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An introduction to the geometry of spaces of operators

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ABSTRACT. This is a write-up of the two lectures I gave at the Workshop on 'Operator theory and operator algebras III' 12-12-05. Most of the material is standard and can be found in the monographs [4], [11] and [8]. Thus most often proofs are not given. The list of references is long, biased and not exhaustive.

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1. LECTURE I

Let X, Y be complex Banach spaces. Let $\mathcal{L}(X,Y)$, $\mathcal{K}(X,Y)$, $\mathcal{W}(X,Y)$, denote the space of bounded linear operators, space of compact operators and the space of weakly compact operators respectively. One has the well-known inclusion $\mathcal{K}(X,Y) \subset \mathcal{W}(X,Y) \subset \mathcal{L}(X,Y)$.

For any $x^* \in X^*$, $y \in Y$, the rank one operator $x^* \otimes y$ defined by $(x^* \otimes y)(x) = x^*(x)y$ is in all the above spaces and $\|x^* \otimes y\| = \|x^*\|\|y\|$. By fixing a x_0^* such that $x_0^*(x_0) = 1 = \|x_0^*\| = \|x_0\|$, it is easy to see that $y \to x_0^* \otimes y$ is an isometric embedding of Y in any of the above spaces and the map $T \to x_0^* \otimes T(x_0)$ is a projection of norm one on the image of Y. Similarly X^* can also be isometrically embedded into the above spaces such that the image is the range of a projection of norm one. Thus a very important aspect of the geometry (isometric theory) of spaces of operators relates to the study of the interplay between the properties of X, Y and the spaces of operators.

Let $\mathcal{F}$ be the closed subspace spanned by $\{x^* \otimes y : x^* \in X^*, y \in Y\}$ (closure of finite rank operators). In general $\mathcal{F}$ can be a proper subspace of $\mathcal{K}(X,Y)$.

For a Banach space X we denote by X_1 the closed unit ball and by $\partial_e X_1$ the set of extreme points (possibly empty).

Definition 1. *X is said to have the metric approximation property (M. A. P) if there is a net $\{P_\alpha\}$ of finite rank operators with $\|P_\alpha\| \leq 1$ such that $P_\alpha \to I$ in the strong operator topology.*

Several classical Banach spaces like the L^p spaces and the space $C(K)$ have this property.

If X^* or Y has the M. A. P, it can be shown that $\mathcal{F} = \mathcal{K}(X,Y)$.

Here is another description of $\mathcal{F}$ in terms of injective tensor product spaces.

We recall that for Banach spaces X, Y, the injective tensor product space $X \otimes_\epsilon Y$ is the completion of the vector space $X \otimes Y$ under the norm $\| \sum_1^n x_i \otimes y_i \| = sup\{ | \sum_1^n x_i^*(x_i) y_i^*(y_i) | : x_i^* \in X_1^*, y_i^* \in Y_1^* \}$.

Thus when X^* or Y has the M. A. P, $\mathcal{K}(X,Y) = X^* \otimes_\epsilon Y$.

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It is some times possible to identify $\mathcal{K}(X,Y)$ with an injective tensor product space, without invoking (at least explicitly) the M. A. P as the following example illustrates.

Example 2. *Let K be a compact Hausdorff space. Let $C(K,X)$ denote the space of X-valued continuous functions on K, equipped with the supremum norm. It is well known the the map $\sum_1^n f_i \otimes x_i \to \sum_1^n f_i x_i$ is an isometry of $C(K) \otimes_\epsilon X$ onto $C(K,X)$.*

Now let $T \in \mathcal{L}(X, C(K))$, let $\delta : K \to C(K)_1^*$ be the canonical embedding. This map is a homeomorphism when $C(K)_1^*$ is equipped with the weak-topology. Consider $T^* \circ \delta : K \to X^*$. This clearly is a continuous map when X^* is equipped with the weak*-topology. Let us denote by $W^*C(K,X^*)$ to be the space of such continuous functions, equipped with the supremum norm. It is easy to see that $T \to T^* \circ \delta$ is an into isometry from $\mathcal{L}(X,C(K))$ to $W^*C(K,X^*)$. Also for $f \in W^*C(K,X^*)$,$T : X \to C(K)$ defined by $T(x)(k) = f(k)(x)$ is a bounded linear map. Therefore $\mathcal{L}(X,C(K))$ is isometric to $W^*C(K,X^*)$.*

It is quite easy to see that under this embedding $\mathcal{K}(X,C(K))$ corresponds to $C(K,X^*)$ and $\mathcal{W}(X,C(K))$ corresponds to $WC(K,X^*)$, functions that are continuous when X^* is equipped with the weak-topology.

We now turn to the study of the duals of spaces of operators.

Let $\Lambda \in X^{**}$, $y^* \in Y^*$. Consider the functional $\Lambda \otimes y^*$ defined by $(\Lambda \otimes y^*)(T) = \Lambda(T^*(y^*))$. It is easy to see that $\|\Lambda \otimes y^*\| = \|\Lambda\|\|y^*\|$. Notice that these functionals are defined on any of the above spaces of operators and have the same norm in any of the dual spaces!. It is easy to see that $\|T\| = \|T^*\| = sup\{(\Lambda \otimes y^*)(T) : \Lambda \in X_1^{**}, y^* \in Y_1^*\}$. In fact by an application of the Krein-Milman theorem one can take $\Lambda \in \partial_e X_1^{**}$, $y^* \in \partial_e Y_1^*$. Thus a simple application of the separation theorem shows that the dual unit ball of any of the spaces of operators above is the weak*-closed convex hull of $\{\Lambda \otimes y^* : \Lambda \in \partial_e X_1^{**}, y^* \in \partial_e Y_1^*\}$. Let us also note that these results are also valid if one considers the set $\{x \otimes y^* : x \in X_1, y^* \in Y_1^*\}$.

Now consider for a dual space Y^*, $\mathcal{L}(X,Y^*)$. Let V be the closed subspace spanned by $\{x \otimes y : x \in X_1, y \in Y_1\} \subset \mathcal{L}(X,Y^*)_1^*$. Our observations so far show that V is a minimal subspace of $\mathcal{L}(X,Y^*)^*$. Thus by the Dixmier-Goldberg-Ruston theorem ([9]) theorem the canonical embedding of $\mathcal{L}(X,Y^*)$ into V^* is an onto isometry.

Another identification of the space V is given by considering projective tensor product spaces.

For Banach spaces X, Y, the projective tensor product space $X \otimes_\pi Y$ is the completion of $X \otimes Y$ under the norm $\|u\| = inf\{\sum_1^n \|x_i\|\|y_i\| : u = \sum_1^n x_i \otimes y_i\}$, for $u \in X \otimes Y$. With this identification we have that $\mathcal{L}(X,Y^*) = (X \otimes_\pi Y)^*$. This gives raise to the following open problem.

Question 3. *Let Z be a Banach space isometric to $\mathcal{L}(X,Y)$. Then is Y isometric to a dual space? In other words if $\mathcal{L}(X,Y)$ is a dual space then is the projective tensor product space one of the preduals?*

When $Y = C(K)$ it was shown in [13] that if $\partial_e X_1 \neq \emptyset$ then $\mathcal{L}(X,C(K))$ is a dual space implies that $C(K)$ is a dual space which by a well-known theorem of Dixmier and Grothendieck (see [10] Section 11) is equivalent to K being hyperstonean. Similar conclusion can also be drawn in the case of $\mathcal{L}(L^1([0,1],C(K))$ even though the domain space in this case lacks extreme points in the unit ball.

We next give the identification of the second dual space of the space of compact operators as a space of operators.

Suppose X or Y^* has the M. A. P. Suppose X^{**} or Y^* has the Radon-Nikodym property (R. N. P). Then $\mathcal{K}(X,Y)^{**} = (X^* \otimes_\epsilon Y)^{**} = (X^{**} \otimes_\pi Y^*)^* = \mathcal{L}(X^{**}, Y^{**})$.

In particular as $C(K)^*$ has the M. A. P, identifying $C(K)^*$ as the space of regular Borel measures $M(K)$, when X^{**} has the R.N. P, $C(K,X^*)^{**} = \mathcal{K}(X,C(K))^{**} = (M(K) \otimes_\pi X^{**})^* = \mathcal{L}(X^{**}, C(K)^{**})$. Now by the Gelfand-Naimark theorem we can identify $C(K)^{**}$ as $C(K')$ for a compact hyperstonean space K'.

Thus if X is a reflexive Banach space, $\mathcal{K}(X,C(K))^{**} = \mathcal{L}(X,C(K'))$.

Question 4. *Given an onto isometry Φ of $\mathcal{K}(X,C(K))$ when does it extend to an onto isometry of $\mathcal{L}(X,C(K))$? For what compact sets K and Banach spaces X, do all isometries $\mathcal{K}(X,C(K))$ extend to isometries of $\mathcal{L}(X,C(K))$?*

Some answers, mostly based on vector-valued versions of Banach-Stone theorems ([1]) can be found in [2].

2. Lecture II

We next recall the Feder-Saphar dichotomy ([7]) for subspaces of $\mathcal{L}(X,Y)$ for reflexive Banach spaces.

Theorem 5. *Let X, Y be reflexive Banach spaces. Let G be a closed subspace such that $X^* \otimes Y \subset G \subset \mathcal{L}(X,Y)$. Then either G is reflexive or it is not a conjugate space.*

We already noted that when X or Y has the M. A. P, since the spaces are reflexive, $\mathcal{L}(X,Y)$ is the bidual of $\mathcal{K}(X,Y)$. Thus if $\mathcal{L}(X,Y)$ is reflexive then $\mathcal{L}(X,Y) = \mathcal{K}(X,Y)$.

A well-known result of Pitt, asserts that this is the case for $X = \ell^p$, $Y = \ell^q$ for $1 < q < p < \infty$. See [15] for a recent proof.

Thus when X or Y is a classical Banach space, if $\mathcal{K}(X,Y) \neq \mathcal{L}(X,Y)$ can $\mathcal{K}(X,Y)$ be isometric to a conjugate space ? (we will not consider the more difficult problem of, can $\mathcal{K}(X,Y)$ be isomorphic to a dual space?, here, see for example [3]).

It is a consequence of the Josefson-Nissenzweig Theorem ([5] Chapter XII) that when K is an infinite compact set and X an infinite dimensional space, there is a non-compact operator from $X \to C(K)$.

Theorem 6. *Let K be an infinite compact set and X an infinite dimensional Banach space. $\mathcal{K}(X,C(K))$ is not isometric to a dual space.*

Proof. We will show the more general statement that $C(K,X)$ is not isometric to a dual space. First we do a general construction to embed c_0 as a subspace of $C(K,X)$ and then define a projection of norm one from $C(K,X)$ onto the range of this embedding.

Using the Josefson-Nissenzweig theorem and the Bishop-Phelps theorem ([9]), since X is infinite dimensional, there exists sequences $\{x_n\}_{n\geq 1} \subset X_1$, $\{x_n^*\}_{n\geq 1} \subset X_1^*$ such that $x_n^*(x_n) = 1$ and $x_n^* \to 0$ in the weak*-topology.

Since K is an infinite set, choose a sequence $\{t_n\}_{n\geq 1} \subset K$ of distinct points and a sequence $\{U_n\}_{n\geq 1}$ of pairwise disjoint open sets such that $t_n \in U_n$. Now by Urysohn' Lemma, we can get a sequence $\{f_n\}_{n\geq 1} \subset C(K)_1$, $0 \leq f_n \leq 1$, $f_n(t_n) = 1$ and $f_n = U^c$ for all n. Define $\Phi : c_0 \to C(K, X)$ by $\Phi(\alpha) = \sum_1^\infty \alpha_i \ f_i \otimes x_i$. By our construction it is easy to see that $\| \sum_k^j \alpha_i \ f_i \otimes x_i\| = max|\alpha_i|$. Thus it follows that Φ is well-defined, linear isometry.

Now define $P : C(K, X) \to \Phi(c_0)$ by $P(f) = \Phi(\{x_n^*(f(t_n))\}_{n\geq 1})$. Since $f(K)$ is a norm compact set , P is well-defined, linear map. Clearly $\|P\| \leq 1$. To show that P is a projection, it is enough to show that $P(f) = P(P(f))$ on $\bigcup_1^\infty U_n$. Now if $k \in U_{n_0}$, then $P(f)(k) = \Phi(\{x_n^*(f(t_n))\}_{n\geq 1})(k) = f_{n_0}(k) = P(P(f))(k)$.

Finally suppose $C(K, X)$ is isometric to a dual space. Consider the sequence of closed balls, $\{B(\Phi(e_n), \frac{1}{2})\}_{n\geq 1}$, where e_n' are the canonical vectors in c_0 . It is easy to see that any finitely many of these balls have a point in common. Since $C(K, X)$ is a dual space, by a weak*-compactness argument, we have, $\bigcap_1^\infty B(\Phi(e_n), \frac{1}{2}) \neq \emptyset$. Let $f \in C(K, X)$ be such that $\|f - \Phi(e_n)\| \leq \frac{1}{2}$ for all n. Let $\alpha \in c_0$ be such that $P(f) = \Phi(\alpha)$. Now for any k, $|\alpha_k - 1| \leq \|\Phi(\alpha) - \Phi(e_k)\| = \|P(f) - P(\Phi(e_k))\| = \|P(f - \Phi(e_k))\| \leq \|f - \Phi(e_k)\| \leq \frac{1}{2}$. Which contradicts, $\alpha_n \to 0$.

Therefore $C(K, X)$ is not a dual space.

□

The corresponding problem, when is $WC(K, X)$ ($\mathcal{W}(X, C(K))$) the space of functions that are continuous when X has the weak topology was settled in [14], [6].

Theorem 7. *$WC(K, X)$ is isometric to a dual space if and only if X is reflexive and K is a hyperstonean space.*

Proof. It was proved in [14] that if $WC(K, X)$ is a dual space then X is reflexive. Thus $\mathcal{L}(X^*, C(K)) = \mathcal{W}(X^*, C(K)) = WC(K, X)$.

Since X^* has an extreme point in the unit ball, it follows from [13] that K is a hyperstonean space.

We have already seen that when K is a hyperstonean space, since $C(K)$ is a dual space, $\mathcal{L}(X, C(K))$ is a dual space. □

These remarks give raise to the following open problem.

Question 8. *Let G be a closed subspace such that $C(K, X) \subset G \subset WC(K, X)$. Suppose G contains a f_0 that is not norm continuous. Also suppose for any $f \in C(K)$, $g \in G$, $fg \in G$ (G is a $C(K)$-module) and $T \circ g \in G$ for all $T \in \mathcal{L}(X)$. Either G is not a dual space or $G = WC(K, X)$?*

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