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BALL PROXIMINALITY IN BANACH SPACES

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ABSTRACT. In this paper we study the notion of ball proximinality. We exhibit several classes of Banach spaces that are ball proximinal.We show that strong proximinality does not imply ball proximinality. We also consider ball proximinality for spaces of vector-valued functions.

1. INTRODUCTION

Let X be a Banach space and let $Y \subset X$ be a closed set. Y is said to be a proximinal set in X if for every $x \in X$ there exists a $y \in Y$ such that $d(x, Y) = \|x - y\|$. Proximinality for subspaces is a well studied concept in Banach space theory, see the monograph [9]. Simple examples of proximinal subspaces include finite dimensional subspaces, reflexive subspaces, weak*-closed subspaces of dual spaces and subspaces of co-dimension one that are kernels of continuous linear functionals that attain their norm. In the case of first three sets of examples by compactness arguments one has also that the closed unit ball Y_1 is a proximinal set in X.

It has been recently proved by Saidi [8] that there are Banach spaces X with a closed proximinal subspace Y of co-dimension one for which Y_1 is not a proximinal set. Motivated by this we consider the following notion of ball proximinality, which was called 'locally proximinal' in [7].

Definition 1. *A closed subspace $Y \subset X$ is said to be ball proximinal if Y_1 is proximinal in X.*

It is easy to see (Lemma 2) that any ball proximinal space is proximinal. In this paper we undertake a detailed study of ball proximinality. We show that several stronger notions of proximinality studied in the literature turnout to be ball proximinal. We also consider two other notions of proximinality closely related to ball proximinality.

Motivated by the notion of strong proximinality of subspaces studied in [2] we introduce the notion of strong ball proximinality and show that it implies strong proximinality. We give an example to show that strong proximinality does not imply ball proximinality. We show that in a strictly convex dual space with the Namioka-Phelps property, any weak*-closed subspace is strong ball proximinal. We also study the continuity property of the metric projection in proximinal and ball proximinal situation.

An interesting question related to the notion of strong ball proximinality is, for what Banach spaces X is X_1 a strong proximinal set? We show that this is the case for locally uniformly convex spaces.

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Saidi'example (Theorem 1 [8]) is a subspace of co-dimension one and the kernel of a projection of norm one (and hence proximinal) whose unit ball fails to be proximinal. Thus it is interesting to investigate situations where proximinal subspaces of finite co-dimension or more generally factor reflexive (i.e., the quotient space X/Y is reflexive) proximinal subspaces are ball proximinal. In this direction we show that for any family of reflexive Banach spaces, for the c_0-direct sum, any factor reflexive proximinal subspace is ball proximinal.

As for the stability of ball proximinality, we note that it is preserved by c_0-direct sums and when the metric projection is single-valued and continuous then it is preserved by vector-valued continuous functions. We do not know if ball proximinality is preserved by ℓ^1 sums. We give some examples from the theory of L-embedded spaces ([3]) where it does get preserved in the space of Bochner integrable functions.

It may be noted that several standard proof techniques from the theory of best approximations by closed subspaces, like translating by a vector from the subspace Y or the mini-max theorem, $d(x, Y) = \|x|Y^{\perp}\|$ are no longer available in the case of ball proximinality.

Our investigations lead to several interesting questions about proximinality that have been mentioned throughout the paper.

2. Main Results

Let X be a Banach space and let $Y \subset X$ be a closed subspace. We first show that proximinality between the unit balls implies proximinality.

Lemma 2. *If the unit ball Y_1 is proximinal in X_1 then Y is proximinal in X.*

Proof. Let $x \in X$.Let $\alpha = d(x, Y)$. It is easy to see that $\alpha = \inf\{\| x - y \| : \| x - y \| \leq \alpha + \delta\}$ for any $\delta > 0$.

Now let $\beta = \alpha + \delta + \| x \|$. Since $\beta Y_1$is proximinal in βX_1, and as $x \in \beta X_1$, $d\,(x, \beta Y_1) = \| x - y_0 \|$ for some $y_0 \in \beta Y_1$.

For any $y \in Y$, $\| y - x \| \leq \alpha + \delta$, we have $\| y \| \leq \beta$ so that $\| x - y \| \geq \| x - y_0 \|$.

Hence $d\,(x, Y) = \| x - y_0 \|$. □

The following Lemma as in [8] shows that several natural summands are ball proximinal. We recall that when $\|x\| \geq 1$, $d(x, X_1) = \|x\| - 1 = \|x - \frac{x}{\|x\|}\|$.

Lemma 3. *Let $Y \subset X$ such that there is a linear onto projection $P : X \to Y$ and a monotone map $f : R^+ \otimes R^+ \to R^+$ with $\|x\| = f(\|P(x)\|, \|x - P(x)\|)$ for all $x \in X$. Then Y is ball proximinal.*

Proof. Let $x \notin Y_1$. We will show that the infimum is attained at $\frac{P(x)}{\|P(x)\|}$. Let $y \in Y_1$, $\|x - y\| = f(\|P(x) - y\|, \|x - P(x)\|) \geq f(\|P(x) - \frac{P(x)}{\|P(x)\|})\|, \|x - P(x)\|) = \|x - \frac{P(x)}{\|P(x)\|}\|$, by the monotonicity of f. □

Remark 4. *It is clear from the above Lemma that if $Y \subset X$ is a complemented subspace then X can be renormed so that Y is ball proximinal in the new norm. We do not know in general how to renorm X (or additional conditions on X or Y) so that the given subspace is ball proximinal in the new norm.*

Summands of the above kind can be used to generate more classes of spaces satisfying the ball proximinality as the following proposition illustrates.

Proposition 5. *Let $Y \subset X$ be the kernel of a norm one projection P. Let $Z \subset P(X)$ be a closed subspace. If $Y + Z$ is ball proximinal then Z is ball proximinal.*

Proof. We note that our assumption identifies $P(X)$ with $X|Y$. Let $x \in P(X)$. $d(x, Z_1) \geq d(x, (Y + Z)_1) = \|x - y_0 - z_0\|$ for some $y_0 + z_0 \in (Y + Z)_1$. Now $\|x - y_0 - z_0\| \geq \|P(x - y_0 - z_0)\| = \|x - z_0\|$. Also $\|z_0\| = \|P(y_0 + z_0)\| \leq 1$. Therefore Z is ball proximinal. □

Remark 6. *Let $Y \subset X$ be a closed subspace and let $\pi : X \to X/Y$ denote the quotient map. Suppose $Z \subset X|Y$ is a closed subspace such that $\pi^{-1}(Z)$ is proximinal. Then it is well-known that Z is proximinal. The above is a ball proximinal analogue of this simple proximinality result, under additional hypothesis. We do not know even if Y is reflexive, ball proximinality of $\pi^{-1}(Z)$ implies the ball proximinality of Z.*

We now recall the notion of local U-proximinality from [4]. Let $[x]$ denote span$\{x\}$.

Definition 7. *A closed subspace $Y \subset X$ is said to be locally U-proximinal if there exists a function $\epsilon : (X \backslash Y) \times R^+ \to R^+$ such that for each fixed x, $\epsilon(x, .)$ is continuous, increasing in ρ, $\epsilon(x, \rho) \to 0$ as $\rho \to 0$ and for $x \in Y$, $\rho > 0$, $(1+\rho)X_1 \cap ([x]_1 + Y) \subset X_1 + \epsilon(x, \rho)Y_1$.*

Well known examples of locally U-proximinal spaces include M-ideals, semi M-ideals, and semi L-summands, see [4] and [3] for the definitions.

Theorem 8. *Suppose Y is locally U-proximinal in X. Then $Y_1$is proximinal.*

Proof. Let $x \in X_1$. Let $0 < \epsilon < 1$. Since Y is locally U-proximinal, there exists a $\delta > 0$ such that $\rho < \delta \Longrightarrow \epsilon(x, \rho) < \epsilon$ as in the definition of U-proximinality. Since $\| x \| < 1 + \rho$, $d(x, Y) \leq 1$, it follows from Proposition 2.3 in [4] with $r = 1$, that there is a best approximation $z \in Y_1$. Thus $\| x - z \| = d(x, Y) \leq d(x, Y_1) \leq \| x - z \|$.

Now suppose $\| x \| > 1$. Let $0 < \epsilon < \frac{1}{\|x\|}$. Apply Proposition 2.3 [4] to $\frac{x}{\|x\|}$, with $r = \frac{1}{\|x\|}$. Then there is a best approximation

$z \in Y$ such that $\| z \| < \frac{1}{\|x\|}$. Thus $\| x \| z \in Y_1$. Now for $m \in Y_1$, $\| x - m \| = \| x \| \| \frac{x}{\|x\|} - \frac{m}{\|x\|} \| \geq \| x \| \| \frac{x}{\|x\|} - z \|$
$= \| x - \| x \| z \|$. Thus $d(x, Y_1) = \| x - \| x \| z \|$. ☐

Remark 9. *In particular it can be shown that if X has a locally uniformly rotund (LUR) norm then for any proximinal subspace Y, Y_1 is proximinal in X. Also for $1 < p < \infty$, since the space of compact operators $\mathcal{K}(\ell^p)$ is a M-ideal in the space of bounded operators (see [3] Chapter VI), we get that $\mathcal{K}(\ell^p)$ is ball proximinal in $\mathcal{L}(\ell^p)$.*

$\mathcal{L}(\ell^p)$.

Let Y be ball proximinal. As in the linear case, let $P_{Y_1} : X \to 2^{Y_1}$ defined by $P_{Y_1}(x) = \{y \in Y_1 : d(x, Y_1) = \|x - y\|\}$, denote the metric projection. Let P_Y denote the metric projection corresponding to the proximinal subspace Y.

In the next proposition we collect several properties of the metric projection.

Proposition 10. (1) *For all $x \in X$, for all $\lambda > 0$, $\lambda P_{Y_1}(\frac{x}{\lambda}) = P_{\lambda Y_1}(x)$.*

(2) *$P_Y(x) = P_{\lambda Y_1}(x)$ for all $\lambda \geq \|x\| + d(x, Y)$.*

(3) *If P_{Y_1} is continuous then P_Y is continuous.*

Proof. 1) Let $x \in X$, $\lambda > 0$, let $y_0 \in P_{Y_1}$. For any $y \in Y_1$, $\|\frac{x}{\lambda} - y_0\| \leq \|\frac{x}{\lambda} - y\| \iff \|x - \lambda y_0\| \leq \|x - \lambda y\|$. Hence $\lambda P_{Y_1}(\frac{x}{\lambda}) = P_{\lambda Y_1}(x)$.

2) Now let $\lambda \geq \|x\| + d(x, Y)$. For $y \in P_Y(x)$, $\|y\| \leq \|x\| + \|x - y\| = \|x\| + d(x, Y) \leq \lambda$. Hence $P_Y(x) \subset P_{\lambda Y_1}(x)$. Since $d(x, Y) \leq d(x, \lambda Y_1)$ the other inclusion is always true.

3) Since P_{Y_1} is continuous if and only if $P_{\lambda Y_1}$ is continuous, the conclusion follows. ☐

Remark 11. *It should be noted that if $y \in P_Y(x)$ then $\|y\| \leq \|x - y\| + \|x\| = d(x, Y) + \|x\| \leq 2\|x\|$.*

We now define the notion of strong ball proximinality analogous to the notion of strong proximinality (same as the definition below with P_Y in the place of P_{Y_1}) studied in [2].

Definition 12. *A closed ball proximinal subspace $Y \subset X$ is said to be strong ball proximinal at a point $x \in X$ if given $\epsilon > 0$, there exists a $\delta > 0$ such that $y \in Y_1$, $\|x - y\| \leq d(x, Y_1) + \delta$ implies there exists a $y' \in P_{Y_1}(x)$ such that $\|y - y'\| \leq \epsilon$. It is said to be strong ball proximinal if it is strong ball proximianl at each point.*

Proposition 13. *If Y is strong ball proximinal in X then it is strongly proximinal in X.*

Proof. Let $x \in X$. We first show that Y_1 is strong proximinal if and only if λY_1 is strong proximinal for any $\lambda > 0$. Suppose Y_1 is strong proximinal. Fix $\lambda > 0$. Let $x \in X$. Since Y_1 is strong proximinal at $\frac{x}{\lambda}$, for $\epsilon > 0$, there exists a $\delta > 0$ such that for all $y \in Y_1$, $\|\frac{x}{\lambda} - y\| \leq d(\frac{x}{\lambda}, Y_1) + \delta$ implies there exists $y' \in P_{Y_1}(\frac{x}{\lambda})$ with $\|y - y'\| \leq \frac{\epsilon}{\lambda}$.

Now let $\delta' = \lambda\delta$. If $y \in \lambda Y_1$ and $\|x - y\| \leq d(x, \lambda Y_1) + \delta'$, then $\|\frac{x}{\lambda} - \frac{y}{\lambda}\| \leq d(\frac{x}{\lambda}, Y_1) + \delta$. Now let $y' \in P_{Y_1}(\frac{x}{\lambda})$ be such that $\|y' - \frac{x}{\lambda}\| \leq \frac{\epsilon}{\lambda}$. Thus $\|\lambda y' - y\| \leq \epsilon$. As $y' \in P_{Y_1}(\frac{x}{\lambda})$ implies that $\lambda y' \in P_{\lambda Y_1}$ we get the desired conclusion.

Finally if Y is strong ball proximinal at x , let $\lambda > \|x\| + d(x, Y)$. By Proposition 10 we have $P_{\lambda Y_1}(x) = P_Y(x)$. Hence Y is strong proximinal at x. □

We next exhibit some examples of strong ball proximinal spaces.

Theorem 14. *Suppose $Y \subset X^*$ is a weak*-closed subspace and $X^*$is a strictly convex space with the Namioka-Phelps property (i.e., weak*-norm topologies coincide on the unit sphere) , then Y is strong ball proximinal and P_{Y_1} is continuous.*

Proof. Let $f \in X^*$ and let $d(f, Y_1) = \| f - P_{Y_1}(f) \|$. Suppose strong proximinality fails at f.

Then there exists a $\epsilon > 0$ and a sequence $\{f_n\}_{n \geq 1} \subset Y_1$ such that $\| f - f_n \| \leq d(f, Y_1) + \frac{1}{n} \leq \| f - f_n \| + \frac{1}{n}$ for all n but $\| f_n - P_{Y_1}(f) \| > \epsilon$.

Going through a subnet if necessary we may assume that $f_n \to f_0 \in Y_1$ in the weak*-topology. By the weak *-lower-semi-continuity of

the norm, $\| f - f_0 \| \leq \liminf \| f - f_n \| \leq d(f, Y_1) = \| f - P_{Y_1}(f) \|$. Thus $\| f - f_0 \| = \lim \| f - f_n \|$. Consequently by the Namioka-Phelps property, $f_n \to f_0$ in the norm and by the strict convexity , $P_{Y_1}(f) = f_0$. This contradicts the choice of the sequence $\{f_n\}$.

Now if $\{f_n\}_{n \geq 1} \subset X$ and $f_n \to f$, then by hypothesis, $\| f_n - P_{Y_1}(f_n) \| = d(f_n, Y_1) \to d(f, Y_1) = \| f - P_{Y_1}(f) \|$. Again we assume w. l. o. g

that $P_{Y_1}(f_n) \to f_0$ in the weak*-topology. $d(f, Y_1) \leq \| f - f_0 \| \leq \liminf \| f - P_{Y_1}(f_n) \| \leq \liminf \| f - f_n \| + \| f_n - P_{Y_1}(f_n) \| \leq d(f, Y_1)$.

Therefore we conclude that $P_{Y_1}(f) = f_0$ and $P_{Y_1}(f_n) \to P_{Y_1}(f)$. □

It follows from arguments similar to the ones given above that any finite dimensional subspace is strong ball proximinal. Similar arguments also yield the following corollary. We recall that a Banach space X is said to be a G-space if every point of the unit sphere is a denting point. Equivalently these are strictly convex spaces such that weak and norm topologies agree on the unit sphere.

Corollary 15. *Suppose X is a G space and $Y \subset X$ is a reflexive subspace then, Y is strong ball proximinal and P_{Y_1} is continuous.*

In the following theorem we use the renorming procedure of Lemma 1 of [8] to show that strong proximinality does not imply ball proximinal.

Theorem 16. *There exists a Banach space X and a strongly proximinal subspace $Y \subset X$ that is not ball proximinal.*

Proof. As in Lemma 1 of [8], let Y be a non-reflexive subspace with a non-proximinal hyperplane H. Let $\|y_0\| = 1 = d(y_0, H)$ and the ball $B(y_0, 1)$ is disjoint from H. Let $a \in 2Y_1 \cap (y_0 + H)$ and let $R = A - a$. Choose $\gamma \in \mathcal{R}$ such that $\frac{1}{\gamma}(\pm R) \subset Y_1$. Let $X = Y \oplus_1 \mathcal{R}$ (ℓ^1-directsum). Let $e_0 = 1$. Let $x = y + \lambda e_0$ then from Lemma 1 in [8] we have, $P'(x) = \{y + \frac{\lambda}{\gamma} R\}$ and $d(x, Y) = |\lambda|$.

To show that Y is strongly proximinal in X, let $\epsilon > 0$ and let $0 < \delta < \epsilon$. Suppose $y' \in Y$ and $\|x - y'\| \leq d(x, Y) + \delta$. Now $\|x - y'\| = \|(y + \lambda e_0 - y'\| = \|(y - y') + \lambda e_0\| = \|y - y'\| + |\lambda| \leq |\lambda| + \delta$. Thus $\|y - y'\| \leq \epsilon$ and as $y \in P'(x)$ we have the required conclusion.

As already noted in [8], Y is not ball proximinal. □

Remark 17. *It follows from these arguments that a Banach space is reflexive if and only if every closed subspace of co-dimension one (and hence every closed subspace) is ball proximinal. In the case of closed subspace of co-dimension one, we do not know a necessary and sufficient condition in terms of the continuous linear functional determining the subspace to ensure ball proximinality.*

Since for a Banach space X the unit ball is always a proximinal set, it is interesting to consider situations when it is a strong proximinal set.

Proposition 18. *Let X be a locally uniformly rotund space then X_1 is a strong proximinal set.*

Proof. Suppose X_1 is not a strong proximinal set. Since X is strictly convex, by the uniqueness of best approximation, there exists a $x \notin X_1$, an $\epsilon > 0$ and a sequence $\{y_n\}_{n \geq 1} \subset X_1$ such that $\|y_n - \frac{x}{\|x\|}\| > \epsilon$, $\|x - y_n\| \leq \|x\| - 1 + \frac{1}{n}$ for all n.

Thus $lim_{n \to \infty} \|x - y_n\| = \|x\| - 1$. Also $\|x\| - 1 \leq \|\frac{y_n + y_m}{2} - x\| \leq \frac{1}{2}(\|y_n - x\| + \|y_m - x\|) \to \|x\| - 1$.

Hence $\|\frac{y_n + y_m}{2} - x\| \to \|x\| - 1$. Now using local uniform convexity on the sphere centered at x and radius $\|x\| - 1$, we see that $\{y_n\}_{n \geq 1}$ is a Cauchy sequence. Let $y_n \to y_o \in X_1$. Since $\|x - y_0\| = \|x\| - 1$ we have $y_0 \in P_{X_1}(x) = \{\frac{x}{\|x\|}\}$. This contradicts the choice of the sequence $\{y_n\}_{n \geq 1}$. Therefore X_1 is a strong proximinal set. □

Recall that Saidi's example [8], is a proximinal closed subspace $Y \subset X$ with a $x \in X$ such that $d(x, Y) = d(x, Y_1)$, the distance is attained at a point of Y, but not at Y_1. Thus it is interesting to consider local conditions on x to ensure that this distance is attained at Y_1 as well. The following proposition leads in a natural way to a newer notion of proximinality.

Proposition 19. *Let $x \in X$. Suppose $\alpha = d(x, Y) = d(x, Y_1)$, assume that Y is strongly proximinal at x and $P_Y(x)$ is a compact set. Then Y_1 is proximinal at x.*

Proof. By strong proximinality , for each n , let $y_n \in Y_1$ with $\| x - y_n \| \leq \alpha + \delta_n$. There exists y'_n with $d(x, Y) = \| x - y'_n \|$ and $\| y'_n - y_n \| \leq \frac{1}{n}$.

Since $P_Y(x)$ is compact we can assume w. l. o. g that $y'_n \to y' \in P_Y(x)$. Thus $\| y' \| \leq 1$, $d(x, Y_1) = \| x - y' \|$. □

We next show the ball proximinality of factor reflexive proximinal subspaces in c_0-direct sums. The following proposition is well-known and is included here to indicate an alternate proof of a result of Garkavi from Chapter III of [9] that we will be using below.

Proposition 20. *Let $Z \subset Y \subset X$ be closed subspaces such that Y/Z is proximinal in X/Z. If Z is proximinal in X then Y is proximinal in X.*

Proof. Let $x \in X$. Suppose $d(\pi(x), Y/Z) = \|\pi(x) - \pi(y_0)\|$ for some $y_0 \in Y$. Now $\|\pi(x) - \pi(y_0)\| = \|\pi(x - y_0)\| = d(x - y_0, Z) = \|x - y_0 - z_0\|$ for some $z_0 \in Z$ as Z is proximinal in X. For $y \in Y$, $\|x - y\| \geq \|\pi(x - y)\| \geq \|x - (y_0 + z_0)\|$. As $y_0 + z_0 \in Y$ we get that Y is proximinal in X. □

Corollary 21. *(Garkavi) Let $Z \subset Y \subset X$ be closed, factor reflexive subspaces of X. If Z is proximinal in X then Y is proximinal in X.*

Proof. By our assumption of factor reflexivity, Y/Z is a proximinal subspace of the reflexive space X/Z. □

Question 22. *We do not know if a ball proximinal versions of the above proposition or corollary is true?*

Remark 23. *There exists a Banach space X with subspaces $Y \subset Z \subset X$, Y is of co-dimension one in Z and Z is of co-dimension one in Y, Z is ball proximinal but Y is not. To see this one starts with a Banach space Z and a subspace Y of co-dimension one that is not ball proximinal in Z as in Lemma 1 of* [8]*. Let $X = Z \oplus_\infty \mathcal{R}$ (ℓ^∞ sum). As noted in Lemma 2, Z is ball proximinal in X. Clearly Y is not ball proximinal in X.*

We next exhibit examples where proximinal factor reflexive subspaces are ball proximinal.

Theorem 24. *Suppose $\{X_\alpha\}_{\alpha \in I}$ is a family of reflexive spaces and let $Y = \oplus_{c_0} X_\alpha$ be the c_0-direct sum. Let $Z \subset Y$ be any proximinal, factor reflexive subspace. Then Z is ball proximinal. The same conclusion holds for any factor reflexive proximinal subspace of Z. Also the intersection of finitely many ball proximinal subspaces of Y is again ball proximinal.*

Proof. Let $f \in Z^\perp$. Since $Z \subset ker(f) \subset X$, Z is proximinal and factor reflexive, by Corollary 20, we get that $ker(f)$ is a proximinal subspace and hence f attains its norm.

Since $Y^* = \oplus_1 X^*_\alpha$ (ℓ^1-direct sum), it is easy to see that f has only finitely many non-zero coordinates. Since $Z^\perp$ is a Banach space, by a simple application of the Baire Category theorem we see that there is a finite set $A \subset I$ such that $f_\alpha = 0$ for all $\alpha \notin A$ and $f \in Z^\perp$.

Thus it is easy to see, $Z_1 = (Z \cap (\oplus_\infty X_\alpha)_{\{\alpha \in A\}})_1 \oplus_\infty (\oplus_{c_0} (X_\alpha)_{\alpha \notin A})_1$.

Consider now the corresponding splitting of $Y = (\oplus_\infty X_\alpha)_{\{\alpha \in A\}} \oplus_\infty \oplus_{c_0} (X_\alpha)_{\alpha \notin A}$. As A is a finite set, the first set in the decomposition of $Z_1$is proximinal by weak compactness of balls in reflexive spaces, in the corresponding space. The second set is the unit ball and hence is proximinal in the corresponding space. Therefore Z_1 is proximinal in Y .

We note the we have in particular proved that any factor reflexive proximinal subspace of Y is again a c_0-sum of reflexive spaces. Thus if $Z' \subset Z$ is a proximinal factor reflexive space, then Z' is ball proximinal in Z.

Also it is clear from the above arguments if $Z \subset Y$ is a closed subspace such that $Z^{\perp} \subset \oplus_1(X_{\alpha}^{*}\ )_{\{\alpha\in A\}}$for a finite set A, then Z is ball proximinal. Note that any set of the form $\oplus_1(X_{\alpha}^{*})_{\{\alpha\in A\}}$ is a weak* closed subset of Y^*. Thus if $Z_1, ..., Z_n$ are ball proximinal subspaces of Y then since $(\bigcap_{1\leq i\leq n} Z_i)^{\perp} \subset \oplus_1(X_{\alpha}^{*})_{\{\alpha\in A\}}$ for some finite set A, we get that $\bigcap_{1\leq i\leq n} Z_i$ is ball proximinal. □

Question 25. *It is well known* [4] *that the space of compact operators,* $\mathcal{K}(\ell^2)$ *is a locally U-proximinal subspace of* $\mathcal{L}(\ell^2)$ *and hence is ball proximinal. We do not know analogous to the case of* c_0*, if every proximinal factor reflexive subspace of* $\mathcal{K}(\ell^2)$ *is also ball proximinal?*

The following Lemma is from [3] Section IV.1.

Lemma 26. *Let* P *be a contractive projection in* X^**. Suppose* $Y \subset P(X^*)$ *is a norm closed subspace such that* $P\left(Y_1^-\right) \subset Y$ *(closure w. r. t weak* topology). Then* Y *is ball proximinal in* $P(X^*)$*.*

Proof. Let $x^* \in P(X^*)$. Using weak* compactness, let $y_0^* \in Y_1^-$ be such that $d\left(x^*, Y_1^-\right) = \parallel x^* - y_0^* \parallel$.

Now $d\left(x^*, Y_1\right) \geq d\left(x^*, Y_1^-\right) = \parallel x^* - y_0^* \parallel \geq \parallel P\left(x^* - y_0^*\right) \parallel = \parallel x^* - P\left(y_0^*\right) \parallel$. As $P\left(y_0^*\right) \in Y_1$ we have the required conclusion. □

We recall from [3] that X is said to be L-embedded if under the canonical embedding of X in X^{**}, there exists a linear onto projection $P : X^{**} \to X$ such that $\|\Lambda\| = \|P(\Lambda)\| + \|\Lambda - P(\Lambda)\|$ for all $\Lambda \in X^{**}$. These spaces are natural analogues of reflexive spaces. See [3] Chapter IV for several examples and geometric properties of these spaces.

Corollary 27. $Y \subset X$*, such that both* Y *and* X *are* L*-embedded. Then* Y *is ball proximinal in* X*.*

Proof. It is known that if $Y \subset X$ and both are L-embedded then the L-projection P in X^{**} with range X has the property that $P(Y^-) \subset Y$ (Proposition IV.1.10 [3]). Hence Y_1 is proximinal in X. □

The importance of these notions lies in the fact that being L-embedded is preserved by ℓ^1-sums and in several cases the space of Bochner integrable functions with values in a L-embedded space is again a L-embedded space.

Corollary 28. *Let* μ*,* ν *be positive measures. Let* $Y \subset L^1(\mu)$ *be a reflexive subspace.* $L^1(\nu, Y)$ *is ball proximinal in* $L^1(\nu \otimes \mu)$*.*

Proof. : It is well-known that $L^1(\nu, Y)$ is a L-embedded subspace (see [3] page 200) of $L^1(\nu, L^1(\mu)) = L^1(\nu \otimes \mu)$. Thus the conclusion follows from the above corollary. □

We next consider the stability property of ball proximinality for spaces of vector-valued function spaces. We follow the notations and terminology of [1].

It is easy to see that if $\{Y_\alpha\}$ is a family of ball proximinal Banach spaces in X_α, then the c_0-direct sum $\oplus_{c_0} Y_\alpha$ is also ball proximinal in $\oplus_{c_0} X_\alpha$.

We first consider the space $C(K, X)$ of X-valued continuous functions on a compact set K.

Proposition 29. *Suppose $Y \subset X$ is a closed ball proximinal subspace and $P : X \to Y_1$ is single-valued and continuous or has a continuous selection. For any compact Hausdorff space K, $C(K, Y)_1$ is proximinal in $C(K, X)$.*

Proof. Let $f \in C(K, X)$, put $f_0 = P \circ f$. Then $f_0 \in C(K, Y)_1$ and for any $g \in C(K, Y)_1$ and $k \in K$, $\| f(k) - f_0(k) \| \leq \| f(k) - g(k) \|$. Thus $d(f, C(K, Y)_1) = \| f - f_0 \|$.

Similar arguments work in the case P has a continuous selection. □

We do not know if ball proximinality is preserved by ℓ^1-sums. In the following theorem for a ball proximinal separable subspace, we show that $L^1(\mu, Y_1)$ is a proximinal subset of $L^1(\mu, X)$. See [5] for a similar result. In the following theorem to keep the measure theory simple we assume that the domain is $[0,1]$ with the Lebsgue measure μ. Standard measure theoretic arguments can be used to extend it to positive measures on completions of countably generated σ-fields.

Theorem 30. *Suppose $Y \subset X$ is a separable subspace such that Y_1 is proximinal . Then for the Lebesgue measure μ on $[0,1]$, $L^1(\mu, Y_1)$ is proximinal in $L^1(\mu, X)$.*

Proof. Fix $f \in L^1([0,1])$. We assume w. l. o. g that the functions are defined every where. Let $G = \{(t, y) \in [0,1] \times Y_1 \colon d(f(t), Y_1) \geq \| f(t) - y \|\}$. Since Y_1 is proximinal the projection of G to the first coordinate is $[0,1]$.

We now show that G is a measurable set. Let $\{y_n\}_{n \geq 1}$ be a dense sequence in Y_1.

$G = \cap_n \{(t, y) \in [0,1] \times Y_1 : \| f(t) - y_n \| \geq \| f(t) - y \|\}$. Therefore since f is a measurable function, G is measurable.

Thus by a consequence of the von Neumann selection theorem ([6] Theorem 7.2) we get a measurable function $g_0 : [0,1] \to Y_1$ such that $(t, g_0(t)) \in G$ for a.e t, in particular

$d(f(t), Y_1) = \| f(t) - g_0(t) \|$, a.e. Since Y is separable, $g_0 \in L^1(\mu, Y_1)$.

For any $g \in L^1(\mu, Y_1)$, $\| f - g \| = \int \| f(\omega) - g(w) \| \, d\mu(\omega) \geq \int_A d(f(\omega), Y_1) \, d\mu(\omega) = \int \| f(\omega) - g_0(\omega) \| \, d\mu(w) = \| f - g_0 \|$.

Therefore $d(f, L^1(\mu, Y)_1) \geq \| f - g_0 \| \geq d(f, L^1(\mu, Y_1))$ as $g_0 \in L^1(\mu, Y_1)$. □

In the following proposition we prove another partial result on the ball proximinality of ℓ^1-sums.

Proposition 31. *Let $\{X_i\}_{i\in I}$ be a family of Banach spaces with reflexive subspaces $Y_i \subset X_i$. $Y = \oplus_1 Y_i$ is ball proximinal in $S = \{s \in \oplus_1 X_i : s(i) = 0 \ for \ i \notin A \ for \ a \ finite \ A \subset I\}$.*

Proof. Let $s \in S$ and let $A \subset I$ be the associated finite set. We claim that $d(s, Y_1) \geq d(s, (\oplus_{i\in A} Y_i)_1)$. Once this claim is proved, since $\oplus_{i\in A} Y_i$ is reflexive there exists a $y_0 \in (\oplus_{i\in I} Y_i)_1 \subset Y_1$ such that $d(s, (\oplus_{i\in A} Y_i)_1) = \|s - y_0\|$. Therefore $d(s, Y_1) = \|s - y_0\|$.

To see the claim, let $y \in Y_1$. $\|s - y\| \geq \sum_{i\in A} \|s(i) - y(i)\| = \|s - y'\| \geq d(s, (\oplus_{i\in A} Y_i))_1$. Where $y'(i) = y(i)$ for $i \in A$ and $y'(i) = 0$ otherwise. □

Remark 32. *If X_i's are also L-embedded then since this property is preserved by ℓ^1-sums (see Proposition IV.1.5* [3]*) we get that Y is ball proximinal in $\oplus_1 X_{i\in I}$ by Corollary 26.*

We conclude the paper with another interesting variation on proximinality of subspaces, namely when is Y_1 proximinal in X_1?

As noted in Lemma 1 this again implies that Y is a proximinal subspace and is implied by ball proximinality. Note that the projection considered in Lemma 2 is a bicontractive projection.

Proposition 33. *If P be a bicontractive projection with range Y, then Y_1 is proximinal in X_1.*

Proof. Let $x \in X_1$, for $y \in Y$, $\| x - y \| \geq \| x - P(x) \|$ and $P(x) \in Y_1$. Thus $d(x, Y) = \| x - P(x) \| = d(x, Y_1)$. □

Remark 34. *It follows from Remark 11 that if Y is proximinal in X then Y_1 is proximinal in $Y_1 \cup \frac{1}{2} X_1$.*

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