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On quasi-isometric embeddings of Lamplighter groups

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ON QUASI-ISOMETRIC EMBEDDINGS OF LAMPLIGHTER GROUPS

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1. Introduction

Recall that if (X, d) and (Y, d') are a metric spaces, then a map $f : X \longrightarrow Y$ is called (λ, ε) quasi-isometric embedding if there exist constants $\lambda, \varepsilon \geq 0$ such that

$$\frac{1}{\lambda}d(x,y) - \varepsilon \leq d'(f(x), f(y)) \leq \lambda(x,y) + \varepsilon.$$

A quasi-isometric embedding f is called a quasi-isometry if there exists a constant $C \geq 0$ such that $d'(y, f(X)) \leq C$ for all $y \in Y$. Associated to any metric space (X, d) is its quasi-isometry group $\mathcal{QI}(X)$. This is the group of all self quasi-isometries of X modulo those which are at a bounded distance from each other.

If Γ is a finitely generated group with a finite generating set $\mathcal{A}$, then the word metric corresponding to $\mathcal{A}$ is denoted by $d_{\mathcal{A}}$. If $\mathcal{B}$ is another finite generating set for Γ, then the metric spaces $(\Gamma, d_{\mathcal{A}})$ and $(\Gamma, d_{\mathcal{B}})$ are quasi-isometric. We can therefore unambiguously talk about two finitely generated groups being quasi-isometric without referring to the word metrics.

Let G be a finite group. A Lamplighter group Γ_G is the wreath product of G and $\mathbb{Z}$. Hence, $\Gamma_G = (\oplus_{i\in\mathbb{Z}}G) \rtimes \mathbb{Z}$. We denote by G_m the 'mth copy' of G and by a_m, a typical element of G_m. These finitely generated groups have been the object of much study in recent times (e.g. [5], [6], [8]).

If G and F are two finite groups such that, $ord(G^k) = ord(F^l)$ for some positive integers k, l, then the groups Γ_G and Γ_F are quasi-isometric (see [3],[4]). We denote by Γ_n the Lamplighter group Γ_G of a finite group G of order n. Not much is known when the orders of the groups do not satisfy the above condition. In fact, it is not known whether the groups Γ_2 and Γ_3 are quasi-isometric. From the point of view of geometric group theory, one is interested in the quasi-isometry classification of the Lamplighter groups.

In this article we prove that for any integers $n, m > 1$, there exists a quasi-isometric embedding from Γ_n to Γ_m. We also study $\mathcal{QI}(\Gamma_n)$, the quasi-isometry group of Γ_n and prove that it contains all finite groups. On the other hand, we also show that the order of an automorphism of Γ_n is either infinite or divides $2n\phi(n)$, where $\phi(n)$ denotes the number of integers which are less than n and are co-prime to n. These two results allow us to show that the group of automorphisms of Γ_n has infinite index in $\mathcal{QI}(\Gamma_n)$.

2. Geometry of Γ_n

In this article, we fix the following generating set of Γ_n.

$$\mathcal{A} = \{a_0, t, t^{-1} \mid a_0 \neq e\}.$$

For an integer $k > 0$, the subgroup H_k of Γ_n generated by $\{a_0, t^k, t^{-k} \mid a_0 \neq e\}$ has index k in Γ_n. As H_k is isomorphic to Γ_{n^k}, we see that Γ_n and Γ_{n^k} are quasi-isometric.

We now describe the metric on Γ_n. A possible normal form of a word $\omega \in \Gamma_n$ is given by $\omega = a_{i_1}a_{i_2}\cdots a_{i_r}t^m$ with $i_1 < i_2 < \cdots < i_r$. If ω has the above form, then ω can also be written as

$$\begin{aligned}\omega &= (t^{i_1}a_0t^{-i_1})(t^{i_2}a_0t^{-i_2})\cdots(t^{i_r}a_0t^{-i_r})t^m \\ &= t^{i_1}a_0t^{i_2-i_1}a_0t^{i_3-i_2}\cdots a_0t^{i_r-i_{r-1}}a_0t^{m-i_r}.\end{aligned}$$

Thus

$$\ell(\omega) \leq |i_1| + r + (i_r - i_1) + |m - i_r|. \tag{1}$$

On the other hand, ω can also be written as $\omega = a_{i_r}a_{i_r-1}\cdots a_{i_1}t^m$ so that

$$\omega = t^{i_r}a_0t^{i_{r-1}-i_r}\cdots a_0t^{i_1-i_2}a_0t^{m-i_1}$$

and hence

$$\ell(\omega) \leq |i_r| + r + (i_r - i_1) + |m - i_1|. \tag{2}$$

We shall show that the length $\ell(\omega)$ of ω is given by the following formula.

Lemma 2.1. *Let $\omega \in \Gamma_n$ be as above, then*

$$\ell(\omega) = r + (i_r - i_1) + \min\{|i_1| + |m - i_r|, |i_r| + |m - i_1|\}.$$

Proof. Let $\omega = a_{i_1}a_{i_2}\cdots a_{i_r}t^m$ with $i_1 < i_2 < \cdots < i_r$. Inequalities 1 and 2 imply that

$$\ell(\omega) \leq r + (i_r - i_1) + \min\{|i_1| + |m - i_r|, |i_r| + |m - i_1|\}. \tag{3}$$

Let $P = t^{k_0}a_0t^{k_1}a_0\cdots t^{k_\ell}a_0t^{k_{\ell+1}}$ be any path from 1 to ω in the Cayley graph. Here $k_i \neq 0$ if $0 < i < \ell+1$. First observe that a_0 occurs at least r times in the path P so that $\ell+1 \geq r$. Next, consider the sequence of the partial sums $k_0, k_0+k_1, \ldots, k_0+k_1+\cdots+k_\ell$ of the exponents of t in the path P. The indices i_1 and i_r must appear as one of the terms of this sequence. Suppose that i_1 appears before i_r in this sequence at the jth place. Then, $k_0+\cdots+k_j = i_1$. If a_{i_r} occurs at sth place in P with $j < s$, we get $k_{j+1}+\cdots k_s = i_r - i_1$. Finally, we must have $k_{s+1}+\cdots+k_\ell = m - i_r$ to achieve the exponent m of t. Thus, we see that in this case the length $\ell(P)$ of the path P satisfies the bound

$$\ell(P) = (\ell+1) + \sum_0^{\ell+1}|k_i| \ \geq \ r + |i_1| + (i_r - i_1) + |m - i_r|. \tag{4}$$

On the other hand, if i_r appears before i_1 in the sequence of partial sums, similar arguments show that

$$\ell(P) \geq r + |i_r| + (i_r - i_1) + |m - i_1|. \tag{5}$$

Inequalities 4 and 5 show that

$$\ell(\omega) \geq r + (i_r - i_1) + \min\{|i_1| + |m - i_r|, |i_r| + |m - i_1|\}. \tag{6}$$

This completes the proof of the lemma. □

Remark 2.2. By Lemma 2.1, the length of the word $a_0\cdots a_r$ is $3r+1$. Thus, in Γ_n, the ball of radius $3r+1$ around identity will have at least $(n-1)^{r+1}$ distinct elements. This proves that Γ_n has exponential growth for all $n > 2$. Since Γ_2 contains Γ_4 as a finite index subgroup, Γ_n has exponential growth for all $n > 1$.

Remark 2.3. Another description of the length of a word in the Lamplighter group can be found in [1].

Given two finitely generated groups it is a difficult problem to decide if one quasi-isometrically embeds into the other. Even if there exists a quasi-isometric embedding in one direction there may not be one in the other. Indeed, the free group F_2 of rank 2 does not quasi-isometrically embed into the infinite cyclic group $\mathbb{Z}$. Using the above description of word length in Γ_n we show that any two Lamplighter groups can be quasi-isometrically embedded into the each other.

Theorem 2.4. *Let u, v be integers greater than 1. Then there exists a quasi-isometric embedding $\Theta_{u,v} : \Gamma_u \longrightarrow \Gamma_v$.*

Proof. Without loss, assume that $1 < u \leq v$. Let $\theta : \mathbb{Z}_u \to \mathbb{Z}_v$ be a (set theoretic) one-one map with $\theta(e) = e$. Define $\Theta_{u,v} : \Gamma_u \to \Gamma_v$ by:

$$\Theta_{u,v}(t^i x t^{-i}) = t^i \theta(x) t^{-i} \text{ and } \Theta_{u,v}(a_{i_1} \cdots a_{i_r} t^m) = \Theta_{u,v}(a_{i_1}) \cdots \Theta_{u,v}(a_{i_r}) t^m$$

where x denotes a non-identity element of $\mathbb{Z}_u$.

If ω, τ are two elements of Γ_u with

$$\begin{aligned} \omega &= a_{i_1} \cdots a_{i_r} t^m \\ \tau &= b_{j_1} \cdots b_{j_s} t^n \end{aligned}$$

then,

$$\Theta_{u,v}(\omega^{-1}\tau) = t^{-m}[\Theta_{u,v}(a_{i_1})]^{-1} \cdots [\Theta_{u,v}(a_{i_r})]^{-1} \Theta_{u,v}(b_{j_1}) \cdots \Theta_{u,v}(b_{j_s}) t^n.$$

Since $\Theta_{u,v}$ takes the ith copy of $\mathbb{Z}_u$ in Γ_u to the ith copy of $\mathbb{Z}_v$ in Γ_v, the indices that get cancelled (or clubbed together) in the expression of $\omega^{-1}\tau$ are the same as the indices that get cancelled (or clubbed together) in $\Theta_{u,v}(\omega)^{-1}\Theta_{u,v}(\tau)$ and hence $\ell(\omega^{-1}\tau) = \ell(\Theta_{u,v}(\omega)^{-1}\Theta_{u,v}(\tau))$. Therefore, $\Theta_{u,v}$ is an isometry.

To construct the quasi-isometric embedding $\Theta_{v,u} : \Gamma_v \longrightarrow \Gamma_u$, choose an integer k such that $u^k \geq v$. As observed before, there exists a quasi-isometry $\chi : \Gamma_{u^k} \longrightarrow \Gamma_u$. Then $\chi \circ \Theta_{v,u^k} : \Gamma_v \longrightarrow \Gamma_u$ is a quasi-isometric embedding. This completes the proof. □

Remark 2.5. As $\Gamma_1 = \mathbb{Z}$, there exists a quasi-isometric embedding of Γ_1 inside Γ_v, for all positive integers v. On the other hand, for all $v > 1$, Remark 2.2 shows that Γ_v has exponential growth so that there does not exist any quasi-isometric embedding from Γ_v to Γ_1.

3. Quasi-isometry group of Γ_n

Let Γ and Γ' be two finitely generated groups. If $\varphi : \Gamma \longrightarrow \Gamma'$ is a group homomorphism, then φ is a quasi-isometry if and only if both its kernel and co-kernel are finite. Thus, for any finitely generated group Γ, we have a canonical homomorphism $\theta : Aut(\Gamma) \longrightarrow \mathcal{QI}(\Gamma)$. We shall denote by $C(g)$ the centralizer of g in Γ. The virtual center $K(\Gamma)$ of Γ is the group

$$K(\Gamma) = \{g \in \Gamma \mid [\Gamma : C(g)] < \infty\}.$$

In [7], it was proved that the canonical homomorphism $\theta : Aut(\Gamma) \longrightarrow \mathcal{QI}(\Gamma)$ is injective if $K(\Gamma) = 0$.

The quasi-isometry groups of Γ_n are not known. Their structure has been conjectured in [9]. In this section we show that, for any $n > 1$, the quasi-isometry group $\mathcal{QI}(\Gamma_n)$ contains all finite groups. Thus, the torsion elements in the groups $\mathcal{QI}(\Gamma_n)$ cannot be used to distinguish the quasi-isometry classes amongst the Lamplighter groups. We begin with the following.

Lemma 3.1. *Let G be a finite group with $|G| > 1$. Then $K(\Gamma_G) = 0$.*

Proof. Note that the virtual center consists of precisely those elements having finitely many conjugates. Since,

$$t^i(a_{i_1}\cdots a_{i_r}t^m)t^{-i} = a_{i_1+i}\cdots a_{i_r+i}t^m$$

and since none of the a_i's commute with t^m, we see that every non-identity element of Γ_G has infinitely many conjugates. □

Lemma 3.2. *Let G be a finite group. Let $\theta : G \longrightarrow G$ be an automorphism. Then θ extends to an automorphism Θ of Γ_G.*

Proof. We define Θ by $\Theta(a_{i_1}\cdots a_{i_r}t^m) = \theta(a)_{i_1}\cdots\theta(a)_{i_r}t^m$. □

Remark 3.3. The extension of θ is not unique. For example, if G is abelian, the identity automorphism of G can be extended as conjugation by any finite order element of Γ_G. Since t does not commute with finite order elements of Γ_G, this extension is not an identity automorphism.

Proposition 3.4. *$\mathcal{QI}(\Gamma_n)$ contains all finite groups.*

Proof. Let k be a positive integer. As before, let H_k denote the index k subgroup of the Lamplighter group Γ_n. If $C_{k,n}$ denotes the direct sum of k copies of $\mathbb{Z}_n$, then $H_k = \Gamma_{C_{k,n}}$. By Lemma 3.2, the permutation group S_k is a subgroup of $Aut(\Gamma_{C_{k,n}})$. By Lemma 3.1, we have $S_k \subset \mathcal{QI}(H_k)$. However, $\mathcal{QI}(\Gamma_n) = \mathcal{QI}(H_k)$ as H_k is quasi-isometric to Γ_n. This proves the proposition. □

We contrast the above proposition with the following result. For a positive integer n, let $\phi(n)$ denote the order of the group of units in $\mathbb{Z}_n$.

Proposition 3.5. *Let $\varphi \in Aut(\Gamma_n)$ be an element of finite order. Then the order of φ divides $2n\phi(n)$.*

Proof. Let $\Gamma_n = \Gamma_{\mathbb{Z}_n}$. As before, let G_m denote the 'mth' copy of $\mathbb{Z}_n$ with generator b_m. A typical element of G_m is denoted by a_m. Any automorphism φ of Γ_n is uniquely determined by its values on b_0 and t. Moreover, $\varphi(b_0)$ has order n and hence is of the type $a_{i_1}\cdots a_{i_r}$ for some $i_1 < \cdots < i_r \in \mathbb{Z}$. Also, as t is in the image, we see that $\varphi(t) = wt^{\pm 1}$ for some finite order element $w \in \Gamma_n$.

Case 1. $\varphi(t) = wt$.

Define w_i by $\varphi(t^i) = w_i t^i$ and assume that $\varphi(b_0) = a_{i_1}\cdots a_{i_r}$. Then, $\varphi(b_l) = \varphi(t^l b_0 t^{-l}) = a_{i_1+l}\cdots a_{i_r+l}$, for any integer l. Thus $\varphi^2(b_0) = a_{2i_1}z_2a_{2i_r}$ where $z_2 = c_{j_1}\cdots c_{j_s}$ with $2i_1 < j_q < 2i_r$ for all j_q. Hence $\varphi^n(b_0) = a_{ni_1}z_na_{ni_r}$ where z_n is the product of terms from G_i with $ni_1 < i < ni_r$.

Therefore, for φ to have finite order, we must have $\varphi(b_0) \in G_0$. Hence any such φ induces an automorphism of G_0.

This implies that $\varphi^{\phi(n)}(b_0) = b_0$ and $\varphi^{\phi(n)}(t) = w_{\phi(n)}t$. Since the order of $w_{\phi(n)}$ divides n, we see that $\varphi^{n\phi(n)} \equiv Id$. This means that the order of φ divides $n\phi(n)$.

Case 2. $\varphi(t) = wt^{-1}$.

Define w_i by $\varphi(t^i) = w_i t^{-i}$ and assume that $\varphi(b_0) = a_{i_1}\cdots a_{i_r}$ with $r > 1$ and $i_1 < \cdots < i_r$. Let $q = i_r - i_1$. Then, $\varphi(b_l) = a_{i_1-l}\cdots a_{i_r-l}$, for any integer l. This means that $\varphi^2(b_0) = a_{-q}z_2a_q$ where $z_2 = c_{j_1}\cdots c_{j_s}$ with $-q < j_a < q$ for all j_a. Hence $\varphi^n(b_0) = a_{-(n-1)q}z_na_{(n-1)q}$ where z_n is the product of terms from G_i with $-(n-1)q < i < (n-1)q$.

Therefore, if the order of φ is finite then we must have $\varphi(b_0) = a_l$ for some l. Therefore, $\varphi^2(b_o) \in G_0$ and hence $\psi = \varphi^2$ induces an automorphism of G_0.

As in the previous case, we see that the order of ψ divides $n\phi(n)$. This means that the order of φ divides $2n\phi(n)$. □

Theorem 3.6. *$Aut(\Gamma_n)$ has infinite index in $\mathcal{Q}I(\Gamma_n)$.*

Proof. For any positive integer m, consider the inclusion $S_{2m} \subset \mathcal{Q}I(\Gamma_n)$ given by Proposition 3.4. For any disjoint m-cycles $\sigma, \tau \in S_{2m}$, the quasi-isometry given by $\sigma^{-1}\tau$ has order m. Therefore, by Proposition 3.5, $\sigma^{-1}\tau \notin Aut(\Gamma_n)$ for large m. Hence any such pair σ, τ must belong to distinct cosets of $Aut(\Gamma_n)$ in $\mathcal{Q}I(\Gamma_n)$. As the number of such pairwise disjoint m-cycles is not bounded in $\mathcal{Q}I(\Gamma_n)$, the theorem follows. □

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