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Extremal structures in ultrapower spaces

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EXTREMAL STRUCTURES IN ULTRAPOWER SPACES

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ABSTRACT. This short note deals with extremal structures in ultrapowers of Banach spaces. We show that a unit vector $x \in X$ is a strong extreme point if and only if for every ultra filter U, it is a strongly extreme point of the unit ball of the ultrapower space $(X)_U$. We also give an application of these ideas to approximation theory.

1. INTRODUCTION

We follow the well-known monograph [10] for all the basic results from the theory of ultrapowers of Banach spaces that we will be using here. For a Banach space X we denote by X_1 its closed unit ball. We only consider ultrapowers over nontrivial ultrafilters . We always consider X as canonically embedded in (X_U) and also consider the canonical embedding of X in X^{**}. For $n > 1$ we denote the dual of order $2n$ of X by $X^{(2n)}$.

We recall that a $x \in X_1$ is said to be a strong extreme point if for sequences $\{x_n\}_{n\geq 1}, \{y_n\}_{\geq 1} \subset X_1$, $\frac{x_n+y_n}{2} \to x$ implies $\|x_n - y_n\| \to 0$.

We show that $x \in X_1$ is a strong extreme point if and only if for any ultrafilter U, it is a strong extreme point of $((X)_U)_1$. As a consequence we recover the well-known Banach space result [7] that any strong extreme is also a strong extreme point of the bidual.

For a compact set K, let $C(K, X)$ denote the space of X-valued continuous functions on K equipped with the supremum norm. Using a characterization of strong extreme points of $C(K, X)_1$ due to [2] we show that any strong extreme point of $C(K, X)_1$ is a strong extreme point of $C(K, (X)_U)_1$. We also formulate the notion of weak*-extreme points studied in [5] in the context of ultra powers.

2. MAIN RESULTS

Our first result is based on the following Lemma from [6] which we reproduce here for the sake of completeness.

Lemma 1. *Let $x \in X_1$ and let U be an ultrafilter. x is a strong extreme point if and only if it is an extreme point of $((X)_U)_1$.*

Proof. Let $x \in X_1$ be a strong extreme point. Suppose $x = \frac{y+z}{2}$ for $y, z \in ((X)_U)_1$. By the definition of the ultrapower we have that there exists $\{y_n\}_{n\geq 1}, \{z_n\}_{n\geq 1} \subset X_1$ such that $\|x - \frac{y_n+z_n}{2}\| \to 0$. Therefore $\|y_n - z_n\| \to 0$. Thus $y = z$ and hence x is an extreme point.

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Conversely suppose $\|x - \frac{y_n+z_n}{2}\| \to 0$ for sequences, $\{y_n\}_{n\geq 1}, \{z_n\}_{n\geq 1} \subset X_1$. Then since $(x - \frac{y_n+z_n}{2}) \in U$ we have that $x = \frac{(y_n)+(z_n)}{2}$ in $(X_U)_1$. As x is extreme we have that $x - (y_n), x - (z_n) \in U$. Therefore $\|y_n - z_n\| \to 0$. □

Theorem 2. *Let $x \in X_1$ and let U be an ultrafilter. x is a strong extreme point if and only if it is a strong extreme point of $((X)_U)_1$.*

Proof. Let $x \in X_1$ be a strong extreme point. For any ultra filter V we will show that x is an extreme point of the unit ball of the space $((X)_U)_V$. It would then follow from the above Lemma that x is a strong extreme point of $((X)_U)_1$. We have from the Corollary on page 103 in [10] that $((X)_U)_V$ is again an ultrapower. Now applying the above Lemma for the inclusion of $X \subset ((X)_U)_V$ we get that x is an extreme point of the unit ball of $((X)_U)_V$. Hence the conclusion follows.

The converse is easy to see. □

Our next corollary follows from Theorem 11.3 of [10] which gives an ultra filter U such that X^{**} is isometrically embedded in $(X)_U$. It is easy to see from the choice of U and the proof of Theorem 11.3 that this embedding coincides on X with the canonical embedding of X in X^{**}. Thus we have the canonical inclusion $X \subset X^{**} \subset (X)_U$.

Corollary 3. *Let $x \in X$ be a strong extreme point then it is a strong extreme point of X_1^{**}.*

We next apply this result to the space $C(K, (X)_U)$. It may be recalled that when K is infinite, this space is different from $(C(K, X))_U$.

Theorem 4. *$f \in C(K, X)_1$ is a strong extreme point if and only if it is a strong extreme point of $C(K, (X)_U)_1$.*

Proof. Let $f \in C(K, X)_1$ be a strong extreme point. Let $k \in K$. It follows from [2] that $f(k)$ is a strong extreme point of X_1. Thus by the previous theorem, $f(k)$ is a strong extreme point of $((X)_U)_1$. Thus by [2] again f is a strong extreme point of $C(K, (X)_U)_1$. □

Remark 5. *Similar arguments can be used to show that $x \in X_1$ is a strongly exposed point if and only if it is a strongly exposed point of $((X)_U)_1$ and hence one can deduce the standard conclusion that a strongly exposed point remains as a strongly exposed point of X_1^{**}.*

We recall that $x \in X_1$ is said to be a weak*-extreme point if under the canonical embedding x is an extreme point of X_1^{**}. Analogous to the definition of a strong extreme point, it was proved in [5] that x is a weak*-extreme point if and only if for any sequences $\{y_n\}_{n\geq 1}, \{z_n\}_{n\geq 1} \subset X_1$ such that $\|x - \frac{y_n+z_n}{2}\| \to 0$ implies $y_n - z_n \to 0$ in the weak topology. It was shown in [3] that weak*-extreme point of X_1 can fail to be a weak*-extreme point of X_1^{**}. Also in [3] there is an example of a weak*-extreme point that is a weak*-extreme point of the unit ball of all the duals of even order of X but is not a strong extreme point.

It was shown in [4] that some notions of extremality (that are weaker than strong extreme) even though are preserved in $(X^{(2n)})_1$ when n is small, lose the extremality for larger n.

Remark 6. *Following the arguments given during the proof of Theorem 11.3 in* [10] *and using the Corollary on page 103, it can be shown that for any* $n > 1$ *there exists an ultrafilter* U *such that* $X^{(2n)}$ *is isometrically embedded in* $(X)_U$ *and restricted to* $X^{(2n-2)}$ *this isometry is the canonical embedding for all* n.

In view of this the following question is of interest.

Question 7. *What is the analogue of weak*-*extreme point for utrapowers ? More generally can a point* $x \in X_1$ *that is an extreme point of* $(X^(2n))_1$ *be described in terms of* X *or its ultrapowers alone? What is the geometric description of a point* $x \in X_1$ *that is a weak*-*extreme point of* $(X_U)_1$*?*

We next consider the notion of a unitary for ultrapowers.

Let $W \subset X^*$ be a norming subspace and $x_0 \in X$ be a unit vector. Let $S = \{x^* \in W_1 : x^*(x_0) = 1\}$. The authors of [1] call x_0 a W-unitary if S spans W. It is called a unitary when $W = X^*$. This notion coincides with that of a unitary in the case of C^*-algebras and several geometric properties of unitaries were derived in [1]. Any unitary is an extreme point. Our next result gives an ultra power proof of some of these properties.

Theorem 8. *Let* $x_0 \in X_1$ *be a unitary. Then for any ultrafilter* U,x_0 *is a unitary in* $(X)_U$. *Hence any unitary is a strong extreme point and remains a unitary in all the duals of even order.*

Proof. Let U be an ultrafilter. We make use of the canonical embedding J of $W = (X^*)_U$ as a norming subspace of $(X)^*_U$ given by Lemma 11.1 in [10]. We next observe that x_0 is a W-unitary. We recall that for $(f_i)_U \in (X^*)_U$, $J(f_i)_U((x_i)_U) = lim f_i(x_i)$. Now it follows as in the proof of Corollary 3.2 in [1], since x_0 is a unitary, the seminorm p_W is an equivalent norm on $(X)_U$ and hence x_0 is a unitary in $(X)_U$.

The rest of the conclusions can now be derived from Theorem 2 and the remarks made above about the duals of even order. □

We next give an application of the above remark to best approximation theory.

We recall that $Y \subset X$ is said to be proximinal if for any $x \in X$ there exists a $y \in Y$ such that $d(x, Y) = \|x - y\|$. X is said to be proxbid if it is a proximinal subspace of X^{**}.

We now give a proof using ultra powers of a result from [9]

Theorem 9. *Let* X *be a finite dimensional uniformly convex space.* $C(K, X)$ *is proximinal in all the duals of even order.*

Proof. We first observe that for any ultra filter U, $C(K, X)$ is proximinal in $C(K, (X)_U)$. By proposition 10. 6 in [10] we have that $(X)_U$ is a uniformly convex space. Thus by Corollary 2.2 from [8] we have that $C(K, X)$ is proximinal in $C(K, (X)_U)$.

Since X is finite dimensional we have $C(K, (X)_U) = (C(K, X))_U$ (see the Exercise on page 86 of [10]). Thus by our above remark, for $n > 1$ there exists an ultrafilter U such that $C(K, X) \subset C(K, X)^{(2n)} \subset (C(K, X))_U = C(K, (X)_U)$ canonically. Hence $C(K, X)$ is proximinal in all the duals of even order. □

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