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An Insurance Network: Brownian Perturbed Continuous Case

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AN INSURANCE NETWORK : BROWNIAN PERTURBED CONTINUOUS CASE

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Abstract: We consider an insurance network whose dynamics is governed by the solution to the Skorokhod problem in an orthant, with time and space dependent drift and reflection terms and the underlying driving process being a d-dimensional Brownian motion. (In [Ra5] it has been shown that such a network may be viewed upon as an optimal risk diversification model). We show that two or more members of the network can not be zero at the same time; in particular, ruin of the network is not possible. Viewing transience as a net profitability condition, and recurrence as a tendency to bankruptcy, sufficient criteria for transience / recurrence of the network are given.

Key words: Brownian motion, Skorokhod problem, Lyapunov function, transience, recurrence, drift, diffusion, reflection, generator, spectral radius, strong Markov property, stopping time.

1 Introduction

In [Ra5] an insurance network model has been introduced involving d companies operating under a treaty to diversify risk. Accordingly if one company needs a certain amount to prevent ruin, other companies of the network give previously-agreed-upon fractions of the amount; any shortfall has to be procured by the concerned company from external sources. The amount obtained from internal sources carry easier repayment terms. The risk / surplus process of each company (in the absence of any control) is assumed to be an r.c.l.l. function with time and space dependent drift (like sample path of a perturbed Levy process). With each company trying to minimize its repayment obligation, the situation can be viewed upon as a deterministic d-person dynamic game with state space constraints. It was shown in [Ra5] that under suitable conditions a (unique) Nash equilibrium is provided by the solution to the Skorokhod problem in an orthant. The framework considered in [Ra5] is a completely deterministic set up. As mentioned in [Ra5] this network can be considered as a reinsurance model. This provides a good justification for considering insurance networks whose dynamics are governed by solutions to appropriate Skorokhod problems.

Skorokhod problem in an orthant has been studied extensively ever since Harrison and Reiman [HR] proposed reflected/ regulated Brownian motion in an orthant as the appropriate heavy traffic limit for certain queueing networks; see [Re], [MP], [Ra3] and the references therein.

Some natural questions arise in the above context.

1. As the network has been formulated to diversify risk, what can be said about possible ruin of the network itself?

2. Can it happen that at the same time two or more companies make demands on other members of the network to avert ruin?

3. What could be a natural analogue of the net profitability condition for the network?

In the present article we address these questions in a probabilistic set up, viz. when the r.c.l.l. functions (in the absence of controls) are continuous and in fact arise as sample paths of a perturbed d-dimensional Brownian motion; for example, diffusion approximations of ceratin classical risk models [Ta]. We assume that the dynamics of the (controlled) network is governed by the solution to the Skorokhod problem in an orthant. The sample paths of the network are continuous and can be obtained by solving the Skorokhod problem in an orthant path by path.

A moment's reflection indicates that the first two questions are related. In fact a negative answer to the second question immediately implies that ruin of the network is not possible. Our analysis, though, proceeds by first establishing non-ruin of the network and then using it to tackle the second question.

Following one-dimensional models of insurance, we take transience as a criterion for net profitability. As the risk processes are confined to the positive orthant, note that transience means essentially wandering off to infinity in the positive direction. Some sufficient conditions for transience of the network are provided. This immediately suggests that recurrence of the network would correspond to coming close to bankruptcy, though ruin may be avoided; some criteria are provided.

Our approach involves the choice of an appropriate Lyapunov function, a common thread connecting the three problems. This is the reason for considering only the Brownian perturbed continuous case. We also use a backward induction procedure as in [RW]; moreover our result gives a generalization of the result of [RW] under our hypotheses.

Most of the results are presented in Section 2; their implications to insurance networks are also discussed. Proofs are given in Section 3; our hypotheses are more general than what may be needed in the insurance context.

2 Results and discussion

Let $G = \{x \in \mathbb{R}^d : x_i > 0, 1 \leq i \leq d\}$ denote the d- dimensional positive orthant, where $d \geq 2$. Let $\{B(t) = (B_1(t), \cdots, B_d(t)) : t \geq 0\}$ denote an $\mathcal{F}_t$-adapted standard d-dimensional Brownian motion. We assume the following.

(A1) $b : [0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ is a bounded measurable function; $(y, z) \mapsto b(t, y, z)$ is Lipschitz continuous uniformly over t; we write
$b(t, y, z) = (b_1(t, y, z), \cdots, b_d(t, y, z))$; denote $\underline{\beta_i} = \inf\{b_i(t, y, z) : t, y, z\}, \overline{\beta_i} = \sup\{b_i(t, y, z) : t, y, z\}$, $\underline{\beta} = (\underline{\beta_1}, \cdots, \underline{\beta_d}), \overline{\beta} = (\overline{\beta_1}, \cdots, \overline{\beta_d})$.

(A2) $R : [0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbf{M}_d(\mathbb{R}^d)$ is a bounded measurable function; we write $R(t, y, z) = ((R_{ij}(t, y, z)))_{1 \leq i,j \leq d}$; $(y, z) \to R_{ij}(t, y, z)$ is Lipschitz continuous uniformly over t; $R_{ii}(\cdots) \equiv 1$ for all i.

(A3) For $i \neq j$ there exists constant W_{ij} such that $|R_{ij}(t, y, z)| \leq W_{ij}$; set $W = ((W_{ij}))$ with $W_{ii} = 0$ for all i; we assume that the spectral radius of W is strictly less than one.

(A4) $\sigma = ((\sigma_{ij}))$ is a $(d \times d)$ symmetric strictly positive definite matrix with real entries.

Let $\{Y(t) = (Y_1(t), \cdots, Y_d(t)) : t \geq 0\}, \{Z(t) = (Z_1(t), \cdots, Z_d(t)) : t \geq 0\}$ be continuous $\mathcal{F}_t$-adapted processes obtained by solving the Skorokhod problem $SP(w, b, R; G)$ path by path as w varies over $w(\cdot) = z + \sigma B(\cdot, \omega)$; note that $\sigma B(\cdot)$ is a d-dimensional Brownian motion with zero drift and covariance matrix $\sigma\sigma^*$. So

$$Z(t) = z + \sigma B(t) + \int_0^t b(r, Y(r), Z(r))dr + \int_0^t R(r, Y(r), Z(r))dY(r), \tag{1}$$

that is, for $i = 1, \cdots, d$

$$Z_i(t) = z_i + \sum_{j=1}^d \sigma_{ij} B_j(t) + \int_0^t b_i(r, Y(r), Z(r))dr + Y_i(t) - Y_i(0) +$$
$$\sum_{j \neq i} \int_0^t R_{ij}(r, Y(r), Z(r))dY_j(r); \tag{2}$$

$Z(t) \in \overline{G}$ for all $t \geq 0$; $Y_i(\cdot)$ is nondecreasing, $Y_i(0) = 0$, and $Y_i(\cdot)$ increases only when $Z_i(\cdot) = 0$, that is

$$Y_i(t) - Y_i(0) = \int_0^t 1_{\{0\}}(Z_i(s))dY_i(s) \tag{3}$$

for $t \geq 0, i = 1, 2, \cdots, d$; here $z \in \overline{G}$. Under (A1)–(A3) it is known that the deterministic Skorokhod problem is well posed; see [Ra3]. Moreover under certain (additional) natural conditions it can be shown that the Y-part of the solution to the Skorokhod problem is a (unique) Nash equilibrium for an appropriate d-person dynamic game with state space constraints; see [Ra4],[Ra5]; so $Z(\cdot)$ is the optimally controlled process.

In addition to (A1)–(A4) assume

(A5) $R_{ji}(\cdots) \leq 0, \ j \neq i, \ \sum\limits_{j \neq i} |R_{ji}| \leq 1, \ i = 1, \cdots, d.$

Then the following interpretation can be given in terms of an insurance network.

Consider d insurance companies whose surplus processes in the absence of any control are perturbed Brownian motions and are continuous. Suppose they have a treaty to diversify risk; accordingly, if

Company i needs at some instance of time an amount dy_i to prevent ruin, then, for $j \neq i$, Company j pitches in $R_{ji}(\cdots)dy_i$; of course, the shortfall $(1 - \sum_{j \neq i} R_{ji}(\cdots))dy_i$ has to be obtained by Company i from 'external' sources. The amount got from internal sources, viz. other members of the network, carry easier repayment terms due to mutual obligations. With each company trying to minimize its repayment liability, the situation can be viewed upon as a d-person dynamic game with state space constraints. In [Ra5] it is proved that under suitable conditions a unique Nash equilibrium is provided by the solution to the Skorokhod problem; so the dynamics of the optimal surplus processes are governed by (2.1)–(2.3) in such a case; (in fact, even the case when $B_i(\cdot)$ are Levy processes has been considered in [Ra5]). Therefore, $Z_i(t)$ = the current surplus of Company i at time t; and, $Y_i(t) - Y_i(0)$ = the cumulative amount obtained by Company i (from internal and external sources) upto time t to prevent ruin. Of course, by (2.3), Company i can seek help to prevent ruin only when its surplus is zero, and the quantum sought should be just sufficient to keep it afloat.

In this paper we shall assume that $Y(\cdot), Z(\cdot)$ are given by (2.1)–(2.3); note that $B(\cdot), Y(\cdot), Z(\cdot)$ are continuous $\mathcal{F}_t$-adapted processes. The conditions assumed , viz. (A1)–(A4), are more general than needed to guarantee that a Nash equilibrium is provided by the Skorokhod problem.

Since the network has been motivated by the desirability to diversify risk, a first question concerns possible ruin of the network itself. Indeed we have

Theorem 1 *Assume (A1)–(A4). Then for any $z \in \overline{G}, z \neq \underline{0}$, (where $\underline{0} = (0, \dots, 0)$),*

$$P(Z(t) = \underline{0} \text{ for some } t \geq 0 \mid Z(0) = z) = 0. \tag{4}$$

If $z = \underline{0}$ then

$$P(Z(t) = \underline{0} \text{ for some } t > 0 \mid Z(0) = z) = 0. \tag{5}$$

In other words, with probability one the network does not get ruined. □

Remarks (i) From the point of view of the insured (the end users of the business) the preceding result is comforting; the network's promise of completely honouring genuine claims will have greater credibility.

(ii) Assumption (A4) is crucial in the above. If σ is not strictly positive definite (so that at least one eigenvalue is zero), the result is not true; the easiest way to see this is to consider the one dimensional reflecting Brownian motion in $[0, \infty)$ as a two or higher dimensional process in non-negative quadrant/ orthant.

(iii) Note that σ is indicative of second order and higher order variations in claim sizes (coming possibly from diffusion type approximation to a compound Poisson process as in [Ta]), as well as volatility factor of stocks in which the companies may invest part of their surplus. In order to ensure that (A4) holds the companies may agree on the following as part of their treaty: (a) each company will have at least one line of insurance business or one geographical region of business which will be independent of other companies' business activities; (b) each company may invest in at least one type of stock that will be independent of other companies' investments. □

In the context of an insurance network, when one company needs an amount to prevent ruin, other companies help out; also as the dynamics is governed by the Skorokhod problem, by (2.3), a company can seek help from other companies only if its surplus is zero. So, what happens if more than one company need help at the same time? In other words, is it possible for more than one company to have zero surplus at the same time? If $d = 2$ the answer is in the negative by the preceding theorem. If $d > 2$, as there is provision to get reinforcements from 'external' sources, thanks to the spectral radius condition (A3), it does not lead to any inconsistency; nevertheless it is an awkward situation to be avoided. Our next result assures that such a contingency does not arise.

Theorem 2 *Assume (A1)–(A4). Let* $J \subseteq \{1, 2, \ldots, d\}$ *with* $|J| \geq 2$. *Then*

$$P(Z_i(t) = 0 \quad \forall i \in J \quad \textit{for some} \quad t \geq 0 \quad | \ Z(0) = z) = 0 \tag{6}$$

for any $z \in \overline{G}$ *with* $z_j > 0$ *for at least one* $j \in J$; *and*

$$P(Z_i(t) = 0 \quad \forall i \in J \quad \textit{for some} \quad t > 0 \quad | \ Z(0) = z) = 0 \tag{7}$$

for any $z \in \overline{G}$. ☐

Under the hypotheses of Theorem 2.2 an immediate corollary is

$$\int_0^\infty \left[\prod_{i \in J} 1_{\{0\}}(Z_i(s)) \right] dY_k(s) = 0, \forall k \in J \quad \text{a.s.} \tag{8}$$

for any starting point $z \in \overline{G}$; (for $k \notin J$ also (2.8) holds because of (2.3) and the above). When b, R are constants, with R being a completely-$\mathcal{S}$ matrix, (2.8) has been proved by Reiman and Williams [RW] in the context of heavy traffic approximation to queueing networks. So when (A1)–(A4) hold, Theorem 2.2 is a generalization of the result of Reiman and Williams. ☐

We next look at the question of transience. In the case of one dimensional insurance models like the compound Poisson process and renewal models, the net profit condition is just the condition for transience in the positive direction of the underlying random walk; see Rolski et al. [RSST], or Mikosch [Mi]. So transience of the network in the orthant may be taken as a good indication of profitability of the network. Note that under (A1)–(A3) the pair $\{(Y(t), Z(t)) : t \geq 0\}$ is a strong Markov process; see [Ra3]; if b, R do not depend on the y- variables then $Z(\cdot)$ is strong Markov. Though $Z(\cdot)$ need not be a Markov process in general, it makes sense to talk of transience of $Z(\cdot)$; in fact, transience of projections of diffusions has been considered in [Ra1]; see also [Ra3]. We say that the process $\{Z(t) : t \geq 0\}$ is *transient* if

$$P(\lim_{t \to \infty} |Z(t)| \quad = \infty \quad | \quad Z(0) = z) = 1 \tag{9}$$

for any $z \in \overline{G}$.

Theorem 3 *Assume (A1)–(A4). In addition assume that* R_{ij}, b_i *are constants, viz.*

$$R_{ij}(\cdots) \equiv -W_{ij}, W_{ij} \geq 0, j \neq i, \quad b_i(\cdots) \equiv \beta_i, \quad 1 \leq i, j \leq d \tag{10}$$

where $\beta_i = \underline{\beta_i} = \overline{\beta_i}, ((W_{ij}))$ *are as in (A1),(A3); write* $\beta = (\beta_1, \ldots, \beta_d), R = I - W$. *If*

$$(R^{-1}\beta)_i > 0 \quad \textit{for some} \quad i = 1, 2, \ldots, d \tag{11}$$

then the process $\{Z(t) : t \geq 0\}$ *is transient.* ☐

By the spectral radius condition (A3), note that

$$R^{-1} = (I - W)^{-1} = I + W + W^2 + W^3 + \cdots \tag{12}$$

is a matrix of nonnegative entries. So if $\beta_i \geq 0$ for all i with strict inequality holding for at least one i then (2.11) is satisfied. Even if some, but not all, of the β_j are negative but (2.11) holds for one

i we get transience. This leads to the following interpretation. Suppose one insurance company in the network specializes in one geographical area or in one line of business. In case of a cataclysmic event there could be massive hardship for the insured and consequently premium income may be zero for the company for some time; it may even be negative if overheads connected with premium collection are taken into account. Thanks to the backing from the healthy network, the company, instead of folding up under the strain, will be able to meet all its obligations. This enhances its prestige and that of the network leading to future expansion of business. Even in a non-insurance context the above scenario is meaningful. If the members of the network represent different wings of the same company, or different sectors of an interdependent economy as in [Ra3] similar interpretations can be given even more readily; in the latter case there can also be nonsurplus welfare sectors like health, education, pollution control, etc. which are beneficial to the economy in the long run.

Recurrence is a notion that is generally considered along with transience; in fact, there is a dichotomy between recurrence and transience for diffusion processes; see [Bh], [BR1]. We say that the process $\{Z(t) : t \geq 0\}$ is *recurrent* if $Z(\cdot)$ visits every nonempty open set infinitely often. (Note that $Z(\cdot)$ need not be Markov.) Thus if $Z(\cdot)$ is recurrent, for any $\epsilon > 0$, the event $|Z(\cdot)| < \epsilon$ can happen infinitely often; that is, the network, even though may not get ruined, can come arbitrarily close to bankruptcy infinitely often. Clearly this is a situation to be avoided. As we shall see in Section 3, under (A1)–(A4) and (2.10), recurrence of the network follows if $\beta_i < 0$ for all $1 \leq i \leq d$. Intuitively this is to be expected, as the network is very 'inward looking'; see also Section 6 of [Ra3].

3 Proofs

We begin with a remark.

Remark 4 *To prove Theorems 2.1 and 2.2, by Theorem 4.1 of [Ra3], it is enough to consider the case when*

$$R_{ij}(\cdots) \equiv -W_{ij}, W_{ij} \geq 0, j \neq i, \quad b_i(\cdots) \equiv \underline{\beta_i}, \quad 1 \leq i, j \leq d \tag{13}$$

where $W = ((W_{ij})), \underline{\beta} = (\underline{\beta_1}, \ldots, \underline{\beta_d})$ *are as in (A3),(A1).* □

Put $a = ((a_{ij})) = \sigma\sigma^{\dagger}$. Define the second order elliptic operator L by

$$Lg(z) = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij} D_{ij} g(z) + \sum_{i=1}^{d} \underline{\beta_i} D_i g(z) \tag{14}$$

where $D_i = \partial/\partial z_i$, $D_{ij} = D_i D_j = D_j D_i$; note that L is the generator of the diffusion $\sigma B(t) + t\underline{\beta}, t \geq 0$. Under (A1)–(A4),(3.1) note that the equation (2.1) becomes

$$Z(t) = z + \sigma B(t) + t\underline{\beta} + (I - W)Y(t). \tag{15}$$

By (3.3) and Ito's formula note that for any sufficiently smooth function g on $\mathbb{R}^d$, we have

$$\begin{aligned} & g(Z(t)) - \int_0^t Lg(Z(s))ds - \int_0^t \nabla g(Z(s)) \cdot (I - W)dY(s) \\ & = \text{ a martingale w.r.t. } \{\mathcal{F}_t\}. \end{aligned} \tag{16}$$

We need a lemma.

Lemma 5 *Assume (A1)–(A4). Let* $U \subset \overline{G}$ *be a bounded open set; let* $\tau_U = \inf\{t \geq 0 : Z(t) \notin U\}$. *Then*

$$\sup\{E(\tau_U \mid Z(0) = z) : z \in U\} < \infty. \tag{17}$$

Proof. Without loss of generality take $U = B \cap \overline{G}$ where B is a ball around 0. So by the comparison theorem (Theorem 4.1) of [Ra3] we may assume (3.1). By the spectral radius condition (A3) note that there exist positive constants $\theta_1, \ldots, \theta_d, 0 < \alpha < 1$ such that

$$\sum_{j \neq i} \theta_j W_{ji} = \sum_{j \neq i} \theta_j |W_{ji}| \leq \alpha \theta_i, 1 \leq i \leq d. \tag{18}$$

Choosing the constant $k_0 > 0$ suitably one can get a function $h \in C_b^2(\mathbb{R}^d)$ such that $h(z) = \exp(k_0 \sum_{i=1}^{d} \theta_i z_i)$, $Lh(z) \geq 1$, $z \in \overline{U}$. Also $(\nabla h(z) \cdot (I - W))_i \geq 0$ for all $z \in \overline{G}, i = 1, 2, \ldots, d$ by (3.6). Using (3.4) for $g = h$, as h is bounded, it is now easy to get (3.5); (cf. see Lemma 2.8 of [BR2]). ■

For a sufficiently smooth function f on $\mathbb{R}^d$, define $g(z) = f(R^{-1} \cdot z), z \in \mathbb{R}^d$; (2.12) assures that g is well defined and smooth. An elementary differentiation gives

$$Lg(z) = \widetilde{L} f(R^{-1} z) \tag{19}$$

where

$$\widetilde{L} f(x) = \frac{1}{2} \sum_{i,j=1}^{d} \widetilde{a}_{ij} \widetilde{D}_{ij} f(x) + \sum_{i=1}^{d} \widetilde{b}_i \widetilde{D}_i f(x) \tag{20}$$

with $((\widetilde{a}_{ij})) = \widetilde{a} = R^{-1} a (R^{-1})^*$, $(\widetilde{b}_1, \ldots, \widetilde{b}_d) = \widetilde{b} = R^{-1} \underline{\beta}$, $\widetilde{D}_i = \partial/\partial x_i, \widetilde{D}_{ij} = \widetilde{D}_i \widetilde{D}_j = \widetilde{D}_j \widetilde{D}_i$; (cf. Lemma 2.5 of [Ra2]). Consequently (3.4),(3.7),(3.8) give

$$\begin{aligned} & f(R^{-1} \cdot Z(t)) - \int_0^t \widetilde{L} f(R^{-1} \cdot Z(s)) ds - \int_0^t \widetilde{\nabla} f(R^{-1} \cdot Z(s)) \cdot dY(s) \\ & = g(Z(t)) - \int_0^t Lg(Z(s)) ds - \int_0^t \nabla g(Z(s)) \cdot R dY(s) \\ & = \text{a} \quad \text{martingale w.r.t. } \{\mathcal{F}_t\}. \end{aligned} \tag{21}$$

where $\widetilde{\nabla} = (\widetilde{D}_1, \ldots \widetilde{D}_d)$.

For $r > 0$, denote $\tau_r = \inf\{t \geq 0 : |Z(t)| = r\}$, $\widetilde{\tau}_r = \inf\{t \geq 0 : |R^{-1} Z(t)| = r\}$; it is possible that these stopping times take the value $+\infty$. Let $0 < c < n$. As R is invertible, taking $\lambda = ||R^{-1}||$ it can be seen that

$$\{\widetilde{\tau}_{\lambda n} < \widetilde{\tau}_{\lambda c}\} \subseteq \{\tau_n < \tau_c\} \tag{22}$$

Proof of Theorem 2.1: We may assume (3.1) in view if Remark 3.1. We shall prove (2.4) first. Let $z \neq 0$ be fixed. To prove (2.4) it is enough to show that

$$\lim_{c \to 0} P(\tau_n < \tau_c \mid Z(0) = z) = 1$$

for any $n > 0$; this will be accomplished, due to (3.10), if we establish

$$\lim_{c \to 0} P(\widetilde{\tau}_n < \widetilde{\tau}_c \mid Z(0) = z) = 1. \tag{23}$$

We need to choose an appropriate Lyapunov function. Our approach is similar to [Ra2]; see also [Fr],[Bh],[Ra1]. For $0 \neq x \in \mathbb{R}^d$ set

$$\widetilde{A}(x) = \frac{1}{2} \sum_{i,j=1}^{d} \widetilde{a}_{ij} x_i x_j / |x|^2, \widetilde{B}(x) = \sum_{i=1}^{d} \widetilde{a}_{ii}, \widetilde{C}(x) = 2 \sum_{i=1}^{d} x_i \widetilde{b}_i.$$

For $r > 0$ set

$$\widetilde{\gamma}(r) = \inf \left\{ \frac{\widetilde{B}(x) - \widetilde{A}(x) + \widetilde{C}(x)}{\widetilde{A}(x)} : |x| = r \right\}$$

Let $c > 0$; for $r > 0$ put

$$\widetilde{I}(r;c) = \int_c^r \frac{1}{u} \widetilde{\gamma}(u) du, \quad \widetilde{F}(r;c) = \int_c^r \exp(-\widetilde{I}(u;c)) du. \tag{24}$$

(Here and elsewhere, for $r < c$, $\int_c^r \cdots$ means $-\int_r^c \cdots$). For $x \in \mathbb{R}^d, |x| > c$ define $f(x) \equiv f_c(x) = \widetilde{F}(|x|;c)$. We may extend f to a suitable smooth function on $\mathbb{R}^d$ and denote the extension also by f. Clearly $\widetilde{D}_i f(x) \geq 0, x \in \overline{G}, |x| \geq c, 1 \leq i \leq d$. By the definition of $\widetilde{\gamma}(\cdot)$ note that $\widetilde{L} f(x) \geq 0, |x| > c$.

By Lemma 3.2 note that $P((\tau_c \wedge \tau_n) < \infty \mid Z(0) = z) = 1$ for any $c \leq |z| \leq n$, and hence

$$P((\widetilde{\tau}_c \wedge \widetilde{\tau}_n) < \infty \mid Z(0) = z) = 1 \tag{25}$$

for any $z \in \overline{G}$ with $c \leq |R^{-1} z| \leq n$. As R^{-1} is a matrix of nonnegative entries note that $\widetilde{D}_i f(R^{-1} Z(s)) \geq 0, 1 \leq i \leq d$ for all s such that $|R^{-1} Z(s)| \geq c$. Consequently putting $g(y) = f(R^{-1} y), y \in \overline{G}$ in (3.9), by optional sampling theorem, (3.13) and standard arguments we get

$$P(\widetilde{\tau}_n < \widetilde{\tau}_c \mid Z(0) = z) \geq \frac{\widetilde{F}(r;c)}{\widetilde{F}(n;c)} \tag{26}$$

for any $z \in \overline{G}$ with $c < r = |R^{-1} z| < n$.

Denote by ν_0, ν_1 respectively the least and the largest eigenvalues of $\widetilde{a}$. By (A4), $\nu_0 > 0$. Consequently

$$\frac{1}{s} \widetilde{\beta}(s) \geq \frac{-2|\widetilde{b}|}{\nu_0} + \frac{(d-1)\nu_0}{s \nu_1} \quad \longrightarrow +\infty, \quad \text{as} \quad s \downarrow 0. \tag{27}$$

It is now easy to see that (cf. Theorem 2.3 of [Ra2]),

$$\lim_{c \downarrow 0} \frac{\widetilde{F}(r;c)}{\widetilde{F}(n;c)} = 1. \tag{28}$$

Now (3.11) follows from (3.14),(3.16). This establishes (2.4).

Finally let $Z(0) = \underline{0} = (0, 0, \ldots, 0)$, and $\eta = \inf\{t > 0 : Z(t) \in G\}$. Since the drift and the reflection coefficients are constants note that the process $\{Z(t) : t \geq 0\}$ is strong Markov. To prove (2.5), in view of (2.4) and the strong Markov property, it is enough to prove that $\eta = 0$ a.s.; that is $Z(\cdot)$ visits G instantaneously. But this is a consequence of Blumenthal 0-1 law and regularity of the domain G for Brownian motion, or more precisely regularity of the boundary point 0 of $[0, \infty)$ for one dimensional Brownian motion; see [KS]. This completes the proof. ■

Proof of Theorem 2.2: As in Theorem 2.1 it is enough to prove the first part, as the second part follows from the first part, regularity and the strong Markov property. Moreover we may assume (3.1).

We shall prove the result by backward induction on $|J|$ as in [RW]. When $|J| = d$, this is just Theorem 2.1. So without loss of generality let $J = \{1, 2, \dots, k\}$ with $2 \leq |J| = k \leq d-1$, and $Z(0) = z \in \overline{G}$ with $z_j > 0$ for at least on $j \leq k$.

Writing $X(t) = \sigma B(t) + t\underline{\beta}, t \geq 0$, clearly $X(\cdot)$ is a d-dimensional Brownian motion with drift vector $\underline{\beta}$ and diffusion matrix $a = \sigma\sigma^{\dagger}$. Set $\widehat{\sigma} = ((\sigma_{i\ell}))_{1\leq i\leq k, 1\leq \ell\leq d}, \widehat{\widehat{\sigma}} = ((\sigma_{i\ell}))_{k+1\leq i\leq d, 1\leq \ell\leq d}, \widehat{b} = (\underline{\beta_1}, \dots, \underline{\beta_k}), \widehat{\widehat{b}} = (\underline{\beta}_{k+1}, \dots, \underline{\beta_d})$; while $\widehat{\sigma}$ is a $k \times d$ matrix, $\widehat{\widehat{\sigma}}$ is a $(d-k) \times d$ matrix. Observe that we may write $X(t) = (\widehat{X}(t), \widehat{\widehat{X}}(t)), t \geq 0$ where $\widehat{X}(\cdot)$ is a k-dimensional Brownian motion with drift vector $\widehat{b}$ and diffusion matrix $\widehat{a} = \widehat{\sigma}\widehat{\sigma}^{\dagger}$, and $\widehat{\widehat{X}}(\cdot)$ is a $(d-k)$-dimensional Brownian motion with drift $\widehat{\widehat{b}}$ and diffusion matrix $\widehat{\widehat{a}} = \widehat{\widehat{\sigma}}\widehat{\widehat{\sigma}}^{\dagger}$.

Note that we may now write (2.1) as

$$Z(t) = z + X(t) + (I - W)Y(t). \tag{29}$$

Set $\widehat{z} = (z_1, \dots, z_k), \widehat{Z}(t) = (Z_1(t), \dots, Z_k(t)), \widehat{\widehat{z}} = (z_{k+1}, \dots, z_d), \widehat{\widehat{Z}}(t) = (Z_{k+1}(t), \dots, Z_d(t))$. By hypothesis $z_i > 0$ for some $i \leq k$. By backward induction hypothesis, as in Lemma 5 of [RW], to prove (2.6) it is enough to prove that

$$P(\widehat{Z}(t) = \widehat{0}, \widehat{\widehat{Z}}(t) > \widehat{\widehat{\eta}}_{\ell}, \forall \ell \geq k+1 \text{ for some } t \geq 0 \mid Z(0) = z) = 0 \tag{30}$$

for any $\widehat{\widehat{\eta}} \in \mathbb{R}^{d-k}$ with $\widehat{\widehat{\eta}}_{\ell} > 0, \ell \geq k+1$.

Fix $\widehat{\widehat{\eta}} \in \mathbb{R}^{d-k}$ with $\widehat{\widehat{\eta}}_{\ell} > 0$ for all $\ell \geq k+1$. Set $T_0 \equiv 0, T_1 = .\inf\{t \geq 0 : Z_{\ell}(t) < \frac{1}{2}\widehat{\widehat{\eta}}_{\ell}$ for *some* $\ell \geq k+1\}$, $T_2 = \inf\{t \geq 0 : Z_{\ell}(t) > \widehat{\widehat{\eta}}_{\ell}$ for *all* $\ell \geq k+1\}, T_{2m+1} = .\inf\{t \geq T_{2m} : Z_{\ell}(t) < \frac{1}{2}\widehat{\widehat{\eta}}_{\ell}$ for *some* $\ell \geq k+1\}$, $T_{2m+2} = \inf\{t \geq T_{2m+1} : Z_{\ell}(t) > \widehat{\widehat{\eta}}_{\ell}$ for *all* $\ell \geq k+1\}$, for $m \geq 1$. By continuity of sample paths $T_m \uparrow \infty$ a.s. Clearly $\{\widehat{Z}(t) = \widehat{0}, \widehat{\widehat{Z}}_{\ell}(t) > \widehat{\widehat{\eta}}_{\ell}, \ell \geq k+1$ for some $t \geq 0\} \subseteq \bigcup_{m=1}^{\infty} \{\widehat{Z}(t) = \widehat{0}$ for some $t \in [T_{2m}, T_{2m+1}]\}$. So it is sufficient to establish for $m = 1, 2, \dots$

$$P(\widehat{Z}(t) = \widehat{0} \text{ for some } t \in [T_{2m}, T_{2m+1}] \mid Z(0) = z) = 0. \tag{31}$$

By applying the strong Markov property of $Z(\cdot)$, successively to stopping times $T_{2m}, m = 1, 2, \dots$ (3.19) can be obtained once the following is proved: Assume $z_{\ell} > \widehat{\widehat{\eta}}_{\ell}$ for all $\ell \geq k+1$; then

$$P(\widehat{Z}(t) = \widehat{0} \text{ for some } 0 \leq t \leq T_1 \mid Z(0) = z) = 0. \tag{32}$$

To establish (3.20), set $R_J = ((R_{i\ell}))_{1\leq i,\ell\leq k}, W_J = ((W_{i\ell}))_{1\leq i,\ell\leq k}$. Note that $R_J = (I_J - W_J)$ where I_J denotes the $k \times k$ identity matrix. By a result given in the appendix of [KT] it follows that the spectral radius of W_J is less than 1. Let $Y^J(\cdot), Z^J(\cdot)$ denote the solution to the Skorokhod problem in k-dimensional orthant so that

$$Z^J(t) = \widehat{z} + \widehat{X}(t) + R_J Y^J(t), \tag{33}$$

$$Y_i^J(t) - Y_i^J(0) = \int_0^t 1_{\{0\}}(Z_i^J(s))\, dY_i^J(s). \tag{34}$$

By Theorem 2.1 it follows that $P(Z^J(t) = \widehat{0}$ for some $t \geq 0) = 0$. But for $0 \leq t \leq T_1$ note that $Y_{\ell}(t) = 0$ for all $\ell \geq k+1$, and hence a comparison of (3.17) and (3.21) gives $\widehat{Z}(t) = Z^J(t), \widehat{Y}(t) = Y^J(t)$ for $0 \leq t \leq T_1$. Consequently (3.20) follows. This completes the proof. ■

Proof of Theorem 2.3: As R^{-1} has only nonnegative entries (by (2.12)), note that $\{Z(t) : t \geq 0\}$ is transient $\iff$ $\{R^{-1}Z(t) : t \geq 0\}$ is transient; also the process $R^{-1}Z(\cdot)$ is transient if

$$P(\lim_{t \longrightarrow \infty} (R^{-1}Z)_i(t) = +\infty \quad | \quad Z(0) = z) = 1 \tag{35}$$

for some $1 \leq i \leq d$.

Without loss of generality take $(R^{-1}\beta)_1 > 0$, that is, $\widetilde{b}_1 > 0$ as our hypothesis. Let $c > 0$. For $r > c$ define

$$\widetilde{F}_1(r; c) = \left[\exp\left(\frac{2c\widetilde{b}_1}{\widetilde{a}_{11}}\right)\right] \int_c^r \exp\left(-\frac{2\widetilde{b}_1 u}{\widetilde{a}_{11}}\right) du,$$

and for $x \in \mathbb{R}^d$ with $x_1 > c$ define $f_1(x) = \widetilde{F}_1(x_1; c)$. Then $\widetilde{L}f_1(x) = 0, \widetilde{D}_1 f_1(x) \geq 0, \widetilde{D}_i f_1(x) = 0, i \geq 2$ for any $x \in \overline{G}$ with $x_1 > c$. Denote $\widetilde{\tau}_s^{(1)} = \inf\{t \geq 0 : (R^{-1}Z)_1(t) = s\}, s > 0$. Taking $g_1(z) = f_1(R^{-1}z), z \in \overline{G}$, making use of (3.9), and proceeding as in the derivation of (3.14) in the proof of Theorem 2.1 we get

$$P(\widetilde{\tau}_c^{(1)} < \widetilde{\tau}_n^{(1)} \quad | \quad Z(0) = z) \leq 1 - \frac{\widetilde{F}_1(r; c)}{\widetilde{F}_1(n; c)} \tag{36}$$

for any $z \in \overline{G}$ such that $c < r = (R^{-1}z)_1 < n$. As $\widetilde{b}_1 > 0$ note that $\lim_{n \longrightarrow \infty} \widetilde{F}_1(n; c) = (\widetilde{a}_{11}/2\widetilde{b}_1) < \infty$. So from (3.24) we get for $0 < c < \widehat{c} < \infty$,

$$\sup\{P(\widetilde{\tau}_c^{(1)} < \infty \quad | \quad Z(0) = z) : (R^{-1}z)_1 \geq \widehat{c}\} < 1. \tag{37}$$

Using (3.25) and the strong Markov property repeatedly, (3.23) can now be established for $i = 1$; (cf. see [Bh]). This completes the proof. ■

In view of the above proof we can have the following criterion for transience; we omit the proof; (cf. see [Bh], [BR1],[BR2]).

Theorem 6 *Let σ, b, R be as in Theorem 2.3. Let $\widetilde{F}$ be defined by (3.12) in the proof of Theorem 2.1. If $\lim_{n \longrightarrow \infty} \widetilde{F}(n; c) < \infty$, then the process $R^{-1}Z(\cdot)$ and hence the process $Z(\cdot)$ are transient.* ■

As mentioned in Section 2, recurrence of the network is indicative of coming arbitrarily close to bankruptcy. Assume that (2.10) holds, in addition to (A1)–(A4). To indicate a criterion for recurrence, we define, for $z \neq 0$,

$$A(z) = \sum_{i,j=1}^{d} a_{ij} \frac{z_i z_j}{|z|^2}, M(z) = \sum_{i=1}^{d} a_{ii}, C(z) = 2\sum_{i=1}^{d} z_i \beta_i.$$

For $r > 0$, put

$$\overline{\zeta}(r) = \sup\{\frac{M(z) - A(z) + C(z)}{A(z)} : |z| = r\}.$$

Let $c > 0$; for $r > c$, put

$$F(r; c) = \int_c^r \exp\left(-\int_c^s \frac{1}{u}\overline{\zeta}(u) du\right) ds. \tag{38}$$

Theorem 7 *Assume (A1)–(A4) and (2.10). Let F be defined as above. If for some $c > 0$, $\lim_{n \longrightarrow \infty} F(n; c) = +\infty$, then the process $\{Z(t) : t \geq 0\}$ is recurrent. In particular, if $b_i(\cdots) \equiv \beta_i < 0, \forall 1 \leq i \leq d$ in (2.10) then $\{Z(t) : t \geq 0\}$ is recurrent.*

Proof: Under our hypotheses $Z(\cdot)$ is a strong Markov process; see [Ra3]. Note that $LF(x;c) \leq 0, |x| > c, x \in \overline{G}, \nabla_x F(x;c) \cdot R \leq 0, |x| > c, x \in \partial G$. So the first assertion can be proved as in Proposition 2.7(a) of [BR1], or Proposition 3.9(a) of [BR2]. The second assertion follows from the first. □

Using the comparison result of [Ra3] note that transience (resp. recurrence) in the case of non-constant drift and reflection can be inferred from transience (resp. recurrence) in the case of appropriate constant b, R.

We conclude with a question. In the course of our proof of Theorem 2.3 we have shown that $(R^{-1}\beta)_i > 0$ implies $(R^{-1}Z)_i(\cdot)$ is transient for any fixed i; this in turn implies transience of $Z(\cdot)$. An even better indicator of the health of the network would be if each component $Z_i(\cdot)$ were transient. Can one get conditions ensuring this? In particular if $\inf\{b_i(s,y,z) : s \geq 0, y, z \in \overline{G}, 1 \leq i \leq d\} > 0$ does it follow that $Z_j(\cdot)$ is transient for each $j = 1, \ldots, d$?

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