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Obstruction Theory and Characteristic Classes

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1. Euler Class

1.1. The Lefschetz Fixed Point Theorem.

Definition 1.1.1 (Lefschetz Number)**.** Let X be a topological space with total rational homology $H_*(X, \mathbb{Q})$ finite dimensional. Let $f : X \to X$ be a continuous self map of X. We define the *Lefschetz number of* f by the formula:

$$L(f) := \sum_{i=0}^{\infty} (-1)^i \, Tr[f_{i*} : H_i(X, \mathbb{Q}) \to H_i(X, \mathbb{Q})]$$

Clearly, by definition, it is a homotopy invariant of f. Also note that $L(f) = \chi(X)$, the Euler characteristic of X, for any map homotopic to id_X.

We now state the following remarkable theorem of Lefschetz.

Theorem 1.1.2 (Lefschetz Fixed Point Theorem)**.** Let X be a finite simplicial complex, and $f : X \to X$ be a continuous self-map of X. If f has no fixed points, then $L(f) = 0$.

Remark 1.1.3. The theorem is sharp. If one drops the finiteness (=compactness) condition, one sees that translation $T_a : \mathbb{R} \to \mathbb{R}$ by $a \neq 0$ has no fixed points, but the Lefschetz number $L(T_a) = 1$. Similarly (Exercise!) one can easily construct a self map f of $X = S^1 \vee S^1$ with the wedge point as the only fixed point, but with $L(f) = 0$, which shows that the converse statement is false.

It is natural to ask whether $L(f)$ actually counts the number of fixed points in some sense. That this is not (naively) so is clear from the following example.

Example 1.1.4. Let $X = D^2 = \{z \in \mathbb{C} :\mid z \mid \leq 1\}$, the closed unit disc, which is a finite simplicial complex. Let $f : D^2 \to D^2$ be the self map $f(z) = z^2$. Note that $F(f) = \{0, 1\}$, but $L(f) = Tr[f_{0*} : \mathbb{Q} \to \mathbb{Q}] = 1$. Worse yet, the map $g(z) = \frac{z^2}{2}$ is homotopic to f, so $L(f) = L(g)$, but $F(g) = \{0\} \neq F(f)$. So $L(f)$ certainly doesn't compute the number of fixed points of f.

As an example where it *does* count the fixed points, we have the following:

Example 1.1.5. Let $X = S^2 = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere, and let $f : X \to X$ be the map $z \mapsto z^2$. Then the set of fixed points $F(f) = \{0, 1, \infty\}$ is of cardinality 3. On the other hand one finds that f_* is multiplication by 2 on $H_2(X, \mathbb{Q}) = \mathbb{Q}$, and identity on $H_0(X, \mathbb{Q}) = \mathbb{Q}$, and $H_i = 0$ for $i \neq 0, 2$. So $L(f) = 3 = \#F(f)$.

So let us review what we need for $L(f)$ to count some number, preferably $\#(F(f))$, the set of fixed points of F, assuming it is finite. We observe that for any topological space X, and continuous map $f : X \to X$ with finite fixed point set $F(f)$, $F(f)$ is the cardinality of the intersection $\Gamma_f \cap \Delta_X$, where $\Gamma_f := \{(x, f(x)) \in X \times X : x \in X\}$ is the *graph of* f in $X \times X$, and $\Delta_X := \Gamma_{Id_X}$ is the *diagonal* in $X \times X$. To compute the cardinality of this set algebraically, one needs a good homological or cohomological *intersection theory* for $X \times X$.

Suppose X is a compact connected n-manifold that is *orientable.* In this situation, $X \times X$ is also compact connected orientable of dimension $2n$, and there is a beautiful (co)-homological intersection theory on X and $X \times X$ due to Poincare. The upshot is that for X as above of dimension n there is a *Poincare duality isomorphism*: $D_X := (-) \cap [X] : H^i(X, \mathbb{Z}) \to H_{n-i}(X, \mathbb{Z})$, where $[X] \in H_n(X, \mathbb{Z})$ is the *fundamental* or *orientation* class of X. To compute the *intersection number* of a homology class $\alpha \in H_i(X, \mathbb{Z})$ with a complementary dimensional $(n-i)$-class $\beta \in H_{n-i}(X, \mathbb{Z})$, one defines:

$$\alpha\#\beta := \langle D_X^{-1}(\alpha) \cup D_X^{-}(\beta),\, [X]\rangle = \langle D_X^{-1}(\beta), \alpha\rangle$$

Similarly, one can define the intersection number of complementary dimensional *cohomology classes* by $\alpha\#\beta := \langle \alpha \cup \beta, [X]\rangle$.

Now for X as above the map $(1, f) : X \to X \times X$ gives the *graph homology class* $[\Gamma_f] := (1, f)_*[X]$ and the *diagonal homology class* $[\Delta_X] := \Delta_*[X]$, both in $H_n(X \times X, \mathbb{Z})$. They are of complemetary dimension in $X \times X$, and one can calculate their intersection number in $X \times X$ as defined above. Indeed:

Theorem 1.1.6 (Lefschetz-Poincare)**.** Let X be a compact connected oriented manifold, and let $f : X \to X$ be any continuous map. Then there is the identity:

$$L(f) = [\Gamma_f]\#[\Delta_X]$$

where the intersection number on the right is computed in $X \times X$. If X is a compact *smooth manifold*, and f is a *smooth map* such that Γ_f intersects Δ_X *transversely* in in the finite set Σ, the intersection number on the right is the sum $\sum_{x \in \Sigma} \epsilon(x)$, where $\epsilon(x) = \pm 1$ is a sign that computes whether the local orientation of Γ_f followed by the local orientation of Δ_X at x is $+$ or $-$ the local orientation of $X \times X$ at x.

Remark 1.1.7. (The transversality assumption) If X is a smooth compact oriented manifold, and $f : X \to X$ is merely a *continuous map*, one can still geometrically interpret the intersection number on the right by first making a small perturbation of f to a smooth map f_1 which is homotopic to f, and then again perturbing f_1 by a small amount to get a smooth f_2 such that Γ_{f_2} meets Δ_X transversely at all points of intersection. (Deep results of Sard and Thom are needed for this). One can construct an easy example of a map $f : S^1 \to S^1$ to see that transversality is essential for the geometric intersection number to agree with the algebraic one. (Make Γ_f meet Δ tangentially at exactly one point. Note $[\Delta]\#[\Delta] = 0$ in the torus $S^1 \times S^1$.)

1.2. **The Euler Characteristic.** We now focus on the smooth case. Suppose X is a smooth compact oriented manifold of dimension n, and suppose we are given a smooth vector field $v : X \to TX$ on X. (Here TX denotes the tangent bundle of X). Then by the theorem on existence and uniqueness of ODE's, there is a 1-parameter group of self-diffeomorphisms $\phi_t : X \to X$, with $\phi_o = Id_X$, and $v(x) = \frac{d\phi_t(x)}{dt}_{|t=0}$ for all $x \in X$.

Further assume that v has an *isolated set of zeros* $Z(v) = \{x \in X : v(x) = 0\}$. Then for all $t \in \mathbb{R}$, $\phi_t(x) = x$ for all $x \in Z(v)$, so $Z(v) \subset F(\phi_t)$ for each t. By using the mean-value theorem one can see that for $\epsilon > 0$ small enough, $Z(v) = F(\phi_\epsilon)$. Also for all t, ϕ_t is homotopic to $\phi_0 = Id_X$. Thus for such small ϵ, it is reasonable to expect that $\#Z(v) = \#F(\phi_\epsilon) = L(\phi_\epsilon) = L(Id_X) = \chi(X)$.

It is clear that if we can guarantee the transversality condition for $f = \phi_\epsilon$, then we would have such an equality by the Theorem 1.1.6 above. To do this one would need a condition on v: i.e. that v have only *non-degenerate* zeros. More precisely

Theorem 1.2.1 (Poincare-Hopf Index Theorem)**.** Let X be a smooth compact connected oriented manifold, and let $v : X \to TX$ be a smooth vector field on X, with zero set $Z(v)$. Assume that for each $x \in Z(v)$, the Jacobian of the map $Dv(x) : T_x(X) \to T_x(X)$ is non-singular (i.e. X is *non-degenerate*). Then $Z(v)$ is a finite set of points, and $\operatorname{Im} v$ is transverse to the zero section. Then we have:

$$\chi(X) = \sum_{x \in Z(v)} ind_x v$$

where $ind_x v := \epsilon(x) = \pm 1$ can be computed by looking at the sign of det $Dv(x)$.

Clearly a smooth vector field with no zeros is non-degenerate and the sum on the right is 0. As a consequence,

Corollary 1.2.2. If X as above has an everywhere non-vanishing smooth vector field then its Euler characteristic is zero. In fact, the converse is also true (that is, the Euler characteristic is the *only* obstruction, though it requires more work to prove).

In particular, every compact connected Lie Group has Euler characteristic 0. The even-dimensional spheres S^{2n} cannot have a nowhere vanishing vector field (called the *Hairy Ball Theorem*), and in particular, cannot be given the structure of a Lie Group compatible with their differentiable structure. On the other hand, the odd-dimensional spheres S^{2n-1} all have Euler characteristic zero, and it is an easy exercise to construct nowhere vanishing vector fields on them (using the embedding into $\mathbb{R}^{2n} = \mathbb{C}^n$ and multiplication by i.) Incidentally, only the spheres S^0, S^1 and S^3 are topological groups with their usual topology. S^7 can be made a 'non-associative' topological group. This is a very deep result due to J.F. Adams.

Thus, we see that the Euler characteristic $\chi(X)$ *is an obstruction to finding a nowhere vanishing vector field* on X. That is, it is an obstruction to finding a trivial vector subbundle of rank 1 in the tangent bundle TX. As we saw above, the Euler characteristic is a homotopy and hence a topological invariant. It is remarkable that the obstruction to the solution of a smooth problem on X is a topological invariant! In fact the result of Poincare-Hopf is true even for *continuous* vector-fields with isolated zeros, provided one defines the index suitably.

1.3. **Thom class and Euler class.** Let X be a compact oriented connected smooth manifold. It is not hard to see that a tubular neighbourhood U of the diagonal Δ in $X \times X$ is diffeomorphic to the closed unit disc bundle $D(TX)$ (with respect to any Riemannian metric) of the tangent bundle TX of X (one again needs the Picard Theorem on ODE's). Under this diffeo, the diagonal $\Delta \subset U$ goes over to the zero-section 0_{TX} of TX. Let $v : X \to TX$ be a smooth vector field, which we can globally scale by $\epsilon > 0$ to assume that $\|v(x)\| < 1$ for all $x \in X$.

Then under the above diffeo, the intersection number $\Gamma_{\phi_\epsilon} \# \Delta$ discussed earlier can be replicated by the intersection number $\mathrm{Im}\,(v) \# 0_{TX}$ of the section v with the zero section 0_{TX}.

Using this as the point of departure, let us consider $\pi : E \to X$ to be any *oriented* vector bundle of rank n on an arbitrary (paracompact, Hausdorff, 2nd countable) topological space X. Saying E is oriented is equivalent to requiring that the top exterior power bundle $\Lambda^n E \to X$ is a trivial line bundle.

Question: Is there a way to use the oriented compact $2n$-manifold $D(E)$ (with boundary the unit sphere bundle $S(E) := \partial\, D(E)$) to do intersection theory and compute the intersection number of the image of a section $s : X \to E$ and the zero-section 0_E?

There is a version of Poincare duality available for an oriented manifold with boundary such as $(D(E), S(E))$, called Alexander-Lefschetz Duality. But the first step is to identify the homology (or cohomology) class in $H^*(D(E), S(E))$ which corresponds to the diagonal class in $X \times X$ discussed earlier. To this end we have the famous:

Theorem 1.3.1 (Thom Isomorphism)**.** Let $\pi : E \to X$ be an oriented rank n-bundle on a topological space X (always assumed paracompact, Hausdorff and 2-nd countable hereafter). Then there is a unique relative cohomology class $U_E \in H^n(D(E), S(E); \mathbb{Z})$ called the *Thom class of* E such that U_E restricts to a generator of $H^n(D(E_x), S(E_x); \mathbb{Z}) = H^n(D^n, S^{n-1}; \mathbb{Z}) \simeq \mathbb{Z}$ for each $x \in X$. Furthermore, the map:

$$\begin{aligned} \phi_E : H^i(X, \mathbb{Z}) &\to H^{i+n}(D(E), S(E); \mathbb{Z}) \\ \alpha &\mapsto (\pi^* \alpha) \cup U_E \end{aligned}$$

is an isomorphism called the *Thom isomorphism.* (The right hand side is defined via the cup product $H^i(D(E), \mathbb{Z}) \otimes H^n(D(E), S(E); \mathbb{Z}) \xrightarrow{\cup} H^{i+n}(D(E), S(E); \mathbb{Z})$). If E is not assumed oriented, the same result is true with $\mathbb{Z}_2$-coefficients replacing $\mathbb{Z}$-coefficients throughout.

The result is not at all difficult to prove. The existence of U_E follows by the definition of orientability, which says that the sheaf or 'bundle' of abelian groups $H(D(E_x), S(E_x); \mathbb{Z})$ on X, (a *local system* or *locally constant sheaf* of abelian groups by local triviality of E) is actually a constant sheaf. Then the isomorphism itself follows from the Kunneth formula for E a trivial bundle, and hence for any locally trivial bundle E by restricting to trivialising open sets and using a Mayer-Vietoris patching argument.

To motivate the forthcoming definition of the Euler class, *assume X is a compact connected oriented topological manifold of dimension n.* Then the orientablility of E implies that $(D(E), S(E))$ is a compact oriented connected $2n$-manifold with boundary. It turns out that this Thom class $U_E \in H^n(D(E), S(E); \mathbb{Z})$ is Alexander-Lefschetz dual to the complementary dimensional homology class $[0_E] \in H_n(D(E); \mathbb{Z})$ under the Alexander-Lefschetz Duality isomorphism $H^i(D(E), S(E); \mathbb{Z}) \to H_{2n-i}(D(E); \mathbb{Z})$. Since a section $s : X \to D(E)$ is homotopic to the 0-section $0 : X \to D(E)$, it follows that the intersection number $s_*[X]\#[0_E]$ is precisely the integer $\chi(E)$ defined by $U_E \cup U_E = \chi(E)\phi_E[X]$. This is because by the Thom isomorphism above $\phi_E[X]$ is the generating (orientation) class in $H^{2n}(D(E), S(E); \mathbb{Z})$. Hence it is reasonable to make the following definition for *any* topological space X.

Definition 1.3.2 (Euler class)**.** Let E be an oriented rank n-bundle on a topological space X. Define the *Euler-class* $e(E) \in H^n(X, \mathbb{Z})$ of E by the formula $e(E) := \phi_E^{-1}(U_E \cup U_E)$. (In the particular case when X is an oriented manifold it satisfies $\langle e(E), [X]\rangle = \chi(E)$, by the above discussion).

In case E is not assumed orientable, the same definition applied with $\mathbb{Z}_2$ coefficients gives the *top Stiefel-Whitney class* $w_n(E) \in H^n(X, \mathbb{Z}_2)$. When E is orientable, $w_n(E)$ is the mod 2 reduction of $e(E)$.

Theorem 1.3.3. Let E, be an oriented rank n bundle on X a compact oriented connected n-manifold. Assume that there is a nowhere vanishing section $s : X \to E$ of E. Then $e(E) = 0$. When E, X are not assumed oriented, $w_n(E) = 0$.

The proof is immediate, since a section s with no zeros implies that the intersection number of $s(X)$ and 0_E is zero. But more remarkably, we have:

Theorem 1.3.4. Let X be any finite simplicial complex of dimension n, and E be an oriented vector bundle on X. Then E admits a nowhere vanishing section s if and only if $e(X) = 0$.

The proof of this stronger fact needs the identification of $e(E)$ via obstruction theory.

The Euler class satisfies the following properties:

Proposition 1.3.5 (Properties of $e(E)$ and $w_n(E)$)**.** The Euler and top Steifel-Whitney class satisfy the following:

- (i): (Functoriality under pullbacks) Let E be an oriented rank n bundle on a topological space X, and $f : Y \to X$ a continuous map. Then $e(f^*E) = f^*e(E)$, where $f^* : H^n(X, \mathbb{Z}) \to H^n(Y, \mathbb{Z})$ is the induced map on cohomology. Analogous statement with $\mathbb{Z}_2$-coefficients and $w_n(E)$ for arbitrary (not necessarily orientable) E.
- (ii): (Trivial bundles) For a trivial bundle $E = \epsilon^n$, $e(E) = 0$. Likewise, $w_n(E) = 0$.
- (iii): (Whitney sum formula) For two oriented bundles E and F on X of ranks n and m respectively, the Euler class of the rank $m+n$ bundle $E \oplus F$ is given by $e(E \oplus F) = e(E) \cup e(F) \in H^{m+n}(X, \mathbb{Z})$. Analogous statement for w_n, when E and F are not assumed orientable.
- (iv): (Euler characteristic) If X is a smooth compact connected oriented manifold of dimension n, then $e(TX) = \chi(X)[X]$. (The last 3 properties are also satisfied for the silly definition $e(E) \equiv 0$ for all bundles E! This property shows that there are bundles with non- zero Euler class).

Note that the only if part of the above Theorem 1.3.4 follows from (ii) and (iii) above.

2. Chern and Stiefel-Whitney classes

2.1. **Real and complex line bundles.** Quite analogously to real vector bundles of rank n, we also have the notion of complex vector bundles of (complex) rank n. By default, they are real vector bundles of (real) rank $2n$. Most importantly, by the fact that $\mid \det A \mid^2 = \det A_{\mathbb{R}}$, where A is a complex linear map (of $\mathbb{C}^n$), and $A_{\mathbb{R}}$ is the corresponding real linear map (of $\mathbb{R}^{2n}$), it follows that *all complex vector bundles are oriented when considered as real bundles.* Furthermore, this real orientation is *uniquely and canonically* determined.

Definition 2.1.1 (First Chern class and first Stiefel-Whitney class)**.** A *complex line bundle*, i.e. a complex vector bundle L of $rk_{\mathbb{C}}L = 1$ on X is automatically a real oriented rank 2 bundle $L_{\mathbb{R}}$ on X (by forgetting the complex structure). We define its first Chern class by:

$$c_1(L) := e(L_{\mathbb{R}}) \in H^2(X, \mathbb{Z})$$

(It turns out that L is a trivial bundle iff $c_1(L) = 0$.) Finally, we define the *total Chern class* $c(L) := 1 + c_1(L) \in H^*(X, \mathbb{Z})$.

Likewise, if L is a real line bundle on X, its first Stiefel-Whitney class is the top Stiefel-Whitney class $w_1(L) \in H^1(X, \mathbb{Z}_2)$ defined earlier. (It turns out that L is a trivial line bundle iff $w_1(L) = 0$). Define the total Stiefel-Whitney class $w(L) := 1 + w_1(L) \in H^*(X, \mathbb{Z}_2)$.

By the Property (i) in Proposition 1.3.5, it follows that the first (resp. total) Chern class and first (resp. total) Stiefel-Whitney class are functorial with respect to pullbacks. We recall the *real projective space* $\mathbb{P}^n(\mathbb{R})$ of all real 1-dimensional subspaces in $\mathbb{R}^{n+1}$. On $\mathbb{P}^n(\mathbb{R})$ there is the *real tautological line bundle* $\lambda_{\mathbb{R}}$, whose fibre over $[v] \in \mathbb{P}^n(\mathbb{R})$ is the line $\mathbb{R}v$. Similarly, there is the *complex tautological line bundle* $\lambda_{\mathbb{C}}$ on the analogously defined complex projective space $\mathbb{P}^n(\mathbb{C})$.

The following proposition is crucial in the sequel.

Proposition 2.1.2. Let $\mathbb{F} = \mathbb{R}$ (resp. $\mathbb{C}$), and let $\mathbb{P}^n(\mathbb{F})$ denote the projective space of dimension n over $\mathbb{F}$. Let c (resp. w) denote the generating cohomology class in $H^2(\mathbb{P}^n(\mathbb{C}), \mathbb{Z})$ evaluating to 1 on $S^2 = \mathbb{P}^1(\mathbb{C}) \subset \mathbb{P}^n(\mathbb{C})$ with its natural orientation as a complex manifold (resp. the unique non-zero generator in $H^1(\mathbb{P}^n(\mathbb{R}), \mathbb{Z}_2)$). Let $\lambda_{\mathbb{F}}$ denote the *tautological* $\mathbb{F}$*-line bundle* on $\mathbb{P}^n(\mathbb{F})$ as defined above. Then $c_1(\lambda_{\mathbb{C}}) = -c$ and $w_1(\lambda_{\mathbb{R}}) = w$.

We recall here that $H^*(\mathbb{P}^n(\mathbb{C}), \mathbb{Z}) = \mathbb{Z}[c]/\langle c^{n+1}\rangle$ as a $\mathbb{Z}$-algebra. Likewise, $H^*(\mathbb{P}^n(\mathbb{R}), \mathbb{Z}_2) = \mathbb{Z}_2[w]/\langle w^{n+1}\rangle$ as a $\mathbb{Z}_2$-algebra.

2.2. **The splitting principle.**

Definition 2.2.1 (Projective bundle)**.** Let $\pi : E \to X$ be a $\mathbb{F}$-vector bundle on a topological space X of $rk_{\mathbb{F}}E = n$. We define the *projective bundle* $P(E)$ of E to be a fibre bundle $p : P(E) \to X$ on X, with fibre $\mathbb{P}(E_x) \simeq \mathbb{P}^{n-1}(\mathbb{F})$ (viz. the $\mathbb{F}$-projective space of the $\mathbb{F}$-vector space E_x) over x, and p the obvious projection mapping this entire fibre to x.

On $P(E)$, there is also a *tautological* $\mathbb{F}$*-line bundle* λ_E, whose fibre over $[v_x] \in \mathbb{P}(E_x)$ is the $\mathbb{F}$-line $\mathbb{F}v_x \subset E_x = (p^*E)_{v_x}$. Thus by definition, we have an exact sequence of bundles on $P(E)$:

$$0 \to \lambda_E \to p^*E \to F \to 0$$

where F is a an $\mathbb{F}$-vector bundle of $rk_{\mathbb{F}}F = n-1$. Now since X is a nice topological space, so is $P(E)$, and all continuous vector bundles on it carry Riemannian/Hermitian metrics. Thus, using such a metric on p^*E, we get that there is a bundle splitting map which is $\mathbb{F}$- linear, and

$$p^*E \simeq \lambda_E \oplus F$$

To summarise, by pulling back the original bundle E of $\mathbb{F}$-rank n to $P(E)$, we have managed to split off a line bundle λ_E, and are left with a bundle F of lower rank $n-1$. Thus, if we can understand what p^* is doing on cohomology, we can define higher Chern (or Stiefel-Whitney) classes by inducting on the rank, and finally appealing to the Definition 2.1.1 for $\mathbb{F}$-line bundles. This is possible by the:

Theorem 2.2.2 (Leray-Hirsch)**.** Let $\mathbb{F} = \mathbb{C}$ (resp. $\mathbb{R}$), and let $\Lambda = \mathbb{Z}$ (resp. $\mathbb{Z}_2$). Let E be an $\mathbb{F}$-vector bundle on the topological space X, with $p : P(E) \to X$ the corresponding ($\mathbb{F}$-) projective bundle. Let λ_E denote the corresponding tautological $\mathbb{F}$-line bundle on $P(E)$ introduced above. Let $c := -c_1(\lambda_E) \in H^2(P(E), \mathbb{Z})$ (resp. $w_1(\lambda_E) \in H^1(P(E), \mathbb{Z}_2)$. Then the ring homomorphism $p^* : H^*(X, \Lambda) \to H^*(P(E), \Lambda)$ is injective, and via this ring homomorphism $H^*(P(E), \Lambda)$ is a free module over $H^*(X, \Lambda)$ with basis $1, c, c^2, .., c^{n-1}$.

Again, the proof is is similar in spirit to the one sketched for the Thom isomorphism theorem. One first notes that since the line bundle λ_E on $P(E)$ restricts to the tautological line bundle on $\mathbb{P}(E_x) \simeq \mathbb{P}^{n-1}(\mathbb{F})$ by definition, the restriction of c to each fibre $\mathbb{P}(E_x)$ gives the cohomology algebra generator $c_x \in H^*(\mathbb{P}(E_x), \Lambda)$ for that fibre (by the Proposition 2.1.2 above). Using the fact that E (and therefore $P(E)$) is locally trivial, hence a locally a product and the Kunneth formula shows the result is true locally. Then one again uses Mayer-Vietoris patching. Alternatively, the classes $1, c, .., c^{n-1}$ can be used to give a morphism of the (degenerate) Kunneth spectral sequence with the Leray spectral sequence for the fibre bundle $P(E)$, and noting that it induces isomorphisms at E_2, it must produce an isomorphism of the limits.

Now we can define the higher Chern and Stiefel-Whitney classes:

Definition 2.2.3 (Chern and Stiefel-Whitney classes)**.** Let E be an $\mathbb{F}$-vector bundle on a topological space X, and let $P(E)$ be its $\mathbb{F}$-projective bundle. Let the class c be defined as in the statement of the Leray-Hirsch Theorem 2.2.2 above. Since $H^*(P(E), \Lambda)$ is free on $H^*(X, \Lambda)$ with basis $1, c, c^2, .., c^{n-1}$, and p^* is injective, (by Leray-Hirsch) there must be *unique* elements $a_1, .., a_n \in H^*(X, \Lambda)$ such that:

$$c^n + (p^*(a_1) \cup c^{n-1}) + ... + (p^*(a_i) \cup c^{n-i}) + ... + p^*(a_n) = 0$$

When $\mathbb{F} = \mathbb{C}$ and $\Lambda = \mathbb{Z}$, then $a_i \in H^{2i}(X, \mathbb{Z})$, and is called the *i-th Chern class of* E denoted $c_i(E)$. When $\mathbb{F} = \mathbb{R}$, and $\Lambda = \mathbb{Z}_2$, then $a_i \in H^i(X, \mathbb{Z}_2)$ and is called the *i-th Stiefel-Whitney class of* E denoted $w_i(E)$. Finally the class $c(E) := 1 + c_1(E) + .. + c_n(E) \in H^*(X, \mathbb{Z})$ (resp. $w(E) := 1 + w_1(E) + ... w_n(E) \in H^*(X, \mathbb{Z}_2)$ is called the *total Chern class* (resp. total Steifel-Whitney class) of E.

Why is this an inductive formula? Say E is complex. The essential point is that in the splitting $p^*E = \lambda_E \oplus F$ following Definition 2.2.1 above, we must have $c(p^*E) = p^*(c(E)) = c(\lambda_E)(c(F)) = (1-c)(c(F))$. Thus $c_n(F)$ is the degree n-homogeneous term of $p^*(1 + c_1(E) + .. + c_n(E))(1 + c + c^2 + ...)$. But F of $\mathbb{C}$-rank $n-1$ must mean that this must be zero, which implies the defining relation in the definition above. Likewise for real bundles E and Whitney classes.

Proposition 2.2.4. These Chern and Stiefel-Whitney characteristic classes defined above have the following properties:

(i): (Functoriality) If E is a complex vector bundle on X, and $f : Y \to X$ is a continuous map, then $c(f^*E) = f^*(E)$. Analogously for a real vector bundle E, and total Stiefel-Whitney class.

(ii): If E is a complex (resp. real) vector bundle of complex (resp. real) rank n, then $c_i(E) = 0$ (resp. $w_i(E) = 0$) for $i > n$. Furthermore $c_n(E) = e(E_{\mathbb{R}})$, the Euler class of $E_{\mathbb{R}}$ the underlying real bundle of E. Likewise, if E is real and orientable, then $w_n(E) = e(E) \mod 2$.

(iii): (Whitney sum formula) $c(E \oplus F) = c(E) \cup c(F)$ for complex bundles. Likewise $w(E \oplus F) = w(E) \cup w(F)$ for real bundles.

(iv): For the tautological line bundle λ on $\mathbb{P}^n(\mathbb{C})$ (resp. $\mathbb{P}^n(\mathbb{R})$), the first Chern class $c_1(\lambda)$ (resp. $w_1(\lambda)$) are as in Proposition 2.1.2

Here are some very interesting applications of Stiefel-Whitney classes:

(i): For $n = 2^k$, the real projective space $\mathbb{P}^n(\mathbb{R})$ cannot be embedded in $\mathbb{R}^{2n-1}$. Hence the *Whitney embedding theorem* that every compact smooth n-manifold embeds in $\mathbb{R}^{2n}$ is sharp.
(ii): A compact connected smooth n-manifold is the boundary of an $(n+1)$-manifold if and only if all the finitely many (Stiefel-Whitney) numbers $\{\langle w_1^{i_1} w_2^{i_2} .. w_n^{i_n}, [X]\rangle : \sum i_j = n\} \subset \mathbb{Z}_2$ vanish.
(iii): Every compact connected 3-manifold is parallelisable (i.e. has trivial tangent bundle).

2.3. **The Chern Character, Todd Class and Riemann-Roch.** Using rational instead of integer cohomology, one can define certain very important characteristic classes of a complex vector bundle E.

Definition 2.3.1 (Chern character)**.** Let E be a complex vector bundle of rank $rk_{\mathbb{C}}E = n$ on a topological space X. Let $c_i := c_i(E) \in H^{2i}(X, \mathbb{Q})$ denote the (images of) the Chern classes of E in rational cohomology. Let $S_k = S_k(\sigma_1, .., \sigma_n)$ be the *k-th Newton Polynomial* (i.e. the expansion of $t_1^k + ...t_n^k$ as a polynomial in the elementary symmetric functions $\sigma_i(t_1, .., t_n)$). Define the *Chern character of* E by the formula:

$$ch(E) = rk_{\mathbb{C}}E + \sum_{k=0}^{\infty} \frac{S_k(c_1, .., c_n)}{k\,!} \in H^*(X, \mathbb{Q})$$

This characteristic class satisfies (a) $ch(L) = \exp(c_1)$ if $rk_{\mathbb{C}}L = 1$, (b) $ch(E \oplus F) = ch(E) + ch(F)$ and (c) $ch(E \otimes F) = ch(E)ch(F)$.

The Chern character turns out to give a very important isomorphism between the rational complex K-ring $K^*(X)\otimes\mathbb{Q}$ and rational cohomology ring $H^*(X,\mathbb{Q})$ (Atiyah-Hirzebruch). An immediate application is that the only even-dimensional spheres which can admit the structure of an almost complex manifold (i.e. a complex vector bundle structure on the tangent bundle) are S^2 and S^6. (S^2 is the well-known complex manifold given as the Riemann sphere. It is not known if any of the many known almost complex structures of S^6 actually arise from a complex manifold structure on it).

Definition 2.3.2. (Todd Class) Let E be a complex vector bundle of $rk_{\mathbb{C}}E = n$ on a topological space X, and let c_i denote the Chern classes in $H^*(X,\mathbb{Q})$ as above. Define the *k-th Todd Polynomial* $T_k(\sigma_1,..,\sigma_n)$ by expressing the k-th degree homogeneous term of $\prod_{i=1}^n \left(\frac{t_i}{1-e^{-t_i}}\right)$ as a polynomial in the elementary symmetric functions $\sigma_i(t_1,..,t_n)$. Then the *total Todd class of* E is defined as:

$$Td(E) := 1 + \sum_{k=0}^{\infty} T_k(c_1,..,c_n) \in H^*(X,\mathbb{Q})$$

It satisfies the following (a) $Td(L) = \left(\frac{c_1(L)}{1-e^{-c_1(L)}}\right)$ for $rk_{\mathbb{C}}L = 1$ and (b) $Td(E\oplus F) = Td(E)Td(F)$

A cornerstone theorem in algebraic geometry is:

Theorem 2.3.3 (Hirzebruch-Riemann-Roch)**.** Let X be a compact connected complex manifold of dimension n and let E a holomorphic vector bundle on X. Let $\mathcal{O}(E)$ denote the sheaf of holomorphic sections of E. Then the *Arithmetic Euler characteristic* or *Arithmetic genus* is defined by the formula:

$$\chi(X,E) := \sum_{i=0}^{n} (-1)^i \dim_{\mathbb{C}} H^i(X,\mathcal{O}(E))$$

$\chi(X,E)$ a priori depends on the holomorphic structures of X and E, but is actually a *topological invariant of* X *and* E. Indeed it can be computed as:

$$\chi(X,E) = \langle Td(\tau(X)) \cup ch(E), [X]\rangle$$

where $\tau(X)$ is the holomorphic tangent bundle of X.

In the instance of X of complex dimension 1 (i.e. a smooth complex curve or Riemann surface) of genus g, this formula was proved by Riemann and Roch. It says that if $D = \sum_{i=1}^{l} n_i p_i$ is a finite integral linear combination of points $p_1,..,p_l$ on X, and $d = \deg\, D := \sum_{i=1}^{l} n_i$, then:

$$\dim_{\mathbb{C}} H^0 - \dim_{\mathbb{C}} H^1 = d - g + 1$$

where H^0 is the vector space of meromorphic functions f on X satisfying $(f) + D \geq 0$ and H^1 is a similarly defined vector space of Abelian differentials (meromorphic 1-forms) ω on X with $(\omega) - D \geq 0$. In particular, if $d > g - 1$, then there are plenty of meromorphic functions on X.