambda g(j)}{\gamma_n}-\mu c_{j+1}\right)\right|d\lambda d\mu$$
$$\le\prod_{j=[l/2]}^{l-1}\left(\int_{\{|\mu|>a,|\lambda|<\infty\}}F\left(\frac{\lambda}{\gamma_n}-\mu,\mu\right)\left|\psi\left(\frac{\lambda g(j)}{\gamma_n}-\mu c_{j+1}\right)\right|^{l-[l/2]}d\lambda d\mu\right)^{\frac{1}{l-[l/2]}} \tag{69}$$

Here note that (making the change of variable $\frac{\lambda g(j)}{\gamma_n} - \mu c_{j+1} \longmapsto \frac{\lambda}{b_{[n\delta]}}$)

$$\begin{aligned}
&\int_{\{|\mu|>a,|\lambda|<\infty\}} F\left(\frac{\lambda}{\gamma_n}-\mu,\mu\right)\left|\psi\left(\frac{\lambda g(j)}{\gamma_n}-\mu c_{j+1}\right)\right|^{l-[l/2]} d\lambda d\mu \\
= &\ \frac{\gamma_n}{|g(j)|\, b_l}\int_{\{|\mu|>a,|\lambda|<\infty\}}\left|F\left(\frac{\lambda}{g(j)b_l}-\mu+\mu\frac{c_{j+1}}{g(j)},\mu\right)\right|\left|\psi\left(\frac{\lambda}{b_l}\right)\right|^{l-[l/2]} d\lambda d\mu \\
\leq &\ C\frac{\gamma_n}{\gamma_l}Q_n(a)\int\left|\psi\left(\frac{\lambda}{b_l}\right)\right|^{l-[l/2]} d\lambda \leq CQ_n(a)\frac{\gamma_n}{\gamma_l}
\end{aligned}$$

where

$$Q(a) = \max_{[n\delta]\leq j\leq n}\sup_v \int_{\{|\mu|>a\}}\left|F\left(v-\mu+\mu\frac{c_{j+1}}{g(j)},\mu\right)\right| d\mu$$

and we have used the facts $\max_{[l/2]\leq j\leq l}\frac{\gamma_l}{|g(j)|b_l}\leq C$ (see (54)) and $\int\left|\psi\left(\frac{\lambda}{b_l}\right)\right|^{l-[l/2]}d\lambda\leq C$ (see (50) and (51)). Note that

$$\begin{aligned}
\int_{\{|\mu|>a\}}\left|F\left(\nu-\mu+\mu\frac{c_{j+1}}{g(j)},\mu\right)\right| d\mu &\leq \sqrt{\int\left|\widehat{w}\left(\nu-\mu+\mu\frac{c_{j+1}}{g(j)},\mu\right)\right|^2 d\mu\int_{\{|\mu|>a\}}|\psi(\mu)|^2 d\mu} \\
&\leq C\left(\int_{\{|\mu|>a\}}|\psi(\mu)|^2 d\mu\right)^{1/2} \qquad (70)
\end{aligned}$$

where in the last step we have used Lemma 14. Thus

$$\begin{aligned}
&\int_{\{|\mu|>a,|\lambda|<\infty\}}\left|F\left(\frac{\lambda}{\gamma_n}-\mu,\mu\right)\prod_{j=[l/2]}^{l-1}\psi\left(\frac{\lambda g(j)}{\gamma_n}-\mu c_{j+1}\right)\right| d\lambda d\mu \\
\leq &\ C\frac{\gamma_n}{\gamma_l}\left(\int_{\{|\mu|>a\}}|\psi(\mu)|^2 d\mu\right)^{1/2}.
\end{aligned}$$

We also have $\prod_{j=\nu_n}^{l-1}\left|\psi\left(\frac{\lambda g(j)}{\gamma_n}-\mu c_{j+1}\right)\right| \leq \prod_{j=[[n\delta]/2]}^{[n\delta]}\left|\psi\left(\frac{\lambda g(j)}{\gamma_n}-\mu c_{j+1}\right)\right|$ because $\nu_n < [n\delta]/2 < l/2$. Hence in the same way as above

$$\begin{aligned}
&\int_{\{|\mu|\leq a,|\lambda|>a\}}\left|F\left(\frac{\lambda}{\gamma_n}-\mu,\mu\right)\prod_{j=[[n\delta]/2]}^{[n\delta]}\psi\left(\frac{\lambda g(j)}{\gamma_n}-\mu c_{j+1}\right)\right| d\lambda d\mu \\
\leq &\ CQ_n^*(a)\int_{\{|\lambda|>d_n a-e_n\}}\left|\psi\left(\frac{\lambda}{b_{[n\delta]}}\right)\right|^{[n\delta]-[[n\delta]/2]} d\lambda
\end{aligned}$$

where

$$e_n = ab_{[n\delta]}\max_{[[n\delta]/2]\leq j\leq[n\delta]}|c_{j+1}| \text{ and } d_n = \min_{[[n\delta]/2]\leq j\leq[n\delta]}\frac{\gamma_n}{|g(j)|\, b_{[n\delta]}}$$

and

$$\begin{aligned} Q_n^*(a) &= \max_{[[n\delta]/2]\le j\le [n\delta]} \sup_{\nu} \int_{\{|\mu|\le a\}} \left| F\left(\nu-\mu+\mu\frac{c_{j+1}}{g(j)},\mu\right)\right| d\mu \\ &\le C\left(\int_{\{|\mu|\le a\}} |\psi(\mu)|^2 d\mu\right)^{1/2} \le C, \qquad \text{similar to (70).} \end{aligned}$$

Note that $d_n \ge d > 0$ for some $d > 0$ (see (54)). In addition $e_n \to 0$. To see this, assume for simplicity that $b_n \sim n^{\frac{1}{\alpha}}$, and $c_j \sim j^{H-1-\frac{1}{\alpha}}$ in the case of assumption (A1). Noting that $H-1-\frac{1}{\alpha}<0$, we then have $e_n \sim Cn^{H-1}$. In the case of Assumption (A2), we have $|e_n| \le Cn^{\frac{1}{\alpha}-1}$ where $\frac{1}{\alpha}-1<0$ because $1<\alpha\le 2$.

Hence there is an n_0 such that $\{|\lambda| > d_n a - e_n\} \subset \left\{|\lambda| > \frac{d}{2}a\right\}$ for all $n \ge n_0$. Hence using (50) and (51)

$$\int_{\{|\lambda|>d_n a-e_n\}} \left|\psi\left(\frac{\lambda}{b_{[n\delta]}}\right)\right|^{[n\delta]-[[n\delta]/2]} d\lambda \le C\int_{\left\{|\lambda|>\frac{d}{2}a\right\}} e^{-a|\lambda|^c} d\lambda + C\rho^{[n\delta]},$$

where $0<\rho<1$. Thus $|R_n(y_1,y_2,a,\delta)|$ is bounded by

$$\le C\left(\int_{\{|\mu|>a\}} |\psi(\mu)|^2 d\mu\right)^{1/2} \left(\frac{\gamma_n}{n}\sum_{l=1}^{n}\frac{1}{\gamma_l}\right) + C\int_{\left\{|\lambda|>\frac{d}{2}a\right\}} e^{-a|\lambda|^c} d\lambda + C\rho^{[n\delta]}.$$

for all $n \ge n_0$. In view of (14) and the fact $\frac{\gamma_n}{n}\sum_{l=1}^{n}\frac{1}{\gamma_l} \le C$, this completes the proof. ■

Lemma 16. *Let $K_n(y_1,y_2,a,\delta)$, $\delta>0$, be the difference between*

$$(2\pi)^2 \frac{\gamma_n}{n}\sum_{l=l_0}^{n} E\left[w\left(y_1+T_l, y_2+T_{l+r}\right)\right] \tag{71}$$

and

$$\frac{1}{n}\sum_{l=[n\delta]}^{n} \int_{\{|\mu|\le a,|\lambda|\le a\}} U_n(\lambda,\mu,y_1,y_2)\, E\left[e^{-i\lambda\gamma_n^{-1}T_l}\right] \psi_{S_r}(-\mu)\,\widehat{w}(-\mu,\mu)\, d\lambda d\mu \tag{72}$$

where $U_n(\lambda,\mu,y_1,y_2) = e^{-i\lambda\gamma_n^{-1}y_1 - i\mu(y_2-y_1)}$ as in Lemma 15 (and $\psi_{S_r}(\mu) = E\left[e^{i\mu S_r}\right]$ as before). Then

$$\lim_{\delta\to 0}\lim_{a\to\infty}\limsup_{n\to\infty}\left(\sup_{y_1,y_2}|K_n(y_1,y_2,a,\delta)|\right) = 0.$$

Proof. According to (65) and (66), we have

$$\begin{aligned} & \left|E\left[w\left(y_1+T_l, y_2+T_{l+r}\right)\right]\right| \\ \le\ & \frac{C}{\gamma_l}\int \prod_{j=[l/2]}^{l-1} \left|\psi\left(\frac{\lambda g(j)}{\gamma_l} - \mu\left(c_{j+1}+\ldots+c_{j+r}\right)\right)\right| \left|\psi(-\mu)\right| \left|\widehat{w}\left(\frac{\lambda}{\gamma_l}-\mu,\mu\right)\right| d\lambda d\mu. \end{aligned}$$

According to the arguments contained in the in the first part of the preceding proof of Lemma 15, this is bounded by $\frac{C}{\gamma_l}$. Thus

$$\sup_{y_1,y_2}\left|\frac{\gamma_n}{n}\sum_{l=1}^{[n\delta]}E\left[f\left(y_1+T_l\right)f\left(y_2+T_{l+r}\right)\right]\right|\leq C\frac{\gamma_n}{n}\sum_{l=1}^{[n\delta]}\frac{1}{\gamma_l}.$$

Clearly this converges to 0 as $n\to\infty$ first and then $\delta\to 0$.

Hence, in view of Lemma 15, letting $R_n^*\left(y_1,y_2,a,\delta\right)$ for the difference between (71) and (68),

$$\lim_{\delta\to 0}\lim_{a\to\infty}\limsup_{n\to\infty}\left(\sup_{y_1,y_2}\left|R_n^*\left(y_1,y_2,a,\delta\right)\right|\right)=0.$$

Therefore, letting $R_n^{**}\left(y_1,y_2,a,\delta\right)$ for the difference between (68) and (72), it is enough to show that

$$\limsup_{n\to\infty}\left(\sup_{y_1,y_2}\left|R_n^{**}\left(y_1,y_2,a,\delta\right)\right|\right)=0 \text{ for each } a,\delta. \tag{73}$$

Note that without loss of generality, we can assume that ν_n, upon which $T_{nl,r}^*$ and T_{nl}^* of Lemma 15 depend, is such that $\nu_n\to\infty$ and $\frac{\nu_n}{n}\to 0$. Then, because $T_l-T_{nl}^*$ and $\sum_{s=0}^{\nu_n-1}g\left(s\right)\xi_s$ have the same distribution,

$$\sup_{[n\delta]\leq l\leq n}P\left(\gamma_n^{-1}\left|T_l-T_{nl}^*\right|>\epsilon\right)=P\left(\left|\gamma_n^{-1}\sum_{s=0}^{\nu_n-1}g\left(s\right)\xi_s\right|>\epsilon\right)\to 0,$$

where we have used the fact that $\gamma_{\nu_n}^{-1}\sum_{s=0}^{\nu_n-1}g\left(s\right)\xi_s$ converges in distribution and $\gamma_n^{-1}\gamma_{\nu_n}\to 0$. Hence

$$\sup_{|\lambda|\leq a,|\mu|\leq a,[n\delta]\leq l\leq n}\left|E\left[e^{-i\frac{\lambda}{\gamma_n}\left(T_l-T_{nl}^*\right)-i\mu\left(T_{l+1}-T_{nl,r}^*-T_l+T_{nl}^*\right)}\right]-E\left[e^{-i\mu\left(T_{l+r}-T_{nl,r}^*-T_l+T_{nl}^*\right)}\right]\right|\to 0.$$

Further, noting that

$$E\left[e^{-i\mu\left(T_{l+r}-T_{nl,r}^*-T_l+T_{nl}^*\right)}\right]=\prod_{j=0}^{\nu_n+r-1}\psi\left(-\left(c_j+...+c_{j-(r-1)}\right)\mu\right)$$

and

$$\prod_{j=0}^{\infty}\psi\left(-\left(c_j+...+c_{j-(r-1)}\right)\mu\right)=\psi_{S_r}\left(-\mu\right) \tag{74}$$

we have (with r being fixed), because $\nu_n\to\infty$,

$$\sup_{|\mu|\leq a}\left|E\left[e^{-i\mu\left(T_{l+r}-T_{nl,r}^*-T_l+T_{nl}^*\right)}\right]-\psi_{S_r}\left(-\mu\right)\right|\to 0.$$

Now$\left|T^*_{nl,r}-T^*_{nl}\right|=\left|\sum_{j=\nu_n}^{l-1}(c_{j+1}+...+c_{j+r})\xi_j\right|$. Let $0<\tau<\alpha$ be suitably close to α such that $\sum_{j=\nu_n}^{\infty}|c_j|^\tau\to 0$. Then

$$\begin{aligned}\sup_{[n\delta]\le l<\infty} P\left(\left|T^*_{nl,r}-T^*_{nl}\right|>\varepsilon\right) &= \sup_{[n\delta]\le l<\infty} P\left(\left|\sum_{j=\nu_n}^{l-1}(c_{j+1}+...+c_{j+r})\xi_j\right|>\varepsilon\right)\\ &\le Cr\left(1+\varepsilon^{-2}\right)\sum_{j=\nu_n}^{\infty}|c_j|^\tau\to 0, \qquad (75)\end{aligned}$$

where the inequality is obtained using for instance Kasahara and Maejima (1988, Theorem 2.2). Hence

$$\sup_{|\lambda|\le b,|\mu|\le a,[n\delta]\le l\le n}\left|E\left[e^{-i\lambda\frac{T^*_{nl}}{\gamma_n}-i\mu\left(T^*_{nl,r}-T^*_{nl}\right)}\right]-E\left[e^{-i\lambda\frac{T_l}{\gamma_n}}\right]\right|\to 0 \text{ as } n\to\infty$$

Hence (73) follows. This completes the proof of the Lemma. ■

The preceding Lemma 16 leads to the next statement where we define

$$\begin{aligned}S\left(\frac{k-1}{m},\frac{t}{m}\right) &= c\int_{-\infty}^{0}\left\{\left(\frac{t}{m}+\frac{k-1}{m}-u\right)^{H-1/\alpha}-(-u)^{H-1/\alpha}\right\}Z_\alpha(du)\\ &\quad +c\int_{0}^{\frac{k-1}{m}}\left(\frac{t}{m}+\frac{k-1}{m}-u\right)^{H-1/\alpha}Z_\alpha(du) \qquad (76)\end{aligned}$$

and

$$T(t)=\int_0^t (t-u)^{H-1/\alpha}Z_\alpha(du). \qquad (77)$$

Note that

$$S\left(\frac{k-1}{m},0\right)=\Lambda_{\alpha,H}\left(\frac{k-1}{m}\right).$$

Lemma 17. *For each integer $m\ge 1$,*

$$\frac{\gamma_n}{n}\sum_{k=1}^{m}\sum_{l=1}^{n_{mk}}E_{\left[n\frac{k-1}{m}\right]}\left[w\left(S_{\left[n\frac{k-1}{m}\right]+l},S_{\left[n\frac{k-1}{m}\right]+l+r}\right)\right]$$

converges in distribution to

$$\left(\frac{1}{2\pi}\int\psi_{S_r}(\mu)\,\widehat{w}(-\mu,\mu)\,d\mu\right)\frac{1}{m^{1-H}}\sum_{k=1}^{m}\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_0^1 e^{-i\lambda m^H S\left(\frac{k-1}{m},\frac{t}{m}\right)}E\left[e^{-i\lambda T(t)}\right]dtd\lambda$$

where $S\left(\frac{k-1}{m},\frac{t}{m}\right)$ and $T(t)$ are as defined above in (76) and (77).

Proof. Because $\frac{\gamma_{n_{mk}}}{n_{mk}}\frac{n}{\gamma_n}\sim m^{1-H}$, it is enough to show that, for each m and k,

$$\frac{\gamma_{n_{mk}}}{n_{mk}}\sum_{l=1}^{n_{mk}}E_{\left[n\frac{k-1}{m}\right]}\left[w\left(S_{\left[n\frac{k-1}{m}\right]+l},S_{\left[n\frac{k-1}{m}\right]+l+r}\right)\right] \qquad (78)$$

converges in distribution to

$$\left(\frac{1}{2\pi}\int\psi_{S_r}(\mu)\,\widehat{w}(-\mu,\mu)\,d\mu\right)\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{0}^{1}e^{-i\lambda m^H S_{mk}\left(\frac{t}{m}\right)}E\left[e^{-i\lambda T(t)}\right]dtd\lambda. \tag{79}$$

Let (y_1, y_2) be as in (64), that is

$$(y_1,y_2)=\left(S_{\left[n\frac{k-1}{m}\right]+l,l},S_{\left[n\frac{k-1}{m}\right]+l+r,l+r}\right). \tag{80}$$

With this (y_1, y_2), let $R_n(a,\delta)$ be the difference between (78) and

$$\frac{1}{n_{mk}(2\pi)^2}\sum_{l=[n\delta]}^{n_{mk}}\int_{\{|\lambda|\le a,|\mu|\le a\}}U_n(\lambda,\mu,y_1,y_2)\,E\left[e^{-i\lambda\gamma_{n_{mk}}^{-1}T_l}\right]\psi_{S_r}(-\mu)\,\widehat{w}(-\mu,\mu)\,d\lambda d\mu, \tag{81}$$

where now

$$U_n(\lambda,\mu,y_1,y_2)=e^{-i\lambda\gamma_{n_{mk}}^{-1}y_1-i\mu(y_2-y_1)}.$$

It follows from Lemma 16 that, for each $\epsilon>0$,

$$\lim_{\delta\to0}\lim_{a\to\infty}\limsup_{n\to\infty}P\left(|R_n(a,\delta)|>\epsilon\right)=0.$$

Therefore it is enough to show that (81) converges in distribution to (79) by taking the limit as $n\to\infty$ first, then $a\to\infty$ and then $\delta\to0$.

To obtain the limit as $n\to\infty$, note that $U_n(\lambda,\mu,y_1,y_2)$ above involves

$$\gamma_{n_{mk}}^{-1}y_1=\gamma_{n_{mk}}^{-1}S_{\left[n\frac{k-1}{m}\right]+l,l}\quad\text{and}\quad y_2-y_1=S_{\left[n\frac{k-1}{m}\right]+l+r,l+r}-S_{\left[n\frac{k-1}{m}\right]+l,l}.$$

Then, with $S_{mk}\left(\frac{t}{m}\right)$ and $T(t)$ as defined in (76) and (77),

$$\left(\gamma_{n_{mk}}^{-1}S_{\left[n\frac{k-1}{m}\right]+[n_{mk}t],[n_{mk}t]},\gamma_{n_{mk}}^{-1}T_{[n_{mk}t]}\right)\Longrightarrow\left(m^H S\left(\frac{k-1}{m},\frac{t}{m}\right),T(t)\right).$$

Further note that

$$S_{\left[n\frac{k-1}{m}\right]+l+r,l+r}-S_{\left[n\frac{k-1}{m}\right]+l,l}=\sum_{j=-\infty}^{\left[n\frac{k-1}{m}\right]}(c_{l+\left[n\frac{k-1}{m}\right]+1-j}+..+c_{l+\left[n\frac{k-1}{m}\right]+r-j})\xi_j,$$

and hence, similar to (75),

$$\begin{aligned}&\sup_{\tau_n<l<\infty}P\left(\left|S_{\left[n\frac{k-1}{m}\right]+l+r,l+r}-S_{\left[n\frac{k-1}{m}\right]+l,l}\right|>\varepsilon\right)\\=&\sup_{\tau_n<l<\infty}P\left(\left|\sum_{i=l}^{\infty}(c_{i+1}+..+c_{i+r})\xi_i\right|>\varepsilon\right)\to0\quad\text{for any }\tau_n\uparrow\infty.\end{aligned}$$

It then follows in the same way as in Jeganathan (2004a, Lemma 8) that (81) with (y_1, y_2) as in (80) converges in distribution to

$$\frac{1}{(2\pi)^2}\int_{\{|\lambda|\le a, |\mu|\le a\}}\left\{\int_\delta^1 e^{-i\lambda m^H S\left(\frac{k-1}{m},\frac{t}{m}\right)} E\left[e^{-i\lambda T(t)}\right] dt\right\}\psi_{S_r}(-\mu)\,\widehat{w}(-\mu,\mu)\,d\lambda d\mu$$

for each a and $\delta > 0$. Let $K(a)$ be the difference between this and

$$\left(\frac{1}{2\pi}\int \psi_{S_r}(\mu)\,\widehat{w}(-\mu,\mu)\,d\mu\right)\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_\delta^1 e^{-i\lambda m^H S\left(\frac{k-1}{m},\frac{t}{m}\right)} E\left[e^{-i\lambda T(t)}\right] dt d\lambda.$$

(Here m, k and δ are fixed.)

Then noting that $\left|e^{-i\lambda m^H S_{mk}\left(\frac{t}{m}\right)}\right| \le 1$ and $|\psi_{S_r}(\mu)| \le |\psi(\mu)|$, we have

$$\begin{aligned}(2\pi)^2 K(a) \;\le\; & \left(\int_{-\infty}^{\infty}|\widehat{w}(-\mu,\mu)|\,|\psi(\mu)|\,d\mu\right)\int_\delta^1\int_{\{|\lambda|>a\}}\left|E\left[e^{-i\lambda T(t)}\right]\right| d\lambda dt\\ &+\left(\int_{\{|\mu|>a\}}|\widehat{w}(-\mu,\mu)|\,|\psi(\mu)|\,d\mu\right)\int_\delta^1\int_{-\infty}^{\infty}\left|E\left[e^{-i\lambda T(t)}\right]\right| d\lambda dt.\end{aligned}$$

Now note that

$$\begin{aligned}\int_{\{|\lambda|>a\}}\left|E\left[e^{-i\lambda T(t)}\right]\right| d\lambda \;&\le\; C\int_{\{|\lambda|>a\}} e^{-c\left|\lambda t^H\right|^\alpha} d\lambda\\ &=\; Ct^{-H}\int_{\{|\lambda|>at^H\}} e^{-c|\lambda|^\alpha} d\lambda\\ &\le\; C\delta^{-H}\int_{\{|\lambda|>a\delta^H\}} e^{-c|\lambda|^\alpha} d\lambda = R(a)\text{, say, if } \delta\le t\le 1.\end{aligned}$$

Hence

$$(2\pi)^2 K(a) \le R(a)\int_{-\infty}^{\infty}|\widehat{w}(-\mu,\mu)|\,|\psi(\mu)|\,d\mu + R(0)\int_{\{|\mu|>a\}}|\widehat{w}(-\mu,\mu)|\,|\psi(\mu)|\,d\mu,$$

where note that $R(a)\to 0$ as $a\to\infty$ and $R(0)<\infty$. In addition

$$\begin{aligned}\int_{\{|\mu|>a\}}|\widehat{w}(-\mu,\mu)|\,|\psi(\mu)|\,d\mu \;&\le\; \sqrt{\int|\widehat{w}(-\mu,\mu)|^2 d\mu\int_{\{|\mu|>a\}}|\psi(\mu)|^2 d\mu}\\ &\le\; C\sqrt{\int_{\{|\mu|>a\}}|\psi(\mu)|^2 d\mu},\end{aligned}$$

where we have used $\int|\widehat{w}(-\mu,\mu)|^2 d\mu \le C$, see Lemma 14. Thus $K(a)\to 0$ as $a\to\infty$. Next note that

$$\begin{aligned}\left|\int_{-\infty}^{\infty}\int_0^\delta e^{-i\lambda m^H S_{mk}\left(\frac{t}{m}\right)} E\left[e^{-i\lambda T(t)}\right] dt d\lambda\right| \;&\le\; C\left(\int_0^\delta t^{-H} dt\right)\left(\int_{-\infty}^{\infty} e^{-c|\lambda|^\alpha} d\lambda\right)\\ &\le\; \frac{C\delta^{1-H}}{1-H}\to 0 \text{ as } \delta\to 0.\end{aligned}$$

This completes the proof of the lemma. ■

To complete the proof of Proposition 13*, it thus remains to obtain

Lemma 18.

$$\frac{1}{m^{1-H}}\sum_{k=1}^{m}\int_0^1\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\lambda m^H S\left(\frac{k-1}{m},\frac{t}{m}\right)}E\left[e^{-i\lambda T(t)}\right]d\lambda\right]dt \Longrightarrow L_1^0 \text{ as } m\to\infty.$$

Proof. We first show that

$$\lim_{\delta\to 0}\lim\sup_{m\to\infty}\int_0^\delta \frac{1}{m^{1-H}}E\left[\left|\sum_{k=1}^{m}\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\lambda m^H S\left(\frac{k-1}{m},\frac{t}{m}\right)}E\left[e^{-i\lambda T(t)}\right]d\lambda\right|\right]dt = 0. \tag{82}$$

To see this note that, in view of (76), $S\left(\frac{k-1}{m},\frac{t}{m}\right)$ is α-stable with scale parameter σ_{tmk} such that

$$\sigma_{tmk}\geq C\left|\frac{t}{m}+\frac{k-1}{m}\right|^H.$$

(See Samorodnitsky and Taqqu (1994, page 345)). Hence

$$\begin{aligned}
E\left[\left|\frac{1}{m^{1-H}}\sum_{k=2}^{m}\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\lambda m^H S\left(\frac{k-1}{m},\frac{t}{m}\right)}d\lambda\right|\right] &\leq \frac{1}{m}\sum_{k=2}^{m}\int\left|E\left[e^{i\lambda S\left(\frac{k-1}{m},\frac{t}{m}\right)}\right]\right|d\lambda\\
&\leq \frac{1}{m}\sum_{k=2}^{m}\frac{1}{\sigma_{mk}}\int e^{-c|\lambda|^\alpha}d\lambda\\
&\leq \frac{C}{m}\sum_{k=2}^{m}\left(\frac{m}{k-1}\right)^H\int e^{-c|\lambda|^\alpha}d\lambda\leq C
\end{aligned}$$

because $\frac{1}{m}\sum_{k=2}^{m}\left(\frac{m}{k-1}\right)^H\leq C$. Here note that in the sum $\sum_{k=2}^{m}$ the leading term corresponding to $k=1$ is left out, but for this we have, in the same way as above, noting $\sigma_{tm1}\geq C\left|\frac{t}{m}\right|^H$,

$$E\left[\left|\frac{1}{m^{1-H}}\int_{-\infty}^{\infty}e^{-i\lambda m^H S\left(0,\frac{t}{m}\right)}d\lambda\right|\right]\leq C\frac{t^{-H}}{m^{1-H}}.$$

Hence,

$$\begin{aligned}
&\int_0^\delta E\left[\left|\frac{1}{m^{1-H}}\sum_{k=1}^{m}\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\lambda m^H S\left(\frac{k-1}{m},\frac{t}{m}\right)}d\lambda\right|\right]dt\\
\leq\ & C\int_0^\delta\left(1+\frac{t^{-H}}{m^{1-H}}\right)dt = C\left(\frac{\delta^{1-H}}{m^{1-H}}+\delta\right).
\end{aligned}$$

Hence, noting that $\left|E\left[e^{-i\lambda T(t)}\right]\right|\leq 1$, (82) follows.

Now consider

$$\begin{aligned}
&\int_\delta^1\frac{1}{m^{1-H}}\sum_{k=1}^{m}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\lambda m^H S\left(\frac{k-1}{m},\frac{t}{m}\right)}E\left[e^{-i\lambda T(t)}\right]d\lambda\right]dt\\
=\ &\int_\delta^1\frac{1}{m^{1-H}}\sum_{k=1}^{m}h_t\left(-m^H S\left(\frac{k-1}{m},\frac{t}{m}\right)\right)dt
\end{aligned}\tag{83}$$

where $h_t(y) \geq 0$ is the density function of $T(t)$, i.e.,

$$h_t(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\lambda y} \widehat{h}_t(\lambda)\, d\lambda \quad \text{where} \quad \widehat{h}_t(\lambda) = E\left[e^{-i\lambda T(t)}\right].$$

Note that for each fixed t, $\left\{S\left(\frac{k-1}{m}, \frac{t}{m}\right), 0 \leq k \leq m\right\}$ has the same structure as that of $\left\{\Lambda_{\alpha,H}\left(\frac{k}{m}\right), 0 \leq k \leq m\right\}$. Hence Jeganathan (2004a, Proposition 6) contains the fact that the difference between the integrand $\frac{1}{m^{1-H}} \sum_{k=1}^{m} h_t\left(-m^H S\left(\frac{k-1}{m}, \frac{t}{m}\right)\right)$ in (83) and

$$\frac{1}{m^{1-H}} \sum_{k=1}^{m} \frac{1}{\sqrt{2\pi}} \int h_t\left(-m^H\left(S\left(\frac{k-1}{m}, \frac{t}{m}\right) + \varepsilon z\right)\right) e^{-z^2/2} dz \tag{84}$$

converges to 0 in mean-square, as $m \to \infty$ first and then $\varepsilon \to 0$. In addition it is easy to see that the arguments in Jeganathan (2004a) also give that this mean-square convergence is uniform over $\delta \leq t \leq 1$. (Note that this is a very specific case so that the steps in Jeganathan (2004a) will take a rather simple and direct form.)

Now, note that $\frac{1}{m^{1-H}} \sum_{k=1}^{m} \int h_t\left(-m^H(y + \varepsilon z)\right) e^{-z^2/2} dz$ is sufficiently smooth in y (see Jeganathan (2004a, Lemma 7)). Hence, for each $\varepsilon > 0$, it can be seen that (84) can be approximated, as $m \to \infty$, by

$$\frac{1}{m^{1-H}} \sum_{k=1}^{m} \frac{1}{\sqrt{2\pi}} \int h_t\left(-m^H\left(S\left(\frac{k-1}{m}, 0\right) + \varepsilon z\right)\right) e^{-z^2/2} dz$$

uniformly over $\delta \leq t \leq 1$, which in turn is approximated by $\frac{1}{m^{1-H}} \sum_{k=1}^{m} \frac{1}{\sqrt{2\pi}} h_t\left(-m^H S\left(\frac{k-1}{m}, 0\right)\right)$ as before as $m \to \infty$ first and then $\varepsilon \to 0$.

Noting that $S\left(\frac{k-1}{m}, 0\right) = \Lambda_{\alpha,H}\left(\frac{k-1}{m}\right)$, we thus have approximated (83) by

$$\int_{\delta}^{1} \frac{1}{m^{1-H}} \sum_{k=1}^{m} h_t\left(-m^H \Lambda_{\alpha,H}\left(\frac{k-1}{m}\right)\right) dt,$$

which in turn is approximated as before, as $m \to \infty$ first and then $\delta \to 0$, by

$$\begin{aligned} & \int_{0}^{1} \frac{1}{m^{1-H}} \sum_{k=1}^{m} h_t\left(-m^H \Lambda_{\alpha,H}\left(\frac{k-1}{m}\right)\right) dt \\ = \; & \frac{1}{m^{1-H}} \sum_{k=1}^{m} g\left(-m^H \Lambda_{\alpha,H}\left(\frac{k-1}{m}\right)\right) \Longrightarrow \left(\int g(y)\, dy\right) L_1^0 = L_1^0 \end{aligned}$$

where $g(y) = \int_0^1 h_t(y)\, dt$. Note that $\int g(y)\, dy = \int_0^1 \int h_t(y)\, dy dt = 1$ because $\int h_t(y)\, dy = 1$ for each t. In obtaining this convergence we have used Jeganathan (2004a, Theorem 4). Note that $\int g^2(y)\, dy \leq \int_0^1 \int h_t^2(y)\, dy dt \leq C \int_0^1 t^{-H} dt \leq C$. This completes the proof of the lemma, and hence the proof of Proposition 13* is completed. ■

Proof of Proposition 12. It is implicit in the proofs of Lemmas 15 and 16 that the difference between $\frac{\gamma_n}{n}\sum_{k=1}^{m}\sum_{l=1}^{n_{mk}} E_{\left[n\frac{k-1}{m}\right]}\left[f^2\left(S_{\left[n\frac{k-1}{m}\right]+l}\right)\right]$ and

$$\widehat{f^2}(0)\frac{1}{m^{1-H}}\sum_{k=1}^{m}\frac{1}{n_{mk}2\pi}\sum_{l=[n\delta]}^{n_{mk}}\int_{\{|\lambda|\le a\}} e^{-i\lambda\gamma_{n_{mk}}^{-1}S_{\left[n\frac{k-1}{m}\right]+l,l}} E\left[e^{-i\lambda n_{mk}^{-H}T_l}\right]d\lambda$$

converges to 0 in probability as $n\to\infty$ first, then $a\to\infty$ and then $\delta\to 0$, which in turn converges in distribution to $\widehat{f^2}(0)\frac{1}{m^{1-H}}\sum_{k=1}^{m}\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_0^1 e^{-i\lambda m^H S\left(\frac{k-1}{m},\frac{t}{m}\right)}E\left[e^{-i\lambda T(t)}\right]dtd\lambda$, see Lemma 17. Hence the proof follows by Lemma 18. ■

Having verified (R2), we now show the same holds for (R*2) also except for some modifications.

Verification of (R*2). To indicate the required modifications, note that

$$\begin{aligned}
& E_{\left[n\frac{k-1}{m}\right]}\left[\left|\zeta_{nmk}^*\right|^2\right] \\
=\; & \frac{\gamma_n}{n}\sum_{l=l_0}^{n_{mk}} E_{\left[n\frac{k-1}{m}\right]}\left[f^2\left(S_{\left[n\frac{k-1}{m}\right]+l}\right)\omega^2_{\left[n\frac{k-1}{m}\right]+l}\right] \\
& +2\frac{\gamma_n}{n}\sum_{l=l_0}^{n_{mk}}\sum_{r=1}^{n_{mk}-l} E_{\left[n\frac{k-1}{m}\right]}\left[f\left(S_{\left[n\frac{k-1}{m}\right]+l}\right)\omega_{\left[n\frac{k-1}{m}\right]+l}f\left(S_{\left[n\frac{k-1}{m}\right]+l+r}\right)\omega_{\left[n\frac{k-1}{m}\right]+l+r}\right]
\end{aligned}$$

where recall that $\omega_q=\sum_{j=q-\nu+1}^{q} d_{q-j}\eta_j=\eta_q+d_1\eta_{q-1}+\ldots+d_{\nu-1}\eta_{q-\nu+1}$. We have

$$E_{\left[n\frac{k-1}{m}\right]}\left[f^2\left(S_{\left[n\frac{k-1}{m}\right]+l}\right)\omega^2_{\left[n\frac{k-1}{m}\right]+l}\right]=E_{\left[n\frac{k-1}{m}\right]}\left[g\left(S_{\left[n\frac{k-1}{m}\right]+l,\nu}\right)\right]$$

where (recall $S_{v,\nu}^*=\sum_{j=0}^{\nu-1}g(j)\xi_{\nu-j}$)

$$g(x)=E\left[f^2\left(x+S_{\nu,\nu}^*\right)\omega_\nu^2\right].$$

Thus, noting that $\int g(x)\,dx=E\left[\omega_\nu^2\right]\int|f(x)|^2dx=E\left[\omega_\nu^2\right]\frac{1}{2\pi}\int\left|\widehat{f}(\mu)\right|^2d\mu$, together with the fact that $S_{\left[n\frac{k-1}{m}\right]+l,\nu}$ has the same structure as that of $S_{\left[n\frac{k-1}{m}\right]+l}$ so that Proposition 12 becomes essentially applicable, we have

$$\frac{\gamma_n}{n}\sum_{l=l_0}^{n_{mk}} E_{\left[n\frac{k-1}{m}\right]}\left[f^2\left(S_{\left[n\frac{k-1}{m}\right]+l}\right)\omega^2_{\left[n\frac{k-1}{m}\right]+l}\right]\Longrightarrow\left(E\left[\omega_\nu^2\right]\frac{1}{2\pi}\int\left|\widehat{f}(\mu)\right|^2d\mu\right)L_1^0.$$

To deal with the remaining sums, suppose that $r\ge\nu$. Let (recall $S_{q+r,\nu}^*=S_{q+r}-S_{q+r,\nu}$.)

$$S_{q+r,\nu}^{\#}=S_{q+r}-\left(S_{q+r,r+\nu}^*-S_{q+r,r}^*\right)-S_{q+r,\nu}^*.$$

Note that $\left(S_{q,\nu},S_{q+r,\nu}^{\#}\right)$ is independent of $\left(S_{q,\nu}^*,\left(S_{q+r,r+\nu}^*-S_{q+r,r}^*\right)+S_{q+r,\nu}^*\right)$. Then

$$\begin{aligned}
& E_{q-\nu}\left[f\left(S_q\right)\omega_q f\left(S_{q+r}\right)\omega_{q+r}\right] \\
=\; & E_{q-\nu}\left[f\left(S_{q,\nu}+S_{q,\nu}^*\right)\omega_q f\left(S_{q+r,\nu}^{\#}+\left(S_{q+r,r+\nu}^*-S_{\nu+r,r}^*\right)+S_{q+r,\nu}^*\right)\omega_{q+r}\right] \\
=\; & E_{q-\nu}\left[w_r\left(S_{q,\nu},S_{q+r,\nu}^{\#}\right)\right]
\end{aligned}$$

where

$$w_r(x,y) = E\left[f\left(x + S^*_{\nu,\nu}\right)\omega_\nu f\left(y + \left(S^*_{\nu+r,r+\nu} - S^*_{\nu+r,r}\right) + S^*_{v+r,\nu}\right)\omega_{\nu+r}\right].$$

Thus, when $r \geq \nu$ and $l > \nu$,

$$\begin{aligned} & E_{\left[n\frac{k-1}{m}\right]}\left[f\left(S_{\left[n\frac{k-1}{m}\right]+l}\right)\omega_{\left[n\frac{k-1}{m}\right]+l} f\left(S_{\left[n\frac{k-1}{m}\right]+l+r}\right)\omega_{\left[n\frac{k-1}{m}\right]+l+r}\right] \\ = \; & E_{\left[n\frac{k-1}{m}\right]}\left[w_r\left(S_{\left[n\frac{k-1}{m}\right]+l,\nu}, S^{\#}_{\left[n\frac{k-1}{m}\right]+l+r,\nu}\right)\right]. \end{aligned} \tag{85}$$

In the case $r < \nu$, the right hand side here takes the form

$$E_{\left[n\frac{k-1}{m}\right]}\left[w^*_r\left(S_{\left[n\frac{k-1}{m}\right]+l,\nu}, S_{\left[n\frac{k-1}{m}\right]+l+r,\nu+r}\right)\right] \tag{86}$$

with

$$w^*_r(x,y) = E\left[f\left(x + S^*_{\nu,\nu}\right)\omega_\nu f\left(y + S^*_{\nu+r,r+\nu}\right)\omega_{\nu+r}\right].$$

Now, in the case $r \geq \nu$, $\left(S^*_{\nu,\nu}, \left(S^*_{\nu+r,r+\nu} - S^*_{\nu+r,r}\right)\right)$ is independent of $S^*_{\nu+r,\nu}$, and hence we have

$$\widehat{w_r}(\lambda,\mu) = \widehat{f}(\lambda)\,\widehat{f}(\mu)\,E\left[\omega_\nu e^{-i\lambda S^*_{\nu,\nu} - i\mu\left(S^*_{\nu+r,r+\nu} - S^*_{\nu+r,r}\right)}\right] E\left[\omega_{\nu+r} e^{-i\mu S^*_{\nu+r,\nu}}\right].$$

Here, noting that $S^*_{\nu,\nu} = \sum_{j=0}^{\nu-1} g(j)\xi_{\nu-j}$,

$$\begin{aligned} E\left[\omega_{\nu+r} e^{-i\mu S^*_{\nu+r,\nu}}\right] &= E\left[\omega_\nu e^{-i\mu S^*_{\nu,\nu}}\right] = \sum_{i=1}^{\nu} d_{\nu-i} E\left[\eta_i e^{-i\mu S^*_{\nu,\nu}}\right] \\ &= \sum_{i=1}^{\nu} d_{\nu-i} E\left[\eta_i e^{-i\mu g(\nu-i)\xi_i}\right] \prod_{j=0, j\neq\nu-i}^{\nu-1} \psi\left(-g(j)\mu\right). \end{aligned}$$

Noting $S^*_{\nu+r,r+\nu} - S^*_{\nu+r,r} = \sum_{j=0}^{\nu+r-1} g(j)\xi_{\nu+r-j} - \sum_{j=0}^{r-1} g(j)\xi_{\nu+r-j} = \sum_{j=0}^{\nu-1} g(j+r)\xi_{\nu-j}$, we similarly have

$$\begin{aligned} & E\left[\omega_\nu e^{-i\lambda S^*_{\nu,\nu} - i\mu\left(S^*_{\nu+r,r+\nu} - S^*_{\nu+r,r}\right)}\right] \\ = \; & \sum_{i=1}^{\nu} d_{\nu-i} E\left[\eta_i e^{(-i\lambda g(\nu-i) - i\mu g(\nu-i+r))\xi_i}\right] \prod_{j=0, j\neq\nu-i}^{\nu-1} \psi\left(-g(j)\lambda - g(j+r)\mu\right). \end{aligned}$$

In the case $\nu = 1$, note that the preceding two quantities give (recall $d_0 = 1$ and $g(0) = 1$)

$$\widehat{w_r}(\lambda,\mu) = \widehat{f}(\lambda)\,\widehat{f}(\mu)\,E\left[\eta_1 e^{-i\mu\xi_1}\right] E\left[\eta_1 e^{(-i\lambda - i\mu g(r))\xi_1}\right].$$

In the case $0 < r < \nu$, we have

$$\begin{aligned} \widehat{w^*_r}(\lambda,\mu) &= \widehat{f}(\lambda)\,\widehat{f}(\mu)\,E\left[\omega_\nu \omega_{\nu+r} e^{-i\lambda S^*_{\nu,\nu} - i\mu S^*_{\nu+r,\nu+r}}\right] \\ &= \widehat{f}(\lambda)\,\widehat{f}(\mu)\,E\left[\omega_\nu \omega_{\nu+r} e^{-i\lambda \sum_{j=1}^{\nu} g(\nu-j)\xi_j - i\mu \sum_{j=1}^{\nu+r} g(\nu+r-j)\xi_j}\right]. \end{aligned}$$

Now, regarding the analogues of Proposition 13* for the sums of (85) and (86) (note that Proposition 13* involves the sum of $E_{\left[n\frac{k-1}{m}\right]}\left[w\left(S_{\left[n\frac{k-1}{m}\right]+l}, S_{\left[n\frac{k-1}{m}\right]+l+r}\right)\right]$), we note that $S_{\left[n\frac{k-1}{m}\right]+l,\nu}$ has the same structure as that of $S_{\left[n\frac{k-1}{m}\right]+l}$, and similarly both $S^{\#}_{\left[n\frac{k-1}{m}\right]+l+r,\nu}$ and $S_{\left[n\frac{k-1}{m}\right]+l+r,v+r}$ have the same structure as that of $S_{\left[n\frac{k-1}{m}\right]+l+r}$. It can be seen from the proof of Proposition 13* that in both cases (85) and (86) the role of $\psi_{S_r}(-\mu)$ (see (74)) is now played by $\Psi_r(\mu)$ defined in Theorem 2. In addition note that both $\widehat{w_r}(\lambda,\mu)$ and $\widehat{w_r^*}(\lambda,\mu)$ contain the factor $\widehat{f}(\lambda)\widehat{f}(\mu)$, which will serve the purpose of $\widehat{w}(\lambda,\mu)\psi(\mu)$ in the proof of Proposition 13*. We thus see that for each $r\geq 1$,

$$\frac{\gamma_n}{n}\sum_{k=1}^{m}\sum_{l=1}^{n_{mk}} E_{\left[n\frac{k-1}{m}\right]}\left[f\left(S_{\left[n\frac{k-1}{m}\right]+l}\right)\omega_{\left[n\frac{k-1}{m}\right]+l} f\left(S_{\left[n\frac{k-1}{m}\right]+l+r}\right)\omega_{\left[n\frac{k-1}{m}\right]+l+r}\right]$$
$$\Longrightarrow \left(\frac{1}{2\pi}\int\left|\widehat{f}(\mu)\right|^2\Psi_r(\mu)\,\Phi_r(\mu)\,d\mu\right)L_1^0$$

as $n\to\infty$ first and then $m\to\infty$. Here $\Psi_r(\mu)$ is as defined earlier in Theorem 2, and $\left|\widehat{f}(\mu)\right|^2\Phi_r(\mu)=\widehat{w_r}(-\mu,\mu)$ in the case $r\geq\nu$ and $\left|\widehat{f}(\mu)\right|^2\Phi_r(\mu)=\widehat{w_r^*}(-\mu,\mu)$ in the case $r<\nu$. Specifically

$$\Phi_r(\mu)=\begin{cases} E\left[\omega_\nu e^{i\mu S^*_{\nu,\nu}-i\mu\left(S^*_{\nu+r,r+\nu}-S^*_{\nu+r,r}\right)}\right]E\left[\omega_\nu e^{-i\mu S^*_{\nu,\nu}}\right] & \text{if } r\geq\nu \\ E\left[\omega_\nu\omega_{\nu+r}e^{i\mu\sum_{j=1}^{\nu}g(\nu-j)\xi_j-i\mu\sum_{j=1}^{\nu+r}g(\nu+r-j)\xi_j}\right] & \text{if } 1\leq r<\nu. \end{cases}$$

(This $\Phi_r(\mu)$ coincides with that involved in Theorem 2.)

Regarding Lemma 10 we shall see in the next section that its proof, under the conditions of Theorem 1, depends crucially on the fact that $\left|\widehat{f}(\lambda)\widehat{f}(\mu)\right|\leq C|\mu||\lambda|$, which holds under the conditions of Theorem 1, see (62). In the present case the role of $\widehat{f}(\lambda)\widehat{f}(\mu)$ is played by $\widehat{w_r}(\lambda,\mu)$ (it is enough to restrict to the case $r\geq\nu$), for which we now obtain the bound

$$|\widehat{w_r}(\lambda,\mu)|\leq C\left(|\mu||\lambda|+|\mu|^2\right). \tag{87}$$

To see this, assume for convenience that $\nu=1$. Then

$$\begin{aligned}|\widehat{w_r}(\lambda,\mu)| &= \left|\widehat{f}(\lambda)\widehat{f}(\mu)E\left[\eta_1 e^{-i\mu\xi_1}\right]E\left[\eta_1 e^{(-i\lambda-i\mu g(r))\xi_1}\right]\right| \\ &\leq C\left|E\left[\eta_1 e^{-i\mu\xi_1}\right]E\left[\eta_1 e^{(-i\lambda-i\mu g(r))\xi_1}\right]\right|\end{aligned}$$

where, using $E[\eta_1]=0$,

$$\left|E\left[\eta_1 e^{-i\mu\xi_1}\right]\right|=\left|E\left[\eta_1\left(e^{-i\mu\xi_1}-1\right)\right]\right|\leq|\mu|\,E\left[|\eta_1\xi_1|\right]\leq C|\mu|$$

and similarly $\left|E\left[\eta_1 e^{(-i\lambda-i\mu g(r))\xi_1}\right]\right|\leq C\left(|\lambda|+|\mu|\right)$.

We shall see later that (87) will give the analogue of Lemma 10, see the arguments at the end of the proof of Lemma 10. ■

We next verify (R4) (where $\alpha = 2$ and hence $E[\xi_1] = 0$ and $E[\xi_1^2] < \infty$).

Verification of (R4): For notational convenience, we take $\gamma_r = r^H$ and $g(r) \sim Cr^{H-1/2}$. Then (χ_{nmk} is as defined in (R4))

$$\zeta_{nmk}\chi_{nmk} = n^{-\frac{1}{2}-\frac{1-H}{2}} \left(I_{1,nmk} + I_{2,nmk} + I_{3,nmk}\right) \tag{88}$$

where

$$I_{1,nmk} = \sum_{l=\left[n\frac{k-1}{m}\right]+1}^{\left[n\frac{k}{m}\right]} \sum_{r=l+1}^{\left[n\frac{k}{m}\right]} f(S_l)\,\xi_r,$$

$$I_{2,nmk} = \sum_{l=\left[n\frac{k-1}{m}\right]+1}^{\left[n\frac{k}{m}\right]} \sum_{r=l+1}^{\left[n\frac{k}{m}\right]} \xi_l f(S_r)$$

and

$$I_{3,nmk} = \sum_{l=\left[n\frac{k-1}{m}\right]+1}^{\left[n\frac{k}{m}\right]} f(S_l)\,\xi_l.$$

Now

$$E_{\left[n\frac{k-1}{m}\right]}\left[f(S_l)\,\xi_l\right] = E_{\left[n\frac{k-1}{m}\right]}\left[f_1(S_{l,1})\right] \text{ with } f_1(y) = E\left[\xi_1 f(y+\xi_1)\right],$$

where $S_{l,1}$ is as in (59). Note that $\int f_1(y)\,dy = 0$ and similarly other restrictions in Theorem 1 stated for $f(y)$ are satisfied for $f_1(y)$, see Lemma 7. It is proved in the next section (see the bound for (101)) that

$$E_{\left[n\frac{k-1}{m}\right]}\left[\left(n^{-\frac{1-H}{2}} \sum_{l=\left[n\frac{k-1}{m}\right]+1}^{\left[n\frac{k}{m}\right]} f_1(S_{l,1})\right)^2\right] \le C.$$

Hence

$$\left|E_{\left[n\frac{k-1}{m}\right]}\left[n^{-\frac{1}{2}-\frac{1-H}{2}} I_{3,nmk}\right]\right| \le Cn^{-\frac{1}{2}}. \tag{89}$$

Clearly

$$E_{\left[n\frac{k-1}{m}\right]}\left[I_{1,nmk}\right] = 0. \tag{90}$$

To deal with $I_{2,nmk}$ we have

$$S_r = S_{r,\left[n\frac{k-1}{m}\right]} + S^*_{r,\left[n\frac{k-1}{m}\right]},$$

where recall (see (58)) that $S^*_{r,\left[n\frac{k-1}{m}\right]} = \sum_{q=0}^{r-\left[n\frac{k-1}{m}\right]-1} g(q)\,\xi_{r-q}$ and is independent of $S_{r,\left[n\frac{k-1}{m}\right]}$. We also have $f(S_r) = \frac{1}{2\pi}\int e^{-i\lambda S_r}\widehat{f}(\lambda)\,d\lambda$. Hence

$$\left|E_{\left[n\frac{k-1}{m}\right]}\left[\xi_l f(S_r)\right]\right| \leq \frac{1}{2\pi}\int \left|E\left[\xi_l e^{-i\lambda\sum_{q=0}^{r-\left[n\frac{k-1}{m}\right]-1} g(q)\xi_{r-q}}\right]\right|\left|\widehat{f}(\lambda)\right|d\lambda,$$

and hence

$$\left|E_{\left[n\frac{k-1}{m}\right]}\left[\xi_{l+\left[n\frac{k-1}{m}\right]} f\left(S_{r+\left[n\frac{k-1}{m}\right]}\right)\right]\right| \leq \frac{1}{\gamma_r}\int \left|E\left[\xi_1 e^{-i\frac{\lambda}{\gamma_r}g(r-l)\xi_1}\right]\right| \prod_{q=0,q\neq r-l}^{r-1}\left|\psi\left(\frac{\lambda}{\gamma_r}g(q)\right)\right|\left|\widehat{f}\left(\frac{\lambda}{\gamma_r}\right)\right|d\lambda.$$

Now, because $E[\xi_1]=0$ and $E[\xi_1^2]<\infty$ ((R4) pertains only to the case $\alpha=2$),

$$\left|E\left[\xi_1 e^{-i\frac{\lambda}{\gamma_r}g(r-l)\xi_1}\right]\right| = \left|E\left[\xi_1\left(e^{-i\frac{\lambda}{\gamma_r}g(r-l)\xi_1}-1\right)\right]\right| \leq C\frac{|\lambda|}{\gamma_r}\left|g(r-l)\right|.$$

Further $\left|\widehat{f}\left(\frac{\lambda}{\gamma_r}\right)\right| \leq C\frac{|\lambda|}{\gamma_r}$. Also $\int\prod_{q=0,q\neq r-l}^{r-1}\left|\psi\left(\frac{\lambda}{\gamma_r}g(q)\right)\right|d\lambda \leq C$ by (55) and (56).

Thus, noting that $\gamma_r = r^H$ and $\sum_{l=1}^{r-1}|g(r-l)| \sim Cr^{H+1-1/2}$ because $g(s)\sim Cs^{H-1/2}$,

$$\begin{aligned}
n^{-\frac{1}{2}-\frac{1-H}{2}}\left|E_{\left[n\frac{k-1}{m}\right]}\left[I_{2,nmk}\right]\right| &= n^{-\frac{1}{2}-\frac{1-H}{2}}\left|\sum_{l=1}^{n_{mk}}\sum_{r=l+1}^{n_{mk}} E_{\left[n\frac{k-1}{m}\right]}\left[\xi_{l+\left[n\frac{k-1}{m}\right]} f\left(S_{r+\left[n\frac{k-1}{m}\right]}\right)\right]\right| \\
&\leq Cn^{-\frac{1}{2}-\frac{1-H}{2}}\sum_{r=1}^{n_{mk}}\sum_{l=1}^{r-1}\gamma_r^{-3}\left|g(r-l)\right| \\
&\leq Cn^{-\frac{1}{2}-\frac{1-H}{2}}n^{-2H+\frac{3}{2}} = Cn^{-\frac{3H-1}{2}}.
\end{aligned}$$

Because $3H-1>0$, this together with (89) and (90) complete the verification of (R4).
■

We next show that the verification of (R4) entails that of (R*4).

Verification of (R*4). We have

$$E_{\left[n\frac{k-1}{m}\right]}\left[\zeta^*_{nmk}\chi_{nmk}\right] = \sqrt{\frac{\gamma_n}{n}}E_{\left[n\frac{k-1}{m}\right]}\left[\left(\sum_{l=\left[n\frac{k-1}{m}\right]+l_0}^{\left[n\frac{k}{m}\right]} f(S_l)\,\omega_l\right)\chi_{nmk}\right].$$

Note that $(f(S_l)\omega_l - E_{l-1}[f(S_l)\omega_l], \xi_l)$, $l \geq 1$, form martingale differences and hence

$$\begin{aligned} & E_{\left[n\frac{k-1}{m}\right]}\left[\left(\sum_{l=\left[n\frac{k-1}{m}\right]+l_0}^{\left[n\frac{k}{m}\right]} (f(S_l)\omega_l - E_{l-1}[f(S_l)\omega_l])\right)\chi_{nmk}\right] \\ = & \frac{1}{\sqrt{n}} E_{\left[n\frac{k-1}{m}\right]}\left[\sum_{l=\left[n\frac{k-1}{m}\right]+l_0}^{\left[n\frac{k}{m}\right]} E_{l-1}\{(f(S_l)\omega_l - E_{l-1}[f(S_l)\omega_l])\xi_l\}\right] \\ = & \frac{1}{\sqrt{n}} E_{\left[n\frac{k-1}{m}\right]}\left[\sum_{l=\left[n\frac{k-1}{m}\right]+l_0}^{\left[n\frac{k}{m}\right]} E_{l-1}[f(S_l)\omega_l\xi_l]\right] \end{aligned}$$

where in the last step we have used $E_{l-1}[\xi_l] = 0$, so that $E_{l-1}[E_{l-1}[f(S_l)\omega_l]\xi_l] = 0$.

Consider

$$E_{\left[n\frac{k-1}{m}\right]}\left[\sum_{l=\left[n\frac{k-1}{m}\right]+l_0}^{\left[n\frac{k}{m}\right]} E_{l-1}[f(S_l)\eta_l\xi_l]\right] = E_{\left[n\frac{k-1}{m}\right]}\left[\sum_{l=\left[n\frac{k-1}{m}\right]+l_0}^{\left[n\frac{k}{m}\right]} g(S_{l,1})\right]$$

where we have used $E_{l-1}[f(S_l)\eta_l\xi_l] = g(S_{l,1})$ for a suitable $g(x)$ with $g(x) + g^2(x)$ Lebesgue integrable. Therefore $\frac{\gamma_n}{n} E_{\left[n\frac{k-1}{m}\right]}\left[\sum_{l=\left[n\frac{k-1}{m}\right]+l_0}^{\left[n\frac{k}{m}\right]} g(S_{l,1})\right]$ is bounded (see the bound for (101) below), so that

$$\frac{1}{\sqrt{n}}\sqrt{\frac{\gamma_n}{n}} E_{\left[n\frac{k-1}{m}\right]}\left[\sum_{l=\left[n\frac{k-1}{m}\right]+l_0}^{\left[n\frac{k}{m}\right]} g(S_{l,1})\right] \leq C\frac{1}{\sqrt{n}}\sqrt{\frac{\gamma_n}{n}}\left(\frac{\gamma_n}{n}\right)^{-1} = \frac{C}{\sqrt{\gamma_n}} \to 0.$$

Thus it remains to show that (R*4) holds for $\zeta_{nmk}^* = \sum E_{l-1}[f(S_l)\omega_l]$. We shall reduce this situation to that of (R4). Recall that $\omega_l = \sum_{j=l-\nu+1}^{l} d_{l-j}\eta_j$, which sum consists of ν terms. We use induction on ν. Suppose that $\nu = 1$, that is, $\omega_l = \eta_l$. Then

$$E_{l-1}[f(S_l)\omega_l] = E_{l-1}[f(S_l)\eta_l] = f_*(S_{l,1}).$$

Here $f_*(x) = E_{l-1}[f(x+\xi_1)\eta_1]$, which satisfies all the conditions of Theorem 1 (see Lemma 7), and hence (R*4) holds for $\zeta_{nmk}^* = \sum E_{l-1}[f(S_l)\eta_l]$ when $\nu = 1$.

Now suppose that (R*4) holds for $\zeta_{nmk}^* = \sum E_{l-1}[f(S_l)\omega_l]$ when $\nu = i-1$. Then, when $\nu = i$, we have $E_{l-1}[f(S_l)\omega_l] = E_{l-1}[f(S_l)(\omega_l - \eta_l)] + E_{l-1}[f(S_l)\eta_l]$, where note that

$$E_{l-1}[f(S_l)(\omega_l - \eta_l)] = E_{l-1}[f(S_l)\omega_l^*] = \omega_l^* E_{l-1}[f(S_l)] = g(S_{l,1})\omega_l^*.$$

Here $\omega_l^* = \omega_l - \eta_l = \sum_{j=l-i+1}^{l-1} d_{l-j}\eta_j$, and hence $g(S_{l,1})\omega_l^*$ has the same structure as that of $f(S_l)\omega_l$ but with $\nu = i-1$ (for which we have assumed that (R*4) holds). Hence one can assume that (R*4) holds for $\zeta^*_{nmk} = \sum g\left(S_{l,1}\right)\omega^*_l$ also. We have already verified (R*4) for $\zeta^*_{nmk} = \sum E_{l-1}\left[f\left(S_l\right)\eta_l\right]$. Thus (R*4) holds for $\zeta^*_{nmk} = \sum E_{l-1}\left[f\left(S_l\right)\omega_l\right]$ when $\nu = i$. This completes the proof of the lemma by induction. ■

5 PROOF OF LEMMA 10 AND THE VERIFICATION OF (R3) AND (R*3)

In the rest of the paper we let

$$g(j,r) = g(j+r) - g(j) = c_{j+1} + \ldots + c_{j+r}.$$

We first isolate some bounds on $g(j,r)$ in the next Lemma 19.

Lemma 19. *Let $\vartheta > 0$ be such that*

$$0 < \vartheta < \begin{cases} \min\left(1-H, H, \left|\frac{1}{\alpha}-H\right|, \frac{1}{\alpha}\right) & \text{if } H \neq \frac{1}{\alpha} \\ \min\left(1-\frac{1}{\alpha}, \frac{1}{\alpha}\right) & \text{if } H = \frac{1}{\alpha}. \end{cases} \tag{91}$$

Then

$$\sup_{[l/2]\leq j\leq l, q\geq 1, r\geq 1}\left|b_l \frac{g\left(j+q,r\right)}{\gamma_r}\right| \leq Cl^{\vartheta} \text{ for all } 1 \leq l \leq n. \tag{92}$$

Proof. First consider the case $H \neq \frac{1}{\alpha}$, in which case the requirement (A1) of Section 2 holds. Let $\delta = \frac{\vartheta}{3}$ so that (91) becomes

$$0 < 3\delta < \min\left(1-H, H, \left|\frac{1}{\alpha}-H\right|, \frac{1}{\alpha}\right). \tag{93}$$

Recall the Potter's inequality, mentioned in Lemma 5 of Section 3 above, that if $G(x)$ is slowly varying at ∞, then there is a $B > 0$ such that $\left|\frac{G(x)}{G(y)}\right| \leq B\max\{(x/y)^{\delta}, (x/y)^{-\delta}\}$ for all $x > 0, y > 0$. Therefore one can assume that

$$\left|\frac{c_i}{i^{H-1-\frac{1}{\alpha}}}\right| \leq Bi^{\delta}, \quad \left|\frac{g(i)}{i^{H-\frac{1}{\alpha}}}\right| \leq Bi^{\delta}, \quad \frac{b_r}{r^{\frac{1}{\alpha}}} \leq Br^{\delta}, \quad \frac{r^H}{\gamma_r} \leq Br^{\delta}.$$

We in particular have

$$\frac{b_l}{\gamma_r} \leq Cl^{\frac{1}{\alpha}+\delta}r^{-H+\delta}. \tag{94}$$

Further, noting $H-1-\frac{1}{\alpha}+\delta < 0$ (see (93)), we have when $j \geq [l/2]$,

$$\begin{aligned} \left|g\left(j+q,r\right)\right| &= \left|c_{j+q+1} + .. + c_{j+q+r}\right| \\ &\leq C\left|\left(j+q+1\right)^{H-1-\frac{1}{\alpha}+\delta} + \ldots + \left(j+q+r\right)^{H-1-\frac{1}{\alpha}+\delta}\right| \\ &\leq Cr\left(j+q\right)^{H-1-\frac{1}{\alpha}+\delta} \leq Cr\left(\min\left(l,q\right)\right)^{H-1-\frac{1}{\alpha}+\delta}, \quad j \geq [l/2]. \end{aligned} \tag{95}$$

Here, in obtaining the second inequality we have used $j \geq [l/2]$ and $H - 1 - \frac{1}{\alpha} + \delta < 0$.

Further, when $H - \frac{1}{\alpha} < 0$ (in which case $H - \frac{1}{\alpha} + \delta < 0$, see (93)), we have

$$|g(j+q,r)| \leq |g(j+q)| + |g(j+q+r)| \leq C(j+q)^{H-\frac{1}{\alpha}+\delta} \leq C(\min(l,q))^{H-\frac{1}{\alpha}+\delta}, \quad j \geq [l/2], \tag{96}$$

and similarly when $H - \frac{1}{\alpha} > 0$,

$$\begin{aligned} |g(j+q,r)| &\leq C(j+q+r)^{H-\frac{1}{\alpha}+\delta} \\ &\leq \begin{cases} \begin{cases} Cl^{H-\frac{1}{\alpha}+\delta} & \text{if } j \leq l,\ r \leq l \\ Cr^{H-\frac{1}{\alpha}+\delta} & \text{if } j \leq l,\ r > l \end{cases} & H - \frac{1}{\alpha} > 0,\ q \leq l \\ \begin{cases} Cq^{H-\frac{1}{\alpha}+\delta} & \text{if } j \leq l,\ r \leq q \\ Cr^{H-\frac{1}{\alpha}+\delta} & \text{if } j \leq l,\ r > q \end{cases} & H - \frac{1}{\alpha} > 0,\ q > l. \end{cases} \end{aligned} \tag{97}$$

First consider the situation

$$q \leq l.$$

Using (94) and (95) and noting $1 - H - 2\delta > 0$ (see (93)),

$$\left| b_l \frac{g(j+q,r)}{\gamma_r} \right| \leq Cl^{\frac{1}{\alpha}+\delta} r^{-H+\delta} r l^{H-1-\frac{1}{\alpha}+\delta} = \left(\frac{r}{l}\right)^{1-H-2\delta} r^{3\delta} \leq Cl^{3\delta}, \text{ if } r \leq l,\ j \geq [l/2].$$

In addition, using (96) and (97) and noting $H - \delta > 0$ and $\frac{1}{\alpha} - 2\delta > 0$ (see (93)), we have

$$\left| b_l \frac{g(j+q,r)}{\gamma_r} \right| \leq \begin{cases} Cl^{\frac{1}{\alpha}+\delta} r^{-H+\delta} l^{H-\frac{1}{\alpha}+\delta} = Cr^{-H+\delta} l^{H-\delta} l^{3\delta} \leq Cl^{3\delta} & H - \frac{1}{\alpha} < 0,\ r > l,\ j \leq l \\ Cl^{\frac{1}{\alpha}+\delta} r^{-H+\delta} r^{H-\frac{1}{\alpha}+\delta} = Cr^{2\delta-\frac{1}{\alpha}} l^{\frac{1}{\alpha}-2\delta} l^{3\delta} \leq Cl^{3\delta} & H - \frac{1}{\alpha} > 0,\ r > l,\ j \leq l. \end{cases}$$

Now consider

$$q > l.$$

From (95) we have,

$$\left| b_l \frac{g(j+q,r)}{\gamma_r} \right| \leq Cl^{\frac{1}{\alpha}+\delta} r^{-H+\delta} r q^{H-1-\frac{1}{\alpha}+\delta} = \left(\frac{r}{q}\right)^{1-H+\delta} \left(\frac{l}{q}\right)^{\frac{1}{\alpha}-2\delta} l^{3\delta} \leq Cl^{3\delta}, \text{ if } r \leq q,\ j \geq [l/2].$$

When $H - \frac{1}{\alpha} < 0$, $r > q$, we obtain from (96) that

$$\left| b_l \frac{g(j+q,r)}{\gamma_r} \right| \leq Cl^{\frac{1}{\alpha}+\delta} r^{-H+\delta} q^{H-\frac{1}{\alpha}+\delta} = \left(\frac{q}{r}\right)^{H-\delta} \left(\frac{l}{q}\right)^{\frac{1}{\alpha}-2\delta} l^{3\delta} \leq Cl^{3\delta}.$$

When $H - \frac{1}{\alpha} > 0$, $r > q$, we have from (97) that

$$\left| b_l \frac{g(j+q,r)}{\gamma_r} \right| \leq Cl^{\frac{1}{\alpha}+\delta} r^{-H+\delta} r^{H-\frac{1}{\alpha}+\delta} = \left(\frac{l}{r}\right)^{\frac{1}{\alpha}-2\delta} l^{3\delta} \leq Cl^{3\delta}$$

because $\frac{1}{\alpha} - 2\delta > 0$ and $l < q < r$. This completes the proof of the lemma when $H \neq \frac{1}{\alpha}$.

Now consider the case $H = \frac{1}{\alpha}$. In this case, by (11), we have $\sup_{i\geq 1} |ic_i| \leq C$. In addition $\sup_{i\geq 1} |g(i)| \leq C$ by (A2). Therefore, the inequalities (94) - (97) hold when $H = \frac{1}{\alpha}$, and hence the remaining arguments also hold with $H = \frac{1}{\alpha}$. This completes the proof of the lemma. ■

Below we assume ϑ of Lemma 19 satisfies (in addition to (91))

$$3H - 6\vartheta > 1. \tag{98}$$

This is possible in view of the restriction $3H > 1$.

We are now in a position to proceed with the proof of Lemma 10 and the verification of (R3). For this purpose, note that, when $0 \leq \nu_n < [l/2]$, using (64) - (66) we have

$$\begin{aligned}
&\left| E_{[n\frac{k-1}{m}]} \left[w \left(S_{[n\frac{k-1}{m}]+l}, S_{[n\frac{k-1}{m}]+l+r} \right) \right] \right| \\
\leq\; & \frac{1}{\gamma_l \gamma_r} \int_{R^2} \left(\prod_{j=[l/2]}^{l-1} \left| \psi \left(\frac{\lambda g(j)}{\gamma_l} - \frac{\mu}{\gamma_r} g(j,r) \right) \right| \right) \\
& \times \left(\prod_{j_1=[r/2]}^{r-1} \left| \psi \left(-g(j_1) \frac{\mu}{\gamma_r} \right) \right| \right) \left| \widehat{w} \left(\frac{\lambda}{\gamma_l} - \frac{\mu}{\gamma_r}, \frac{\mu}{\gamma_r} \right) \right| d\lambda d\mu \\
=\; & \frac{1}{\gamma_l \gamma_r} \int_{R^2} \left(\prod_{j=[l/2]}^{l-1} \left| \psi \left(\frac{\lambda g(j)}{\gamma_l} \right) \right| \right) \left(\prod_{j_1=[r/2]}^{r-1} \left| \psi \left(-g(j_1) \frac{\mu}{\gamma_r} \right) \right| \right) \\
& \times \left| \widehat{w} \left(\frac{\lambda}{\gamma_l} \left(1 + \frac{\gamma_l g(j,r)}{\gamma_r g(j)} \mu \right) - \frac{\mu}{\gamma_r}, \frac{\mu}{\gamma_r} \right) \right| d\lambda d\mu.
\end{aligned} \tag{99}$$

Here note that the right hand side is nonrandom.

Before giving the proof of Lemma 10, we note the following useful fact that follows from (99) (recall $n_{mk} = [n\frac{k}{m}] - [n\frac{k-1}{m}]$):

$$E\left[\left(\frac{\gamma_n}{n} \sum_{j=1}^{n} h(S_j) - \frac{\gamma_n}{n} \sum_{k=1}^{m} \sum_{l=1}^{n_{mk}} E_{[n\frac{k-1}{m}]} \left[h \left(S_{[n\frac{k-1}{m}]+l} \right) \right] \right)^2 \right] \to 0 \tag{100}$$

as $n \to \infty$ first and then $m \to \infty$, for any $h(x)$ for which both $h(x)$ and $h^2(x)$ are Lebesgue integrable.

Remark. This fact together with the approximation contained in the proof of Proposition 12 has been used in the proof of Theorem 1 given in Section 2. In addition, essentially the same arguments will be used to deduce Theorem 3 from Proposition 13*.
■

To see that (100) holds note that $\sum_{l=1}^{n_{mk}} h\left(S_{[n\frac{k-1}{m}]+l}\right) - \sum_{l=1}^{n_{mk}} E_{[n\frac{k-1}{m}]}\left[h\left(S_{[n\frac{k-1}{m}]+l}\right)\right]$, $1 \leq k \leq m$, form an array of martingale differences, and hence the expected value in

(100) is bounded by

$$\begin{aligned} & \left(\frac{\gamma_n}{n}\right)^2 \sum_{k=1}^{m} E\left[\left(\sum_{l=1}^{n_{mk}} h\left(S_{[n\frac{k-1}{m}]+l}\right)\right)^2\right] \\ \leq & \left(\frac{\gamma_n}{n}\right)^2 \left\{\sum_{l=1}^{n} E\left[h^2\left(S_l\right)\right] + 2\sum_{l=1}^{n}\sum_{r=1}^{n_{mk}} E\left[h\left(S_l\right)h\left(S_{l+r}\right)\right]\right\}. \end{aligned} \tag{101}$$

In the situation where $|\widehat{w}(\lambda,\mu)| \leq C$, note that (99) gives, $|E[w(S_l, S_{l+r})]| \leq \frac{C}{\gamma_l\gamma_r}$ using (55) and (56) when l, $r \geq l_0$. Similarly, if $\left|\widehat{h}(\lambda)\right| \leq C$, then $|E[h(S_l)]| \leq \frac{C}{\gamma_l}$ when $l \geq l_0$. Hence taking $w(x,y) = h(x)h(y)$, and using in addition (57) when $l \leq l_0$ and/or $r \leq l_0$ we see that (101) is bounded by

$$C\left(\frac{\gamma_n}{n}\right)^2 \left\{\sum_{l=1}^{n} \frac{1}{\gamma_l} + \sum_{l=1}^{n}\sum_{r=1}^{n_{mk}} \frac{1}{\gamma_l\gamma_r}\right\}.$$

Here $\sum_{l=1}^{n} \frac{1}{\gamma_l} \sim C\frac{n}{\gamma_n}$ and, using $n_{mk} \sim \frac{n}{m}$ and $\gamma_{n_{mk}} \sim \gamma_n m^{-H}$, $\max_{1\leq k\leq m} \sum_{r=1}^{n_{mk}} \frac{1}{\gamma_r} \sim C\frac{n}{\gamma_n}\left(\frac{1}{m}\right)^{1-H}$. Thus (100) holds. ■

Proof of Lemma 10. We shall apply (99) with $\widehat{w}(\lambda,\mu) = \widehat{f}(\lambda)\widehat{f}(\mu)$. The fact that $\left|\widehat{f}(\lambda)\right| \leq C|\lambda|$ will now be crucially used (whereas (101) uses only $\left|\widehat{h}(\lambda)\right| + \left|\widehat{h^2}(\lambda)\right| \leq C$). Here note that, for any ϑ satisfying (91),

$$\left|\frac{\gamma_l g(j,r)}{\gamma_r g(j)}\right| = \left|\frac{\gamma_l}{b_l g(j)}\right|\left|\frac{b_l g(j,r)}{\gamma_r}\right| \leq Cl^\vartheta, \quad [l/2] \leq j \leq l, \quad r \geq 1 \tag{102}$$

by (54) and Lemma 19. Therefore

$$\left|\widehat{f}\left(\frac{\lambda}{\gamma_l}\left(1 + \frac{\gamma_l g(j,r)}{\gamma_r g(j)}\mu\right) - \frac{\mu}{\gamma_r}\right)\widehat{f}\left(\frac{\mu}{\gamma_r}\right)\right| \leq C\left(\frac{|\lambda|}{\gamma_l} + \frac{|\lambda\mu|\, l^\vartheta}{\gamma_l} + \frac{|\mu|}{\gamma_r}\right)\frac{|\mu|}{\gamma_r}.$$

Hence (99) is bounded by (when $\widehat{w}(\lambda,\mu) = \widehat{f}(\lambda)\widehat{f}(\mu)$)

$$\begin{aligned} & \frac{1}{\gamma_l\gamma_r}\int_{R^2} \left(\frac{|\lambda|}{\gamma_l} + \frac{|\lambda\mu|\, l^\vartheta}{\gamma_l} + \frac{|\mu|}{\gamma_r}\right)\frac{|\mu|}{\gamma_r}\left(\prod_{j=[l/2]}^{l-1}\left|\psi\left(\frac{\lambda g(j)}{\gamma_l}\right)\right|\right)\prod_{j_1=[r/2]}^{r-1}\left|\psi\left(-g(j_1)\frac{\mu}{\gamma_r}\right)\right| d\lambda d\mu \\ \leq & \; C\frac{1}{\gamma_l\gamma_r}\left(\frac{l^\vartheta}{\gamma_l} + \frac{1}{\gamma_r}\right)\frac{1}{\gamma_r} \end{aligned} \tag{103}$$

when $l, r \geq l_0$, where in obtaining the inequality we have used (55) and (56). Further using (57), the same bound (103) holds for (99) when $l \leq l_0$ and/or $r \leq l_0$ also. Thus we need to show that

$$\frac{\gamma_n}{n}\sum_{l=1}^{n}\sum_{r=q}^{n}\frac{1}{\gamma_l\gamma_r}\left(\frac{l^\vartheta}{\gamma_l} + \frac{1}{\gamma_r}\right)\frac{1}{\gamma_r} \to 0 \tag{104}$$

as $n \to \infty$ first and then $q \to \infty$. To see that this is true, take for convenience that

$$\gamma_n = n^H \quad \text{for all } n \geq 1.$$

First note that, using the restriction $1 < 3H$,

$$\frac{\gamma_n}{n} \sum_{l=1}^{n} \sum_{r=q}^{n} \frac{1}{\gamma_l \gamma_r^3} = \left(\frac{\gamma_n}{n} \sum_{l=1}^{n} \frac{1}{\gamma_l} \right) \sum_{r=q}^{n} \frac{1}{\gamma_r^3} \leq C q^{1-3H} \to 0 \quad \text{as } q \to \infty,$$

where we have used $\frac{\gamma_n}{n} \sum_{l=1}^{n} \frac{1}{\gamma_l} \leq C$ and $\sum_{r=q}^{n} \frac{1}{\gamma_r^3} = \sum_{r=q}^{n} \frac{1}{r^{3H}} \leq C q^{1-3H}$. Next

$$\sum_{l=1}^{n} \frac{l^{\vartheta}}{\gamma_l^2} = \sum_{l=1}^{n} \frac{1}{l^{2H-\vartheta}} \leq \begin{cases} C \log n & \text{if } 2H - \vartheta \geq 1 \\ C n^{1-2H+\vartheta} & \text{if } 2H - \vartheta < 1. \end{cases}$$

Also, $\frac{\gamma_n}{n} = n^{H-1}$. Hence

$$\frac{\gamma_n}{n} \sum_{l=1}^{n} \sum_{r=q}^{n} \frac{l^{\vartheta}}{\gamma_l^2 \gamma_r^2} \leq \frac{\gamma_n}{n} \left(\sum_{l=1}^{n} \frac{l^{\vartheta}}{\gamma_l^2} \right)^2 \leq \begin{cases} C n^{H-1} (\log n)^2 & \text{if } 2H - \vartheta \geq 1 \\ C n^{H-1+2-4H+2\vartheta} = C n^{1-3H+2\vartheta} & \text{if } 2H - \vartheta < 1 \end{cases} \tag{105}$$

where note that $1 - 3H + 2\vartheta < 0$ in view of (98) Thus (104) holds and hence the proof of Lemma 10 is complete.

Now, regarding the Lemma 10 for the situation of Theorem 2, it was indicated earlier (see the end of the Verification of (R*2)) that the only difference essential difference is that in place of $\widehat{w}(\lambda, \mu) = \widehat{f}(\lambda) \widehat{f}(\mu)$ in the above arguments, $\widehat{w_r}(\lambda, \mu)$ as defined in the Verification of (R*2) will be involved, for which we have the inequality (67). Thus, in place of (104), we need to verify that $\frac{\gamma_n}{n} \sum_{l=1}^{n} \sum_{r=q}^{n} \frac{1}{\gamma_l \gamma_r} \left\{ \left(\frac{l^{\vartheta}}{\gamma_l} + \frac{1}{\gamma_r} \right) \frac{1}{\gamma_r} + \frac{1}{\gamma_r^2} \right\} \to 0$ as $n \to \infty$ first and then $q \to \infty$, but this has been done above. ■

We next verify (R3).

VERIFICATION OF (R3). We show that (recall $n_{mk} = \left[n \frac{k}{m} \right] - \left[n \frac{k-1}{m} \right]$)

$$E\left[\zeta_{nmk}^4\right] \leq \frac{C}{n^{\delta}} + C \left(\frac{\gamma_n}{n} \sum_{l=\left[n \frac{k-1}{m}\right]+1}^{\left[n \frac{k}{m}\right]} \frac{1}{\gamma_l} \right) \left(\frac{\gamma_n}{n} \sum_{j=1}^{n_{mk}} \frac{1}{\gamma_j} \right), \quad \text{for some } \delta > 0. \tag{106}$$

This will verify (R3), because then

$$\sum_{k=1}^{m} E\left[\zeta_{nmk}^4\right] \leq \frac{Cm}{n^{\delta}} + C \left(\frac{\gamma_n}{n} \sum_{l=1}^{n} \frac{1}{\gamma_l} \right) \left(\max_{1 \leq k \leq m} \frac{\gamma_n}{n} \sum_{j=1}^{n_{mk}} \frac{1}{\gamma_j} \right)$$

where $\frac{\gamma_n}{n} \sum_{l=1}^{n} \frac{1}{\gamma_l} \leq C$ and $\max_{1 \leq k \leq m} \frac{\gamma_n}{n} \sum_{j=1}^{n_{mk}} \frac{1}{\gamma_j} \sim \frac{1}{1-H} \left(\frac{1}{m} \right)^{1-H}$ as $n \to \infty$.

We shall show in detail that

$$\left(\frac{\gamma_n}{n}\right)^2 \sum_{l=\left[n\frac{k-1}{m}\right]+1}^{\left[n\frac{k}{m}\right]} \sum_{r=1}^{n_{mk}} \sum_{q=1}^{n_{mk}} \left|E\left[f\left(S_l\right) f\left(S_{l+r}\right) f^2\left(S_{l+r+q}\right)\right]\right| \tag{107}$$

and

$$\left(\frac{\gamma_n}{n}\right)^2 \sum_{l=\left[n\frac{k-1}{m}\right]+1}^{\left[n\frac{k}{m}\right]} \sum_{r=1}^{n_{mk}} \sum_{q=1}^{n_{mk}} \sum_{s=1}^{n_{mk}} \left|E\left[f\left(S_l\right) f\left(S_{l+r}\right) f\left(S_{l+r+q}\right) f\left(S_{l+r+q+s}\right)\right]\right| \tag{108}$$

is bounded by r.h.s. of (106). The same can be similarly shown to be true for the remaining analogues in the expansion of $E\left[\zeta_{nmk}^4\right] = \left(\frac{\gamma_n}{n}\right)^2 E\left[\left(\sum_{l=\left[n\frac{k-1}{m}\right]+1}^{\left[n\frac{k}{m}\right]} f\left(S_l\right)\right)^4\right]$. We shall use Lemma 19 in the manner similar to the proof of Lemmas 10 above.

We first deal with (107). Using $S_{l,l}$ as defined in (58) and T_l, T_{nl}^* and $T_{nl,r}^*$ as defined in Section 4 (recall that $S_l = S_{l,l} + T_l$ with $S_{l,l}$ independent of T_l) we have, similar to (65),

$$\begin{aligned}
&(2\pi)^3 E\left[f\left(S_l\right) f\left(S_{l+r}\right) f^2\left(S_{l+r+q}\right)\right] \\
= &\int E\left[e^{-i\lambda_1 S_{l,l} - \lambda_2 S_{l+r,l+r} - \lambda_3 S_{l+r+q,l+r+q}}\right] E\left[e^{-i(\lambda_1+\lambda_2+\lambda_3)T_{nl}^* - i(\lambda_2+\lambda_3)\left(T_{nl,r}^* - T_{nl}^*\right) - i\lambda_3\left(T_{nl,r+q}^* - T_{nl,r}^*\right)}\right] \\
&\times E\left[e^{-i\lambda_1\left(T_l - T_{nl}^*\right) - i\lambda_2\left(T_{l+r} - T_{nl,r}^*\right) - i\lambda_3\left(T_{l+r+q} - T_{nl,r+q}^*\right)}\right] \widehat{f}\left(\lambda_1\right) \widehat{f}\left(\lambda_2\right) \widehat{f^2}\left(\lambda_3\right) d\lambda_1 d\lambda_2 d\lambda_3.
\end{aligned}$$

Using this and using exactly the same ideas as in (99), we have (noting $\left|\widehat{f^2}\left(\lambda\right)\right| \leq C$)

$$\begin{aligned}
&(2\pi)^3 \left|E\left[f\left(S_l\right) f\left(S_{l+r}\right) f^2\left(S_{l+r+q}\right)\right]\right| \\
\leq\ &\frac{C}{\gamma_l \gamma_r \gamma_q} \int \left(\prod_{j_1=[l/2]}^{l-1} \left|\psi\left(\frac{\lambda_1 g(j_1)}{\gamma_l} + \frac{\lambda_2 g(j_1, r)}{\gamma_r} + \frac{\lambda_3 g(j_1 + r, q)}{\gamma_q}\right)\right|\right) \\
&\times \left(\prod_{j_2=[r/2]}^{r-1} \left|\psi\left(\frac{\lambda_2 g\left(j_2\right)}{\gamma_r} + \frac{\lambda_3 g(j_2, q)}{\gamma_q}\right)\right|\right) \left(\prod_{j_3=[q/2]}^{q-1} \left|\psi\left(\frac{\lambda_3 g\left(j_3\right)}{\gamma_q}\right)\right|\right) \\
&\times \left|\widehat{f}\left(\frac{\lambda_1}{\gamma_l} - \frac{\lambda_2}{\gamma_r}\right)\right| \left|\widehat{f}\left(\frac{\lambda_2}{\gamma_r} - \frac{\lambda_3}{\gamma_q}\right)\right| d\lambda_1 d\lambda_2 d\lambda_3,
\end{aligned} \tag{109}$$

where recall that $g(j,r) = g(j+r) - g(j)$. We make the transformation

$$\frac{\lambda_3 g\left(j_3\right)}{\gamma_q} \to \frac{\lambda_3 g\left(j_3\right)}{\gamma_q},$$

$$\frac{\lambda_2 g\left(j_2\right)}{\gamma_r} + \frac{\lambda_3 g(j_2, q)}{\gamma_q} = \frac{g\left(j_2\right)}{\gamma_r}\left(\lambda_2 + \lambda_3 \frac{\gamma_r g(j_2, q)}{\gamma_q g\left(j_2\right)}\right) \to \frac{\lambda_2 g\left(j_2\right)}{\gamma_r},$$

$$\begin{aligned}
\frac{\lambda_1 g(j_1)}{\gamma_l} + \frac{\lambda_2 g(j_1,r)}{\gamma_r} + \frac{\lambda_3 g(j_1+r,q)}{\gamma_q} &= \frac{g(j_1)}{\gamma_l}\left(\lambda_1 + \lambda_2 \frac{\gamma_l g(j_1,r)}{\gamma_r g(j_1)} + \lambda_3 \frac{\gamma_l g(j_1+r,q)}{\gamma_q g(j_1)}\right) \\
&\to \frac{\lambda_1 g(j_1)}{\gamma_l}.
\end{aligned}$$

Here note that, in the same way as in (102) using (54) and Lemma 19, we have

$$\left|\frac{\gamma_r g(j_2,q)}{\gamma_q g\left(j_2\right)}\right| \leq C r^{\vartheta}, \quad \left|\frac{\gamma_l g(j_1,r)}{\gamma_r g(j_1)}\right| \leq C l^{\vartheta}, \quad \left|\frac{\gamma_l g(j_1+r,q)}{\gamma_q g(j_1)}\right| \leq C l^{\vartheta}$$

uniformly in the variables involved. (For instance, using (54) and Lemma 19, $\left|\frac{\gamma_l g(j_1+r,q)}{\gamma_q g(j_1)}\right| = \left|\frac{\gamma_l}{b_l g(j_1)}\right|\left|\frac{b_l g(j_1+r,q)}{\gamma_q}\right| \leq C l^{\vartheta}$, $[l/2] \leq j_1 \leq l$, $r,q \geq 1$.) Therefore, in the same way (99) is bounded by (103), the right hand side in (109) is bounded by

$$\begin{aligned}
&\frac{C}{\gamma_l\gamma_r\gamma_q}\int\left(\frac{1}{\gamma_l}\left(|\lambda_1|\left(1+|\lambda_2|\,l^{\vartheta}+|\lambda_2\lambda_3|\,l^{\vartheta}r^{\vartheta}+|\lambda_3|\,l^{\vartheta}\right)\right)+\frac{1}{\gamma_r}\left(|\lambda_2|+|\lambda_2\lambda_3|\,r^{\vartheta}\right)\right) \\
&\times\left(\frac{1}{\gamma_r}\left(|\lambda_2|+|\lambda_2\lambda_3|\,r^{\vartheta}\right)+\frac{|\lambda_3|}{\gamma_q}\right)\left(\prod_{j_1=[l/2]}^{l-1}\left|\psi\left(\frac{\lambda_1 g(j_1)}{\gamma_l}\right)\right|\right) \\
&\times\left(\prod_{j_2=[r/2]}^{r-1}\left|\psi\left(\frac{\lambda_2 g\left(j_2\right)}{\gamma_r}\right)\right|\right)\left(\prod_{j_3=[q/2]}^{q-1}\left|\psi\left(\frac{\lambda_3 g\left(j_3\right)}{\gamma_q}\right)\right|\right) d\lambda_1 d\lambda_2 d\lambda_3 \\
\leq\;& \frac{C}{\gamma_l\gamma_r\gamma_q}\left(\frac{l^{2\vartheta}+l^{\vartheta}r^{\vartheta}}{\gamma_l}+\frac{r^{\vartheta}}{\gamma_r}\right)\left(\frac{r^{\vartheta}}{\gamma_r}+\frac{1}{\gamma_q}\right).
\end{aligned}$$

Thus we need to consider

$$\left(\frac{\gamma_n}{n}\right)^2 \sum_{l=\left[n\frac{k-1}{m}\right]+1}^{\left[n\frac{k}{m}\right]} \sum_{r=1}^{n_{mk}}\sum_{q=1}^{n_{mk}} \frac{1}{\gamma_l\gamma_r\gamma_q}\left(\frac{l^{\vartheta}+l^{\vartheta}r^{\vartheta}}{\gamma_l}+\frac{r^{\vartheta}}{\gamma_r}\right)\left(\frac{r^{\vartheta}}{\gamma_r}+\frac{1}{\gamma_q}\right). \tag{110}$$

We have, similar to (105),

$$\left(\frac{\gamma_n}{n}\right)^2 \sum_{l=\left[n\frac{k-1}{m}\right]+1}^{\left[n\frac{k}{m}\right]} \sum_{r=1}^{n_{mk}}\sum_{q=1}^{n_{mk}} \frac{l^{\vartheta}r^{\vartheta}}{\gamma_l^2\gamma_r^2\gamma_q} \leq \begin{cases} Cn^{H-1}(\log n)^2 & \text{if } 2H-\vartheta \geq 1 \\ Cn^{1-3H+2\vartheta} & \text{if } 2H-\vartheta < 1. \end{cases}$$

Essentially the same holds for all other terms in (110) except for

$$\left(\frac{\gamma_n}{n}\right)^2 \sum_{l=\left[n\frac{k-1}{m}\right]+1}^{\left[n\frac{k}{m}\right]} \sum_{r=1}^{n_{mk}}\sum_{q=1}^{n_{mk}} \frac{r^{2\vartheta}}{\gamma_l\gamma_r^3\gamma_q} = \left(\left(\frac{\gamma_n}{n}\right)^2 \sum_{l=\left[n\frac{k-1}{m}\right]+1}^{\left[n\frac{k}{m}\right]} \sum_{q=1}^{n_{mk}} \frac{1}{\gamma_l\gamma_q}\right)\left(\sum_{r=1}^{n_{mk}}\frac{r^{2\vartheta}}{\gamma_r^3}\right),$$

which is of the form (106) because $\sum_{r=1}^{n_{mk}} \frac{r^{2\vartheta}}{\gamma_r^3} < \infty$ in view of $3H-2\vartheta > 1$ (see (98)). Thus the bound (106) holds for (107).

We next consider (108). The ideas involved are the same as those used for (107). First,

$$
\begin{aligned}
&(2\pi)^4 \left|E\left[f\left(S_l\right) f\left(S_{l+r}\right) f\left(S_{l+r+q}\right) f\left(S_{l+r+q+s}\right)\right]\right| \\
\leq\ & \frac{C}{\gamma_l\gamma_r\gamma_q\gamma_s} \int \prod_{j_1=[l/2]}^{l-1} \left|\psi\left(\frac{\lambda_1 g(j_1)}{\gamma_l} + \frac{\lambda_2 g(j_1,r)}{\gamma_r} + \frac{\lambda_3 g(j_1+r,q)}{\gamma_q} + \frac{\lambda_4 g(j_1+r+q,s)}{\gamma_s}\right)\right| \\
&\times \prod_{j_2=[r/2]}^{r-1} \left|\psi\left(\frac{\lambda_2 g\left(j_2\right)}{\gamma_r} + \frac{\lambda_3 g\left(j_2,q\right)}{\gamma_q} + \frac{\lambda_4 g\left(j_2+q,s\right)}{\gamma_s}\right)\right| \\
&\times \prod_{j_3=[q/2]}^{q-1} \left|\psi\left(\frac{\lambda_3 g\left(j_3\right)}{\gamma_q} + \frac{\lambda_4 g\left(j_3,s\right)}{\gamma_s}\right)\right| \prod_{j_4=[s/2]}^{s-1} \left|\psi\left(\frac{\lambda_4 g\left(j_4\right)}{\gamma_s}\right)\right| \\
&\times \left|\widehat{f}\left(\frac{\lambda_1}{\gamma_l} - \frac{\lambda_2}{\gamma_r}\right) \widehat{f}\left(\frac{\lambda_2}{\gamma_r} - \frac{\lambda_3}{\gamma_q}\right) \widehat{f}\left(\frac{\lambda_3}{\gamma_q} - \frac{\lambda_4}{\gamma_s}\right) \widehat{f}\left(\frac{\lambda_4}{\gamma_s}\right)\right| d\lambda_1 d\lambda_2 d\lambda_3 d\lambda_4. \qquad (111)
\end{aligned}
$$

This is obtained using the same arguments used in obtaining the bound (109). In exactly the same way as in (109), we first make suitable transformations and then see that, using $\left|\widehat{f}(\lambda)\right| \leq C|\lambda|$, (111) is bounded by

$$
\frac{C}{\gamma_l\gamma_r\gamma_q\gamma_s}\left(\frac{l^\vartheta + l^\vartheta r^\vartheta + l^\vartheta r^\vartheta q^\vartheta}{\gamma_l} + \frac{r^\vartheta + r^\vartheta q^\vartheta}{\gamma_r}\right)\left(\frac{r^\vartheta + r^\vartheta q^\vartheta}{\gamma_r} + \frac{q^\vartheta}{\gamma_q}\right)\left(\frac{q^\vartheta}{\gamma_q} + \frac{1}{\gamma_s}\right)\frac{1}{\gamma_s}.
$$

In the same way as in (110) it is easy to show, using (98), that the sum

$$
\begin{aligned}
&\left(\frac{\gamma_n}{n}\right)^2 \sum_{l=\left[n\frac{k-1}{m}\right]}^{\left[n\frac{k}{m}\right]} \sum_{r=1}^{n_{mk}} \sum_{q=1}^{n_{mk}} \sum_{s=1}^{n_{mk}} \frac{1}{\gamma_l\gamma_r\gamma_q\gamma_s}\left(\frac{l^\vartheta + l^\vartheta r^\vartheta + l^\vartheta r^\vartheta q^\vartheta}{\gamma_l} + \frac{r^\vartheta + r^\vartheta q^\vartheta}{\gamma_r}\right) \\
&\times \left(\frac{r^\vartheta + r^\vartheta q^\vartheta}{\gamma_r} + \frac{q^\vartheta}{\gamma_q}\right)\left(\frac{q^\vartheta}{\gamma_q} + \frac{1}{\gamma_s}\right)\frac{1}{\gamma_s}
\end{aligned}
$$

is bounded by (106). This completes the verification of (R3). ■

Verification of (R*3). We start with the **remark** that when we verified (R3) for $f(S_l)$, it was clear that the same verification will hold for $f(S_{l,\nu})$ also for any $\nu \geq 1$, where $S_{l,\nu}$ is as in (59), because S_l and $S_{l,\nu}$ have the same structural form. Recall that $\zeta^*_{nmk} = \sqrt{\frac{\gamma_n}{n}} \sum_{l=\left[n\frac{k-1}{m}\right]+l_0}^{\left[n\frac{k}{m}\right]} f(S_l)\,\omega_l$, where ω_l is a sum of ν terms, $\nu \geq 1$. It is convenient to prove the claim by induction on ν. Therefore we shall use the notation

$$
\omega_{l,\nu} = \sum_{j=l-\nu+1}^{l} d_{l-j}\eta_j.
$$

For $\nu = 1$, we have $\omega_{l,1} = \eta_l$, and

$$
\sum f(S_l)\,\eta_l = \sum \left(f(S_l)\,\eta_l - f_*(S_{l,1})\right) + \sum f_*(S_{l,1}), \qquad (112)
$$

where

$$f_*(S_{l,1}) = E_{l-1}[f(S_l)\eta_l]$$

is as in (60), corresponding to $\nu = 1$. (Here and in the rest of the proof the sum $\sum$ stands for $\sum_{l=[n\frac{k-1}{m}]+l_0}^{[n\frac{k}{m}]}$.) According to Lemma 7, $f_*(x)$ satisfies the conditions of $f(x)$ of Theorem 1. Therefore, in view of the remark made above, we are implicitly assuming that (R3) is verified for $f_*(S_{l,1})$.

For the remaining term in (112) note that $f(S_l)\eta_l - f_*(S_{l,1})$ form martingale differences, and hence (see Hall and Heyde (1980, Theorem 2.11))

$$\begin{aligned} & E\left[\left(\sum\left(f(S_l)\eta_l - f_*(S_{l,1})\right)\right)^4\right] \\ \leq\ & CE\left[\left(\sum E_{l-1}\left[(f(S_l)\eta_l)^2\right]\right)^2\right] + C\sum E\left[(f(S_l)\eta_l)^4\right]. \end{aligned} \tag{113}$$

We have $E_{l-1}\left[(f(S_l)\eta_l)^2\right] = f_*^{(2)}(S_{l,1})$ where $f_*^{(2)}(x) = E[f^2(x+\xi_l)\eta_l^2]$. That is,

$$\left(\frac{\gamma_n}{n}\right)^2 E\left[\left(\sum E_{l-1}\left[(f(S_l)\eta_l)^2\right]\right)^2\right] = \left(\frac{\gamma_n}{n}\right)^2 E\left[\left(\sum f_*^{(2)}(S_{l,1})\right)^2\right].$$

In addition $\int f_*^{(2)}(x)\,dx < \infty$, which will imply (see the bound in (101) or the Remark at the end of Jeganathan (2004a)) that

$$\left(\frac{\gamma_n}{n}\right)^2 E\left[\left(\sum_{l=[n\frac{k-1}{m}]+1}^{[n\frac{k}{m}]} f_*^{(2)}(S_{l,1})\right)^2\right] \leq C\left(\frac{\gamma_n}{n}\right)^2 \sum_{l=[n\frac{k-1}{m}]+1}^{[n\frac{k}{m}]} \frac{1}{\gamma_l}\left(1 + \sum_{r=1}^{n_{mk}} \frac{1}{\gamma_r}\right) \tag{114}$$

and hence, because $\max_{1\leq k\leq m} \frac{\gamma_n}{n}\sum_{r=1}^{n_{mk}} \frac{1}{\gamma_r} \sim C\left(\frac{1}{m}\right)^{1-H}$ and $\frac{\gamma_n}{n}\sum_{l=1}^{n}\frac{1}{\gamma_l} \leq C$,

$$\left(\frac{\gamma_n}{n}\right)^2 \sum_{k=1}^{m} E\left[\left(\sum_{l=[n\frac{k-1}{m}]+1}^{[n\frac{k}{m}]} f_*^{(2)}(S_{l,1})\right)^2\right] \to 0$$

as $n \to \infty$ first and then $m \to \infty$.

Similarly $\frac{\gamma_n}{n}\sum_{l=1}^{n} E\left[(f(S_l)\eta_l)^4\right] \leq C$, that is, $\left(\frac{\gamma_n}{n}\right)^2\sum_{l=1}^{n} E\left[(f(S_l)\eta_l)^4\right] \leq C\frac{\gamma_n}{n} \to 0$.

Thus (R*3) holds for the case $\nu = 1$. We remark that the same arguments show that (R*3) holds also for $f(S_{l,1})\eta_{l-1} = f(S_{l,1})\omega_{l-1,1}$.

Now suppose that $\nu = i$, $i \geq 2$, and that (R*3) holds for $\nu = i-1$. Taking into account the preceding remark, this means we can assume that (R*3) holds for $f(S_{l,1})\omega_{l,i-1}^*$, where

$$\omega_{l,i-1}^* = \omega_{l,i} - \eta_l = \sum_{j=l-i+1}^{l-1} d_{l-j}\eta_j.$$

We have

$$\sum f(S_l)\,\omega_{l,i} = \sum \left(f(S_l)\,\omega_{l,i} - E_{l-1}\left[f(S_l)\,\omega_{l,i}\right]\right) + \sum E_{l-1}\left[f(S_l)\,\omega_{l,i}\right],$$

where

$$\sum E_{l-1}\left[f(S_l)\,\omega_{l,i}\right] = \sum E_{l-1}\left[f(S_l)\left(\omega_{l,i}-\eta_l\right)\right] + \sum E_{l-1}\left[f(S_l)\,\eta_l\right].$$

Here $E_{l-1}\left[f(S_l)\,\eta_l\right] = f_*(S_{l,1})$ is as before, for which as noted earlier the verification of (R3) will be the same as that for S_l. Also,

$$E_{l-1}\left[f(S_l)\left(\omega_{l,i}-\eta_l\right)\right] = E_{l-1}\left[f(S_l)\,\omega^*_{l,i-1}\right] = \omega^*_{l,i-1}E_{l-1}\left[f(S_l)\right] = g(S_{l,1})\,\omega^*_{l,i-1},$$

where $g(x) = E\left[f(x+\xi_l)\right]$. This form is the same as that of $f(S_{l,1})\,\omega^*_{l,i-1}$, for which we have assumed the induction hypothesis that (R*3) holds.

Regarding the remaining term $\sum \left(f(S_l)\,\omega_{l,i} - E_{l-1}\left[f(S_l)\,\omega_{l,i}\right]\right)$, which is a sum of martingale differences, we have the bound analogous to (113), in which the second term is treated in the same way as the second term in (113). The first term is $E\left[\left(\sum E_{l-1}\left[\left(f(S_l)\,\omega_{l,i}\right)^2\right]\right)^2\right]$, where (recall $\omega_{l,i} = \omega^*_{l,i-1} + \eta_l$)

$$E_{l-1}\left[\left(f(S_l)\,\omega_{l,i}\right)^2\right] \le 2\left|\omega^*_{l,i-1}\right|^2 E_{l-1}\left[f^2(S_l)\right] + 2E_{l-1}\left[f^2(S_l)\,\eta_l^2\right].$$

Letting $g(S_{l,1}) = E_{l-1}\left[f^2(S_l)\right]$, it is implicit in the arguments of the verification of (R*2) that the bound in (114) holds for

$$\left(\frac{\gamma_n}{n}\right)^2 E\left[\left(\sum_{l=\left[n\frac{k-1}{m}\right]+1}^{\left[n\frac{k}{m}\right]} \left|\omega^*_{l,i-1}\right|^2 g(S_{l,1})\right)^2\right]$$

also. Also the term $E_{l-1}\left[f^2(S_l)\,\eta_l^2\right] = f^{(2)}_*(S_{l,1})$ has already been treated. It thus follows that (R*3) holds for $\sum f(S_l)\,\omega_l$ when $\nu = i$. This completes the verification of (R*3). ■

Proof of Theorem 3. First consider the case $r = 1$. Then, in view of Proposition 13*, it is enough to show that

$$E\left[\left(\frac{\gamma_n}{n}\sum_{j=1}^{n} f(S_j, S_{j+1}) - \frac{\gamma_n}{n}\sum_{k=1}^{m}\sum_{l=1}^{n_{mk}} E_{\left[n\frac{k-1}{m}\right]}\left[f\left(S_{\left[n\frac{k-1}{m}\right]+l}, S_{\left[n\frac{k-1}{m}\right]+l+1}\right)\right]\right)^2\right] \to 0. \tag{115}$$

The proof of this is the same as that of (100) but now the inequalies (99) and (111) will be used. To see this note that the inequality (111) holds with $f(S_l, S_{l+r})\, f(S_{l+r+q}, S_{l+r+q+s})$

in place of $f(S_l)\, f(S_{l+r})\, f(S_{l+r+q})\, f(S_{l+r+q+s})$, and hence in particular (taking $r = 1, q = i-1, s = 1$)

$$\left|E\left[f\left(S_l, S_{l+1}\right) f\left(S_{l+i}, S_{l+i+1}\right)\right]\right| \leq \frac{C}{\gamma_l \gamma_i}.$$

Similarly (99) gives $\left|E\left[f^2\left(S_l, S_{l+1}\right)\right]\right| \leq \frac{C}{\gamma_l}$. Hence the proof of (115) is the same as that of (100). The proof of the general case $r \geq 2$ is similar; using the statement of Lemma 14 for the general case $r \geq 2$. We omit the details. ■

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