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The Diagonal as the Zero Locus of a Section

Vishwambhar Pati

Indian Statistical Institute, Bangalore Centre
8th Mile Mysore Road, Bangalore, 560059 India

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1. Introduction

1.1. **Statement of the Problem.** Let M be a compact connected oriented smooth manifold, and let $\Delta \subset M \times M$ be the diagonal submanifold.

Definition 1.1.1. Say that M has the *property* Z if there exists a smooth real vector bundle E of rank n on $M \times M$ and a smooth section s of E such that (i) s is transverse to the 0-section 0_E of E and (ii) $\Delta = s^{-1}(0_E)$. If further the bundle E is orientable, say that M has the *property* Z_o. Finally, if $\dim_{\mathbb{R}} M = 2m$, and E can be chosen to be a smooth complex vector bundle of $\mathrm{rk}_{\mathbb{C}} E = m$, say that M has the property Z_c.

Question: When does M as above have the property Z (resp. Z_o, resp. Z_c)?

Remark 1.1.2. Clearly $Z_o \Rightarrow Z$. Further, if M satisfies Z_c, then since $E_{|\Delta}$ is forced to be isomorphic to the normal bundle of Δ by the transversality condition, and this normal bundle is known to be isomorphic to the tangent bundle τ_M. It follows that E a complex vector bundle forces τ_M to be a complex complex bundle, so M will necessarily have to be an *almost complex manifold.* Of course, Z_o and Z will follow from Z_c whenever it holds.

Remark 1.1.3. If M has the property Z, then M must carry a bundle H of rank n and a smooth section $\sigma \pitchfork 0_H$ with $\sigma^{-1}(0_H) = \{p\}$, for $p \in M$ any given point. Indeed set $H := i^*F$, and the section $\sigma = i^*s$ where $i : M \to M \times M$ is the inclusion $x \mapsto (x, p)$. Then it is easy to check that $\sigma \pitchfork 0_H$ and $\sigma^{-1}(0_H) = \{p\}$.

This forces the top Whitney class $w_n(H)$ to be 1. If M has Z_o and E is orientable, this forces the Euler characteristic e_H of H to be ± 1.

In this note we prove the following:

Theorem 1.1.4.

(i): The only spheres with the property Z are S^1, S^2, S^4 and S^8. All but the first have Z_o.

(ii): No odd-dimensional manifold M has the property Z_o. If further $H_1(M, \mathbb{Z}_2) = 0$, then M does not have the property Z.

(iii): If $\dim M = 4$ and M is an almost complex manifold, then M has the property Z_c (in particular Z_o and Z).

(iv): A compact almost complex manifold M with $\dim_{\mathbb{C}} M = 3$ with $w_2(M) = 0$ ($\Leftrightarrow$ mod 2 reduction of $c_1(M)$ vanishes $\Leftrightarrow$ M is spin) satisfies Z_c.

(v): Let M be an almost complex manifold of $\dim_{\mathbb{C}} M = 3$. Assume $H^1(M, \mathbb{Z}) = 0$ and $H^2(M, \mathbb{Z}) = \mathbb{Z}$. Then M satisfies Z_c if and only if it is spin.

(vi): Let $M \subset \mathbb{P}^N(\mathbb{C})$ be a smooth projective variety of $\dim_{\mathbb{C}} M = 3$ which is a set-theoretically complete intersection. Assume $H_1(X, \mathbb{Z}) = 0$ (or alternatively assume M is a strict complete intersection). Then M has the property Z_c if and only if M is spin.

(vii): Let $M \subset \mathbb{P}^n(\mathbb{C})$ be a smooth strict complete intersection of $\dim_{\mathbb{C}} M = 3$ obtained as $M = X_1 \cap X_2 .. \cap X_{n-3}$ where X_i are smooth hypersurfaces of degree d_i. Then M has the property Z_c if and only if the number $(n+1-\sum_{i=1}^{n-3} d_i)$ is even. In particular, a smooth degree d hypersurface in $\mathbb{P}^4(\mathbb{C})$ has Z_c if and only if d is odd. In particular, the smooth quadric hypersurface in $\mathbb{P}^4(\mathbb{C})$ does not satisfy Z_c.

(viii): For $m \geq 2$, the smooth odd-dimensional quadric hypersurface $Q_{2m-1} \subset \mathbb{P}^{2m}(\mathbb{C})$ does not satisfy Z_c.

These results will be proved in the next section. Meanwhile we note some examples.

1.2. **Some Examples.**

Example 1.2.1 (The circle)**.** S^1 has the property Z. For let γ denote the Mobius bundle on S^1, i.e. the tautological line bundle with S^1 being regarded as $\mathbb{P}^1(\mathbb{R})$. Then clearly it has a section σ transverse to 0_γ and having a single zero at $z = 1$ (regarding S^1 as unit complex mumbers). Let $\mu : S^1 \times S^1 \to S^1$ denote the map $(z, w) \mapsto zw^{-1}$, set $E := \mu^*\gamma$ and s to be the pullback section $s(z, w) = \sigma(zw^{-1})$. This clearly does the job.

Since all orientable bundles of rank 1 on S^1 are trivial, S^1 supports no orientable line bundles H of $e(H) = \pm 1$, so by Remark 1.1.3 S^1 does not have Z_o. This also follows from (ii) of the Theorem 1.1.4 stated above.

Example 1.2.2 (Products)**.** If two manifolds M and N have the property Z, then so does $M \times N$. For, let (E_1, s_1) and (E_2, s_2) be the bundle-section pairs on $M \times M$ and $N \times N$ respectively, realising the property Z for M and N. Then the bundle $F := E_1 \times E_2$ has the section $\sigma := s_1 \times s_2$ being transverse to 0_F, and $\sigma^{-1}(0_F) = \Delta_M \times \Delta_N$. Now take the shuffle diffeo: $\phi : M \times N \times M \times N \to M \times M \times N \times N$ which flips the middle two variables, and take $E := \phi^* F$ and $s := \phi^*\sigma$. Analogous considerations apply for Z_o and Z_c.

Remark 1.2.3 (Tori)**.** By the preceding examples, $\mathbb{T}^n := S^1 \times S^1 .. \times S^1$ has the property Z. It has the property Z_o (indeed even Z_c) only for even n, as will follow from (ii) of the Theorem 1.1.4, Examples 1.2.6 below and 1.2.2 above. Thus examples of manifolds with the property Z exist in all dimensions.

Example 1.2.4 (Lie Groups)**.** Let G be a compact Lie group, and assume that G supports a rank n bundle H with a section $\sigma \pitchfork 0_H$ and $\sigma^{-1}(0_H) = e$, the identity element of G. Then define $\mu : G \times G \to G$ by $\mu(xy) = xy^{-1}$. It is easily checked from the bundle $E := \mu^* H$, and the section $s := \mu^*\sigma$, that G satisfies Z. (This is exactly the construction of the Example 1.2.1 above.)

Likewise if H can be chosen orientable, G satisfies Z_o. Thus the necessary condition described in the Remark 1.1.3 above becomes sufficient for Lie groups.

Example 1.2.5. The space $\mathbb{P}^3(\mathbb{R})$ has the property Z but not Z_o. The rank 3 non-orientable bundle $H = \mathcal{O}(1) \oplus \mathcal{O}(1) \oplus \mathcal{O}(1)$ has a section which vanishes at a single given point. Identifying $\mathbb{P}^3(\mathbb{R})$ with the Lie group $SO(3)$ and the given point to be $e \in SO(3)$, and appealing to the previous Example 1.2.4 we have Z.

However, it follows from (ii) of the Theorem 1.1.4 stated above that $\mathbb{P}^3(\mathbb{R})$ *does not* have property Z_o. In particular, it follows from 1.2.4 above that $\mathbb{P}^3(\mathbb{R})$ supports no orientable rank 3 bundles of Euler class ± 1. (I am unaware if this fact is independently known).

Example 1.2.6 (Compact Riemann Surfaces)**.** Let M be a compact connected Riemann surface. Then the diagonal Δ is an irreducible curve in the complex surface $M \times M$ of multiplicity 1, and by the well-known correspondence between divisors and holomorphic line bundles, is exactly the zero-locus of a holomorphic complex line bundle $\mathcal{L}$, which comes with a natural section defining this divisor Δ. Thus M has Z_c.

Example 1.2.7 (Complex Projective Line)**.** For $\mathbb{P}^1(\mathbb{C}) \simeq S^2$, let $E := \pi_1^*\mathcal{O}(1) \otimes \pi_2^*\mathcal{O}(1)$, where $\pi_i : \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ are the coordinate projections . Define the section $s := z_1w_2 - z_2w_1$. It is easily checked that s and E do the job, and $\mathbb{P}^1(\mathbb{C}) = S^2$ has the property Z_c.

Example 1.2.8 (The spheres S^4, S^8)**.** Define $\mathbb{P}^1(\mathbb{H})_l \simeq S^4$ as the quotient $\mathbb{H}^2\backslash 0$ by the left multiplication of $\mathbb{H}^*$, and likewise $\mathbb{P}^1(\mathbb{H})_r$ as the quotient by the right action. On the former, we have the hyperplane bundle $L_1 := \mathcal{O}(1)_l$ which consists of linear functionals on the tautological bundle γ_l^1 on $\mathbb{P}^1(\mathbb{H})_l$. Since only right multiplication by a quaternion is a linear with respect to left $\mathbb{H}$-module structure on $\mathbb{H}$, this bundle has a *right* "$\mathbb{H}$-vector bundle" structure (It does not have even a left $\mathbb{C}$-vector bundle structure!). Similarly define $L_2 := \mathcal{O}(1)_r$ on $\mathbb{P}^1(\mathbb{H})_r$, which has a left $\mathbb{H}$-bundle structure. Then take the external tensor product $E := \pi_1^*L_1 \otimes_{\mathbb{H}} \pi_2^*L_2$, where π_i are the coordinate projections on $\mathbb{P}^1(\mathbb{H})_l \times \mathbb{P}^1(\mathbb{H})_r$ (this bundle has no further structure than that of a real rank 4 bundle!), and the section defined by $s := z_1w_2 - z_2w_1$. Note that the identification of $\mathbb{P}^1(\mathbb{H})_l$ with S^4 is by $[z_1 : z_2] \mapsto z_2^{-1}z_1$, and of $\mathbb{P}^1(\mathbb{H})_r$ is by $[w_1 : w_2] \mapsto w_1w_2^{-1}$. The section above is transverse to 0_E, and its zero-locus is Δ.

There is a completely analogous construction for S^8, by viewing it as $\mathbb{O} \cup \{\infty\}$, where $\mathbb{O}$ are the octonions. On the two charts $U = S^8 \setminus \{0\}$ and $V = S^8 \setminus \{\infty\}$, one can paste the trivial octonionic line by the transition function $s(u) = u$ for $u \in U \cap V$, and thereby get a bundle with a transverse section vanishing exactly at a point. The remaining details (hinging on the fact that the operation $(u, v) \mapsto v^{-1}.u$ makes sense on the octonions) are omitted.

Example 1.2.9 (Complex projective spaces and Grassmannians)**.** It is known that all complex projective
spaces and grassmannians have Z_c by a construction of Kirillov-Beilinson. (See [Srin], Ch. 5).

2. Reduction to Homotopy Theory

2.1. **Notation and preliminaries.** Put some suitable Riemannian metric on M, which induces one on $M \times M$ as well. Bundle metrics will then result on $\tau_{M\times M}$, and all its subbundles. Let U be a closed ϵ-tubular neighbourhood of Δ in $M \times M$. Via the tubular neighbourhood theorem, there exists a smooth diffeomorphism $\phi : U \to D_\epsilon(\nu)$ of U with the ϵ- disc bundle $D_\epsilon(\nu)$ of the normal bundle $\rho : \nu \to \Delta$ of Δ in $M \times M$. The map $r := \rho \circ \phi : U \to \Delta$ is then a strong deformation retraction of U to its core Δ. Under the diffeo ϕ, the boundary ∂U of U is mapped diffeomorphically to the ϵ-sphere bundle $S_\epsilon(\nu)$.

Hence, under the diffeo ϕ, the pullback bundle $r^*(\nu) \to U$ on U is carried over to the pullback bundle $\rho^*(\nu) \to D_\epsilon(\nu)$, where we continue to denote the restricted bundle projection $D_\epsilon(\nu) \to \Delta$ by ρ. Indeed, there is a bundle diagram:

$$\begin{array}{ccc} r^*(\nu) & \xrightarrow{D\phi} & \rho^*(\nu) \\ \rho \downarrow & & \downarrow \rho \\ U & \xrightarrow{\phi} & D_\epsilon(\nu) \end{array} \tag{1}$$

Note that the restricted bundle $\rho^*(\nu)_{|S_\epsilon(\nu)} \to S_\epsilon(\nu)$ has a tautological section s defined by $v \mapsto v$, which satisfies $\|s(v)\| = \epsilon$ for all $v \in S_\epsilon(\nu)$. Thus we have an orthogonal direct sum decomposition of bundles on $S_\epsilon(\nu)$:

$$\rho^*(\nu)_{|S_\epsilon(\nu)} = \xi \oplus L$$

where L is trivial line subbundle spanned by s. Note that since $\nu \simeq \tau_M$ is orientable, so is ξ.

It is also well known that $\rho : \nu \to \Delta$ is the tangent bundle $\rho : \tau_\Delta \to \Delta$ of Δ, which in turn is isomorphic to the tangent bundle $\rho : \tau_M \to M$. Hence, under this identification, ξ is isomorphic to the quotient bundle $\rho^*(\tau_M)/L \to S_\epsilon(\tau_M)$, where $L \to S_\epsilon(\tau_M)$ is the trivial tautological bundle spanned by the tautological section of $\rho^*(\tau_M)$ over $S_\epsilon(\tau_M)$.

Let $F := \phi^*(\xi)$, a rank $(n-1)$ subbundle of $r^*(\nu)_{|\partial U}$. By the above, it is isomorphic to the quotient rank $(n-1)$ bundle $\rho^*(\tau_M)/L \to S_\epsilon(\tau_M)$ under the above identifications. Note that F is an orientable bundle on ∂U.

Remark 2.1.1. The restriction of the bundle ξ above to each fibre $S_\epsilon(\nu_x)$ of the sphere bundle $\rho : S_\epsilon(\nu) \to M$ is the tangent bundle τ_{n-1} of $S_\epsilon(\nu_x)$. Consequently, the bundle F, when restricted to any fibre $r^{-1}(x)$ of the fibre bundle $r : \partial U \to \Delta$, is isomorphic to τ_{n-1}. This is clear, since the fibre of L at a point $v \in S_\epsilon(\nu_x)$ is precisely $L_v = \mathbb{R}v$. Thus $\xi_v = (\mathbb{R}v)^\perp = T_v(S_\epsilon(\nu_x))$ inside ν_x. The second statement follows from the first by the definition of F.

Since the $n = 1$ case is completely settled by Example 1.2.1, we will assume henceforth that $n \geq 2$.

Lemma 2.1.2. Let M, U, Δ, be as above. Set $X := (M \times M) \setminus U^\circ$. Then M has the property Z if and only if the rank $(n-1)$ bundle $F \to \partial U$ defined above is isomorphic to the restriction to $\partial U = \partial X$ of a smooth rank $(n-1)$ bundle G on X. Further, if M has the property Z_o, the bundle G can be chosen to be orientable.

Proof: We first prove the only if part. Suppose there exists a rank n smooth real vector bundle $\pi : E \to M \times M$, and s a smooth section transverse to the zero-section 0_E such that the diagonal $\Delta = s^{-1}(0_E)$. The retraction map $r : U \to \Delta$ is a strong deformation retraction, so $E_{|U}$ is isomorphic to $r^*(E_{|\Delta})$. Consequently, $E_{r(y)}$ is identified with E_y for all $y \in U$. Furthermore the section s is nowhere vanishing on $X = (M \times M) \setminus U^\circ$, and hence defines a trivial line sub-bundle Λ of $E_{|X}$, and a splitting of bundles on X:

$$E_{|X} = G \oplus \Lambda$$

where G is a rank $(n-1)$ bundle on X. We claim that $G_{|\partial U} = G_{|\partial X}$ is iomorphic to F.

The normal bundle ν of $s^{-1}(0) = \Delta$ in $M \times M$ is isomorphic to $E_{|\Delta}$ via the derivative Ds, by well-known results on transversality. Indeed, for each $x \in \Delta$, $v \mapsto Ds_x(v)$ is an isomorphism of ν_x with E_x by the transversality hypothesis. Since M is compact, the inverse function theorem then identifies an ϵ-tubular neighbourhood U of Δ as above with an ϵ-disc bundle $D_\epsilon(E_{|\Delta})$ of $E_{|\Delta}$, via the map s. Also s maps Δ diffeomorphically to the zero-section of $E_{|\Delta}$.

Consequently there is a bundle diagram:

$$\begin{array}{ccc} r^*(\nu) & \xrightarrow{Ds} & \pi^*(E_{|\Delta}) \\ \rho \downarrow & & \downarrow \pi \\ U & \xrightarrow{s} & D_\epsilon(E_{|\Delta}) \end{array} \tag{2}$$

Putting the two diagrams (1) and (2) together, we have an identification of the bundles $\rho^*(\nu) \to D_\epsilon(\nu)$, $\pi^*(E_{|\Delta}) \to D_\epsilon(E_{|\Delta})$ and $r^*(\nu) \to U$. Under these identifications, the the tautological everwhere $\neq 0$ sections of the first two bundles defined over their respective sphere bundles $S_\epsilon(\nu)$ and $S_\epsilon(E_{|\Delta})$ correspond to each other, and to an everywhere $\neq 0$ section σ of the third bundle $r^*(\nu) \to \partial U$.

It follows that the line bundle L on $S_\epsilon(\nu)$ corresponds with the tautologically defined trivial line subundle of $r^*E_{|\Delta} \to \partial U$. This last bundle is isomorphic to $E_{|\partial U} \to \partial U$ (since r is a strong deformation retraction), and the tatuological trivial line bundle is mapped isomorphically to $\Lambda_{|\partial U}$. Hence, the quotient bundle F defined above is isomorphic to the bundle $G_{|\partial U}$, under the identifications above.

If M had property Z_o, and E was chosen to be an orientable bundle realising Z_o to being with, then G is nothing but $E_{|X}/\epsilon^1$, where ϵ^1 is the trivial subundle spanned by $s_{|X}$ which does not vanish anywhere on X. Thus G will also be orientable. This proves the only if part of the first statement and the last statement.

For the if part, let G be given on X as in the statement. Construct the bundle E, by taking $r^*(\nu)$ on U, and gluing it to the bundle $G \oplus \epsilon^1_X$ on X, after ensuring that the decomposition $r^*(\nu) = F \oplus \phi^*(L)$ on ∂U is preserved, viz. the first summand $G_{|\partial U}$ is glued to the first summand $F_{|\partial U}$ via the given isomorphism, and the second trivial summand ϵ^1_X is glued to the second trivial summand $\phi^*L_{|\partial U}$ on ∂U by matching the section σ of the trivial bundle $\phi^*L \to \partial U$ defined above with any everywhere $\neq 0$ section of ϵ^1_X which extends σ to X (An everwhere non-zero smooth function $f : \partial X \to \mathbb{R}$ always extends to an everwhere non-zero smooth function $g : X \to \mathbb{R}$. For, $n \geq 2$ implies ∂X is connected, so assume w.l.o.g. that $f > 0$, use Tietze to extend the smooth function $\log f$ to X, and exponentiate the extension to get g). The section σ of $r^*(\nu) \to \partial U$ above is just the restriction of the tautological section, also denoted σ, of $r^*(\nu) \to U$ which is transverse to the zero section. Thus the matched section s of the whole bundle E vanishes exactly on Δ, with $s \pitchfork 0_E$. The lemma follows. $\square$.

One would like to have an extension of the result for the property Z_c. Assume (in view of Remark 1.1.2) that M is an almost complex manifold of real dimension $n = 2m$. Then in the notation of the last section, the normal bundle $\nu \simeq \tau_M$ is a complex vector bundle, and $r^*(\nu)$ splits off a *complex line subbundle* ϵ_c^1 defined by the complex span of the tautological section of $r^*(\nu)$. Thus we may write:

$$r^*(\nu)_{|\partial U} = F_c \oplus \epsilon_c^1$$

where F_c is now a *complex vector bundle* with $\mathrm{rk}_{\mathbb{C}} F_c = m - 1$.

Lemma 2.1.3. Let M be an almost complex manifold of dimension $n = 2m$. Then M has the property Z_c iff the bundle F_c of complex rank $m-1$ on ∂X is isomorphic as a complex bundle to the restriction of a complex vector bundle G_c on X.

Proof: The only if part follows entirely as before, after noting that if the complex vector bundle E realises property Z_c for M, then $E_{|X}$ splits off the trivial complex line subbundle Λ_c spanned by s, where s was the chosen section that is everywhere non-zero on X. That is $\Lambda_c = \Lambda \oplus i\Lambda$ where Λ is as in the last lemma. Clearly G_c is a rank $m-1$ complex bundle which restricts on ∂X to a bundle isomorphic to F_c . For the if part, take $r^*(\nu) = F_c \oplus \epsilon_c^1$ on $\partial U = \partial X$, where ϵ_c^1 is the complex span of the real tautological section s on ∂U. Match up $\epsilon_{\mathbb{R}}^1$ with its tautological section on ∂X to a real trivial line bundle ϵ_X^1 on X with an extended everywhere non-vanishing section on X as in the previous Lemma 2.1.2. This automatically matches up ϵ_c^1 with the trivial complex line bundle $\epsilon_{c,X}^1$ by a complex isomorphism, viz by complexification. Also match up F_c with $G_{|\partial X}$ via the given complex isomorphism. The lemma follows. □.

Corollary 2.1.4 (Compact Riemann surfaces again)**.** Let M be a compact Riemann surface. Then M has the property Z_c.

Proof: We saw this in Example 1.2.6. It also immediately follows from Lemma 2.1.3, since F_c is a complex vector bundle of rank 0! □.

2.2. **The homotopy problem.** With the notation of the preceding Lemma 2.1.2, and as before let $\nu \simeq \tau_M$ the normal bundle of Δ in $M \times M$, U a closed ϵ-tubular neighbourhood of Δ in $M \times M$ diffeo to $D_\epsilon(\nu)$, $X = (M \times M) \setminus U^\circ$, with $\partial X = \partial U$ diffeomorphic to $S_\epsilon(\nu)$ ($\simeq S_\epsilon(\tau_M)$). Let $F \to \partial X$ be the $(n-1)$-plane bundle introduced before Lemma 2.1.2. Then we have the following:

Proposition 2.2.1. Let $f : \partial X \to G_{n-1}(\mathbb{R}^\infty)$ be the classifying map for the $(n-1)$-plane bundle F. Then M has the property Z iff there is an extension of f (in the homotopy category) to a smooth map $\widetilde{f} : X \to G_{n-1}(\mathbb{R}^\infty)$. This happens iff an inductively defined sequence of obstructions which lie in the cohomology groups:

$$H^{i+1}(M \times M, \Delta; \pi_i(G_{n-1}(\mathbb{R}^\infty))$$

all vanish. (Here $G_{n-1}(\mathbb{R}^\infty)$ denotes the universal classifying space $BO(n-1)$ for real $(n-1)$-plane bundles).

Proof: The first part is clear, by Lemma 2.1.2 and the classification theorem for $(n-1)$-plane bundles.

For the second part, obstruction theory tells us that there are inductively defined obstructions in the cohomology groups:

$$H^{i+1}(X, \partial X; \pi_i(G_{n-1}(\mathbb{R}^\infty))$$

which vanish iff a continuous extension $\widetilde{f} : X \to G_{n-1}(\mathbb{R}^\infty)$ of f exists in the homotopy category. Since continuous maps are homotopic to sufficiently close smooth approximations on any smooth manifold, we may assume $\widetilde{f}$ is smooth. Since $X \cup U^\circ = M \times M$ and $\partial X \cup U^\circ = U$, these cohomology groups are isomorphic to $H^{i+1}(M \times M, U; \pi_i(G_{n-1}(\mathbb{R}^\infty))$ by excision. Since U has Δ as a strong deformation retract, these groups are in turn isomorphic to the ones in the statement. □

Since it isn't very easy in general to apply the Proposition above directly except in very simple instances, in the next section we will be using elementary algebraic topology to establish the main result Theorem 1.1.4.

3. The Main Theorem

3.1. **Spheres.** The following proposition addresses (i) of the Theorem 1.1.4

Proposition 3.1.1. S^n has the property Z if and only if $n = 1, 2, 4,$ or 8.

Proof: We need to identify the bundle F on ∂U, in order to appeal to the foregoing Lemma 2.1.2. The normal bundle ν of $\Delta \subset S^n \times S^n$ is the tangent bundle $\rho : \tau_n \to S^n$, as remarked earlier. The unit sphere bundle $S(\tau_n)$ is the set of pairs $(x, v) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ such that $v \in T_x(S^n) = (\mathbb{R}x)^\perp$ and $| v |= 1$. But this is precisely the Stiefel manifold:

$$V_2(\mathbb{R}^{n+1}) = \{(x, v) : (x, v) \text{ is an orthonormal 2- frame in } \mathbb{R}^{n+1}\}$$

The bundle projection $\rho : S(\tau_n) \to S^n$ is just projection into the first factor, and thus we have the spherical fibre bundle

$$S_x^{n-1} \xrightarrow{j_x} V_2(\mathbb{R}^{n+1}) \xrightarrow{\rho} S^n \tag{3}$$

where $S_x^{n-1} = \rho^{-1}(x)$ is the fibre over x.

The tautological section of $\rho^*(\tau_n) \to V_2(\mathbb{R}^{n+1})$ is given by $(x, v) \mapsto v$. Thus the bundle $F = \rho^*(\tau_n)/L \to V_2(\mathbb{R}^{n+1})$ defined earlier has the fibre over $y = (x, v)$ given by:

$$F_{(x,v)} = T_x(S^n)/\mathbb{R}v = \mathbb{R}^{n+1}/\gamma_{(x,v)}$$

where $\gamma_{x,v} = \mathbb{R}x + \mathbb{R}v \subset \mathbb{R}^{n+1}$. In other words, $\gamma \to V_2(\mathbb{R}^{n+1})$ is the rank 2 tautological trivial bundle whose fibre at (x, v) is the subspace spanned by x and v in $\mathbb{R}^{n+1}$, and $F = \epsilon^{n+1}/\gamma$, where ϵ^{n+1} is the trivial $(n+1)$-plane bundle on $V_2(\mathbb{R}^{n+1})$. (Since $\gamma \simeq \epsilon^2$, in particular, it follows that F is stably trivial. Hence all resort to stable characteristic classes is futile, and the Euler class is all that works).

The disc bundle of the normal bundle ν of Δ in $S^n \times S^n$ is just the disc bundle of the tangent bundle of Δ. Letting the ϵ-tubular neighbourhood of Δ be denoted by U as above, we note that the complement $X = (S^n \times S^n) \setminus U^\circ$ is precisely a tubular neighbourhood of the antidiagonal Γ which is the graph of the antipodal map $A : S^n \to S^n$ defined by $Ax = -x$.

Now the map $1 \times A$ is an involution of $S^n \times S^n$ mapping Δ diffeomorphically to Γ, and carrying the ϵ-tubular neighbourhood U of Δ to a ϵ-tubular neighbourhood of Γ. Hence, if we set $\epsilon = \pi$, the length of a semicircle, $(1 \times A)$ will achieve a diffeomorphic identification of U with $X = (S^n \times S^n) \setminus U^\circ$. The common boundary $\partial U = \partial X$ is diffeo to the π-sphere bundle $S_\pi(\tau_n) = V_2(\mathbb{R}^{n+1})$. This is the bundle over Δ whose fibre over $(x, x) \in \Delta$ is the $(n-1)$ dimensional equatorial sphere cut by the plane $(\mathbb{R}x)^\perp$. Also X becomes a disc bundle $D_\pi(\nu_\Gamma) = D_\pi(\tau_n)$, just like U. It contains the antidiagonal Γ as a strong deformation retract. Let $\theta : X \to \Gamma$ denote the retraction, coming from the bundle projection $\nu_\Gamma \simeq \tau_n \to \Gamma$. Then, with the identification $\partial X = \partial U = V_2(\mathbb{R}^{n+1})$, the map $\theta((x, v)) = (x, Ax) \in \Gamma$. That is $\theta : \partial U = S_\pi(\tau_n) \to \Gamma$ is again the bundle projection. If we identify Γ with S^n via the first coordinate, then θ is just the map $\rho : V_2(\mathbb{R}^{n+1}) \to S^n$.

Because θ is a deformation retraction, every bundle G on X is the θ-pullback of a bundle on Γ, equivalently a ρ-pullback of a bundle on S^n. Hence it follows, by the Lemma 2.1.2 above, that S^n has property Z iff the bundle F defined above on $V_2(\mathbb{R}^{n+1})$ is isomorphic to the pullback under ρ of some bundle on S^n.

Since by (3), $\rho \circ j_x : S_x^{n-1} \to S^n$ is the constant map to x, it follows that the ρ-pullback of any bundle G on S^n will be trivial when restricted to a fibre S_x^{n-1}. Hence, if F is isomorphic to the ρ-pullback of a bundle on S^n, it follows that $F_{|S_x^{n-1}}$ is isomorphic to a trivial bundle on S_x^{n-1}.

By the Remark 2.1.1 above, $F_{|S_x^{n-1}}$ is isomorphic to the tangent bundle τ_{n-1} of S_x^{n-1}. Hence F will be a ρ-pullack of a bundle on S^n only if τ_{n-1} is trivial. In particular, only if $n - 1 = 0, 1, 3,$ or 7. That is, only if $n = 1, 2, 4$ or 8.

This proves the only if part of the proposition. For the 'if' part, refer to the Examples 1.2.7 and 1.2.8. Presumably the same thing works for $n = 8$ (??) and the octonions. □

3.2. **Odd dimensional manifolds.** We continue with the proof of (ii) of Theorem 1.1.4. All homologies and cohomologies hereafter are with $\mathbb{Z}$ coefficients unless otherwise stated.

Proposition 3.2.1. Let M be a compact orientable manifold of odd dimension. Then M does not have the property Z_o. If further $H_1(M, \mathbb{Z}_2) = 0$, then it does not have the property Z.

Proof: Let E be an orientable bundle of odd dimension $2k+1$ on $M \times M$, where dim $M = 2k+1$. Then the Euler class of $H := E_{|M \times \{p\}}$ must be zero, since H is of odd rank $= 2k+1 =$ dim M (this is because the automorphism $v \mapsto -v$ on H reverses orientation, and hence the sign of E, and top homology of M is $\mathbb{Z}$, devoid of 2-torsion). On the other hand we remarked in Remark 1.1.3 that H must have euler clas ± 1. This is a contradiction.

For the second assertion, note that if $H_1(M, \mathbb{Z}_2) = 0$, then $H^1(M, \mathbb{Z}_2) = 0$ and $H^1(M \times M, \mathbb{Z}_2) = 0$ also. In particular every bundle on $M \times M$ is orientable. Thus if M satisfies Z, it automatically satisfies Z_o. But this contradicts the first statement. □

Remark 3.2.2. We note from the Example 1.2.5 of $\mathbb{P}^3(\mathbb{R})$ above that this is a sharp result. That is, if the $H_1(M, \mathbb{Z}_2) = 0$ condition is dropped, then it is possible for an odd dimensional M to have property Z.

Remark 3.2.3. The part (i) of Theorem 1.1.4 for spheres of odd dimension $n \geq 2$ follows from the Proposition above.

3.3. **Complex dimension 2.** We now prove (iii) of the Theorem 1.1.4.

Proposition 3.3.1. Let M be an almost complex manifold of $\dim_{\mathbb{C}} M = 2$. Then M has the property Z_c ($\Rightarrow Z_o \Rightarrow Z$).

Proof: We appeal to the Lemma 2.1.3. The bundle F_c on ∂X defined there is a complex line bundle, and hence it extends to X iff its first Chern class (=Euler class) $e_{F_c} \in H^2(\partial X)$ lifts to $H^2(X)$. So it is enough to show that the restriction homomorphism:

$$H^2(X) \to H^2(\partial X)$$

is surjective. We do this by observing the commutative diagram induced by inclusions:

$$\begin{array}{ccc} H^2(X) & \to & H^2(\partial X) \\ j^* \uparrow & & \uparrow l_* \\ H^2(M \times M) & \to & H^2(U) \end{array}$$

Note that:

(a): The bottom restriction map is the same as the map $H^2(M \times M) \xrightarrow{\Delta^*} H^2(M)$ by deforming U to its core Δ. But this map is a split surjection because $\Delta^* f^* = id_{H^2(M)}$, where $f : M \times M \to M$ is $(x, y) \mapsto x$. Hence the bottom horizontal map is a surjection.

(b): The left vertical map is an isomorphism, since $H^i(M \times M, X) \simeq H^i(D_\epsilon(\nu), S_\epsilon(\nu))$ by excision, and this vanishes for $0 \leq i \leq 3$ by the Thom isomorphism ($\nu \simeq \tau_M$ is a bundle of real rank 4).

(c): the right vertical arrow is an isomorphism because of $H^i(U, \partial X) = H^i(U, \partial U) = H^i(D_\epsilon(\nu), S_\epsilon(\nu)) = 0$ for $0 \leq i \leq 3$ again by the Thom isomorphism.

Hence the top horizontal map is a surjection, and we are done. $\square$.

3.4. **Complex dimension 3.** We now proceed with the proof of (iv) of the main Theorem 1.1.4.

Proposition 3.4.1. Let M be an almost complex manifold of complex dimension 3. Assume that the second Stiefel-Whitney class $w_2(M) = 0$ in $H^2(M, \mathbb{Z}_2)$ ($\Leftrightarrow$ mod 2 reduction of $c_1(M)$ vanishes $\Leftrightarrow$ M has a spin structure). Then M satisfies the property Z_c.

Proof: Since the mod 2 reduction of $c_1(M) = c_1(\tau_M)$ is 0, it follows that the class $c_1(M) \in H^2(M, \mathbb{Z})$ is divisible by 2 (from the Bockstein cohomology exact sequence), so there exists a complex line bundle L on M with $2c_1(L) = c_1(M)$.

Since $H^2(M \times M, \mathbb{Z}) \xrightarrow{\Delta^*} H^2(M)$ is a split surjection, it follows that there is a cohomology class $\alpha \in H^2(M \times M)$ such that $\Delta^* \alpha = c_1(L)$. Since complex line bundles are completely classified by c_1, there is a complex line bundle Γ on $M \times M$ whose restriction to the diagonal is L, i.e. $\Delta^* \Gamma = L$.

Now we appeal to the Lemma 2.1.3. The bundle F_c of complex rank 2 on the sphere bundle $\partial U = \partial X$ defined by:

$$r^*(\tau_M) = F_c \oplus \epsilon_c^1$$

needs to be extended to a complex rank 2 bundle G_c on $X := (M \times M) \setminus U^\circ$ which has boundary $\partial U = \partial X$. Since the line bundle Γ on $M \times M$ is an extension of the line bundle $L \to \Delta$, it is an extension of $r^*L \to \partial X$ to $M \times M$, and *a fortiori* to X. Thus to extend F_c to X, it is enough to extend the twisted rank 2 complex bundle $F_1 := F_c \otimes r^*L^{-1} \to \partial X$ to a complex rank 2 bundle $G_1 \to X$. (Then the required bundle $G_c \to X$ extending $F_c \to \partial X$ will be $G_1 \otimes \Gamma_{|X}$).

We claim that $c_1(F_1) = 0$. This is because F_c being of rank 2,

$$c_1(F_1) = c_1(F_c \otimes r^*L^{-1}) = c_1(F_c) + 2c_1(r^*L^{-1}) = r^*(c_1(M)) - 2r^*(c_1(L)) = r^*(c_1(M) - 2c_1(L)) = 0$$

by the definition of L above and the fact that $r^*\nu = r^*\tau_M = F_c \oplus \epsilon^1$.

Now it is an easy matter to extend F_1 to X. Since we have a fibration:

$$S^1 \to BSU(2) \to BU(2)$$

the only obstruction to lifting the classifying map $f : \partial X \to BU(2)$ of the 2-bundle F_1 to $BSU(2)$ is a solitary obstruction in $H^2(\partial X, \pi_1(S^1)) = H^2(\partial X, \mathbb{Z})$. By the functoriality of such obstruction, it is f^*y, where $y \in H^2(BU(2), \mathbb{Z})$ is the corresponding obstruction for the universal 2-plane complex bundle γ^2 on $BU(2)$. Since $H^2(BU(2), \mathbb{Z})$ is generated by $c_1(\gamma^2)$, this obstruction for F_1 is a multiple of $c_1(F_1)$, which we have seen to be 0 in the last para.

Consequently, $f : \partial X \to BU(2)$ lifts to $BSU(2) = BSp_1 = \mathbb{P}^\infty(\mathbb{H})$, and F_1 becomes a *quaternionic line bundle*. It is well known that the $\mathbb{Z}$-cohomology ring of $\mathbb{P}^\infty(\mathbb{H})$ is the polynomial ring $\mathbb{Z}[c_2]$, where $c_2 \in H^4(\mathbb{P}^\infty(\mathbb{H}))$ is the second Chern class of the universal quaternionic line bundle on $\mathbb{P}^\infty(\mathbb{H})$. Also, since $\mathbb{P}^\infty(\mathbb{H}) = K(\mathbb{Z}, 4)$, all quaternionic line bundles on any space Y are classified by $H^4(Y, \mathbb{Z})$, i.e. by their 2nd Chern class.

Since F_1 above is a quaternionic line bundle, all we need to do is check that the map $H^4(X, \mathbb{Z}) \to H^4(\partial X, \mathbb{Z})$ is surjective. Consider the commutative diagram:

$$\begin{array}{ccc} H^4(X) & \to & H^4(\partial X) \\ j^* \uparrow & & \uparrow l_* \\ H^4(M \times M) & \to & H^4(U) \end{array}$$

Note that:

(a): The bottom restriction map is the same as the map $H^4(M \times M) \xrightarrow{\Delta^*} H^4(M)$ by deforming U to its core Δ. But this map is a split surjection. Hence the bottom horizontal map is a surjection.

(b): The left vertical map is an isomorphism, since $H^i(M \times M, X) \simeq H^i(D_\epsilon(\nu), S_\epsilon(\nu))$ by excision, and this vanishes for $0 \leq i \leq 5$ by the Thom isomorphism ($\nu \simeq \tau_M$ is a bundle of real rank 6).

(c): the right vertical arrow is an isomorphism because of $H^i(U, \partial X) = H^i(U, \partial U) = H^i(D_\epsilon(\nu), S_\epsilon(\nu)) = 0$ for $0 \leq i \leq 5$ again by the Thom isomorphism .

Thus the restriction $H^4(X,\mathbb{Z}) \to H^4(\partial X,\mathbb{Z})$ is surjective, and F_1 extends to a bundle on X. By Lemma 2.1.3 we are done. $\square$.

We now proceed to prove (v) of Theorem 1.1.4. We first need a lemma.

Lemma 3.4.2. Let M be an almost complex manifold with $\dim_{\mathbb{C}} M = 3$. Let E be a smooth complex vector bundle on M of any rank, with Chern classes $c_i(E) \in H^{2i}(M,\mathbb{Z})$, for $1 \leq i \leq 3$. Then these Chern classes satisfy:

$$c_3(E) - c_2(E)(c_1(M) + c_1(E)) = 2m\mu, \quad \text{for some } m \in \mathbb{Z}$$

where $\mu \in H^6(M,\mathbb{Z}) \simeq \mathbb{Z}$ is a generator.

Proof: We will need to use the generalised Riemann-Roch Theorem for M, with coefficients in E. Note first that for E of any rank on M, the Chern character of E is well known to be:

$$ch(E) = \mathrm{rk}_{\mathbb{C}}(E) + c_1(E) + \left(\frac{c_1(E)^2 - 2c_2(E)}{2}\right) + \left(\frac{c_1(E)^3 - 3c_1(E)c_2(E) + 3c_3(E)}{6}\right)$$

with no higher terms since dim $M = 6$. It is also well known that on a smooth complex manifold M of $\dim_{\mathbb{C}} M = 3$ we have the formula for the total Todd class of M:

$$Td(M) = 1 + \frac{c_1(M)}{2} + \left(\frac{c_1(M)^2 + c_2(M)}{12}\right) + \frac{c_1(M)c_2(M)}{24}$$

Thus the Todd characteristic class of M with coefficients in E (see also Ch. III §12 of [Hir] for the definition of the Todd characteristic class) obtained by taking the degree 6 term in $Td(M)ch(E)$ is:

$$\begin{aligned} T(M,E) &= \left[\mathrm{rk}_{\mathbb{C}}E\left(\frac{c_1(M)c_2(M)}{24}\right) + c_1(E)\left(\frac{c_1(M)^2 + c_2(M)}{12}\right) + \left(\frac{c_1(E)^2 c_1(M)}{4}\right) + \frac{c_1(E)^3}{6}\right] \\ &+ \left[\frac{c_3(E) - c_2(E)(c_1(M) + c_1(E))}{2}\right] \end{aligned} \tag{4}$$

It is known that if M is any almost complex manifold, it has a Spin$_c$ reduction. For any smooth complex vector bundle E on M, the generalised Riemann-Roch Theorem (i.e. the Atiyah-Singer Theorem applied to the Dirac operator of the elliptic Spin$_c$ complex of M twisted by E) implies that $T(M,E)$ defined above is an integral multiple of the fundamental class $\mu \in H^6(M,\mathbb{Z})$, the integer being the index of this Dirac operator. For a reference to this result, see Theorem 3.5.5 of §3.5 in [Gil] or Theorem 24.5.4 in Appendix 1 of [Hir].

For a rank k bundle E, $c_1(E) = c_1(\wedge^k E) = c_1(\wedge^k E \oplus \epsilon^{k-1})$, and $c_2(\wedge^k E \oplus \epsilon^{k-1}) = c_3(\wedge^k E \oplus \epsilon^{k-1}) = 0$, ($\epsilon^{k-1}$ being the trivial rank $k-1$ bundle). Thus the first boxed term in equation (4) is equal to $T(M, \wedge^k E \oplus \epsilon^{k-1})$, which is an integer multiple of the fundamental class μ, the integer being the index of the Dirac operator for the elliptic Spin$_c$ complex twisted by $\wedge^k(E) \oplus \epsilon^{k-1}$. Hence the second boxed term of (4) is also an integer multiple of the fundamental class. That is,

$$c_3(E) - c_2(E)(c_1(M) + c_1(E)) = 2m\mu$$

for some integer m, and the lemma follows. $\square$

Now we are ready to prove (v) of Theorem 1.1.4.

Proposition 3.4.3. Let M be an almost complex manifold of $\dim_{\mathbb{C}} M = 3$. Assume $H^1(M, \mathbb{Z}) = 0$ and $H^2(M, \mathbb{Z}) = \mathbb{Z}$. Then if M satisfies Z_c, the second Stiefel Whitney class $w_2(M) = 0$. Thus, in view of Proposition 3.4.1, for such an M the property Z_c is equivalent to spinnability.

Proof: Let E be a smooth complex rank 3 bundle on $M \times M$ realising Z_c. Let x denote a generator of $H^2(M, \mathbb{Z})$. Since $H^1(M, \mathbb{Z}) = 0$, and $H^2(M, \mathbb{Z}) = \mathbb{Z}x$, we have by the Kunneth formula that $H^2(M \times M) = \mathbb{Z}(x \times 1) \oplus \mathbb{Z}(1 \times x)$ where "$\times$" denotes the cohomology cross product. Let us denote the first Chern class of E by

$$c_1(E) = a_1(x \times 1) + a_2(1 \times x) \quad \in H^2(M \times M, \mathbb{Z}); \qquad a_1, a_2 \in \mathbb{Z}$$

Since E restricted to the diagonal Δ is isomorphic as a complex vector bundle to the normal bundle ν of Δ, i.e. τ_M, it follows that

$$\Delta^*(c_1(E)) = a_1(x.1) + a_2(1.x) = (a_1 + a_2)x = c_1(\tau_M)$$

where $\Delta : M \to M \times M$ is the diagonal inclusion. Thus we have:

$$(a_1 + a_2)x = c_1(M) \tag{5}$$

We noted in the Remark 1.1.3, that the restriction of E to the slices $M \times \{p\}$ and $\{q\} \times M$ will have Euler class ± 1 times the generator. Thus the 3rd Chern classes of the rank 3 bundles $E_1 := E_{|M \times \{p\}}$ and $E_2 := E_{|\{q\} \times M}$ are both equal to $\pm\mu$, where μ is the fundamental class in $H^6(M, \mathbb{Z})$. The first Chern classes of E_i are clearly:

$$c_1(E_1) = a_1 x; \qquad c_1(E_2) = a_2 x$$

since $1 \times x$ (resp. $x \times 1$) restricts to 0 on the slice $M \times \{p\}$ (resp. $\{q\} \times M$).

From the Lemma 3.4.2 applied to E_1, and (5) above it follows that:

$$c_2(E_1)(c_1(M) + c_1(E_1)) = c_2(E_1)(2a_1 + a_2)x = c_3(E_1) + 2m_1\mu = (2m_1 \pm 1)\mu; \qquad m_1 \in \mathbb{Z}$$

and similarly

$$c_2(E_2)(2a_2 + a_1)x = (2m_2 \pm 1)\mu; \quad m_2 \in \mathbb{Z}$$

Denote the image of $y \in H^*(M, \mathbb{Z})$ in $H^*(M, \mathbb{Z}_2)$ by $\widetilde{y}$. Then the above equations imply:

$$a_2[\widetilde{c_2(E_1)}.\widetilde{x}] = \widetilde{\mu} = a_1[\widetilde{c_2(E_2)}.\widetilde{x}]$$

Since $\widetilde{\mu}$ generates $H^6(M, \mathbb{Z}_2) = \mathbb{Z}_2$, we have $a_1 \equiv a_2 \equiv 1 \mod 2$. Thus $a_1 + a_2 \equiv 0$ modulo 2.

Thus $c_1(M) = (a_1 + a_2)x$ is an even multiple of x. Since $w_2(M)$ is the modulo 2 reduction of $c_1(M)$, it follows that $w_2(M) = 0$. The proposition follows. □

Now we can prove (vi) of the main Theorem 1.1.4.

Corollary 3.4.4. Let $M \subset \mathbb{P}^N(\mathbb{C})$ be a smooth projective variety of $\dim_{\mathbb{C}} M = 3$. Assume that M is a set-theoretically complete intersection with $H_1(M, \mathbb{Z}) = 0$ (or alternatively that M is a strict complete intersection). Then M has the property Z_c if and only if M is spin (i.e. $c_1(M)$ is an even multiple of the hyperplane class in $H^2(M, \mathbb{Z})$).

Proof: It is known (see Cor. 7.6 on p. 149 of [Har]) that for M as above $H^{1,0}(M) = H^{0,1}(M) = 0$. This shows by Hodge decomposition that $H^1(M, \mathbb{C}) = 0$. Since $H^1(M, \mathbb{Z})$ is a finitely generated free abelian group, it follows that $H^1(X, \mathbb{Z}) = 0$.

The same result cited above shows that $H^{0,2}(M) = H^{2,0}(M) = 0$, and that $H^{1,1}(M) = \mathbb{C}$. It follows again that $H^2(M, \mathbb{C}) = \mathbb{C}$. Since by hypothesis $H_1(X, \mathbb{Z}) = 0$, it follows that $H^2(X, \mathbb{Z})$ has no torsion, and is $= \mathbb{Z}$, generated by the hyperplane class. Similarly, for a strict smooth complete intersection M with $\dim_{\mathbb{C}} M \geq 3$, it is known that $H^2(M, \mathbb{Z}) = \mathbb{Z}$ by a Theorem of Grothendieck and Lefschetz (see [Har], Cor. 3.2 on p. 179). So our corollary follows from the Propositions 3.4.1 and 3.4.3. $\square$.

We now prove (vii) of Theorem 1.1.4

Corollary 3.4.5. Let M be a smooth strict complete intersection of $\dim_{\mathbb{C}} = 3$ in $\mathbb{P}^n(\mathbb{C})$, with $M = X_1 \cap ... \cap X_{n-3}$ with X_i smooth hypersurfaces of degree d_i. Then M has property Z_c iff $(n + 1 - \sum_i d_i)$ is even. In particular, a smooth hypersurface M in $\mathbb{P}^4(\mathbb{C})$ has the property Z_c if and only if it is of odd degree. In particular, the non-singular quadric in $\mathbb{P}^4(\mathbb{C})$ does not have the property Z_c.

Proof: The previous corollary implies that M has Z_c iff M is spin.

It is well known that the first Chern class of the tangent bundle of M is $(n + 1 - \sum_i d_i)$ times the hyperplane class (since the normal bundle of M in $\mathbb{P}^n(\mathbb{C})$ is the direct sum $\sum_i \mathcal{O}(d_i)$). Thus $c_1(M)$ is an even multiple of the hyperplane class iff $(n + 1 - \sum_i d_i)$ is even. Thus M is spin iff this number is even.

When $n = 4$, and M is a hypersurface, $(5 - d)$ is even iff d is odd. In particular, for example, the smooth quadric in $\mathbb{P}^4(\mathbb{C})$ does not have Z_c. $\square$

Remark 3.4.6. The condition $H^2(M, \mathbb{Z}) = \mathbb{Z}$ cannot be dispensed with in Proposition 3.4.3. For example, we know that Z_c holds for $\mathbb{P}^2(\mathbb{C})$ and $\mathbb{P}^1(\mathbb{C})$ (by Example 1.2.9) and hence for $M = \mathbb{P}^2(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ (by Example 1.2.2). However the first Chern class of M is $(3x \times 1) + (1 \times 2y) \in H^2(M, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$, and M is not spin. Thus 3.4.3 does not hold good for M. Whether the vanishing of $H^1(X, \mathbb{Z})$ can be relaxed is not entirely clear.

3.5. **Odd-dimensional smooth quadrics.** We remark that since the quadrics of dimension 1 and 2 are respectively $\mathbb{P}^1(\mathbb{C})$ and $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$, both satisfy Z_c. The quadric of dimension 3 does not satisfy Z_c by the Corollary 3.4.5. This last fact generalises to all smooth projective quadric hypersurfaces of odd complex dimension ≥ 3. We first need some lemmas.

Proposition 3.5.1 (Cohomology of Q_{2m-1}). Let $Q_{2m-1} \subset \mathbb{P}^{2m}(\mathbb{C})$ denote the smooth odd-dimensional quadric hypersurface $V(X_0^2 + ... + X_{2m}^2)$, and let $m \geq 2$. Then the integral cohomology ring of Q_{2m-1} is given by:

$$H^*(Q_{2m-1}, \mathbb{Z}) = \mathbb{Z}[x, y]/\langle x^m - 2y, y^2 \rangle$$

where $x := c_1(\mathcal{O}_{Q_{2m-1}}(1))$ is the generator of $H^2(Q_{2m-1},\mathbb{Z})$, and y is the generator of $H^{2m}(Q_{2m-1},\mathbb{Z})$. In particular:

$$\begin{aligned} H^{2k+1}(Q_{2m-1},\mathbb{Z}) &= 0 \quad \text{for all} \quad k \\ H^{2k}(Q_{2m-1},\mathbb{Z}) &= \mathbb{Z}x^k \quad \text{for all} \quad 0 \leq k \leq m-1 \\ &= \mathbb{Z}x^{k-m}y \quad \text{for all} \quad m \leq k \leq 2m-1 \end{aligned}$$

Proof: We first note that there is an inclusion:

$$j : \mathbb{P}^{m-1}(\mathbb{C}) \hookrightarrow Q_{2m-1}$$

where $\mathbb{P}^{m-1}(\mathbb{C})$ is the linear subspace of $\mathbb{P}^{2m}(\mathbb{C})$ defined by:

$$\mathbb{P}^{m-1}(\mathbb{C}) = \{[x_0 : x_1 : ... : x_{2m}] \in \mathbb{P}^{2m}(\mathbb{C}) : x_0 + ix_1 = x_2 + ix_3 = ... = x_{2m-2} + ix_{2m-1} = x_{2m} = 0\}$$

If we let $i : Q_{2m-1} \hookrightarrow \mathbb{P}^{2m}(\mathbb{C})$ denote the natural inclusion we have the composite inclusion:

$$i \circ j : \mathbb{P}^{m-1}(\mathbb{C}) \hookrightarrow \mathbb{P}^{2m}(\mathbb{C})$$

which is the inclusion of a linear projective subspace in $\mathbb{P}^{2m}(\mathbb{C})$. Therefore, if we let $H := c_1(\mathcal{O}_{\mathbb{P}^{2m}(\mathbb{C})}(1))$, $x := c_1(\mathcal{O}_{Q_{2m-1}}(1)) = i^*H$ and $h := c_1(\mathcal{O}_{\mathbb{P}^{m-1}(\mathbb{C})}(1)) = j^*x = j^*i^*H$ denote the respective hyperplane classes, we have that the composite homomorphism:

$$H^r(\mathbb{P}^{2m}(\mathbb{C}),\mathbb{Z}) \xrightarrow{i^*} H^r(Q_{2m-1},\mathbb{Z}) \xrightarrow{j^*} H^r(\mathbb{P}^{m-1}(\mathbb{C}),\mathbb{Z})$$

is an isomorphism for $0 \leq r \leq 2m-2$, since it is induced by the linear inclusion $i \circ j$. It follows that $H^r(Q_{2m-1},\mathbb{Z}) = 0$ for r odd and $0 \leq r \leq 2m-2$. Furthermore, for $r = 2k$, and $0 \leq k \leq m-1$, the extreme left cohomology is $\mathbb{Z}H^k$, and the extreme right one is $\mathbb{Z}h^k$. Thus j^* and i^* are both isomorphisms for $0 \leq r = 2k \leq 2m-2$ and $H^{2k}(Q_{2m-1},\mathbb{Z}) = \mathbb{Z}x^k$ for $0 \leq k \leq m-1$.

Again since $i \circ j$ is a linear inclusion, the homology map given by the composite:

$$H_r(\mathbb{P}^{m-1}(\mathbb{C}),\mathbb{Z}) \xrightarrow{j_*} H_r(Q_{2m-1},\mathbb{Z}) \xrightarrow{i_*} H_r(\mathbb{P}^{2m}(\mathbb{C}),\mathbb{Z})$$

is an isomorphism for $0 \leq r \leq 2m-2$. Again by the same reasoning as given above for cohomology, it follows that $H_r(Q_{2m-1},\mathbb{Z}) = 0$ for $0 \leq r \leq 2m-2$ and r odd. Thus by Poincare duality on Q_{2m-1}, we have $H^i(Q_{2m-1},\mathbb{Z}) = 0$ for all odd i, $0 \leq i \leq 4m-2$. Again, by the above, $j_* : H_{2k}(\mathbb{P}^{m-1}(\mathbb{C}),\mathbb{Z}) \to H_{2k}(Q_{2m-1},\mathbb{Z})$ is an isomorphism of infinite cyclic groups for $0 \leq k \leq m-1$.

Setting D_i to be the Poincare duality isomorphisms for $\mathbb{P}^{m-1}(\mathbb{C})$ and Q_{2m-1} respectively, it follows by the preceding para that the composite:

$$H^{2k}(\mathbb{P}^{m-1}(\mathbb{C}),\mathbb{Z}) \xrightarrow{D_1} H_{2m-2-2k}(\mathbb{P}^{m-1}(\mathbb{C}),\mathbb{Z}) \xrightarrow{j_*} H_{2m-2-2k}(Q_{2m-1},\mathbb{Z}) \xrightarrow{D_2^{-1}} H^{2m+2k}(Q_{2m-1},\mathbb{Z})$$

is an isomorphism for $0 \leq k \leq m-1$. This composite map is the integral cohomology Gysin homomorphism denoted $j_!$, so, setting $k = 0$, we find that $H^{2m}(Q_{2m-1},\mathbb{Z})$ is a cyclic group generated by $y = j_!1$. Also $j_!$ is a $H^*(Q_{2m-1},\mathbb{Z})$-module homomorphism, so $H^{2m+2k}(Q_{2m-1},\mathbb{Z})$ is a cyclic group generated by $j_!(h^k) = x^k j_! 1 = x^k y$ for all $0 \leq k \leq m-1$.

Now we need only determine the algebra relations. Since $H^{4m}(Q_{2m-1},\mathbb{Z}) = 0$, it follows that $y^2 = 0$. Furthermore, since Q_{2m-1} is a degree 2 hypersurface in $\mathbb{P}^{2m}(\mathbb{C})$, we have that $\langle x^{2m-1}, [Q]\rangle = 2$, where $[Q] \in H_{4m-2}(Q_{2m-1},\mathbb{Z})$ is the fundamental homology class of Q_{2m-1}. Thus $\langle x^m.x^{m-1}, [Q]\rangle = 2$.

Since by Poincare duality the generators y of H^{2m} and x^{m-1} of H^{2m-2} are dually paired, we have $\langle y.x^{m-1}, [Q]\rangle = 1$. Thus $x^m = 2y$ and the proposition is proved. $\square$.

Corollary 3.5.2 (mod 2 cohomology)**.** The cohomology ring $H^*(Q_{2m-1}, \mathbb{Z}_2)$ (where $m \geq 2$) is given by:

$$H^*(Q_{2m-1}, \mathbb{Z}_2) = \mathbb{Z}_2[\xi, \eta]/\langle \xi^m, \eta^2\rangle$$

where ξ (resp. η) is the mod 2 reduction of x (resp. y) of the last proposition. Alternatively, $\xi = w_2(\mathcal{O}_{Q_{2m-1}}(1))$, the second Steifel-Whitney class of the canonical bundle on Q_{2m-1} considered as a real 2-plane bundle, and $\eta = j_!1$, where $j_! : H^*(\mathbb{P}^{m-1}(\mathbb{C}), \mathbb{Z}_2) \to H^{*+2m}(Q_{2m-1}, \mathbb{Z}_2)$ is the $\mathbb{Z}_2$-cohomology Gysin homormorphism. In particular :

$$\begin{aligned} H^{2k+1}(Q_{2m-1}, \mathbb{Z}_2) &= 0 \ \text{ for all } \ k \\ H^{2k}(Q_{2m-1}, \mathbb{Z}_2) &= \mathbb{Z}_2\xi^k \ \text{ for all } \ 0 \leq k \leq m-1 \\ &= \mathbb{Z}_2\xi^{k-m}\eta \ \text{ for all } \ m \leq k \leq 2m-1 \end{aligned}$$

Proof:. Everything follows immediately from the last proposition. It is known that for a complex vector bundle, the total Stiefel-Whitney class is the mod 2 reduction of the total Chern class (see [Mil-S], Problem 14-B on p. 171). Hence $\xi = w_2(\mathcal{O}_{Q_{2m-1}}(1))$. $\square$.

Lemma 3.5.3 (Steenrod squares)**.** Let $m \geq 2$. Then the second Steenrod squaring operation Sq^2 on $H^*(Q_{2m-1}, \mathbb{Z}_2)$ is determined by:

$$\begin{aligned} Sq^2(\xi) &= \xi^2 \\ Sq^2(\eta) &= (m-1)\,\xi\eta \pmod 2 \end{aligned}$$

where ξ and η are the algebra generators of Corollary 3.5.2 above.

Proof: Since $Sq^i x = x^2$ for $x \in H^i$ (see [Mil-S], part (3) on p. 90), and $\xi \in H^2(Q_{2m-1}, \mathbb{Z}_2)$, it follows that $Sq^2(\xi) = \xi^2$.

For the second formula, one notes that the Gysin homomorphism $j_!$ used above is well known to be the composite:

$$H^i(\mathbb{P}^{m-1}(\mathbb{C}), \mathbb{Z}_2) \xrightarrow{\phi} H^{i+2m}(D(\nu), S(\nu); \mathbb{Z}_2) \xrightarrow{(l^*)^{-1}} H^{i+2m}(Q_{2m-1}, Q_{2m-1}\backslash\mathbb{P}^{m-1}; \mathbb{Z}_2) \to H^{i+2m}(Q_{2m-1}, \mathbb{Z}_2)$$

where ν is the real rank $2m$ normal bundle of $\mathbb{P}^{m-1}(\mathbb{C})$ in Q_{2m-1}, $D(\nu)$ its disc bundle, $S(\nu)$ its sphere bundle, ϕ the $\mathbb{Z}_2$ Thom isomorphism for ν, $(l^*)^{-1}$ is an excision isomorphism, and the last arrow is restriction. For brevity's sake, denote the composite of the last two maps by α. Then $\eta = j_!1 = \alpha(\phi(1)) = \alpha(U_\nu)$ where $U_\nu \in H^{2m}(D(\nu), S(\nu); \mathbb{Z}_2)$ is the $\mathbb{Z}_2$ Thom class of ν.

Since α is the composite of maps induced by restriction (and the inverse of a restriction), the functorial operation Sq^2 commutes with α (see (2). p. 91 *op cit*). Thus

$$Sq^2(\eta) = Sq^2(\alpha(U_\nu)) = \alpha(Sq^2 U_\nu)$$

So now it remains to determine $Sq^2 U_\nu$. By Thom's identity for Stiefel-Whitney classes (p.91, *op cit*), we have $\phi(w_i(E)) = Sq^i U_E$ for any real bundle E, so $Sq^2 U_\nu = \phi(w_2(\nu))$. Now since the normal bundle of Q_{2m-1} in $\mathbb{P}^{2m}(\mathbb{C})$ is $\mathcal{O}_{Q_{2m-1}}(2)$, and the normal bundle of the linear subspace $\mathbb{P}^{m-1}(\mathbb{C})$ in $\mathbb{P}^{2m}(\mathbb{C})$ is the sum of $m+1$ copies of $\mathcal{O}_{\mathbb{P}^{m-1}(\mathbb{C})}(1)$, it follows that :

$$\nu \oplus \mathcal{O}_{\mathbb{P}^{m-1}(\mathbb{C})}(2) = \left[\mathcal{O}_{\mathbb{P}^{m-1}(\mathbb{C})}(1)\right]^{m+1}$$

Thus $c_1(\nu) = (m+1)h - 2h = (m-1)h$, where h is the hyperplane class of $\mathbb{P}^{m-1}(\mathbb{C})$.

Since $w_2(\nu)$ is the mod 2 reduction of $c_1(\nu)$, it follows that $Sq^2(U_\nu) = \phi(w_2(\nu)) = (m-1)\phi(h)$ mod 2. Thus

$$\begin{aligned} Sq^2(\eta) &= \alpha(Sq^2 U_\nu) = (m-1)\alpha(\phi(h)) \mod 2 \\ &= (m-1)j_!(h) = (m-1)\xi . j_!(1) = (m-1)\xi\eta \mod 2 \end{aligned}$$

since $j^*\xi = h$ mod 2 and $j_!$ is a $H^*(Q_{2m-1}, \mathbb{Z}_2)$-module homomorphism. This proves the lemma. □

Corollary 3.5.4. In the setting of above, the Steenrod square Sq^2 on the generator of $H^{4m-4}(Q_{2m-1}, \mathbb{Z}_2)$ is given by:

$$Sq^2(\xi^{m-2}\eta) = \xi^{m-1}\eta$$

Proof: By the derivational identity for Steenrod squares $Sq^k(a.b) = \sum_{i+j=k} Sq^i(a) Sq^j(b)$ (see (4), p. 91, *op cit*), and the fact that Sq^1 acts as 0 on $H^*(Q_{2m-1}, \mathbb{Z}_2)$ (since it lands in odd degree cohomologies which are 0), we have by Lemma 3.5.3 above that:

$$\begin{aligned} Sq^2(\xi^{m-2}\eta) &= Sq^2(\xi^{m-2})\eta + \xi^{m-2} Sq^2(\eta) \\ &= (m-2)\xi^{m-3} Sq^2(\xi)\eta + \xi^{m-2}.\xi\eta \\ &= (m-2)\xi^{m-3}\xi^2\eta + (m-1)\xi^{m-1}\eta \mod 2 \\ &= (2m-3)\xi^{m-1}\eta \mod 2 \\ &= \xi^{m-1}\eta \end{aligned}$$

which proves the corollary. □

Proposition 3.5.5. Let E be a continuous complex vector bundle of any rank on Q_{2m-1} (where $m \geq 2$). In terms of the generators of H^2, H^{4m-4} and H^{4m-2} determined in Proposition 3.5.1, define the Chern numbers $c_j \in \mathbb{Z}$ by:

$$c_{2m-1}(E) = c_{2m-1}(x^{m-1}y); \quad c_{2m-2}(E) = c_{2m-2}(x^{m-2}y); \quad c_1(E) = c_1.x$$

Then:

$$c_{2m-1} = c_{2m-2}(c_1 + 1) \mod 2$$

Proof: Since for a complex bundle E, the Stiefel-Whitney class $w_{2j}(E)$ is the mod 2 reduction of $c_j(E)$ (and odd Stiefel-Whitney classes vanish) as remarked above, what we have by definition is:

$$w_{4m-2}(E) = c_{2m-1}\xi^{m-1}\eta; \quad w_{4m-4}(E) = c_{2m-2}.\xi^{m-2}\eta; \quad w_2(E) = c_1\xi \quad \text{all mod } 2$$

where ξ and η are the mod 2 reductions of x and y respectively, as in Corollary 3.5.2. Hence, by Wu's formula for Stiefel-Whitney classes of a real bundle (see Problem 8-B on p. 94, *op cit*):

$$Sq^2 w_n = w_2 w_n + (2-n) w_1 w_{n+1} + \frac{(2-n)(2-n-1)}{2!} w_0 w_{n+2}$$

For a complex vector bundle E, $w_1(E) = 0$ and applying the formula above for $n = 4m-4$ we have:

$$\begin{aligned} Sq^2(w_{4m-4}(E)) &= w_2(E) w_{4m-4}(E) + \frac{(2-4m+4)(2-4m+3)}{2} w_{4m-2}(E) \mod 2 \\ &= w_2(E) w_{4m-4}(E) + (1-2m)(1-4m) w_{4m-2}(E) \mod 2 \\ &= w_2(E) w_{4m-4}(E) + w_{4m-2}(E) \end{aligned}$$

Hence substituting from the first paragraph, and using the Corollary 3.5.4 we have:

$$\begin{aligned} Sq^2(c_{2m-2}.\xi^{m-2}\eta) &= (c_1\xi).(c_{2m-2}\xi^{m-2}\eta) + c_{2m-1}\xi^{m-1}\eta \mod 2 \\ c_{2m-2}(\xi^{m-1}\eta) &= (c_1 c_{2m-2} + c_{2m-1})\xi^{m-1}\eta \mod 2 \end{aligned}$$

which implies the proposition. □.

We can now prove (viii) of the main theorem.

Proposition 3.5.6. Let $m \geq 2$. Then the smooth quadric hypersurface $Q_{2m-1} \subset \mathbb{P}^{2m}(\mathbb{C})$ does not possess the property Z_c.

Proof: The proof proceeds exactly as in the proof of 3.4.3. First note that since the normal bundle of Q_{2m-1} in $\mathbb{P}^{2m}(\mathbb{C})$ is $\mathcal{O}(2)$, the first Chern class of Q_{2m-1} is $c_1(\tau_{Q_{2m-1}}) = (2m+1)x - 2x = (2m-1)x$. Thus $w_2(\tau_{Q_{2m-1}}) = \xi \in H^2(Q_{2m-1}, \mathbb{Z}_2)$.

Now let E be a complex vector bundle of complex rank $2m-1$ on $Q_{2m-1} \times Q_{2m-1}$ realising the property Z_c. Let its first Chern class be $c_1(E) = a_1(x \times 1) + a_2(1 \times x) \in H^2(Q_{2m-1} \times Q_{2m-1}, \mathbb{Z})$. Then since Z_c implies that $\Delta^*(E) \simeq \tau_{Q_{2m-1}}$, we have $\Delta^*(c_1(E)) = (a_1 + a_2)x = c_1(\tau_{Q_{2m-1}}) = (2m-1)x$, so that $a_1 + a_2 \equiv 1 \mod 2$.

On the other hand, we find that the restrictions E_i, $i = 1, 2$ of E to the horizontal and vertical slices $Q_{2m-1} \times \{p\}$ and $\{q\} \times Q_{2m-1}$ respectively must have top Chern number $c_{2m-1} \equiv 1 \mod 2$. This implies, by the Proposition 3.5.5 that the first Chern numbers c_1 of E_i are both even. That is a_1 and a_2 are both $\equiv 0$ mod 2. This contradicts the last paragraph. The proposition follows. □.

Remark 3.5.7. For the case of $m = 2$, that smooth complex vector bundles on $Q_3 \subset \mathbb{P}^4(\mathbb{C})$ satisfy the above mod 2 identity for Chern numbers (and hence Z_c fails for Q_3) follows from Corollary 3.4.5.

Remark 3.5.8. It is not clear what happens for quadrics of even complex dimension. We note that $Q_2 = \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ and $Q_4 = G_2(\mathbb{C}^4)$ both satisfy Z_c by Example 1.2.9 and 1.2.2.

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