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Limit Theorems in Network Traffic Models with Very Heavy Tails

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Abstract. We consider a data communication network model in which a source starts a process of transmission of some data at the renewal time point $S_k, k \geq 1$, and continues for a period of random length T_k. Transmissions are assumed to occur either at unit rate or at the random rate ξ_k corresponding to the source that starts at time S_k. The processes of interest are the number of active sources at time t, the cumulative number of active sources upto time t, the amount of work inputted at time t, and the cumulative amount of work inputted upto time t. We present a detailed study of the asymptotic behaviors of all these processes in a suitable unified framework, under the assumption that the common distribution of the iid T_k is very heavy tailed and further appropriate assumptions on the common distributions of inter-renewal times and the transmission rates ξ_k.

1 INTRODUCTION

The present paper is directly related to Mikosch and Resnick (2004), as will be made clear below (but see further the references given in that paper). We begin by recalling the framework in that work. Consider

$$S_0 = 0, \qquad S_k = \sum_{i=1}^{k} X_i, \ k \geq 1,$$

where $\{X_k, k \geq 1\}$ is a sequence of iid nonnegative random variables with common distribution F. Here S_k has the interpretation of the renewal time point at which an event begins and continues for a period of random length T_k, that is, the event terminates at $S_k + T_k$. It is assumed that $T_k, k \geq 1$, are iid (nonnegative) random variables with common distribution G. It is further assumed, without further mentioning, that

$$X_k \text{ and } T_k \text{ are independent for each } k \geq 1.$$

In the context of communication network, S_k is the renewal time at which a source starts a process of transmission of some data, which process is continued for the period of length T_k. Then the number of active sources of transmission at the time z is given by

$$M_\infty(z) = \sum_{k=1}^{\infty} \mathbb{I}_{\{S_k \leq z < S_k + T_k\}}.$$

(Here and below $\mathbb{I}_A$ stands for the indicator function of the event A.) Also

$$M_n(z) = \sum_{k=1}^{n} \mathbb{I}_{\{S_k \leq z < S_k + T_k\}}$$

is the number of active sources at the time z among the first n sources. (Here $M_\infty(z)$ and $M_n(z)$ are called activity rates.)

In the insurance context, if S_k is the time of occurrence of an accident and if T_k is the length of time it takes to settle the insurance claim, then $M_n(z)$ is the number of active claims at time z among the first n claims.

Note that if we let

$$N(x) = \sup\{l \geq 0 : S_l \leq x\} = \inf\{l \geq 0 : S_{l+1} > x\} = \sum_{k=1}^{\infty} \mathbb{I}_{\{S_k \leq x\}},$$

then

$$M_\infty(z) = M_{N(z)}(z).$$

Further, the processes

$$A_\infty(z) = \int_0^z M_\infty(u)\, du, \qquad A_n(z) = \int_0^z M_n(u)\, du$$

have the interpretation of cumulative input processes upto the time z. We note that

$$\begin{aligned} A_n(z) &= \sum_{k=1}^{n} \int_0^z \mathbb{I}_{\{S_k \leq u < S_k + T_k\}} du = \sum_{k=1}^{n} \mathbb{I}_{\{S_k \leq z\}} \left(\min(S_k + T_k, z) - S_k\right) \\ &= \sum_{k=1}^{n} \mathbb{I}_{\{S_k \leq z\}} \min(T_k, z - S_k) = \sum_{k=1}^{n} \min\left(T_k, (z - S_k)^+\right) \end{aligned}$$

where we define $x^+ = x$ if $x \geq 0$, $= 0$ if $x < 0$. Further,

$$A_\infty(z) = A_{N(z)}(z) = \sum_{k=1}^{N(z)} \min(T_k, z - S_k).$$

Above, it is assumed that the data transmissions occur at unit rate. Now assume more generally that the rate corresponding to the source that starts at time S_k is ξ_k, where ξ_k are iid nonnegative random variables, with

$$\xi_k \text{ independent of } X_k \text{ and } T_k \text{ for each } k \geq 1.$$

Then the cumulative input process upto the time z takes the form

$$A_{\infty}^{*}(z) = \sum_{k=1}^{N(z)} \min(T_k, z - S_k)\,\xi_k = \int_0^z M^{*}(u)\,du$$

where

$$M_{\infty}^{*}(u) = \sum_{k=1}^{\infty} \mathbb{I}_{\{S_k \leq u < S_k + T_k\}} \xi_k,$$

and that upto the time z among the first n sources takes the form

$$A_n^{*}(z) = \sum_{k=1}^{n} \min\left(T_k, (z - S_k)^{+}\right) \xi_k = \int_0^z M_n^{*}(u)\,du$$

with

$$M_n^{*}(u) = \sum_{k=1}^{n} \mathbb{I}_{\{S_k \leq u < S_k + T_k\}} \xi_k.$$

Here $M_{\infty}^{*}(u)$ has the interpretation of the rate of work inputted at time u, and that $M_n^{*}(u)$ as the rate at time u among the first n sources.

It will be assumed throughout that the distribution function G of T_1 is *very heavy tailed* in the sense that

$$\overline{G}(x) = 1 - G(x) \sim x^{-\beta} L_G(x), \quad x \to \infty, \, 0 \leq \beta < 1,$$

for some slowly varying function $L_G(x)$. Regarding the distribution F of X_1, we shall assume that either

$$\overline{F}(x) = 1 - F(x) \sim x^{-\alpha} L_F(x), \, x \to \infty, \, 0 < \alpha < 1, \tag{1}$$

for some slowly varying function $L_F(x)$, or $E[X_1] < \infty$.

As in Mikosch and Resnick (2004) we shall consider the following three cases separately.

Case I: F and G are very heavy tailed with comparable tails: $0 < \beta = \alpha < 1$ and $\overline{F}(x) \sim c\overline{G}(x)$ as $x \to \infty$ for some $c > 0$. For simplicity, we shall assume that $c = 1$.

Case II: F and G are very heavy tailed with G heavier tailed: $0 \leq \beta < \alpha < 1$, or $0 < \beta = \alpha < 1$ with $\frac{\overline{F}(x)}{\overline{G}(x)} \to 0$ as $x \to \infty$.

Case III: F has finite mean and G is very heavy tailed: $0 < \beta < 1$ and $E[X_1] < \infty$.

In addition, appropriate moment conditions on ξ_1 will also be imposed when we deal with $M_n^*(z)$, $M_\infty^*(z)$, $A_n^*(z)$ and $A_\infty^*(z)$.

The results obtained in Mikosch and Resnick (2004) for the preceding three cases may be roughly summarized as follows, where the limit is taken as $s \to \infty$.

Some Results of Mikosch and Resnick (2004)

Case I $\quad M_\infty(sz) \Longrightarrow$ random limit.

Case II $\quad \begin{cases} \frac{\overline{F}(s)}{\overline{G}(s)} M_\infty(sz) \Longrightarrow \text{ random limit,} \\ \frac{\overline{F}(s)}{s\overline{G}(s)} A_\infty(sz) \Longrightarrow \text{ random limit.} \end{cases}$

Case III $\quad \begin{cases} \frac{1}{s\overline{G}(s)} M_\infty(sz) \Longrightarrow \text{ constant,} \\ \sqrt{\frac{1}{s\overline{G}(s)}} \left(M_\infty(sz) - \text{ random center}\right) \Longrightarrow \text{ Gaussian.} \end{cases}$

(These results correspond to (28) of Section 3, (38) and (39) of Section 4 and (52) and (53) of Section 5 below.)

In the present paper, we obtain the limiting behaviors of the processes

$$(t,z) \longmapsto M_{[nt]}(a_n z), \; M_{[nt]}^*(a_n z), \; A_{[nt]}(a_n z) \text{ and } A_{[nt]}^*(a_n z) \tag{2}$$

as $n \to \infty$ (here in the Cases I and II a_n is such that $n\overline{F}(a_n) \to 1$ and in the Case III, $a_n = n$.), which will also allow us to deduce the limiting behaviors of those of

$$z \longmapsto M_\infty(sz), \; M_\infty^*(sz), \; A_\infty(sz) \text{ and } A_\infty^*(sz) \tag{3}$$

as $s \to \infty$ in a continuous manner. In addition, in Cases II and III we also obtain second order type limit results (such a result for $M_\infty(sz)$ in the Case III is already in Mikosch and Resnick (2004, Proposition 3.2); see the table above with Gaussian limit.)

We would like to mention that many of the problems dealt with in the present paper are, in some form or other, either posed or alluded to (in the form of unresolved problems) in Mikosch and Resnick (2004, Section 5). However, the approach taken in the present paper is different and is based on certain general results obtained in Jeganathan (2006), which framework appears to be particularly suitable for the present situation.

The plan of the paper is as follows. In section 2 we recall some preliminary results, in particular a result from Jeganathan (2006), specialized in a form directly applicable in the present context. This result will be a main framework to obtain the limit theorems. Sections 3 - 5 respectively treat the Cases I - III. We shall deal with the Cases I and II in detail, dealing each of the four quantities in (2) or (3) separately. But it will become clear that the results, as well as the technical details, of Case III are almost identical to those of Case II, and therefore its detailed treatment will become unnecessary.

Among the three cases, the Case I appears to be more relevant for practical purposes than the other two in the context of communication network or in the insurance context, because in Cases II and III, the activity rate $M_\infty(z)$ introduced earlier increases without bound when z does so.

For the detailed references together with some discussions to the empirical works that motivate the present model with very heavy tails, see Mikosch and Resnick (2004) and Resnick (2003). It may be noted that the present work, following Mikosch and Resnick (2004), assumes the independence of the sequences $\{X_k, k \geq 1\}$, $\{T_k, k \geq 1\}$ and $\{\xi_k, k \geq 1\}$. We do not know to what extent this restriction can be relaxed, but see Maulik, Resnick and Rootzén (2002).

Notations. In addition to the natations $\mathbb{I}_A$ for the indicator function of the event A, $c(s) \sim d(s)$ to mean $\frac{c(s)}{d(s)} \to 1$ as $s \to \infty$, $\overline{F}(x) = 1 - F(x)$ and $\overline{G}(x) = 1 - F(x)$ that we have already used above, we shall also use in what follows the following additional notations.

$\xrightarrow{p}$ stands for the convergence in probability. Further, by $\xi_{n,\epsilon} \xrightarrow{p} 0$ as $n \to \infty$ first and then $\epsilon \to 0$, we mean $\lim_{\epsilon\to 0} \text{limsup}_{n\to\infty} P[|\xi_{n,\epsilon}| > \eta] = 0$ for every $\eta > 0$. Convergence in distribution of a sequence of random vectors (of the same order) will be denoted by $\Longrightarrow$. The notation $\stackrel{fdd}{\Longrightarrow}$ stands for the convergence in distribution of a sequence of random processes in the sense of convergence in distribution of all finite dimensional distributions. Also, if the index set of the processes involved is not clear from the context it will be explicitly indicated; for example in the form $\Lambda_n(t) \stackrel{fdd}{\Longrightarrow} \Lambda(t)$, $t \in J$, instead of $\Lambda_n(t) \stackrel{fdd}{\Longrightarrow} \Lambda(t)$.

The notation " $\Longrightarrow$ in $D_{\mathbb{R}^q}[0,1]$ ", for $q \geq 1$, means the convergence in distribution of a sequence of random processes in the Skorokhod space $D_{\mathbb{R}^q}[0,1]$ ($D_{\mathbb{R}^q}[0,1]$ is by definition the collection of all functions from $[0,1]$ to $\mathbb{R}^q$ that are right continuous and admit left-hand limits, equipped with the Skorokhod topology, see Billingsley (1968, Ch. 3) for $D_{\mathbb{R}}[0,1]$ and Jacod and Shiryayev (1987, Ch. ?) for $D_{\mathbb{R}^q}[0,1]$, $q \geq 1$.)

Corresponding to the arrays $\{\mathcal{F}_{nk}, k = 1, 2, ...\}$ of σ-fields defined in (13) below, the abbreviations $E_{k-1}\left[\ .\ \right]$ and $P_{k-1}\left[\ .\ \right]$ stand respectively for the conditional expectation and the conditional probability given the σ-field $\mathcal{F}_{n,k-1}$, that is, $E\left[\ .\ \middle|\mathcal{F}_{n,k-1}\right]$ and $P\left[\ .\ \middle|\mathcal{F}_{n,k-1}\right]$.

2 PRELIMINARY RESULTS

Consider the situation in which (1) is satisfied, where recall that $0<\alpha<1$. Then recall that the constants a_n are such that

$$n\overline{F}\left(xa_n\right)\rightarrow x^{-\alpha},\quad x>0, \tag{4}$$

and (see Lemma 2.4 below)

$$na_n^{-1}E\left[X_1\mathbb{I}_{\{X_1<\tau a_n\}}\right]\rightarrow\frac{\alpha}{1-\alpha}\tau^{1-\alpha}. \tag{5}$$

Let

$$X_{n,k}=a_n^{-1}X_{k+1},\ \ k\geq 0, \tag{6}$$

with X_{k+1} as specified earlier. Note that, with $S_k=\sum_{j=1}^{k}X_j$ as defined earlier,

$$\sum_{k=0}^{k}X_{nk}=a_n^{-1}S_{k+1}.$$

We first recall the familiar facts that underlie the convergence of the process $\sum_{k=0}^{[nt]}X_{nk}$. (These facts will motivate the conditions (C1) and (C2) below.) In view of (4), we have, for all $x\in(0,\infty)$ and $t\in[0,\infty)$,

$$\sum_{k=1}^{[nt]}P\left[X_{nk}>x\right]=[nt]\,\overline{F}\left(xa_n\right)\rightarrow tx^{-\alpha}=L\left(x,t\right),\text{ say.} \tag{7}$$

Also, because $1-\alpha>0$, (5) implies, for all $t>0$,

$$\sum_{k=1}^{[nt]}E\left[X_{nk}\mathbb{I}_{\{X_{nk}<\epsilon\}}\right]=[nt]\,a_n^{-1}E\left[X_1\mathbb{I}_{\{X_1<\epsilon a_n\}}\right]\rightarrow 0\qquad\text{as } n\rightarrow\infty\text{ first and then }\epsilon\rightarrow 0. \tag{8}$$

The preceding two facts entail that

$$\sum_{k=1}^{[nt]}E\left[X_{nk}\mathbb{I}_{\{X_{nk}<\tau\}}\right]\rightarrow\int_0^{\tau}xL\left(dx,t\right)<\infty.$$

In addition, because in the preceding convergence both sides are monotone in t (recall $X_{nk}\geq 0$) and the limit is continuous in t,

$$\sup_{0\leq t\leq M}\left|\sum_{k=1}^{[nt]}E\left[X_{nk}\mathbb{I}_{\{X_{nk}<\tau\}}\right]-\int_0^{\tau}xL\left(dx,t\right)\right|\rightarrow 0 \tag{9}$$

for all $M > 0$. Further (noting $X_{nk} \geq 0$),

$$\sum_{k=1}^{[nt]} \left| E\left[X_{nk} \mathbb{I}_{\{X_{nk}<\tau\}} \right] \right|^2 \to 0 \tag{10}$$

for all $\tau > 0$ because of (9) and because $\sup_{1\leq k\leq [nt]} E\left[X_{nk}\mathbb{I}_{\{X_{nk}<\tau\}}\right] \to 0$ which follows from the fact $\sup_{1\leq k\leq [nt]} P\left[X_{nk} \geq \eta\right] \to 0$ for all $\eta > 0$. In addition, in view of (8),

$$\sum_{k=1}^{[nt]} E\left[X_{nk}^2 \mathbb{I}_{\{X_{nk}<\epsilon\}}\right] \leq \epsilon \sum_{k=1}^{[nt]} E\left[X_{nk}\mathbb{I}_{\{X_{nk}<\epsilon\}}\right] \to 0 \quad \text{as } n \to \infty \text{ first and then } \epsilon \to 0. \tag{11}$$

As is well-known (see for instance Gikhman and Skorokhod (1969)), (7) and (9) -(11) entail that

$$\sum_{k=0}^{[nt]} X_{nk} = a_n^{-1} S_{[nt]+1} \Longrightarrow S(t) \text{ in } D_{\mathbb{R}}[0,\infty) \tag{12}$$

where $S(t)$ is a stable process with index α, $0 < \alpha < 1$, that is, a process with stationary independent increments such that

$$\log E\left[e^{ivS(t)}\right] = t\int_0^\infty \left(e^{ivx} - 1\right)\alpha x^{-\alpha-1}dx.$$

We next present a specialization of a result from Jeganathan (2006), which will be the main framework in Sections 3 -5 below. For this purpose, we let $\mathcal{F}_{nk}$ to be the σ-field generated by $(X_1, ..., X_{k+1}, T_1, ..., T_k, \xi_1, ..., \xi_k)$, that is,

$$\mathcal{F}_{n0} = \sigma(X_1), \qquad \mathcal{F}_{nk} = \sigma(X_1, ..., X_{k+1}, T_1, ..., T_k, \xi_1, ..., \xi_k), \; k \geq 1. \tag{13}$$

with X_j, T_j and ξ_j as in Section 1. For each $n \geq 1$, consider the array of random variables $\{X_{nk}, Y_{nk}, Z_{nk}, k = 1, 2, ...\}$ adapted to the array $\{\mathcal{F}_{nk}, k = 1, 2, ...\}$ of σ-fields.

Here $X_{nk} = a_n^{-1}X_{k+1}$ is as defined in (6) but Y_{nk} and Z_{nk} will depend on the particular quantity in (2) under consideration. Further, for the present purpose it is enough to confine to the restricted situations where

$$Y_{nk} \geq 0, \quad E\left[Z_{nk}^2\right] < \infty, \quad E_{k-1}\left[Z_{nk}\right] = 0. \quad \text{(Note } X_{nk} \geq 0 \text{ already.)}$$

(Recall that $E_{k-1}\left[\ .\ \right] = E\left[\ .\ \middle|\mathcal{F}_{n,k-1}\right]$ and $P_{k-1}\left[\ .\ \right] = P\left[\ .\ \middle|\mathcal{F}_{n,k-1}\right]$.)

We assume, for each $t \in [0, \infty)$,

$$\sup_{1\leq k\leq [nt]} P_{k-1}\left[Y_{nk} \geq \eta\right] \xrightarrow{p} 0 \text{ for all } \eta > 0. \tag{14}$$

We next list the further assumptions on these arrays, where the process

$$S = (S(t), 0 \le t \le 1)$$

is as in (12).

(C1). There are families $\{\gamma(y,t,S) : y > 0, t > 0\}$ and $\{B(t,S) : t > 0\}$ of functionals of the process S and a dense subset J of $(0,\infty)$ such that

$$\left(\sum_{k=1}^{[nt]} X_{nk}, \ \sum_{k=1}^{[nt]} P_{k-1}[Y_{nk} > y], \ \sum_{k=1}^{[nt]} E_{k-1}\left[Z_{nk}^2\right]\right) \stackrel{fdd}{\Longrightarrow} (S(t), \gamma(y,t,S), B(t,S)), \quad (t,y) \in [0,\infty) \times J_2 \tag{15}$$

as $n \to \infty$, where

$$P\left[t \longmapsto B(t,S) \text{ and } t \longmapsto \gamma(y,t,S) \text{ are continuous}\right] = 1$$

for each $y > 0$ and, for all $t > 0$,

$$\gamma(\infty,t,S) = 0 \text{ a.s.}, \qquad \int_0^\infty y\gamma(dy,t,S) < \infty \text{ a.s.} \tag{16}$$

(C2). For each $t \in [0,\infty)$,

$$\sum_{k=1}^{[nt]} E_{k-1}\left[Y_{nk}\mathbb{I}_{\{Y_{nk}<\epsilon\}}\right] \xrightarrow{p} 0 \qquad \text{as } n \to \infty \text{ first and then } \epsilon \to 0.$$

(C3). For every bounded closed intervals $I_1, I_2 \subset (0,\infty)$,

$$\sum_{k=1}^{[nt]} P_{k-1}[X_{nk} \in I_1, Y_{nk} \in I_2] \xrightarrow{p} 0.$$

(C4). For each $t \in [0,\infty)$,

$$\sum_{k=1}^{[nt]} E_{k-1}\left[Z_{nk}^2\mathbb{I}_{\{|Z_{nk}|\ge\eta\}}\right] \xrightarrow{p} 0 \qquad \text{for all } \eta > 0.$$

Note that (C4) entails that

$$\sum_{k=1}^{[nt]} P_{k-1}[|Z_{nk}| \ge \eta] \xrightarrow{p} 0 \qquad \text{for all } \eta > 0,$$

and hence for every closed intervals $I_1 \subset (0,\infty)$ and $I_2 \subset (0,\infty) \cup (0,\infty)$,

$$\sum_{k=1}^{[nt]} P_{k-1}\left[X_{nk} \in I_1, Z_{nk} \in I_2\right] \xrightarrow{p} 0, \qquad \sum_{k=1}^{[nt]} P_{k-1}\left[Y_{nk} \in I_1, Z_{nk} \in I_2\right] \xrightarrow{p} 0. \tag{17}$$

(C4) also entails that, in view of $E_{k-1}\left[Z_{nk}\right] = 0$,

$$\sup_{0\le t\le M}\left|\sum_{k=1}^{[nt]} E_{k-1}\left[Z_{nk}\mathbb{I}_{\{|Z_{nk}|<\tau\}}\right]\right| \xrightarrow{p} 0 \tag{18}$$

for every $\tau > 0$. Furthermore, (15) together with (C2) entails that

$$\sum_{k=1}^{[nt]} E_{k-1}\left[Y_{nk}\mathbb{I}_{\{Y_{nk}<\tau\}}\right] \overset{fdd}{\Longrightarrow} \int_0^\tau y\gamma\left(dy,t,S\right), \quad \text{jointly with (15)}, \tag{19}$$

and because the left hand side in this convergence is monotone in t and the limit is continuous in t with probability one,

$$t \longmapsto \sum_{k=1}^{[nt]} E_{k-1}\left[Y_{nk}\mathbb{I}_{\{Y_{nk}<\tau\}}\right] \text{ is tight in } D_{\mathbb{R}}\left[0,\infty\right) \tag{20}$$

(see Jacod and Shiryayev (1987, Ch. VI, Theorem 3.37 (Statement (a)), page 318.) Further, (19) entails (noting $Y_{nk} \ge 0$), similar to (10),

$$\sum_{k=1}^{[nt]} \left|E_{k-1}\left[Y_{nk}\mathbb{I}_{\{Y_{nk}<\tau\}}\right]\right|^2 \xrightarrow{p} 0 \tag{21}$$

for all $\tau > 0$ and (C2) implies

$$\sum_{k=1}^{[nt]} E_{k-1}\left[Y_{nk}^2\mathbb{I}_{\{Y_{nk}<\epsilon\}}\right] \xrightarrow{p} 0 \quad \text{as } n \to \infty \text{ first and then } \epsilon \to 0. \tag{22}$$

Before stating the result it is convenient to recall a criterion for the tightness of a sequence of processes in $D_{\mathbb{R}^q}\left[0,\infty\right)$. The criterion is due to Aldous (1978) (as modified in Jacod and Shiryayev (1987, Ch. VI, Section 4a, page 320)).

Aldous Criterion: *The sequence of processes* $\left(\Lambda_n(t) = \left(\Lambda_{n,1}(t), ..., \Lambda_{n,q}(t)\right), 0 \le t < \infty\right)$ *taking values in* $D_{\mathbb{R}^q}\left[0,M\right]$ *adapted to an increasing and right-continuous family of* σ*-fields* $\left\{\mathcal{F}_{nk_n(t)}; t \in [0,M]\right\}$ *is tight if the following two requirements hold:*

$$\lim_{\upsilon\to\infty} \operatorname{limsup}_{n\to\infty} P\left[\sup_{0\le t\le M} \max_{1\le i\le q} \left|\Lambda_{n,i}(t)\right| > \upsilon\right] = 0 \tag{23}$$

and for every $\eta > 0$,

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \sup_{T^*\leq T\leq T^*+\delta} P\left[\max_{1\leq i\leq q}\left|\Lambda_{n,i}(T)-\Lambda_{n,i}(T^*)\right|>\eta\right]\to 0 \tag{24}$$

where the supremum $\sup_{T^*\leq T\leq T^*+\delta}$ *is with respect to all stopping times* T *and* T^* *satisfying* $T^*\leq T\leq T^*+\delta$ *and adapted to* $\left\{\mathcal{F}_{nk_n(t)}; t\in[0,M]\right\}$.

Note that (23) and (24) hold if they hold for each of the components $\Lambda_{n,i}(t)$ separately (but it is important to note that the class of stopping times are the same for all the components.) *This fact will be used repeatedly below in establishing tightness.*

Thus we obtain the following Theorem 2.1, as a corollary to Theorem 2 and Remark 10 in Jeganathan (2006). The result involves functions $g_j(u,v)$, $j=1,...,l$, each of which are assumed to be continuous (jointly in u and v) such that, for each $\kappa>0$, $j=1,...,l$,

$$g_j(u,0)=0, \qquad \sup_{|u|\leq\kappa}\sup_{0<|v|\leq\eta}\frac{1}{v^2}\left|g_j(u,v)-vg'_{u,j}-\frac{v^2}{2}g''_{u,j}\right|\to 0$$

as $\eta\to 0$, for suitable $u\longmapsto g'_{u,j}$, $g''_{u,j}$ that are continuous in u. Here $g'_{u,j}$ and $g''_{u,j}$ may respectively be viewed as first and second partial derivatives of $g_j(u,v)$ with respect to v at $v=0$.

We recall that $\sum_{k=1}^{[nt]}X_{nk}=a_n^{-1}S_{[nt]+1}$.

Theorem 2.1. *Assume that the assumptions (C1) - (C3) stated above are satisfied. Let the functions* $g_j(u,v)$, $j=1,...,l$, *be as above. Then*

$$\left(\sum_{k=1}^{[nt]}X_{nk},\ \sum_{k=1}^{[nt]}Z_{nk},\ \sum_{k=1}^{[nt]}g_j\left(\sum_{j=1}^{k-1}X_{nj},Y_{nk}\right),\ j=1,...,l\right)$$
$$\Longrightarrow\ \left(S(t),Z(t),R_j^*(t),\ j=1,...,l\right)\ \text{in } D_{\mathbb{R}^{2+l}}[0,\infty)$$

where $S(t)$ *is as in (12) and, conditionally on* $S=(S(t),0\leq t\leq 1)$, *the process* $\left(R_j^*(t), j=1,...,l\right)$ *is independent of the process* $Z(t)$ *such that for reals* $\theta_1,...,\theta_l$, $\sum_{j=1}^l\theta_jR_j^*(t)$ *is (conditionally) additive such that, with* $\gamma(x,z,S)$ *as in (C1),*

$$\log E\left[e^{iv\sum_{j=1}^l\theta_jR_j^*(t)}\Big|S\right]=\int_0^t\int_0^\infty\left(e^{iv\sum_{j=1}^l\theta_jg_j(S(r),x)}-1\right)\gamma(dx,dr,S),$$

and $Z(t)$ *is (conditionally) Gaussian additive with mean* 0 *and, with* $B(t,S)$ *as in (C1),*

$$E\left[Z^2(t)\big|S\right]=B(t,S).$$

Proof. By taking $g(u,v)=\sum_{j=1}^l\theta_jg_j(u,v)$, with $g'_u=\sum_{j=1}^l\theta_jg'_{u,j}$ and $g''_u=\sum_{j=1}^l\theta_jg''_{u,j}$, the conditions of Theorem 2 (and Remark 10) in Jeganathan (2006) hold for

$g(u,v)$, in view of (7), (9) - (11), (C1), (C3), (C4) and (17) - (22). Hence the convergence of finite dimensional distributions follows (though the convergence in $D_{\mathbb{R}^{2+l}}[0,\infty)$ itself is not). The limiting form for the characteristic function for $\sum \theta_j R_j^*(t)$ in the statement is obtained by noting that the terms (in Theorem 2 in Jeganathan (2006)) that involve g_u'' will vanish because of (21) and (22), and those that involve g_u' will cancel out because of (16).

It only remains to show the tightness in $D_{\mathbb{R}^{2+l}}[0,\infty)$, that is, (23) and (24) with $q = 2+l$. As noted earlier, it is enough to verify (23) and (24) for each one of these $2+l$ components separately. Now each $g_j(u,v)$ is of the form $g(u,v)$ of Theorem 2 in Jeganathan (2006), for which the required tightness has been verified (in Sections 4.4 and 5.2 of that paper). Hence the proof is complete. ■

Remark 1. Note that when $l = 1$ and $g_1(u,v) \equiv v$, the preceding result in particular gives the convergence $\left(\sum_{k=1}^{[nt]} X_{nk}, \sum_{k=1}^{[nt]} Z_{nk}, \sum_{k=1}^{[nt]} Y_{nk}\right) \Longrightarrow (S(t), Z(t), R(t))$ in $D_{\mathbb{R}^3}[0,\infty)$. The limiting processes $R(t)$ and $Z(t)$ are, given $(S(t), 0 \le t \le 1)$, conditionally additive and independent. Further $R(t)$ induces a random measure ν on the Borel σ-field of $(0,\infty) \times (0,\infty)$ which is conditionally Poisson with intensity measure $E[\nu(dxdr,S)|S] = \gamma(dxdr,S)$ where $\gamma(x,t,S)$ is as in (C1). In addition, because of (16), $R(t)$ has the well-known stochastic integral representation with respect to ν, in the form (see for instance Sato (1999, Chapter 4, Theorem 19.3, page 121))

$$R(t) = \int_0^t \int_0^\infty x\nu(dxdr,S).$$

In terms of this representation, the limits $R_j^*(t)$ in Theorem 2.1 have the representations

$$R_j^*(t) = \int_0^t \int_0^\infty g_j(S(r), x)\,\nu(dxdr,S), \; j = 1, ..., l,$$

by noting that the conditional joint characteristic function of this coincides with that given in the statement of Theorem 2.1. ■

Remark 2. Note that in the case $Y_{nk} = b_n^{-1} T_k$, where T_k are as before and b_n is such that $n\overline{G}(b_n) = nP[T_1 > b_n] \to 1$, the $\gamma(y,t,S)$ in the condition (C1) will take the form $\gamma(y,t,S) = ty^{-\beta}$. In addition, in this case when the component $\sum_{k=1}^{[nt]} Z_{nk}$ is absent, all the requirements of Theorem 2.1 are satisfied.

Remark 3. We shall need this remark in order to obtain results for the quantities in (3). Theorem 2.1 holds also when the index n is replaced by $[s]$ where $s \to \infty$ in a continuous manner. In such a situation, $\sum_{j=1}^{k-1} X_{nj} = a_{[s]}^{-1} S_k$ and the first component takes the form $\sum_{k=1}^{[[s]t]} X_{nk} = a_{[s]}^{-1} S_{[[s]t]+1}$. Here the normalizing constant $a_{[s]}$ can be replaced by any d_s such that $\frac{d_s}{a_{[s]}} \to 1$ without affecting the limiting behaviors in Theorem 2.1, as is easily seen, because the change pertains only to the normalizing constant $a_{[s]}$. In the case $Y_{nk} = b_{[s]}^{-1} T_k$ as in the preceding Remark 2, the same remark applies to the normalizing constant $b_{[s]}$. ■

We shall also need the following result. It is essentially a restatement of Theorem 1.1, Chapter III, in Borodin and Ibragimov (1995, pages 80 - 83), which itself is related to Theorem 1, Section 7, Chapter 9 in Gikhman and Skorokhod (1969).

Proposition 2.2. *Suppose that the processes* $U_n(t)$, $Y_n(t)$, $n \geq 1$, *and* $U(t)$, $Y(t)$, $t \in [0, \infty)$, *are possibly vector valued such that* $(U_n(t), Y_n(t)) \overset{fdd}{\Longrightarrow} (U(t), Y(t))$ *and such that, for every* $M > 0$ *and* $\eta > 0$,

$$\sup_{|u-v| \leq \delta,\ |u| \leq M, |v| \leq M} P\left[|Y_n(v) - Y_n(u)| > \eta\right] \to 0 \tag{25}$$

as $n \to \infty$ *first and then* $\delta \to 0$. *Assume that* $Y_n(t)$, $n \geq 1$, *and* $Y(t)$, $t \in [0, \infty)$, *are measurable processes.*

In addition, assume that for each $t \geq 0$, *the distribution of* $Y(t)$ *is absolutely continuous with respect to Lebesgue measure.*

Then for any compactly supported Riemann integrable function $h(u)$,

$$\left(U_n(t), Y_n(t), \int_0^t h(Y_n(v))\, dv\right) \overset{fdd}{\Longrightarrow} \left(U(t), Y(t), \int_0^t h(Y(v))\, dv\right).$$

This conclusion holds also for any locally Riemann integrable function $h(u)$ *provided that either there is a compact set outside which* $h(u)$ *is uniformly bounded and uniformly continuous or* $\sup_{0 \leq t \leq M} |Y_n(t)|$ *is stochastically bounded for every* $M > 0$.

Proof. For convenience, we restrict to showing that

$$\left(U_n(t), Y_n(t), \int_0^1 h(Y_n(v))\, dv\right) \overset{fdd}{\Longrightarrow} \left(U(t), Y(t), \int_0^1 h(Y(v))\, dv\right). \tag{26}$$

First suppose that $h(u)$ is compactly supported and continuous. Then it is also uniformly continuous, so that given $\delta > 0$ there is a η such that $|h(u) - h(v)| \leq \delta$ whenever

$|u-v|\leq\eta$. In addition $|h(u)|\leq C$ for some constant $C>0$. Hence

$$\begin{aligned}
& E\left[\left|\int_0^1 h\left(Y_n(v)\right)dv-\frac{1}{l}\sum_{j=0}^{l-1}h\left(Y_n\left(\frac{j}{l}\right)\right)\right|\right] \\
\leq & \sup_{|u-v|\leq\frac{1}{l},\ |u|\leq 1,|v|\leq 1} E\left[\left|h\left(Y_n(v)\right)-h\left(Y_n(v)\right)\right|\right] \\
\leq & \ \delta+2C\sup_{|u-v|\leq\frac{1}{l},\ |u|\leq 1,|v|\leq 1} P\left[\left|h\left(Y_n(v)\right)-h\left(Y_n\left(\frac{j}{l}\right)\right)\right|>\delta\right] \\
\leq & \ \delta+2C\sup_{|u-v|\leq\frac{1}{l},\ |u|\leq 1,|v|\leq 1} P\left[\left|Y_n(v)-Y_n\left(\frac{j}{l}\right)\right|>\eta\right].
\end{aligned}$$

Thus

$$\lim_{l\to\infty}\ \text{limsup}_{n\to\infty}\ E\left[\left|\int_0^1 Y_n(v)\,dv-\frac{1}{l}\sum_{j=0}^{l-1}Y_n\left(\frac{j}{l}\right)\right|\right]=0.$$

Also, (25) entails $\sup_{|u-v|\leq\delta,\ |u|\leq M,|v|\leq M} P\left[\left|Y(v)-Y(u)\right|>\eta\right]\to 0$ as $\delta\to 0$ by Fatou's lemma, and hence as above

$$\lim_{l\to\infty} E\left[\left|\int_0^1 Y(v)\,dv-\frac{1}{l}\sum_{j=0}^{l-1}Y\left(\frac{j}{l}\right)\right|\right]=0.$$

The preceding two facts together give (26) because, for each l,

$$\left(U_n(t),Y_n(t),\frac{1}{l}\sum_{j=0}^{l-1}Y_n\left(\frac{j}{l}\right)\right)\overset{fdd}{\Longrightarrow}\left(U(t),Y(t),\frac{1}{l}\sum_{j=0}^{l-1}Y\left(\frac{j}{l}\right)\right).$$

In the general case where $h(u)$ is compactly supported Riemann integrable, one can find, for each $\epsilon>0$, functions $h_{1,\epsilon}(u)$ and $h_{2,\epsilon}(u)$ that are continuous and compactly supported such that

$$h_{1,\epsilon}(u)\leq h(u)\leq h_{2,\epsilon}(u)$$

and such that

$$\int_{-\infty}^{\infty}\left(h_{2,\epsilon}(u)-h_{1,\epsilon}(u)\right)du\to 0 \text{ as } \epsilon\to 0. \tag{27}$$

Without loss of generality we can take $\sup_{\epsilon,u}\left(\left|h_{1,\epsilon}(u)\right|+\left|h_{2,\epsilon}(u)\right|\right)\leq C$ for some constant $C>0$, for instance by changing $h_{1,\epsilon}(u)$ to $\min\left(\max\left(h_{1,\epsilon}(u),-2M\right),2M\right)$, and the same change for $h_{2,\epsilon}(u)$, where the constant M is such that $|h(u)|\leq 2M$. (Recall that Riemann integrability of $h(u)$ entails that $h(u)$ is bounded.). Because we have proved that (26) holds for $h_{1,\epsilon}(u)$ and $h_{2,\epsilon}(u)$,

$$E\left[\int_0^1\left(h_{2,\epsilon}\left(Y_n(v)\right)-h_{1,\epsilon}\left(Y_n(v)\right)\right)dv\right]\to E\left[\int_0^1\left(h_{2,\epsilon}\left(Y(v)\right)-h_{1,\epsilon}\left(Y(v)\right)\right)dv\right]$$

as $n \to \infty$, where, with $f_v(u)$ the Lebesgue density of $Y(v)$,

$$\begin{aligned} & E\left[\int_0^1 \left(h_{2,\epsilon}(Y(v)) - h_{1,\epsilon}(Y(v))\right) dv\right] \\ = & \int_0^1 \int_{-\infty}^{\infty} \left(h_{2,\epsilon}(u) - h_{1,\epsilon}(u)\right) f_v(u)\, du dv \\ \leq & M \int_{-\infty}^{\infty} \left(h_{2,\epsilon}(u) - h_{1,\epsilon}(u)\right) du + C \int_0^1 \int_{-\infty}^{\infty} I_{\{f_v(u) > M\}} f_v(u)\, du dv \to 0 \end{aligned}$$

as $\epsilon \to 0$ first and then $M \to \infty$, because of (27) and because $\int_{-\infty}^{\infty} f_v(u)\, du = 1$ for all v so that Lebesgue dominated convergence becomes applicable for $\int_0^1 \int_{-\infty}^{\infty} I_{\{f_v(u)>M\}} f_v(u)\, du dv$ as $M \to \infty$. We have thus shown that $\int_0^1 \left(h_{2,\epsilon}(Y_n(v)) - h_{1,\epsilon}(Y_n(v))\right) dv \xrightarrow{p} 0$. This establishes (26) for the compactly supported Riemann integrable function $h(u)$, completing the proof of the first part of the conclusion of the proposition.

Regarding the second part, note that the argument used earlier for compactly supported continuous function $h(u)$ holds also when $h(u)$ is assumed to be uniformly bounded and uniformly continuous. Also when $\sup_{0 \le t \le M} |Y_n(t)|$ is stochastically bounded, $h(u)$ can be assumed without loss of generality to have a compact support. Hence the second part of the conclusion follows from that of the first, completing the proof of the proposition. ■

Next, let

$$U(\tau) = \sum_{k=0}^{\infty} P[S_k \le \tau].$$

Note that (recall $\overline{G}(y) = P[T_1 > y]$)

$$E[M_\infty(\tau)] = \int_0^\tau U(dx)\, \overline{G}(\tau - x).$$

As a further preliminary, it is convenient to reproduce the following result (together with a sketch of its proof for completeness), established in Mikosch and Resnick (2004, Section 1.2).

Lemma 2.3. *(i). In the Cases I and II, for every* $0 \le \delta < 1$,

$$\frac{\overline{F}(\tau)}{\overline{G}(\tau)} \int_{\tau\delta}^{\tau} U(dx)\, \overline{G}(\tau - x) \sim \int_\delta^1 (1-s)^{-\beta} \alpha s^{\alpha-1} ds, \qquad \tau \to \infty.$$

(ii). In the Case III, for every $0 \le \delta < 1$,

$$\frac{1}{\tau \overline{G}(\tau)} \int_{\tau\delta}^{\tau} U(dx)\, \overline{G}(\tau - x) \sim (E[X_1])^{-1} \int_\delta^1 (1-s)^{-\beta} ds, \qquad \tau \to \infty$$

Proof. According to Feller (1971, page 471), when $0 < \alpha < 1$,

$$U(\tau)\overline{F}(\tau) \sim (\Gamma(1-\alpha)\Gamma(1+\alpha))^{-1}, \quad x \to \infty.$$

Hence, in the Cases I and II,

$$\begin{aligned}\int_{\tau\delta}^{\tau} U(dx)\overline{G}(\tau - x) &= \int_{\delta}^{1} U(\tau ds)\overline{G}(\tau(1-s)) = \overline{G}(\tau)U(\tau)\int_{\delta}^{1}\frac{\overline{G}(\tau(1-s))}{\overline{G}(\tau)}\frac{U(\tau ds)}{U(\tau)} \\ &\sim (\Gamma(1-\alpha)\Gamma(1+\alpha))^{-1}\frac{\overline{G}(\tau)}{\overline{F}(\tau)}\int_{\delta}^{1}(1-s)^{-\beta}\alpha s^{\alpha-1}ds, \quad \tau \to \infty.\end{aligned}$$

In the Case III, $U(\tau) \sim \tau(E[X_1])^{-1}$. Hence

$$\begin{aligned}\int_{\tau\delta}^{\tau} U(dx)\overline{G}(\tau - x) &= \tau\overline{G}(\tau)\int_{\delta}^{1}\frac{\overline{G}(\tau(1-s))}{\overline{G}(\tau)}\frac{U(\tau ds)}{\tau} \\ &\sim \tau\overline{G}(\tau)(E[X_1])^{-1}\int_{\delta}^{1}(1-s)^{-\beta}ds.\end{aligned}$$

Hence the proof is complete. ■

The following fact (Feller (1971, (5.22), page 579)) will be needed below in a few places below. For convenience we state it separately.

Lemma 2.4. *Suppose that $Y \geq 0$ is a random variable such that $x \longmapsto P[Y > x]$ is regularly varying with index $-\gamma$, $0 < \gamma < 2$. Then for any $\theta > \gamma$,*

$$E\left[Y^{\theta}\mathbb{I}_{\{Y<\tau\}}\right] \sim \frac{\gamma}{\theta - \gamma}\tau^{\theta}P[Y > \tau] \qquad \text{as } \tau \to \infty.$$

3 THE CASE I

Recall that in this case $0 < \beta = \alpha < 1$ and $\overline{F}(x) \sim \overline{G}(x)$ as $x \to \infty$. Also recall that the constants a_n are such that $n\overline{F}(a_n x) \to x^{-\alpha}$. In view of $\overline{F}(x) \sim \overline{G}(x)$, we also have $n\overline{G}(a_n x) \to x^{-\alpha}$. Then, with $S_{[nt]} = \sum_{k=1}^{k} X_k$ as in Section 2 and

$$R_{[nt]} = \sum_{k=1}^{[nt]} T_k,$$

one has, because of the independence of $(X_k, k \geq 1)$ and $(T_k, k \geq 1)$,

$$\begin{aligned}\left(a_n^{-1}S_{[nt]}, a_n^{-1}R_{[nt]}\right) &\Longrightarrow (S(t), R(t)) \text{ in } D_{\mathbb{R}^2}[0,\infty), \\ \left(a_n^{-1}S_{[nt]+1}, a_n^{-1}R_{[nt]}\right) &\Longrightarrow (S(t), R(t)) \text{ in } D_{\mathbb{R}^2}[0,\infty),\end{aligned}$$

where $R(t)$ is also a stable process. In addition the limits $S(t)$ and $R(t)$ are independent.

In this section we shall apply Theorem 2.1 with the component Z_{nk} is absent and with $Y_{nk} = a_n^{-1}T_k$. Thus we are in the situation of Remark 2, so that in Theorem 2.1 $\gamma(y,t,S) = ty^{-\beta}$. Further, in view of Remark 1, the Poisson random measure ν on the Borel σ-field of $(0,\infty)\times(0,\infty)$ induced by $R(t)$ has the intensity measure (because $\overline{F}(x) \sim \overline{G}(x)$)

$$E[\nu(dxdr)] = \beta x^{-\beta-1}dxdr = \alpha x^{-\alpha-1}dxdr.$$

As noted earlier, the second statement $M_\infty(sz) \overset{fdd}{\Longrightarrow} \mathbb{M}_\infty(z)$ as $s\to\infty$ in the next result (see (28)) is already contained in Mikosch and Resnick (2004, Corollary 2.3).

Theorem 3.1. *Let $z_{n1} < ... < z_{nl}$ be such that $(z_{n1},...,z_{nl}) \to (z_1,...,z_l)$ with $z_1 > 0$. Then*

$$\left(a_n^{-1}S_{[nt]+1}, M_{[nt]}(a_nz_{n1}), ...M_{[nt]}(a_nz_{nl})\right) \Longrightarrow \left(S(t), \mathbb{M}_t(z_1), ..., \mathbb{M}_t(z_l)\right) \text{ in } D_{\mathbb{R}^{1+l}}[0,\infty)$$

where $M_n(z)$ is as defined in Section 1 and

$$\mathbb{M}_t(z) = \int_0^t \mathbb{I}_{\{S(r)\le z\}} \int_{z-S(r)}^\infty \nu(dxdr).$$

In addition, for $z_{s1} < ... < z_{sl}$ such that $(z_{s1},...,z_{sl}) \to (z_1,...,z_l)$ as $s\to\infty$ with $z_1 > 0$,

$$\left(M_\infty(sz_{s1}), ...M_\infty(sz_{sl})\right) \Longrightarrow \left(\mathbb{M}_\infty(z_1), ..., \mathbb{M}_\infty(z_l)\right) \text{ as } s\to\infty, \tag{28}$$

(where $\mathbb{M}_\infty(z) = \int_0^\infty I_{\{S(r)\le z\}} \int_{z-S(r)}^\infty \nu(dxdr)$.)

Proof. We first consider the proof of the first statement, which consists of reducing to the situation where Theorem 2.1 becomes applicable. For convenience we obtain this reduction for the case $\left(a_n^{-1}S_{[nt]+1}, M_{[nt]}(a_nz_n)\right)$ with $z_n \to z > 0$. It is enough to take $z_n \equiv z$ for all $n \ge 1$, for otherwise we simply need to replace $a_n^{-1}S_k$ by $a_n^{-1}S_k + z - z_n$ in the arguments below.

Then, recall that

$$M_{[nt]}(a_nz) = \sum_{k=1}^{[nt]} \mathbb{I}_{\left\{a_n^{-1}S_k \le z < a_n^{-1}S_k + a_n^{-1}T_k\right\}}.$$

Because $z > 0$ is fixed, without loss of generality assume that $z > 2\eta$ for some $\eta > 0$. Then note that the expected value of the difference between $\sum \mathbb{I}_{\left\{a_n^{-1}S_k \le z < a_n^{-1}S_k + a_n^{-1}T_k\right\}}$ and $\sum \mathbb{I}_{\left\{a_n^{-1}S_k \le z-\eta\right\}} \mathbb{I}_{\left\{z - a_n^{-1}S_k < a_n^{-1}T_k\right\}}$ is bounded by

$$\begin{aligned} &\sum_{k=0}^\infty E\left[\mathbb{I}_{\left\{z-\eta < a_n^{-1}S_k \le z\right\}} \overline{G}\left(a_n\left(z - a_n^{-1}S_k\right)\right)\right] \\ =& \int_{a_nz\left(1-\frac{\eta}{z}\right)}^{a_nz} U(dx)\, G(a_nz - x) \to 0, \quad \text{by part (i) of Lemma 2.3,} \end{aligned}$$

as $n \to \infty$ first and then $\eta \to 0$. In addition $\int_0^t \mathbb{I}_{\{z-\eta<S(r)\le z\}} \int_{z-S(r)}^{\infty} \nu(dxdr) \xrightarrow{p} 0$.

Therefore, it is enough to show, for each $\eta > 0$, that

$$\left(a_n^{-1} S_{[nt]+1}, \sum_{k=1}^{[nt]} \mathbb{I}_{\left\{a_n^{-1} S_k \le z-\eta\right\}} \mathbb{I}_{\left\{z-a_n^{-1} S_k < a_n^{-1} T_k\right\}}\right)$$

$$\Longrightarrow \left(S(t), \int_0^t \mathbb{I}_{\{S(r)\le z-\eta\}} \int_{z-S(r)}^{\infty} \nu(dxdr)\right) \text{ in } D_{\mathbb{R}^2}[0,\infty).$$

For this purpose, note that

$$\sum \mathbb{I}_{\left\{a_n^{-1} S_k \le z-\eta\right\}} \mathbb{I}_{\left\{z-a_n^{-1} S_k < a_n^{-1} T_k\right\}} = \sum \mathbb{I}_{\left\{\eta\le z-a_n^{-1} S_k < a_n^{-1} T_k\right\}} = \sum g\left(a_n^{-1} S_k, a_n^{-1} T_k\right)$$

where

$$g(u,v) = \mathbb{I}_{\{\eta\le z-u<v\}}.$$

This function satisfies the conditions stated in Theorem 2.1 (with $g_u' = 0 = g_u''$) except that $(u,v) \longmapsto g(u,v)$ is not continuous, but for each $\epsilon > 0$, it is easily seen that there are continuous functions $g_{1,\epsilon}(u,v)$ and $g_{2,\epsilon}(u,v)$ and satisfying the remaining conditions of Theorem 2.1, such that

$$g_{1,\epsilon}(u,v) \le g(u,v) \le g_{2,\epsilon}(u,v), \quad g_{2,\epsilon}(u,v) - g_{1,\epsilon}(u,v) \to 0$$

as $\epsilon \to 0$. This proves the first statement, because the intensity measure of the Poisson measure associated with $R(t)$ has the Lebesgue density $\beta x^{-\beta-1}$.

Regarding the second statement, note that a_n defined for integers n can be extended to $s \in (0,\infty)$ such that

$$a_s \sim \begin{cases} \inf\left\{x : \overline{F}(x) < \frac{1}{s}\right\} & \text{Cases I and II} \\ s & \text{Case III.} \end{cases}.$$

Further a_s, $s \in (0,\infty)$, can always be chosen such that $s \longmapsto a_s$ is continuous and strictly increasing (see Mikosch and Resnick (2004, Section 1.3, for the references)). Also observe that $\frac{a_s}{a_{[s]}} \to 1$. Hence in view of Remark 3 in Section 2 and in view of what we have proved above, for any $z_s \to z$ as $s \to \infty$, $M_{[[s]\kappa]}(a_s z_s) \Longrightarrow \mathbb{M}_\kappa(z)$ as $s \to \infty$ for each $\kappa > 0$. In addition $\mathbb{M}_\kappa(z) \xrightarrow{p} \mathbb{M}_\infty(z)$ as $\kappa \to \infty$. Thus

$$M_{[[s]\kappa]}(a_s z_s) \Longrightarrow \mathbb{M}_\infty(z) \tag{29}$$

as $s \to \infty$ first and then $\kappa \to \infty$. Now

$$\begin{aligned} M_\infty (a_s z_s) - M_{[[s]\kappa]} (a_s z_s) &= \sum_{k=[[s]\kappa]+1}^{\infty} \mathbb{I}_{\left\{a_s^{-1} S_k \le z_s < a_n^{-1} S_k + a_s^{-1} T_k\right\}} \\ &= \sum_{k=[[s]\kappa]+1}^{\infty} \mathbb{I}_{\left\{a_s^{-1} S_k \le z_s\right\}} \mathbb{I}_{\left\{z_s - a_s^{-1} S_k < a_s^{-1} T_k\right\}} \\ &= \sum_{k=[[s]\kappa]+1}^{N(a_s z_s)} \mathbb{I}_{\left\{z - a_s^{-1} S_k < a_s^{-1} T_k\right\}} = 0 \text{ on } \left\{N (a_s z_s) \le [[s] \kappa]\right\}, \end{aligned}$$

where $N(a_s z_s) = \sup \{l : S_l \le a_s z_s\}$. Now recall that $s^{-1} N(a_s z_s)$ converges in distribution (see Feller (1971, page 373)), so that

$$P[N(a_s z_s) \le [[s] \kappa]] = P\left[s^{-1} N(a_s z_s) \le s^{-1} [[s] \kappa]\right] \to 1$$

as $s \to \infty$ first and then $\kappa \to \infty$. Hence

$$M_\infty (a_s z_s) - M_{[[s]\kappa]} (a_s z_s) \xrightarrow{p} 0$$

as $s \to \infty$ first and then $\kappa \to \infty$. In view of (29), this gives the second statement of the theorem because $s \longmapsto a_s$ is continuous and strictly increasing, completing the proof. ■

Remark 4. It may be noted that the situation considered in Theorem 3.1 is a particular case of Theorem 3.3 below, because $M^*_{[nt]}(z)$ reduces to $M_{[nt]}(z)$ when the rate $\xi_k \equiv 1$ for all $k \ge 1$. However, a direct application of Theorem 2.1 as done above allows us to express the limits in terms of the Poisson measure ν of Remark 1, whereas this does not seem to be possible for the limits in Theorem 3.3. The same remark applies regarding the relationship between Theorems 3.2 and 3.4 below. ■

In the next result note that the quantities involved are the integrals of those in Theorem 3.1, but unfortunately we are unable to exploit this fact in order to directly deduce the result from Theorem 3.1, though it is possible to use Proposition 2.2 to deduce the convergence of finite dimensional distributions.

Theorem 3.2. *For any $z_{n1} < ... < z_{nl}$ such that $(z_{n1}, ..., z_{nl}) \to (z_1, ..., z_l)$ with $z_1 > 0$,*

$$\begin{aligned} t &\longmapsto \left(a_n^{-1} S_{[nt]+1}, a_n^{-1} A_{[nt]} (a_n z_{n1}), ..., a_n^{-1} A_{[nt]} (a_n z_{nl})\right) \\ &\Longrightarrow (S(t), \mathbb{A}_t (z_1), ..., \mathbb{A}_t (z_l)) \text{ in } D_{\mathbb{R}^{1+l}} [0, \infty), \end{aligned}$$

where $A_n(z)$ is as defined in Section 1 and

$$\mathbb{A}_t (z) = \int_0^z \mathbb{M}_t (u)\, du = \int_0^t \int_0^\infty \mathbb{I}_{\{S(r) \le z\}} \min (x, z - S(r))\, \nu (dx dr).$$

In addition

$$z \longmapsto a_n^{-1} A_{[nt]}(a_n z) \Longrightarrow \mathbb{A}_t(z) \text{ in } D_{\mathbb{R}}[0,\infty)$$

for each $0 \le t < \infty$*, and*

$$z \longmapsto s^{-1} A_\infty(sz) \Longrightarrow \mathbb{A}_\infty(z) \text{ in } D_{\mathbb{R}}[0,\infty) \;\; s \to \infty.$$

Proof. The proof is similar to that of Theorem 3.1. First consider the first statement. Recall that

$$a_n^{-1} A_{[nt]}(a_n z) = a_n^{-1} \int_0^{a_n z} M_{[nt]}(u)\, du = \sum_{k=1}^{[nt]} I_{\left\{a_n^{-1} S_k \le z\right\}} \min\left(a_n^{-1} T_k, z - a_n^{-1} S_k\right).$$

For simplicity we restrict to $\left(a_n^{-1} S_{[nt]+1}, a_n^{-1} A_{[nt]}(a_n z)\right)$. For $0 < \eta < z$,

$$\begin{aligned} & \sum I_{\left\{z-\eta < a_n^{-1} S_k \le z\right\}} \min\left(a_n^{-1} T_k, z - a_n^{-1} S_k\right) \\ = \; & \sum I_{\left\{z-\eta < a_n^{-1} S_k \le z\right\}} \int_0^z \mathbb{I}_{\left\{a_n^{-1} S_k \le u < a_n^{-1} S_k + a_n^{-1} T_k\right\}} du \\ = \; & \int_{z-\eta}^z \sum I_{\left\{z-\eta < a_n^{-1} S_k \le u\right\}} \mathbb{I}_{\left\{u < a_n^{-1} S_k + a_n^{-1} T_k\right\}} du, \end{aligned}$$

the expected value of which is

$$\begin{aligned} & \int_{z-\eta}^z \sum_{k=0}^\infty E\left[I_{\left\{z-\eta < a_n^{-1} S_k \le u\right\}} \mathbb{I}_{\left\{u < a_n^{-1} S_k + a_n^{-1} T_k\right\}}\right] = \int_{z-\eta}^z \int_{a_n(z-\eta)}^{a_n u} U(dx)\, \overline{G}(a_n u - x)\, du \\ = \; & \int_{z-\eta}^z \int_{\frac{z-\eta}{u}}^1 U(a_n u dy)\, \overline{G}(a_n u (1-y))\, du \\ \le \; & \int_{z-\eta}^z \int_{1-\frac{\eta}{z}}^1 U(a_n u dy)\, \overline{G}(a_n u (1-y))\, du \to 0 \end{aligned}$$

as $n \to \infty$ first and then $\eta \to 0$, by part (i) of Lemma 2.3. Further,

$$\int_0^t \int_0^\infty \mathbb{I}_{\{z-\eta < S(r) \le z\}} \min(r, z - S(r))\, J(drdx) \xrightarrow{p} 0 \text{ as } \eta \to 0.$$

On the other hand, in the same manner as was done in the proof of Theorem 3.1, Theorem 2.1 becomes applicable to obtain, for each $\eta > 0$,

$$\begin{aligned} & \left(a_n^{-1} S_{[nt]+1}, \sum_{k=1}^{[nt]} I_{\left\{a_n^{-1} S_k \le z-\eta\right\}} \min\left(a_n^{-1} T_k, z - a_n^{-1} S_k\right)\right) \\ \Longrightarrow \; & \left(S(t), \int_0^t \int_0^\infty \mathbb{I}_{\{S(r) \le z-\eta\}} \min(r, z - S(r))\, J(dxdr)\right) \text{ in } D_{\mathbb{R}^2}[0,\infty) \end{aligned}$$

because $g(u,v) = I_{\{u \le z-\eta\}} \min(v, z-u)$ satisfies the conditions stated in Theorem 2.1, with $g'_u \equiv 1$ and $g''_u \equiv 0$, except that $(u,v) \longmapsto g(u,v)$ is not continuous but it can be remedied in a way similar to that in the proof of Theorem 3.1. Hence the first statement follows follows.

Regarding the second statement, it follows from the first statement that $z \longmapsto a_n^{-1} A_{[nt]}(a_n z) \overset{fdd}{\Longrightarrow} \mathbb{A}_t(z)$. Because both $a_n^{-1} A_{[nt]}(a_n z)$ and $\mathbb{A}_t(z)$ are monotone in z and because $z \longmapsto \mathbb{A}_t(z)$ is continuous, this will entail convergence in $D_{\mathbb{R}}[0,\infty)$, in view of Jacod and Shiryayev (1987, Ch. VI, Theorem 3.37 (Statement (a)), page 318)). In the same way, using the same arguments in the proof of Theorem 3.1, the last statement $z \longmapsto s^{-1} A_\infty(sz) \Longrightarrow \mathbb{A}_\infty(z)$ in $D_{\mathbb{R}}[0,\infty)$ $s \to \infty$ also follows, completing the proof of the theorem. ■

We remark that, conditionally on $\{S(t), 0 \le t < \infty\}$, the Lévy measure L_t on $[0,\infty)^l - \{0\}$ of the conditional distribution of the process $t \longmapsto (\mathbb{A}_t(z_1), ..., \mathbb{A}_t(z_l))$ is given by, for $(x_1 ..., x_l) \in [0,\infty)^l - \{0\}$ (note that $\max(x_1 ..., x_l) \neq 0$)

$$L_t([x_1,\infty) \times ... \times [x_l,\infty)) = (\max(x_1 ..., x_l))^{-\alpha} \int_0^t \prod_{j=1}^l I_{\{(z_j - S(v))^+ \ge x_j\}} dv.$$

This will be a special case of Theorem 3.3 below, but can also be verified directly.

Next consider limiting behavior of $M^*_{[nt]}(a_n z)$. Recall that

$$\begin{aligned} M^*_{[nt]}(a_n z) &= \sum_{k=1}^{[nt]} I_{\{a_n^{-1} S_k \le z < a_n^{-1} S_k + a_n^{-1} T_k\}} \xi_k, \\ M^*_\infty(a_n z) &= M^*_{N(a_n z)}(a_n z) = \sum_{k=1}^{\infty} I_{\{a_n^{-1} S_k \le z < a_n^{-1} S_k + a_n^{-1} T_k\}} \xi_k \end{aligned}$$

where ξ_k are iid nonnegative random variables. We shall assume that

$$E\left[\xi_1^\theta\right] < \infty \quad \text{for some } \theta > \alpha. \tag{30}$$

Theorem 3.3 *Let the sequence $\xi_k, k \ge 1$, be as above. Then for any $z_{n1} < ... < z_{nl}$ such that $(z_{n1}, ..., z_{nl}) \to (z_1, ..., z_l)$ with $z_1 > 0$,*

$$\begin{aligned} t &\longmapsto \left(a_n^{-1} S_{[nt]+1}, M^*_{[nt]}(a_n z_{n1}), ..., M^*_{[nt]}(a_n z_{nl})\right) \\ &\Longrightarrow (S(t), \mathbb{M}^*_t(z_1), ..., \mathbb{M}^*_t(z_l)) \text{ in } D_{\mathbb{R}^{1+l}}[0,\infty), \end{aligned}$$

where, conditional on $S(t)$, the process $t \longmapsto (\mathbb{M}^*_t(z_1), ..., \mathbb{M}^*_t(z_l))$ is conditionally additive with the Lévy measure L_t on $[0,\infty)^l - \{0\}$ given by, for $(x_1 ..., x_l) \in [0,\infty)^l - \{0\}$,

$$L_t([x_1,\infty) \times ... \times [x_l,\infty)) = P[\xi_1 \ge \max(x_1, ..., x_l)] \int_0^t (z_l - S(r))^{-\alpha} dr.$$

In addition, for $z_{s1} < ... < z_{sl}$ *such that* $(z_{s1}, ..., z_{sl}) \to (z_1, ..., z_l)$ *as* $s \to \infty$ *with* $z_1 > 0$*,*

$$\left(M_\infty^* (sz_{s1}), ... M_\infty^* (sz_{sl})\right) \Longrightarrow \left(\mathbb{M}_\infty^* (z_1), ..., \mathbb{M}_\infty^* (z_l)\right) \text{ as } s \to \infty,$$

Proof. We verify the conditions (C1) - (C3) of Section 2 above, where note that in the present context the component Z_{nk} in the array $\{X_{nk}, Y_{nk}, Z_{nk}, k = 1, 2, ...\}$ is absent and we take $Y_{nk} = \sum_{j=1}^{l} \theta_j I_{\left\{a_n^{-1} S_k \le z_j < a_n^{-1} S_k + a_n^{-1} T_k\right\}} \xi_k$ for reals θ_j.

First consider (C1). Define, for $(x_1 ..., x_l) \in [0, \infty)^l - \{0\}$,

$$\begin{aligned} L_{[nt]}(x_1, .., x_l) &= \sum_{k=1}^{[nt]} P_{T_1, \xi_1} \left[I_{\left\{a_n^{-1} S_k \le z_j < a_n^{-1} S_k + a_n^{-1} T_1\right\}} \xi_1 \ge x_j, \ j = 1, ..., l \right] \\ &= \frac{1}{n} \sum_{k=1}^{[nt]} g_n\left(a_n^{-1} S_k\right) \end{aligned}$$

where

$$\begin{aligned} g_n(r) &= nP\left[I_{\left\{r \le z_j < r + a_n^{-1} T_1\right\}} \xi_1 \ge x_j, \ j = 1, ..., l \right] \\ &= nP\left[\xi_1 > 0, I_{\left\{r \le z_j < r + a_n^{-1} T_1\right\}} \xi_1 \ge x_j, \ j = 1, ..., l \right]. \end{aligned}$$

Here the second equality follows because $(x_1 ..., x_l) \in [0, \infty)^l - \{0\}$ implies $x_i > 0$ for some $1 \le i \le l$, so that the event $\left\{ I_{\left\{r \le z_j < r + a_n^{-1} T_1\right\}} \xi_1 \ge x_j, j = 1, ..., l \right\}$ entails $\xi_1 > 0$.

First suppose that $x_i > 0$ for all $1 \le i \le l$. Then,

$$\begin{aligned} & P\left[I_{\left\{r \le z_j < r + a_n^{-1} T_1\right\}} \xi_1 \ge x_j, \ j = 1, ..., l \right] \\ = \ & P\left[r \le z_j < r + a_n^{-1} T_1, \ \xi_1 \ge \max(x_1, ..., x_l) \right] \\ = \ & P\left[\xi_1 \ge \max(x_1, ..., x_l), \ r \le z_j < r + a_n^{-1} T_1, \ j = 1, ..., l \right] \\ = \ & P\left[\xi_1 \ge \max(x_1, ..., x_l) \right] P\left[r \le z_j < r + a_n^{-1} T_1, \ j = 1, ..., l \right] \end{aligned}$$

where, noting that $\min(z_1, ..., z_l) = z_1$, $\max(z_1, ..., z_l) = z_l$,

$$\begin{aligned} & P\left[r \le z_j < r + a_n^{-1} T_1, \ j = 1, ..., l \right] \\ = \ & P\left[r \le \min(z_1, ..., z_l), \ \max(z_1, ..., z_l) < r + a_n^{-1} T_1 \right] \\ = \ & P\left[a_n^{-1} T_1 > z_l - r \right] = \overline{F}\left(a_n(z_l - r)\right). \end{aligned}$$

Therefore,

$$\sup_r |g_n(r) - h(r)| \to 0 \tag{31}$$

where we let

$$h(r) = (z_l - r)^{-\alpha} P[\xi_1 \geq \max(x_1, ..., x_l)].$$

(Note that $r \leq z_1 < z_l$.)

Now consider the general case in which some of $x_i = 0$ but $\max(x_1, ..., x_l) > 0$. For concreteness, suppose that $x_1 = 0$ and $x_2 > 0, ..., x_l > 0$. Then

$$P\left[I_{\left\{r \leq z_j < r + a_n^{-1} T_1\right\}} \xi_1 \geq x_j,\ j = 1, ..., l\right] = P\left[I_{\left\{r \leq z_j < r + a_n^{-1} T_1\right\}} \xi_1 \geq x_j,\ j = 2, ..., l\right].$$

Hence (31) holds because $\max(x_2, ..., x_l) = \max(x_1, ..., x_l)$.

Thus

$$\sup_{0 \leq t \leq M} \left| \frac{1}{n} \sum_{k=1}^{[nt]} g_n\left(a_n^{-1} S_k\right) - \frac{1}{n} \sum_{k=1}^{[nt]} h\left(a_n^{-1} S_k\right) \right| \leq \frac{[nM]}{n} \sup_r |g_n(r) - h(r)| \to 0.$$

Now note that in view of Proposition 2.2,

$$\left(a_n^{-1} S_{[nt]}, \frac{1}{n} \sum_{k=1}^{[nt]} h\left(a_n^{-1} S_k\right)\right) \stackrel{fdd}{\Longrightarrow} \left(S(t), \int_0^t h(S(z))\, dz\right).$$

Here we have used the fact that $(z_l - r) \geq (z_l - z_1) > 0$, and therefore $h(r)$ is uniformly continuous and uniformly bounded. This completes the verification of (C1).

Next, we note that in the present context (C3) follows from (C1). To see this note that, with $Y_{nk} = I_{\left\{a_n^{-1} S_k \leq z < a_n^{-1} S_k + a_n^{-1} T_k\right\}} \xi_k$ for simplicity,

$$\begin{aligned} P_{k-1}[X_{nk} \geq \eta, Y_{nk} \geq \eta] &= P\left[a_n^{-1} X_{k+1} \geq \eta\right] P_{k-1}\left[I_{\left\{a_n^{-1} S_k \leq z < a_n^{-1} S_k + a_n^{-1} T_k\right\}} \xi_k \geq \eta\right] \\ &= P\left[a_n^{-1} X_1 \geq \eta\right] P_{T_1, \xi_1}\left[I_{\left\{a_n^{-1} S_k \leq z < a_n^{-1} S_k + a_n^{-1} T_1\right\}} \xi_1 \geq \eta\right] \end{aligned}$$

for any $\eta > 0$, where $P[a_n^{-1} X_1 \geq \eta] \to 0$ and $\sum P_{T_1, \xi_1}\left[I_{\left\{a_n^{-1} S_k \leq z < a_n^{-1} S_k + a_n^{-1} T_1\right\}} \xi_1 \geq \eta\right]$ is stochastically bounded by (C1). Hence $\sum P_{k-1}[X_{nk} \in I_1, Y_{nk} \in I_2] \stackrel{p}{\to} 0$ for any bounded closed I_1 and I_2 in $(0, \infty)$.

It remains to verify (C2). It is enough to consider $Y_{nk} = I_{\left\{a_n^{-1} S_k \leq z < a_n^{-1} S_k + a_n^{-1} T_k\right\}} \xi_k$. Here $Y_{nk} < \epsilon$ means either $Y_{nk} = 0$ or $\xi_k < \epsilon$. Hence

$$E_{k-1}\left[Y_{nk} \mathbb{I}_{\{Y_{nk} < \epsilon\}}\right] \leq E\left[I_{\left\{a_n^{-1} S_k \leq z < a_n^{-1} S_k + a_n^{-1} T_k\right\}} \xi_k \mathbb{I}_{\{\xi_k < \epsilon\}}\right] \leq \epsilon P\left[a_n^{-1} S_k \leq z < a_n^{-1} S_k + a_n^{-1} T_1\right]$$

where in view of Lemma 2.3, $\sum P[a_n^{-1} S_k \leq z < a_n^{-1} S_k + a_n^{-1} T_1]$ is bounded. This verifies (C2) and hence the proof of the first statement is completed. As in the proof of Theorem

3.1, the second statement follows from the first statement, completing the proof of the theorem. ■

Next consider limiting behavior of $A^*_{[nt]}(a_n z)$. Recall that

$$A^*_{[nt]}(a_n z) = \sum_{k=1}^{[nt]} I_{\{S_k \le a_n z\}} \min(T_k, a_n z - S_k)\, \xi_k = \sum_{k=1}^{[nt]} \min\left(T_k, (a_n z - S_k)^+\right) \xi_k$$

where ξ_k are as previously satisfying (30).

Theorem 3.4 *Let the sequence $\xi_k, k \ge 1$, be as in (30). Then for any $z_{n1} < \ldots < z_{nl}$ such that $(z_{n1}, \ldots, z_{nl}) \to (z_1, \ldots, z_l)$ with $z_1 > 0$,*

$$\begin{aligned} t &\longmapsto \left(a_n^{-1} S_{[nt]+1}, a_n^{-1} A^*_{[nt]}(a_n z_{n1}), \ldots, a_n^{-1} A^*_{[nt]}(a_n z_{nl})\right) \\ &\Longrightarrow \left(S(t), \mathbb{A}^*_t(z_1), \ldots, \mathbb{A}^*_t(z_l)\right) \text{ in } D_{\mathbb{R}^{1+l}}[0,\infty), \end{aligned}$$

where, conditional on $S(t)$, the process $t \longmapsto (\mathbb{A}^*_t(z_1), \ldots, \mathbb{A}^*_t(z_l))$ is conditionally additive with the Lévy measure L_t on $[0,\infty)^l - \{0\}$ given by, for $(x_1 \ldots, x_l) \in [0,\infty)^l - \{0\}$,

$$\begin{aligned} & L_t([x_1, \infty) \times \ldots \times [x_l, \infty)) \\ = \; & (\max(x_1, \ldots, x_l))^{-\alpha} \int_0^t E_{\xi_1}\left[\xi_1^\alpha I_{\left\{(z_j - S(v))^+ u \ge x_j,\ j=1,\ldots,l\right\}}\right] dv, \end{aligned}$$

where E_{ξ_1} stands for the expectation with respect to ξ_1.

In addition

$$z \longmapsto a_n^{-1} A^*_{[nt]}(a_n z) \Longrightarrow \mathcal{A}^*_t(z) \text{ in } D_{\mathbb{R}}[0,\infty)$$

for each $0 \le t < \infty$, and

$$z \longmapsto s^{-1} A^*_\infty(sz) \Longrightarrow \mathbb{A}^*_\infty(z) \text{ in } D_{\mathbb{R}}[0,\infty) \text{ as } s \to \infty.$$

Proof. As in the proof of Theorem 3.3, we need to verify (C1) -(C3) of Section 2 above. First consider (C1). Define, for $(x_1 \ldots, x_l) \in [0,\infty)^l - \{0\}$,

$$\begin{aligned} L_{[nt]}(x_1, .., x_l) &= \sum_{k=1}^{[nt]} P_{T_1,\xi_1}\left[\min\left(a_n^{-1} T_1, \left(z_j - a_n^{-1} S_k\right)^+\right) \xi_1 \ge x_j,\ j = 1, \ldots, l\right] \\ &= \sum_{k=1}^{[nt]} g_n\left(a_n^{-1} S_k\right) \end{aligned}$$

where

$$\begin{aligned} g_n(r) &= P_{T_1,\xi_1}\left[a_n^{-1} T_1 \xi_1 \ge x_j,\ (z_j - r)^+ \xi_1 \ge x_j,,\ j = 1, \ldots, l\right] \\ &= P_{T_1,\xi_1}\left[a_n^{-1} T_1 \xi_1 \ge \max(x_1, \ldots, x_l),\ (z_j - r)^+ \xi_1 \ge x_j,,\ j = 1, \ldots, l\right]. \end{aligned}$$

Let

$$J \subset [0,\infty)^l - \{0\} \text{ be the set of all continuity points of}$$
$$(x_1..., x_l) \longmapsto L_t\left([x_1,\infty)\times ... \times [x_l,\infty)\right).$$

It is enough to show that

$$\left(a_n^{-1}S_{[nt]}, L_{[nt]}(x_1,..,x_l)\right) \overset{fdd}{\Longrightarrow} \left(S(t), L_t\left([x_1,\infty)\times ... \times [x_l,\infty)\right)\right),$$

where $(t,(x_1...,x_l)) \in [0,\infty)\times J$.

Now, for any $x>0$,

$$\begin{aligned} & P\left[a_n^{-1}T_1\xi_1 \geq x,\ a_n^{-1}T_1\xi_1 > \eta,\ (z_j - r)^+\xi_1 \geq x_j,\ j=1,...,l\right] \\ = & \int P\left[a_n^{-1}T_1 \geq \max\left(\frac{x}{u},\eta\right)\right] I_{\left\{(z_j-r)^+u\geq x_j,\ j=1,...,l\right\}} dF_{\xi_1}(du) \end{aligned}$$

where note that $\left\{(z_j-r)^+u \geq x_j,\ j=1,...,l\right\}$ entails $u>0$, because $\max(x_1,...,x_l) > 0$, that is, $x_i > 0$ for some $1\leq i\leq l$. Further

$$\sup_u \left| nP\left[a_n^{-1}T_1 \geq \max\left(\frac{x}{u},\eta\right)\right] - \left(\max\left(\frac{x}{u},\eta\right)\right)^{-\alpha}\right| \to 0.$$

because $\max\left(\frac{x}{u},\eta\right) \geq \eta$ and because $\sup_{y\geq\eta}\left|nP\left[a_n^{-1}T_1\geq y\right] - y^{-\alpha}\right| \to 0$. Thus

$$\sup_r \int \left| nP\left[a_n^{-1}T_1 \geq \max\left(\frac{x}{u},\eta\right)\right] - \left(\max\left(\frac{x}{u},\eta\right)\right)^{-\alpha}\right| I_{\left\{(z_j-r)^+u\geq x_j,\ j=1,...,l\right\}} dF_{\xi_1}(du) \to 0,$$

where

$$\begin{aligned} & \int \left(\max\left(\frac{x}{u},\eta\right)\right)^{-\alpha} I_{\left\{(z_j-r)^+u\geq x_j,\ j=1,...,l\right\}} dF_{\xi_1}(du) \\ \to\ & x^{-\alpha}\int^{-\alpha} u^{\alpha} I_{\left\{(z_j-r)^+u\geq x_j,\ j=1,...,l\right\}} dF_{\xi_1}(du) \text{ as } \eta\to 0, \quad \text{uniformly in } r. \end{aligned}$$

Further (recall $\alpha<\theta$),

$$\begin{aligned} nP\left[a_n^{-1}T_1\xi_1 > x, a_n^{-1}T_1\leq\eta\right] &\leq nx^{-\theta}E\left[\left|a_n^{-1}T_1\xi_1\right|^{\theta} I_{\left\{a_n^{-1}T_1\leq\eta\right\}}\right] \\ &= na_n^{-\theta}x^{-\theta}E\left[\left|\xi_1\right|^{\theta}\right]E\left[\left|T_1\right|^{\theta} I_{\left\{a_n^{-1}T_1\leq\eta\right\}}\right] \\ &\sim \frac{\alpha}{\theta-\alpha}x^{-\theta}E\left[\left|\xi_1\right|^{\theta}\right]\eta^{\theta-\alpha} \to 0 \text{ as } \eta\to 0. \end{aligned}$$

where we have used Lemma 2.4, according to which

$$E\left[\left|T_1\right|^{\theta} I_{\left\{a_n^{-1}T_1\leq\eta\right\}}\right] \sim \frac{\alpha}{\theta-\alpha}a_n^{\theta}\eta^{\theta}P\left[T_1 > a_n\eta\right] \sim \frac{\alpha}{\theta-\alpha}a_n^{\theta}\eta^{\theta}n^{-1}\eta^{-\alpha}.$$

Thus, if we let

$$h(r) = (\max(x_1, ..., x_l))^{-\alpha} \int u^\alpha I_{\{(z_j - r)^+ u \geq x_j,\, j=1,...,l\}} dF_{\xi_1}(du),$$

then $\sup_r |ng_n(r) - h(r)| \to 0$, and hence

$$\sup_{0 \leq t \leq M} \left| \sum_{k=1}^{[nt]} g_n\left(a_n^{-1} S_k\right) - \frac{1}{n} \sum_{k=1}^{[nt]} h\left(a_n^{-1} S_k\right) \right| \leq \frac{[nM]}{n} \sup_r |ng_n(r) - h(r)| \to 0.$$

Now, the function $r \longmapsto h(r)$ has a compact support. In addition it is continuous because $(x_1, ..., x_l) \in J$. Therefore, an application of Proposition 2.2 gives the convergence with the required limit

$$\begin{aligned} \int_0^t h(S(v))\, dv &= (\max(x_1, ..., x_l))^{-\alpha} \int_0^t \int_0^\infty u^\alpha I_{\{(z_j - S(v))^+ u \geq x_j,\, j=1,...,l\}} dF_{\xi_1}(du)\, dv \\ &= (\max(x_1, ..., x_l))^{-\alpha} \int_0^t E_{\xi_1}\left[\xi_1^\alpha I_{\{(z_j - S(v))^+ u \geq x_j,\, j=1,...,l\}}\right] dv. \end{aligned}$$

This completes the verification of (C1).

In exactly the same way as in the proof of Theorem 3.3, (C3) follows from (C1).

It remains to verify (C2). It is enough to consider $Y_{nk} = \min\left(a_n^{-1} T_k, (z - a_n^{-1} S_k)^+\right) \xi_k$. We have

$$\begin{aligned} & E_{k-1}\left[Y_{nk} \mathbb{I}_{\{Y_{nk} < \epsilon\}}\right] \\ \leq\ & E_{T_1, \xi_1}\left[\min\left(a_n^{-1} T_1, \left(z - a_n^{-1} S_k\right)^+\right) \xi_1 \left(\mathbb{I}_{\{a_n^{-1} T_1 \xi_1 < \epsilon\}} + \mathbb{I}_{\{(z - a_n^{-1} S_k)^+ \xi_1 < \epsilon\}}\right)\right]. \end{aligned}$$

Here

$$\begin{aligned} & \sum E_{T_1, \xi_1}\left[\min\left(a_n^{-1} T_1, \left(z - a_n^{-1} S_k\right)^+\right) \xi_1 \mathbb{I}_{\{a_n^{-1} T_1 \xi_1 < \epsilon\}}\right] \\ \leq\ & n E_{T_1, \xi_1}\left[a_n^{-1} T_1 \xi_1 \mathbb{I}_{\{a_n^{-1} T_1 \xi_1 < \epsilon\}}\right] \to 0 \end{aligned}$$

as $n \to \infty$ first and then $\epsilon \to 0$, because $y \longmapsto P[T_1 \xi_1 > y]$ is regularly varying with index $0 < \alpha < 1$. In the same way, for every $\epsilon, \eta > 0$,

$$\begin{aligned} & \sum E_{T_1, \xi_1}\left[\min\left(a_n^{-1} T_1, \left(z - a_n^{-1} S_k\right)^+\right) \xi_1 \mathbb{I}_{\{(z - a_n^{-1} S_k)^+ \xi_1 < \epsilon\}}\right] \\ \leq\ & n E_{T_1, \xi_1}\left[a_n^{-1} T_1 \xi_1 \mathbb{I}_{\{a_n^{-1} T_1 \xi_1 < \eta\}}\right] + \epsilon n P_{T_1, \xi_1}\left[a_n^{-1} T_1 \xi_1 \geq \eta\right] \end{aligned}$$

where, for each $\eta > 0$, $\epsilon n P_{T_1, \xi_1}\left[a_n^{-1} T_1 \xi_1 \geq \eta\right] \to 0$ as $n \to \infty$ first and then $\epsilon \to 0$, and $n E_{T_1, \xi_1}\left[a_n^{-1} T_1 \xi_1 \mathbb{I}_{\{a_n^{-1} T_1 \xi_1 < \eta\}}\right] \to 0$ as $n \to \infty$ first and then $\eta \to 0$. Hence (C2) is verified, completing the proof of the first statement of the theorem. The remaining statements also follow in the same way as in Theorem 3.2. This completes the proof of the theorem. ■

4 THE CASE II

In this case $0 \leq \beta < \alpha < 1$, or $0 < \beta = \alpha < 1$ with $\frac{\overline{F}(a_n)}{\overline{G}(a_n)} \to 0$ as $x \to \infty$ (these requirements are assumed in this section without further mentioning). Throughout this section we let

$$\kappa(a_n) = \frac{\overline{F}(a_n)}{\overline{G}(a_n)}.$$

We first consider the limiting behavior of $M_{[nt]}(a_n z)$. For this purpose, we let

$$\begin{aligned} M^{\#}_{[nt]}(a_n z) &= \kappa(a_n) M_{[nt]}(a_n z) \\ &= V_{[nt]}(a_n z) + \sqrt{\kappa(a_n)} W_{[nt]}(a_n z), \end{aligned}$$

where

$$V_{[nt]}(a_n z) = \kappa(a_n) \sum_{k=1}^{[nt]} \mathbb{I}_{\left\{a_n^{-1} S_k \leq z\right\}} \overline{G}\left(a_n\left(z - a_n^{-1} S_k\right)\right)$$

and

$$W_{[nt]}(a_n z) = \sqrt{\kappa(a_n)} \sum_{k=1}^{[nt]} \zeta_k(a_n z)$$

with

$$\zeta_k(a_n z) = \left(\mathbb{I}_{\left\{a_n^{-1} S_k \leq z <\right\}} \mathbb{I}_{\left\{z < a_n^{-1} S_k + a_n^{-1} T_k\right\}} - \mathbb{I}_{\left\{a_n^{-1} S_k \leq z\right\}} \overline{G}\left(a_n\left(z - a_n^{-1} S_k\right)\right)\right).$$

Note that

$$M^{\#}_{\infty}(a_n z) = M^{\#}_{N(a_n z)}(a_n z), \quad W_{\infty}(a_n z) = W_{N(a_n z)}(a_n z). \tag{32}$$

We define $M^{\#}_{\infty}(sz)$ and $W_{\infty}(sz)$ to be the same as $M^{\#}_{\infty}(a_n z)$ and $W_{\infty}(a_n z)$ with s substituted for a_n.

Theorem 4.1 *For any $z_{n1} < ... < z_{nl}$ such that $(z_{n1}, ..., z_{nl}) \to (z_1, ..., z_l)$ with $z_1 > 0$,*

$$\begin{aligned} &\left(a_n^{-1} S_{[nt]+1}, M^{\#}_{[nt]}(a_n z_{nq}), W_{[nt]}(a_n z_{nq}), q = 1, ..., l\right) \\ \Longrightarrow &\left(S(t), \mathbb{M}^{\#}_t(z_q), \mathbb{W}_t(z_q), q = 1, ..., l\right) \text{ in } D_{\mathbb{R}^{2l+1}}[0, \infty), \end{aligned}$$

where

$$\mathbb{M}^{\#}_t(z) = \int_0^t \mathbb{I}_{\{S(r) \leq z\}} (z - S(r))^{-\beta} dr$$

and, conditionally on $S = (S(t); 0 \le t \le 1)$, *the process* $(t,z) \longmapsto \mathbb{W}_t(z)$ *is conditionally Gaussian with*

$$E[\mathbb{W}_t(z)|S] = 0, \quad E[\mathbb{W}_{t_1}(z_i)\mathbb{W}_{t_2}(z_j)|S] = \int_0^{\min(t_1,t_2)} \mathbb{I}_{\{S(r)\le z_i\}}(z_j - S(r))^{-\beta}\,dr, \;\; z_i < z_j.$$

In addition, with $M_\infty^\#(sz)$ *and* $W_\infty(sz)$ *as defined above,*

$$z \longmapsto \left(M_\infty^\#(sz), W_\infty(sz)\right) \overset{fdd}{\Longrightarrow} \left(\mathbb{M}_\infty^\#(z), \mathbb{W}_\infty(z)\right).$$

Proof. We first show the convergence of finite dimensional distributions of the first statement. Because $\kappa(a_n) \to 0$, this is equivalent to

$$\begin{aligned}&\left(a_n^{-1}S_{[nt]+1}, V_{[nt]}(a_n z_{nq}), W_{[nt]}(a_n z_{nq}), q = 1,...,l\right)\\ &\overset{fdd}{\Longrightarrow} \left(S(t), \mathbb{M}_t^\#(z_q), \mathbb{W}_t(z_q), q = 1,...,l\right),\end{aligned} \tag{33}$$

which we now establish. (Note that the preceding convergence, as well as that in the statement of the theorem, entails $\kappa(a_n) M_{[nt]}(a_n z) - V_{[nt]}(a_n z) \overset{p}{\to} 0$.) As in the proof of Theorem 3.1, without loss of generality we can take $z_{nq} = z_q$ for all $n \ge 1$, $1 \le q \le l$.

We first prove, with the help of Proposition 2.2 in particular, that

$$\begin{aligned}&\left(a_n^{-1}S_{[nt]}, \kappa(a_n)\sum_{k=1}^{[nt]} \mathbb{I}_{\left\{a_n^{-1}S_k \le z_i\right\}}\overline{G}\left(a_n\left(z_j - a_n^{-1}S_k\right)\right),\ 1 \le i \le j \le l\right)\\ &\overset{fdd}{\Longrightarrow} \left(S(t), \int_0^t \mathbb{I}_{\{S(r)\le z_i\}}(z_j - S(r))^{-\beta}\,dr,\ 1 \le i \le j \le l\right).\end{aligned} \tag{34}$$

Consider, for $\delta > 0$,

$$\kappa(a_n)\sum_{k=1}^{[nt]} \mathbb{I}_{\left\{a_n^{-1}S_k \le z_i - \delta\right\}}\overline{G}\left(a_n\left(z_j - a_n^{-1}S_k\right)\right) = \frac{1}{n}\sum_{k=1}^{[nt]} g_{n,\delta}^{(ij)}\left(a_n^{-1}S_k\right)$$

where we let

$$g_{n,\delta}^{(ij)}(u) = n\overline{F}(a_n)\,\mathbb{I}_{\{u \le z_i - \delta\}}\frac{\overline{G}(a_n(z_j - u))}{\overline{G}(a_n)}.$$

We have $n\overline{F}(a_n) \to 1$ and

$$\frac{\overline{G}(a_n(z_j - u))}{\overline{G}(a_n)} \to (z_j - u)^{-\beta} \text{ uniformly over } z_j - u > \delta.$$

Thus, noting that $u \le z_i - \delta$ entails $z_j - u > \delta$ because $z_i < z_j$,

$$\begin{aligned}&\sup_{0\le t\le M}\left|\frac{1}{n}\sum_{k=1}^{[nt]} g_{n,\delta}^{(ij)}\left(a_n^{-1}S_k\right) - \frac{1}{n}\sum_{k=1}^{[nt]}\mathbb{I}_{\{u\le z_i-\delta\}}\left(z_j - a_n^{-1}S_k\right)^{-\beta}\right|\\ \le\ & M\sup_u\left|g_{n,\delta}^{(ij)}(u) - \mathbb{I}_{\{u\le z_i-\delta\}}(z_j - u)^{-\beta}\right| \to 0\end{aligned}$$

for every $M>0$. Hence applying Proposition 2.2 we have

$$\begin{aligned} &\left(a_n^{-1}S_{[nt]}, \frac{1}{n}\sum_{k=1}^{[nt]} g_{n,\delta}^{(ij)}\left(a_n^{-1}S_k\right),\ 1\le i\le j\le l\right) \\ &\quad \overset{fdd}{\Longrightarrow} \left(S\left(t\right), \int_0^t \mathbb{I}_{\{u\le z_i-\delta\}}\left(z_j - S\left(r\right)\right)^{-\beta} dr,\ 1\le i\le j\le l\right). \end{aligned} \tag{35}$$

In addition, according to part (i) of Lemma 2.3, for all $1\le i\le j\le l$,

$$E\left[\kappa\left(a_n\right)\sum_{k=1}^{[nt]}\mathbb{I}_{\left\{z_i-\delta<a_n^{-1}S_k\le z_i\right\}}\overline{G}\left(a_n\left(z_j-a_n^{-1}S_k\right)\right)\right]\to 0 \tag{36}$$

as $n\to\infty$ first and then $\delta\to 0$, and similarly $\int_0^t \mathbb{I}_{\{z_i-\delta<S(r)\le z_i\}}\left(z_j-S\left(r\right)\right)^{-\beta} dr \xrightarrow{p} 0$ as $\delta\to 0$. Thus, in view of (35), (34) holds.

Now, note that

$$\begin{aligned} &\sum_{k=1}^{[nt]} E_{k-1}\left[\zeta_{nk}\left(z_i\right)\zeta_{nk}\left(z_j\right)\right] - \kappa\left(a_n\right)\sum_{k=1}^{[nt]}\mathbb{I}_{\left\{a_n^{-1}S_k\le z_i\right\}}\overline{G}\left(a_n\left(z_j-a_n^{-1}S_k\right)\right) \\ = &\ \kappa\left(a_n\right)\sum_{k=1}^{[nt]}\mathbb{I}_{\left\{a_n^{-1}S_k\le z_i\right\}}\left\{\overline{G}\left(a_n\left(z_j-a_n^{-1}S_k\right)\right)\right\}^2 \xrightarrow{p} 0 \end{aligned} \tag{37}$$

because of (35) and, in view of (36) (note $\overline{G}\left(a_n\delta\right)\to 0$),

$$\begin{aligned} &\kappa\left(a_n\right)\sum_{k=1}^{[nt]}\mathbb{I}_{\left\{a_n^{-1}S_k\le z_i-\delta\right\}}\left\{\overline{G}\left(a_n\left(z_j-a_n^{-1}S_k\right)\right)\right\}^2 \\ \le &\ \overline{G}\left(a_n\delta\right)\kappa\left(a_n\right)\sum_{k=1}^{[nt]}\mathbb{I}_{\left\{a_n^{-1}S_k\le z-\delta\right\}}\overline{G}\left(a_n\left(z-a_n^{-1}S_k\right)\right) \xrightarrow{p} 0. \end{aligned}$$

Having obtained (34) and (37), we next invoke Theorem 2.1. For this purpose, let $Z_{nk}=\sqrt{\kappa\left(a_n\right)}\sum_{q=1}^{l}\theta_q\zeta_k\left(a_n z_q\right)$, so that

$$\sum_{q=1}^{l}\theta_q W_{[nt]}\left(a_n z_q\right)=\sum_{k=1}^{[nt]} Z_{nk},$$

where $\theta_q, q=1,...,l$, are reals. Clearly Z_{nk} are martingale differences such that

$$\sum P_{k-1}\left[\left|Z_{nk}\right|\ge\eta\right]\xrightarrow{p} 0 \quad \text{for every } \eta>0$$

because $\kappa\left(a_n\right)\to 0$. This implies, because Z_{nk} are uniformly bounded in absolute value by a constant, that the condition (C4) holds. In view of this together with (34) and

(37), the conditions of Theorem 2.1 are satisfied (with the component $\sum_{k=1}^{[nt]} Y_{nk}$ absent, and hence (C2) and (C3) are not required). Hence

$$\left(a_n^{-1} S_{[nt]+1}, W_{[nt]}\left(a_n z_q\right), q=1, \ldots, l\right) \stackrel{fdd}{\Longrightarrow} \left(S\left(t\right), W_t\left(z_q\right), q=1, \ldots, l\right).$$

This in turn implies the convergence (33), in view of the same arguments involved in obtaining (34) based on Proposition 2.2 using $a_n^{-1} S_{[nt]} \stackrel{fdd}{\Longrightarrow} S\left(t\right)$.

The tightness in $D_{\mathbb{R}^{2l+1}}\left[0, \infty\right)$ follows in exactly the same way as was done in the proof of Theorem 2.1 based on Aldous criterion. We note that this criterion holds for the component $M_{[nt]}^{\#}\left(a_n z_q\right)$ because it is increasing in t and its limit $\mathbb{M}_t^{\#}\left(z_q\right)$ is increasing and continuous in t, see Jacod and Shiryayev (1987, Ch. VI, Theorem 3.37 (Statement (a)), page 318)). This completes the proof of the first statement of the theorem. Using the same arguments in the proof of Theorem 3.1, and in view of (32), the last statement $z \longmapsto \left(M_{\infty}^{\#}\left(sz\right), W_{\infty}\left(sz\right)\right) \stackrel{fdd}{\Longrightarrow} \left(\mathbb{M}_{\infty}^{\#}\left(z\right), \mathbb{W}_{\infty}\left(z\right)\right)$ also follows, completing the proof of the theorem. ■

Note that last statement of Theorem 4.1 above contains the statement

$$M_{\infty}^{\#}\left(sz\right) \stackrel{fdd}{\Longrightarrow} \mathbb{M}_{\infty}^{\#}\left(z\right) \qquad \text{as } s \rightarrow \infty. \tag{38}$$

Similarly the last statement Theorem 4.2 below contains the statement

$$\int_0^z M_{\infty}^{\#}\left(su\right) du \Longrightarrow \int_0^z \mathbb{M}_{\infty}^{\#}\left(u\right) du \text{ in } D_{\mathbb{R}}\left[0, \infty\right) \qquad \text{as } s \rightarrow \infty. \tag{39}$$

These two results are already contained in Mikosch and Resnick (2004, Corollary 2.7 and Theorem 4.4). ■

In the next result we obtain both the first order and the second order limiting behaviors for $A_{[nt]}\left(a_n z\right)$, in parallel to Theorem 4.1. To state the result, let

$$A_{[nt]}^{\#}\left(a_n z\right) = a_n^{-1} \kappa\left(a_n\right) A_{[nt]}\left(a_n z\right).$$

Recall that, with $V_{[nt]}\left(a_n u\right)$ and $W_{[nt]}\left(a_n u\right)$ as defined earlier,

$$\begin{aligned} A_{[nt]}^{\#}\left(a_n z\right) &= \int_0^z M_{[nt]}^{\#}\left(a_n u\right) du \\ &= \int_0^z V_{[nt]}\left(a_n u\right) du + \sqrt{\kappa\left(a_n\right)} \int_0^z W_{[nt]}\left(a_n u\right) du \\ &= \int_0^z V_{[nt]}\left(a_n u\right) du + \sqrt{\kappa\left(a_n\right)} W_{[nt]}^{\#}\left(a_n z\right), \end{aligned} \tag{40}$$

where

$$W_{[nt]}^{\#}\left(a_n z\right) = \int_0^z W_{[nt]}\left(a_n u\right) du = a_n^{-1} \int_0^{a_n z} W_{[nt]}\left(u\right) du.$$

Further we define $A_{\infty}^{\#}(sz)$ and $W_{\infty}^{\#}(sz)$ to be the same as $A_{\infty}^{\#}(a_n z) = A_{N(a_n z)}^{\#}(a_n z)$ and $W_{\infty}^{\#}(a_n z) = W_{N(a_n z)}^{\#}(a_n z)$ with s substituted for a_n.

Note that the limits occurring in the statement are the appropriate integrals of those of Theorem 4.1, as is to be expected. Indeed it is possible to obtain the finite dimensional convergence as a consequence of Theorem 4.1 by verifying the conditions of the Proposition 2.2, but establishing tightness in the Skorokhod space for the component $W_{[nt]}^{\#}(a_n z)$ requires some extra work, which work essentially constitutes the proof given below.

Theorem 4.2. *For any $z_{n1} < ... < z_{nl}$ such that $(z_{n1}, ..., z_{nl}) \to (z_1, ..., z_l)$ with $z_1 > 0$,*

$$\left(a_n^{-1} S_{[nt]+1}, A_{[nt]}^{\#}(a_n z_{nq}), W_{[nt]}^{\#}(a_n z_{nq}), q = 1, ..., l\right)$$
$$\Longrightarrow \left(S(t), \mathbb{A}_t^{\#}(z_q), \mathbb{W}_t^{\#}(z_q), q = 1, ..., l\right) \text{ in } D_{\mathbb{R}^{2l+1}}[0, \infty)$$

where $\mathbb{W}_t^{\#}(z) = \int_0^z \mathbb{W}_t(u)\, du$ and

$$\mathbb{A}_t^{\#}(z) = \frac{1}{1-\beta} \int_0^t \mathbb{I}_{\{S(r) \le z\}} (z - S(r))^{1-\beta} dr = \int_0^z \mathbb{M}_t^{\#}(u)\, du,$$

with $\mathbb{M}_t^{\#}(u)$ and $\mathbb{W}_t(u)$ as defined in Theorem 4.1. Further

$$E\left[\mathbb{W}_{t_1}^{\#}(z_i)\, \mathbb{W}_{t_2}^{\#}(z_j) \middle| S\right] = \int_0^{\min(t_1,t_2)} \mathbb{I}_{\{S(r) \le z_i\}} h_{(z_i,z_j)}(S(r))\, dr$$

where

$$h_{(z_i,z_j)}(u) = \frac{1}{1-\beta} \left((z_i - u)(z_j - u)^{1-\beta} - \frac{\beta}{(2-\beta)}(z_i - u)^{2-\beta}\right).$$

In addition, for each $t > 0$,

$$z \longmapsto A_{[nt]}^{\#}(a_n z) \Longrightarrow \mathbb{A}_t^{\#}(z) \text{ in } D_{\mathbb{R}}[0, \infty),$$

and, with $A_{\infty}^{\#}(sz)$ and $W_{\infty}^{\#}(sz)$ as defined above,

$$z \longmapsto \left(A_{\infty}^{\#}(sz), W_{\infty}^{\#}(sz)\right) \overset{fdd}{\Longrightarrow} \left(\mathbb{A}_{\infty}^{\#}(z), \mathbb{W}_{\infty}^{\#}(z)\right), \quad A_{\infty}^{\#}(sz) \Longrightarrow \mathbb{A}_{\infty}^{\#}(z) \text{ in } D_{\mathbb{R}}[0, \infty).$$

Proof. The main steps of the proof parallel those of the proof of Theorem 4.1. We have

$$W_{[nt]}^{\#}(a_n z_q) = \sum_{k=1}^{[nt]} \zeta_{nk}^{*}(z_q)$$

where, with $\zeta_k(a_n u)$ as defined earlier,

$$\begin{aligned}\zeta^*_{nk}(z) &= \sqrt{\kappa(a_n)}\int_0^z \zeta_k(a_n u)\,du \\ &= \sqrt{\kappa(a_n)}\mathbb{I}_{\left\{a_n^{-1}S_k\le z\right\}}\left(\min\left(a_n^{-1}T_k, z-a_n^{-1}S_k\right)-E_{T_1}\left[\min\left(a_n^{-1}T_1, z-a_n^{-1}S_k\right)\right]\right).\end{aligned}$$

Now, for reals $\theta_q, q=1,...,l$, consider $\sum_{q=1}^{l}\theta_q W^{\#}_{[nt]}(a_n z_q)=\sum_{k=1}^{[nt]} Z_{nk}$ where

$$Z_{nk}=\sum_{q=1}^{l}\theta_q\zeta^*_{nk}(z_q).$$

Clearly Z_{nk} are martingale differences. Because $\kappa(a_n)\to 0$ and because $\zeta^*_{nk}(z_q)$ is bounded by $2z_q\sqrt{\kappa(a_n)}$, it is also clear that

$$\sum P_{k-1}\left[|Z_{nk}|\ge\eta\right]\xrightarrow{p}0 \quad \text{for every } \eta>0.$$

We next show that

$$\left(a_n^{-1}S_{[nt]+1}, \sum_{k=1}^{[nt]}E_{k-1}\left[Z^2_{nk}\right]\right)\overset{fdd}{\Longrightarrow}\left(S(t), \sum_{i=1}^{l}\sum_{i=1}^{l}u_i u_j\int_0^t \mathbb{I}_{\{S(r)\le z_i\}}h_{(z_i,z_j)}(S(r))\,dr\right) \tag{41}$$

where $h_{(z_i,z_j)}(u)$ is as in the statement of the theorem. For this purpose, note that

$$\sum_{k=1}^{[nt]}E_{k-1}\left[Z^2_{nk}\right]=\sum_{i=1}^{l}\sum_{i=1}^{l}\theta_i\theta_j\sum_{k=1}^{[nt]}E_{k-1}\left[\zeta^*_{nk}(z_i)\,\zeta^*_{nk}(z_j)\right]$$

where

$$\sum_{k=1}^{[nt]}E_{k-1}\left[\zeta^*_{nk}(z_i)\,\zeta^*_{nk}(z_j)\right]=n\overline{F}(a_n)\frac{1}{n}\sum_{k=1}^{[nt]}\mathbb{I}_{\left\{a_n^{-1}S_k\le z_i\right\}}\left(g^{(i,j)}_{n1}\left(a_n^{-1}S_k\right)-g^{(i,j)}_{n2}\left(a_n^{-1}S_k\right)\right)$$

with

$$g^{(i,j)}_{n1}(u)=\frac{1}{\overline{G}(a_n)}E_{T_1}\left[\min\left(a_n^{-1}T_1, z_i-u\right)\min\left(a_n^{-1}T_1, z_j-u\right)\right]$$

and

$$g^{(i,j)}_{n2}(u)=\frac{1}{\overline{G}(a_n)}E_{T_1}\left[\min\left(a_n^{-1}T_1, z_i-u\right)\right]E_{T_1}\left[\min\left(a_n^{-1}T_1, z_j-u\right)\right].$$

We have

$$\begin{aligned}\overline{G}(a_n)\,g^{(i,j)}_{n1}(u) &= a_n^{-2}E_{T_1}\left[\mathbb{I}_{\left\{a_n^{-1}T_1\le z_i-u\right\}}T_1^2\right]+a_n^{-1}(z_i-u)\,E_{T_1}\left[\mathbb{I}_{\left\{z_i-u<a_n^{-1}T_1\le z_j-u\right\}}T_1\right] \\ &\quad +(z_i-u)(z_j-u)\,\overline{G}(a_n(z_j-u)).\end{aligned}$$

According to Lemma 2.4 of Section 2, for each constant $\eta > 0$, the following approximations hold uniformly over $z_i - u \geq \eta$,

$$\frac{1}{\overline{G}(a_n)} a_n^{-2} E_{T_1}\left[\mathbb{I}_{\left\{a_n^{-1}T_1 \leq z_i - u\right\}} T_1^2\right] - \frac{\beta}{2-\beta}(z_i - u)^{2-\beta} \to 0,$$

$$\frac{1}{\overline{G}(a_n)} a_n^{-1} E_{T_1}\left[\mathbb{I}_{\left\{z_i - u < a_n^{-1}T_1 \leq z_j - u\right\}} T_1\right] - \frac{\beta}{1-\beta}\left((z_j - u)^{1-\beta} - (z_i - u)^{1-\beta}\right) \to 0.$$

In addition, uniformly over $z_i - u \geq \eta$,

$$\frac{1}{\overline{G}(a_n)} \overline{G}(a_n(z_j - u)) - (z_j - u)^{-\beta} \to 0.$$

Thus for each $\eta > 0$ we have, uniformly over u,

$$\sup_r \left| \frac{1}{\overline{G}(a_n)} \mathbb{I}_{\{u \leq z_i - \eta\}} \left\{ E_{T_1}\left[\min\left(a_n^{-1}T_1, z_i - u\right) \min\left(a_n^{-1}T_1, z_j - u\right)\right] - h_{(z_i,z_j)}(u) \right\} \right| \to 0$$

where

$$\begin{aligned} h_{(z_i,z_j)}(u) &= \frac{\beta}{2-\beta}(z_i - u)^{2-\beta} + \frac{\beta(z_i - u)}{1-\beta}\left((z_j - u)^{1-\beta} - (z_i - u)^{1-\beta}\right) \\ &\quad + (z_i - u)(z_j - u)^{1-\beta}. \end{aligned}$$

(This expression for $h_{(z_i,z_j)}(u)$ coincides with that given in the statement of the theorem.) Thus by invoking Proposition 2.2 exactly as in the proof of Theorem 4.1, we have

$$\begin{aligned} &\left(a_n^{-1} S_{[nt]+1}, \sum_{i=1}^{l} \sum_{i=1}^{l} n\overline{F}(a_n) \frac{1}{n} \sum_{k=1}^{[nt]} \mathbb{I}_{\left\{a_n^{-1}S_k \leq z_i - \eta\right\}} g_{n1}^{(i,j)}\left(a_n^{-1}S_k\right) \right) \\ &\overset{fdd}{\Longrightarrow} \left(S(t), \sum_{i=1}^{l} \sum_{i=1}^{l} u_i u_j \int_0^t \mathbb{I}_{\{S(r) \leq z_i - \eta\}} h_{(z_i,z_j)}(S(r))\, dr \right) \qquad (42) \end{aligned}$$

Next note that

$$\begin{aligned} &\kappa(a_n) \sum_{k=1}^{[nt]} \mathbb{I}_{\left\{z_i - \eta < a_n^{-1}S_k \leq z_i\right\}} E_{T_1}\left[\min\left(a_n^{-1}T_1, z_i - a_n^{-1}S_k\right) \min\left(a_n^{-1}T_1, z_j - a_n^{-1}S_k\right)\right] \\ &\leq z_i \kappa(a_n) \sum_{k=1}^{[nt]} \mathbb{I}_{\left\{z_i - \eta < a_n^{-1}S_k \leq z_i\right\}} E_{T_1}\left[\min\left(a_n^{-1}T_1, z_i - a_n^{-1}S_k\right)\right] \to 0 \qquad (43) \end{aligned}$$

as $n \to \infty$ first and then $\eta \to 0$, where $\to 0$ is obtained using the arguments (based on Lemma 2.3) contained in the proof of Theorem 3.2. In the same way, for some constant $C > 0$,

$$\int_0^t \mathbb{I}_{\{z_i - \eta < S(r) \leq z_i\}} h_{(z_i,z_j)}(S(r))\, dr \leq \int_0^t \mathbb{I}_{\{z_i - \eta < S(r) \leq z_i\}} \left(C + (z_j - S(r))^{-\beta}\right) dr \xrightarrow{p} 0 \qquad (44)$$

as $\eta \to 0$. Moreover,

$$\kappa\left(a_n\right) \sum_{k=1}^{[nt]} \mathbb{I}_{\left\{z_i-\eta<a_n^{-1} S_k \leq z_i\right\}} E_{T_1}\left[\min\left(a_n^{-1} T_1, z_i-a_n^{-1} S_k\right)\right] E_{T_1}\left[\min\left(a_n^{-1} T_1, z_j-a_n^{-1} S_k\right)\right] \to 0 \tag{45}$$

because the same bound in (43) holds here also. In addition

$$\begin{aligned} & \kappa\left(a_n\right) \sum_{k=1}^{[nt]} \mathbb{I}_{\left\{a_n^{-1} S_k \leq z_i-\eta\right\}} E_{T_1}\left[\min\left(a_n^{-1} T_1, z_i-a_n^{-1} S_k\right)\right] E_{T_1}\left[\min\left(a_n^{-1} T_1, z_j-a_n^{-1} S_k\right)\right] \\ \leq\ & E_{T_1}\left[\min\left(a_n^{-1} T_1, \eta\right)\right] \kappa\left(a_n\right) \sum_{k=1}^{[nt]} \mathbb{I}_{\left\{a_n^{-1} S_k \leq z_i-\eta\right\}} E_{T_1}\left[\min\left(a_n^{-1} T_1, z_i-a_n^{-1} S_k\right)\right] \to 0 \quad (46) \end{aligned}$$

because $E_{T_1}\left[\min\left(a_n^{-1} T_1, z_i\right)\right] \to 0$ and because, using the same arguments involved in obtaining (42), $\kappa\left(a_n\right) \sum_{k=1}^{[nt]} \mathbb{I}_{\left\{a_n^{-1} S_k \leq z_i-\eta\right\}} E_{T_1}\left[\min\left(a_n^{-1} T_1, z_i-a_n^{-1} S_k\right)\right]$ is stochastically bounded. Thus, in view of (42) - (46), (41) follows.

Now (41) together with $\sum P_{k-1}\left[\left|Z_{nk}\right| \geq \eta\right] \xrightarrow{p} 0$ entail, using Theorem 2.1 in the same way as in the proof of Theorem 4.1, that

$$\left(a_n^{-1} S_{[nt]+1}, W_{[nt]}^{\#}\left(a_n z_q\right), q=1, \ldots, l\right) \stackrel{fdd}{\Longrightarrow}\left(S(t), \mathbb{W}_t^{\#}\left(z_q\right), q=1, \ldots, l\right)$$

where, conditionally on S, the process $(t, z) \longmapsto \mathbb{W}_t^{\#}(z)$ is conditionally Gaussian as specified in the statement of the theorem. This implies, using the same arguments (with the help of Proposition 2.2 in particular) involved in (42) - (46) that led to this convergence, that

$$\begin{aligned} & \left(a_n^{-1} S_{[nt]+1}, \int_0^{z_q} V_{[nt]}\left(a_n u\right) du, W_{[nt]}^{\#}\left(a_n z_q\right), q=1, \ldots, l\right) \\ & \stackrel{fdd}{\Longrightarrow}\left(S(t), \int_0^{z_q} \mathbb{M}_t^{\#}(u) du, \mathbb{W}_t^{\#}\left(z_q\right), q=1, \ldots, l\right) \end{aligned}$$

holds, where note that

$$\int_0^z V_{[nt]}\left(a_n u\right) du=\kappa\left(a_n\right) \sum_{k=1}^{[nt]} \mathbb{I}_{\left\{a_n^{-1} S_k \leq z\right\}} E_{T_1}\left[\min\left(a_n^{-1} T_1, z-a_n^{-1} S_k\right)\right]$$

and

$$\int_0^z \mathbb{M}_t^{\#}(u) du=\frac{1}{1-\beta} \int_0^t \mathbb{I}_{\{S(r) \leq z\}}(z-S(r))^{1-\beta} dr.$$

The convergence of $W_{[nt]}^{\#}\left(a_n z_q\right)$ implies, in view of (40) and because $\kappa\left(a_n\right) \to 0$,

$$A_{[nt]}^{\#}\left(a_n z_q\right)-\int_0^{z_q} V_{[nt]}\left(a_n u\right) du \xrightarrow{p} 0.$$

This then gives the $\stackrel{fdd}{\Longrightarrow}$ part of the first statement of the theorem.

Thus to complete the proof of the first statement of the theorem it only remains to obtain the tightness in $D_{\mathbb{R}^{2l+1}}[0,\infty)$, but this is done in the same way as is indicated in the proof of Theorem 4.1 (note that the component $A^{\#}_{[nt]}(a_n z_q)$ is increasing in t and its limit $\mathbb{A}^{\#}_t(z_q)$ is increasing and continuous in t.)

Using the same arguments in the proof of Theorem 3.2, the remaining statements are also obtained. (Note that $A^{\#}_{[nt]}(a_n z)$ and $A^{\#}_{\infty}(sz)$ are monotone in z and their limits are continuous in z.) This completes the proof of the theorem. ■

As in the earlier Case I, we next consider $M^*_{[nt]}(a_n z)$ and $A^*_{[nt]}(a_n z)$. Recall that these involve nonnegative iid weights ξ_k. The next result will assume a further moment restriction on ξ_1, and gives what may be called the first order approximations.

Theorem 4.3. *Assume that* $\mu_\xi = E[\xi_1] < \infty$. *Let* $V_{[nt]}(a_n z)$ *and* $A_{[nt]}(a_n z)$ *be as in Theorem 4.1. Then, for each* $t > 0$,

$$\kappa(a_n)\, M^*_{[nt]}(a_n z) - \mu_\xi V_{[nt]}(a_n z) \xrightarrow{p} 0 \tag{47}$$

and

$$a_n^{-1}\kappa(a_n)\, A^*_{[nt]}(a_n z) - \mu_\xi \int_0^z V_{[nt]}(a_n u)\, du \xrightarrow{p} 0. \tag{48}$$

(The limiting behaviors of $V_{[nt]}(a_n z)$ *and* $a_n^{-1}A_{[nt]}(a_n z)$ *are given in Theorem 4.1.)*

Proof. In view of Theorem 4.1, $\kappa(a_n)\sum_{k=1}^{[nt]} \mathbb{I}_{\left\{a_n^{-1}S_k \le z < a_n^{-1}S_k + a_n^{-1}T_k\right\}} - V_{[nt]}(a_n z) \xrightarrow{p} 0$, and hence to prove (47) it is enough to show that

$$\kappa(a_n)\sum_{k=1}^{[nt]} \mathbb{I}_{\left\{a_n^{-1}S_k \le z < a_n^{-1}S_k + a_n^{-1}T_k\right\}}(\xi_k - \mu_\xi) \xrightarrow{p} 0. \tag{49}$$

Define

$$\xi_{nk} = \xi_k \mathbb{I}_{\left\{\xi_k \le (\kappa(a_n))^{-\lambda}\right\}},\quad 0 < \lambda < 1.$$

Note that $\kappa(a_n)\,\mathbb{I}_{\left\{a_n^{-1}S_k \le z < a_n^{-1}S_k + a_n^{-1}T_k\right\}}(\xi_{nk} - E[\xi_{n1}])$ are martingale differences and the sum of their conditional variances is

$$\begin{aligned} &(\kappa(a_n))^2 \sum_{k=1}^{[nt]} \mathbb{I}_{\left\{a_n^{-1}S_k \le z < a_n^{-1}S_k + a_n^{-1}T_k\right\}} E\left[(\xi_{nk} - E[\xi_{n1}])^2\right] \\ \le\ & (\kappa(a_n))^2 (\kappa(a_n))^{-\lambda} \mu_\xi \sum_{k=1}^{[nt]} \mathbb{I}_{\left\{a_n^{-1}S_k \le z < a_n^{-1}S_k + a_n^{-1}T_k\right\}} \xrightarrow{p} 0 \end{aligned}$$

where we have used $E\left[\left(\xi_{nk}-E\left[\xi_{n1}\right]\right)^2\right]\leq E\left[\xi_{nk}^2\right]\leq\left(\kappa\left(a_n\right)\right)^{-\lambda}\mu_\xi$ and $\left(\kappa\left(a_n\right)\right)^{1-\lambda}\to 0$, together with the fact that $\kappa\left(a_n\right)\sum_{k=1}^{[nt]}\mathbb{I}_{\left\{a_n^{-1}S_k\leq z<a_n^{-1}S_k+a_n^{-1}T_k\right\}}$ is stochastically bounded. Thus

$$\kappa\left(a_n\right)\sum_{k=1}^{[nt]}\mathbb{I}_{\left\{a_n^{-1}S_k\leq z<a_n^{-1}S_k+a_n^{-1}T_k\right\}}\left(\xi_{nk}-E\left[\xi_{n1}\right]\right)\xrightarrow{p}0.$$

This implies (47) because $\xi_k-\xi_{nk}$ is nonnegative with the expected value $\mu_\xi-E\left[\xi_{n1}\right]\to 0$.

The proof of (48) is similar. We need to show that

$$\kappa\left(a_n\right)\sum_{k=1}^{[nt]}\left(\int_0^z\mathbb{I}_{\left\{a_n^{-1}S_k\leq u<a_n^{-1}S_k+a_n^{-1}T_k\right\}}du\right)\left(\xi_k-\mu_\xi\right)\xrightarrow{p}0.$$

Because $\left(\int_0^z\mathbb{I}_{\left\{a_n^{-1}S_k\leq u<a_n^{-1}S_k+a_n^{-1}T_k\right\}}du\right)^2\leq z\int_0^z\mathbb{I}_{\left\{a_n^{-1}S_k\leq u<a_n^{-1}S_k+a_n^{-1}T_k\right\}}du$, the required proof is the same as that of (47). This completes the proof of the theorem. ■

The statements in Theorem 4.3 may be viewed as first order approximations to $M_{[nt]}^*\left(a_nz\right)$ and $A_{[nt]}^*\left(a_nz\right)$. We next obtain second order approximations analogous to those in Theorems 4.1 and 4.2, under the condition $E\left[\xi_1^2\right]<\infty$ (or more generally if ξ_1 is in the domain of attraction of a normal distribution). Unfortunately we are unable to obtain analogous result under more general moment conditions. We note that the conditional covariance function of the limiting process $(t,z)\longmapsto\widetilde{\mathbb{W}}_t\left(z\right)$ in the statement below is the same as that in Theorem 4.1 except for a multiplicative constant.

Theorem 4.4. *Assume that $E\left[\xi_1^2\right]<\infty$. Then for any $z_{n1}<...<z_{nl}$ such that $(z_{n1},...,z_{nl})\to(z_1,...,z_l)$ with $z_1>0$,*

$$\begin{aligned}&\left(a_n^{-1}S_{[nt]+1},\sqrt{\kappa\left(a_n\right)}\left(M_{[nt]}^*\left(a_nz_{nq}\right)-\mu_\xi\left(\kappa\left(a_n\right)\right)^{-1}V_{[nt]}\left(a_nz_{nq}\right)\right),q=1,...,l\right)\\ \Longrightarrow\ &\left(S\left(t\right),\widetilde{\mathbb{W}}_t\left(z_q\right),q=1,...,l\right)\ \text{ in } D_{\mathbb{R}^{l+1}}\left[0,\infty\right)\end{aligned}$$

where, conditionally on S, the process $(t,z)\longmapsto\widetilde{\mathbb{W}}_t\left(z\right)$ is conditionally Gaussian with $E\left[\widetilde{\mathbb{W}}_t\left(z\right)\middle|S\right]=0$ and the conditional covariance

$$E\left[\widetilde{\mathbb{W}}_{t_1}\left(z_i\right)\widetilde{\mathbb{W}}_{t_2}\left(z_j\right)\middle|S\right]=E\left[\xi_1^2\right]\int_0^{\min(t_1,t_2)}\mathbb{I}_{\left\{S(r)\leq z_i\right\}}\left(z_j-S\left(r\right)\right)^{-\beta}dr,\ z_i<z_j.$$

In addition,

$$z\longmapsto\sqrt{\kappa\left(s\right)}\left(M_\infty^*\left(sz\right)-\frac{\mu_\xi}{\kappa\left(s\right)}V_\infty\left(sz\right)\right)\overset{fdd}{\Longrightarrow}\widetilde{\mathbb{W}}_\infty\left(z\right)\qquad\text{as } s\to\infty.$$

Proof. Recalling $V_{[nt]}\left(a_nz\right)=\kappa\left(a_n\right)\sum_{k=1}^{[nt]}\mathbb{I}_{\left\{a_n^{-1}S_k\leq z\right\}}\overline{G}\left(a_n\left(z-a_n^{-1}S_k\right)\right)$, we have

$$\sqrt{\kappa\left(a_n\right)}\left(M_{[nt]}^*\left(a_nz\right)-\frac{\mu_\xi}{\kappa\left(s\right)}V_{[nt]}\left(a_nz\right)\right)=\sum_{k=1}^{[nt]}\zeta_{nk}\left(z\right)$$

where

$$\zeta_{nk}(z) = \sqrt{\kappa(a_n)} \left\{ \mathbb{I}_{\left\{a_n^{-1}S_k \le z < a_n^{-1}S_k + a_n^{-1}T_k\right\}} \xi_k - \mu_\xi \mathbb{I}_{\left\{a_n^{-1}S_k \le z\right\}} \overline{G}\left(a_n\left(z - a_n^{-1}S_k\right)\right) \right\}.$$

We first verify that, for each $z > 0$,

$$\sum_{k=1}^{n} P_{k-1}\left[\left|\zeta_{nk}(z)\right| > \eta\right] \xrightarrow{p} 0, \tag{50}$$

and, for reals $\theta_1, ..., \theta_l$,

$$\begin{aligned} &\left(a_n^{-1}S_{[nt]}, \sum_{i=1}^{l}\sum_{j=1}^{l}\sum_{k=1}^{[nt]} \theta_i\theta_j E_{k-1}\left[\zeta_{nk}(z_i)\,\zeta_{nk}(z_j)\right]\right) \\ &\overset{fdd}{\Longrightarrow} \left(S(t), \sum_{i=1}^{l}\sum_{j=1}^{l} \theta_i\theta_j E\left[\widetilde{\mathbb{W}}_t(z_i)\,\widetilde{\mathbb{W}}_t(z_j)\middle| S\right]\right) \end{aligned} \tag{51}$$

where $E\left[\widetilde{\mathbb{W}}_t(z_i)\,\widetilde{\mathbb{W}}_t(z_j)\middle| S\right]$ is as defined in the statement of the theorem. Because

$$\begin{aligned} &\sum_{k=1}^{[nt]} E_{k-1}\left[\zeta_{nk}(z_i)\,\zeta_{nk}(z_j)\right] \\ = \; & E\left[\xi_1^2\right]\kappa(a_n)\sum_{k=1}^{[nt]} \mathbb{I}_{\left\{a_n^{-1}S_k \le z_i\right\}} \overline{G}\left(a_n\left(z_j - a_n^{-1}S_k\right)\right) \\ & -\mu_\xi^2 \kappa(a_n) \sum_{k=1}^{[nt]} \mathbb{I}_{\left\{a_n^{-1}S_k \le z_i\right\}} \overline{G}\left(a_n\left(z_i - a_n^{-1}S_k\right)\right) \overline{G}\left(a_n\left(z_j - a_n^{-1}S_k\right)\right), \end{aligned}$$

(51) follows from the arguments contained in the proof of Theorem 4.1.

Regarding (50), note that according to Theorem 4.3, by identifying ξ_k with ξ_k^2, and by using Theorem 4.1,

$$A_n(t) = \kappa(a_n)\sum_{k=1}^{[nt]} \mathbb{I}_{\left\{a_n^{-1}S_k \le z < a_n^{-1}S_k + a_n^{-1}T_k\right\}} \xi_k^2 \overset{fdd}{\Longrightarrow} A(t)$$

where we let $A(t) = E\left[\xi_1^2\right]\mathbb{M}_t^{\#}(z) = E\left[\xi_1^2\right]\int_0^t \mathbb{I}_{\{S(r)\le z\}}(z - S(r))^{-\beta}\,dr$. ($\mathbb{M}_t^{\#}(z)$ is defined in Theorem 4.1.) In addition note that, for each $n \ge 1$, $A_n(t)$ is nondecreasing in t and its limit $A(t)$ is increasing and continuous in t. These facts will entail that, for each $M > 0$,

$$\sup_{|u-v|\le\delta, |u|\le M, |v|\le M} \left|A_n(u) - A_n(v)\right| \xrightarrow{p} 0$$

as $n \to \infty$ first and then $\delta \to 0$. Hence $\sup_{1\le k\le[nt]} \kappa(a_n)\,\mathbb{I}_{\left\{a_n^{-1}S_k\le z<a_n^{-1}S_k+a_n^{-1}T_k\right\}}\xi_k^2 \xrightarrow{p} 0$ for every $t>0$, and hence

$$\sup_{1\le k\le[nt]}\sqrt{\kappa(a_n)\,\mathbb{I}_{\left\{a_n^{-1}S_k\le z<a_n^{-1}S_k+a_n^{-1}T_k\right\}}\xi_k^2} = \sup_{1\le k\le[nt]}\sqrt{\kappa(a_n)}\mathbb{I}_{\left\{a_n^{-1}S_k\le z<a_n^{-1}S_k+a_n^{-1}T_k\right\}}\xi_k \xrightarrow{p} 0.$$

In addition we have $\sup_{1\le k\le[nv]}\sqrt{\kappa(a_n)}\mathbb{I}_{\left\{a_n^{-1}S_k\le z\right\}}\overline{G}\left(a_n\left(z-a_n^{-1}S_k\right)\right)\le\sqrt{\kappa(a_n)}\to 0$. Hence

$$\sup_{1\le k\le[nt]}\left|\zeta_{nk}(z)\right|\xrightarrow{p}0.$$

It can be seen that this is equivalent to (50). As in the proof of Theorem 4.1, this completes the proof. ■

The next statement gives the result for $A^*_{[nt]}(a_nz)$. It is completely analogous to Theorem 4.4, by invoking Theorem 4.2 in place of Theorem 4.1, and therefore its proof is omitted.

Theorem 4.5. *Assume that* $E\left[\xi_1^2\right]<\infty$. *Then for any* $z_{n1}<...<z_{nl}$ *such that* $(z_{n1},...,z_{nl})\to(z_1,...,z_l)$ *with* $z_1>0$,

$$\left(a_n^{-1}S_{[nt]+1},\sqrt{\kappa(a_n)}\left(a_n^{-1}A^*_{[nt]}(a_nz_{nq})-\mu_\xi\kappa(a_n)\int_0^{z_{nq}}V_{[nt]}(a_nu)\,du\right),q=1,...,l\right)$$
$$\Longrightarrow\ \left(S(t),\widetilde{\mathbb{W}}^\#_t(z_q),q=1,...,l\right)\ \text{in } D_{\mathbb{R}^{l+1}}[0,\infty)$$

where, conditionally on S, *the process* $(t,z)\longmapsto\widetilde{\mathbb{W}}^\#_t(z)$ *is conditionally Gaussian with* $E\left[\widetilde{\mathbb{W}}^\#_t(z)\middle|S\right]=0$ *and the conditional covariance*

$$E\left[\widetilde{\mathbb{W}}^\#_{t_1}(z_i)\widetilde{\mathbb{W}}^\#_{t_2}(z_j)\middle|S\right]=E\left[\xi_1^2\right]\int_0^{\min(t_1,t_2)}\mathbb{I}_{\{S(r)\le z_i\}}h_{(z_i,z_j)}(S(r))\,dr,\ i<j$$

with $h_{(z_i,z_j)}(u)$ *as in Theorem 4.2.*

In addition,

$$z\longmapsto\sqrt{\kappa(s)}\left(a_n^{-1}A^*_\infty(sz)-\mu_\xi\kappa(s)\int_0^zV_\infty(su)\,du\right)\overset{fdd}{\Longrightarrow}\widetilde{\mathbb{W}}^\#_\infty(z).$$

5 THE CASE III

Here $0<\beta<1$ and $\mu_X=E[X_1]<\infty$. The results as well as their proofs of this case may be viewed in close analogy with those of the Case II, to the extent that it would be sufficient to illustrate the statement of one result.

More specifically, we now take

$$a_n = n, \qquad \kappa(a_n) = \frac{1}{n\overline{G}(n)},$$

and let all the processes $V_{[nt]}(nz)$, $M^{\#}_{[nt]}(nz)$, $W_{[nt]}(nz)$, $A^{\#}_{[nt]}(nz)$, $W^{\#}_{[nt]}(nz)$ be as defined in Section 4, that is, the only difference being that a_n is replaced by n.

Then all the statements of the results of Section 4 (with a_n and $\kappa(a_n)$ taken to be as above) remain true for the present case. The limiting forms occurring in these limit theorems will also remain the same except that the process $S(r)$ will need to be replaced by the constant $r\mu_X$, in view of the fact that

$$\sup_{0\le r\le M}\left|n^{-1}S(r) - r\mu_X\right| \xrightarrow{p} 0 \quad \text{for all } M>0,$$

which follows because $n^{-1}S(r)$ is increasing in r and $n^{-1}S(r) \xrightarrow{p} r\mu_X$ for each $r>0$.

As an illustration, we now state the result in analogy with Theorem 4.1 (and similar restatements of the remaining results of Section 4 are omitted.): *For each $z>0$ and $t>0$,*

$$M^{\#}_{[nt]}(nz) \xrightarrow{p} \mathbb{M}^{\#}_t(z) = \int_0^t \mathbb{I}_{\{r\mu_X\le z\}}(z - r\mu_X)^{-\beta}\,dr,$$

and in fact, for each $z>0$ and $M>0$,

$$\sup_{0\le t\le M}\left|M^{\#}_{[nt]}(nz) - \mathbb{M}^{\#}_t(z)\right| \xrightarrow{p} 0.$$

In addition

$$M^{\#}_{\infty}(sz) \xrightarrow{p} \mathbb{M}^{\#}_{\infty}(z) \qquad \text{as } s\to\infty. \tag{52}$$

Further, for any $z_{n1} < ... < z_{nl}$ such that $(z_{n1},...,z_{nl}) \to (z_1,...,z_l)$ with $z_1>0$,

$$\left(W_{[nt]}(nz_{nq}),\, q=1,...,l\right) \Longrightarrow \left(\mathbb{W}_t(z_q),\, q=1,...,l\right) \text{ in } D_{\mathbb{R}^l}[0,\infty),$$

where the process $(t,z)\longmapsto \mathbb{W}_t(z)$ is Gaussian with

$$E[\mathbb{W}_t(z)] = 0, \quad E[\mathbb{W}_{t_1}(z_i)\mathbb{W}_{t_2}(z_j)] = \int_0^{\min(t_1,t_2)} \mathbb{I}_{\{r\mu_X\le z_i\}}(z_j - r\mu_X)^{-\beta}\,dr, \quad z_i<z_j.$$

In addition,

$$z\longmapsto W_{\infty}(sz) \overset{fdd}{\Longrightarrow} \mathbb{W}_{\infty}(z) \qquad \text{as } s\to\infty. \tag{53}$$

where

$$E\left[\mathbb{W}_\infty\left(z\right)\right]=0,\quad E\left[\mathbb{W}_\infty\left(z_i\right)\mathbb{W}_\infty\left(z_j\right)\right]=\int_0^{\mu_X^{-1}z_i}\mathbb{I}_{\{r\mu_X\le z_i\}}\left(z_j-r\mu_X\right)^{-\beta}dr,\quad z_i<z_j.$$

(As noted earlier, the statements (52) and (53) are already contained in Mikosch and Resnick (2004, Propositions 3.1 and 3.2).) The arguments of the proofs are also almost identical except that the second part of Lemma 2.3 will need to be invoked in those places where the first part was invoked in Case II.

Note that

$$\begin{aligned}\mathbb{M}_t^{\#}\left(z\right) &= \mu_X^{-1}\int_0^{\min(t\mu_X,z)}\left(z-r\right)^{-\beta}dr \\ &= \mu_X^{-1}\left(1-\beta\right)^{-1}\left(z^{1-\beta}-\left(z-t\mu_X\wedge z\right)^{1-\beta}\right),\end{aligned} \tag{54}$$

$$\begin{aligned}E\left[\mathbb{W}_{t_1}\left(z_1\right)\mathbb{W}_{t_2}\left(z_2\right)\right] &= \int_0^{t_1}\mathbb{I}_{\{r\mu_X\le z_1\}}\left(z_2-r\mu_X\right)^{-\beta}dr \quad \text{for } t_1\le t_2,\ z_1\le z_2. \\ &= \mu_X^{-1}\left(1-\beta\right)^{-1}\left\{z_2^{1-\beta}-\left(z_2-t_1\mu_X\wedge z_1\right)^{1-\beta}\right\}\end{aligned}$$

and

$$E\left[\mathbb{W}_\infty\left(z_1\right)\mathbb{W}_\infty\left(z_2\right)\right]=\mu_X^{-1}\left(1-\beta\right)^{-1}\left\{z_2^{1-\beta}-\left(z_2-z_1\right)^{1-\beta}\right\},\quad z_1\le z_2.$$

6 REFERENCES

1. Aldous, D. (1978). Stopping times and tightness. *Annals of Probab.*, **6**, 335 - 340.

2. Billingsley, P. (1968). *Convergence of probability measures.* Wiley, New York.

3. Borodin, A.N. and Ibragimov, I.A. (1995). *Limit theorems for functionals of random walks. Proceedings of the Steklov Institute of Mathematics.* **195**(2).

4. Feller, W. (1971). *An introduction to probability theory and its applications. Vol. II. Second Edition.* Wiley, New York.

5. Gikhman, I.I and Skorokhod, A.V. (1969). *Introduction to the theory of random processes.* W.B.Saunders Company, Philadelphia.

6. Jacod, J. and Shiryayev, A.N. (1987)). *Limit theorems for stochastic processes.* Springer-Verlag, Berlin.

7. Jeganathan, P. (2006). Convergence in distribution of row sum processes to mixtures of additive processes. Available at www.isibang.ac.in/~statmath/eprints/

8. Maulik, K., Resnick, S. and Rootzén, H. (2002). Asymptotic independence and a network traffic model. *Journal of Applied Probability*, **39**, 671-699. (Also available at www.orie.cornell.edu/~sid)

9. Mikosch, T. and Resnick, S. (2004). Activity rates with very heavy tails. Available at www.orie.cornell.edu/~sid

10. Resnick, S. (2003). Modeling data networks. In B.Finkenstadt and H.Rootzén, Editors, *SemStat: Seminaire European de Statistique, Extreme Values in Finance, Telecommunications, and the Environment*, pages 287 - 372. Chapman-Hall, London. (Also available at www.orie.cornell.edu/~sid)

11. Sato, K. (1999). *Lévy processes and infinitely divisible distributions.* Cambridge University Press.