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Some Game Theoretic Aspects of the Skorokhod problem in an orthant

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SOME GAME THEORETIC ASPECTS OF THE SKOROKHOD PROBLEM IN AN ORTHANT

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Abstract: Motivated by the Skorokhod problem in an orthant with time and space dependent coefficients, we propose a general d-person dynamic game with state space constraints. Two types of optimality, viz. Nash equilibrium and utopian equilibrium are discussed, along with the respective dynamic programming principles. In the absolutely continuous case, the framework of HJB equations for differential games and constrained viscosity solutions is invoked. Conditions are given for the solution to the Skorokhod problem to provide Nash/utopian equilibrium to the d-person game. In the case of utopian equilibrium, each value function turns out to be the unique constrained viscosity solution to the appropriate HJB equation; optimality for d control problems is attained at the same point. In the case of Nash equilibrium, the Hamiltonian can be discontinuous; there are d interlinked control problems with state constraints; each value function is a constrained viscosity solution to the appropriate discontinuous HJB equation. Uniqueness does not hold in this case. An interesting feature is that the pushing part of the solution to the Skorokhod problem, being the set of value functions of an appropriate d-person game, can be considered as constrained viscosity solution to the appropriate system of HJB equations.

Key words and phrases: d-person dynamic/differential game, state space constraints, Nash equilibrium, utopian equilibrium, dynamic programming principle, system of HJB equations, constrained viscosity solution, deterministic Skorokhod problem, pushing part of solution to Skorokhod problem, drift, reflection.

1 Introduction

Skorokhod problem plays a fundamental role in the stochastic differential equation formulation of reflected Brownian motion, or more generally reflected diffusions. Naturally diffusions in smooth domains with normal reflection at the boundary were studied first, influenced by heat flow in an insulated domain; see [IW], [KS]. However, once Brownian motion in an orthant with oblique reflection at the boundary was recognised in the 1980's as a heavy traffic approximation to queueing networks (see [HR], [Re], [H]), there has been a lot of interest in stochastic differential equations in nonsmooth domains with oblieque reflection; most general results in this direction are perhaps due to [DI2]. Thanks to the impetus from queueing theory, the deterministic Skorokhod problem has also attracted a lot of attention, beginning with [HR], [Re] dealing with constant coefficients. Deterministic Skorokhod problem with time and space dependent drift and reflection terms has been studied by [Sh], [MMP], [MP], [Ra]; see also the references therein.

In this paper we shall be dealing only with the deterministic Skorokhod problem, SP for short; we shall now briefly recall the deterministic Skorokhod problem in an orthant, following the approach in [Ra].

Let b, R respectively be vector valued, matrix valued functions on $[0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$. Let G denote the d-dimensional positive orthant. Let $w : [0,\infty) \to \mathbb{R}^d$ be a continuous function with $w(0) \in \bar{G}$. The SP consists in finding two continuous functions $Y(\cdot) = (Y_1(\cdot), \dots, Y_d(\cdot)),\ Z(\cdot) = (Z_1(\cdot), \dots, Z_d(\cdot))$ such that the following hold.
(i) For $t \geq 0$,

$$Z(t) = w(t) + \int_0^t b(r, Y(r), Z(r))dr + \int_0^t R(r, Y(r), Z(r))dY(r) \tag{1.1}$$

(ii) $Z(\cdot)$ is $\bar{G}$-valued.
(iii) Each $Y_i(\cdot)$ is nondecreasing, $Y_i \geq 0$.
(iv) $Y_i(\cdot)$ can increase only when $Z_i(\cdot) = 0$, that is,

$$Y_i(t) - Y_i(s) = \int_s^t 1_{\{0\}}(Z_i(r))dY_i(r). \tag{1.2}$$

Under suitable conditions (see Sections 2 and 4) it is known that the SP is well posed. Condition (iv), viz. (1.2) is recognised as a minimality condition.

If $d = 1, b \equiv 0, R \equiv 1$ the solution can be explicitly given, viz.

$$Y(t) = \sup_{0 \leq s \leq t} \max\{0, -w(s)\}.$$

If the evolution $w(\cdot)$ tends to go out of $\mathbb{R}_+$ then the function $Y(\cdot)$ "pushes" it inside $\mathbb{R}_+$. An interesting feature is that the push $Y(\cdot)$ is minimal in the following sense: If $\xi(\cdot) \geq 0$ is a nondecreasing function such that $w(t) + \xi(t) \geq 0$ for all t, then $Y(t) \leq \xi(t)$ for all t; see [H].

In Section 5 of [Ra], an attempt was made to elaborate the above remark to higher dimensions, generalising earlier results of [Re], [CM]. Various natural notions of minimality were defined and their relations to the minimality condition (1.2) were analysed. Some interpretations in terms of the subsidy-surplus model introduced in [Ra] were also given.

The present paper is a continuation of that effort. Here the framework of dynamic game theory is used. Treating $y_i(\cdot)$ as the control exercised by the i-th player, we consider only those controls such that $z(t) \in \bar{G}$ for all t, where $z(\cdot)$ satisfies the state equation

$$z(t) = w(t) + \int_0^t b(r, y(r), z(r))\ dr + \int_0^t R(r, y(r), z(r))\ dy(r)$$

and $y(\cdot) = (y_1(\cdot), \ldots, y_d(\cdot))$. Cost function for each player in this d-person (non zero sum) game is defined. Besides the usual terminal and running costs, another type of cost, which we call control cost, is also natural in our context; this cost is incurred only for the duration when control is applied. See Section 2 below where a general framework is presented. Two notions of optimality are considered, viz. *utopian equilibrium* and *Nash equilibrium*; (the first one owes its name to a comment in [L]). Corresponding dynamic programming principles are also given. The section ends on an optimistic note that a utopia can exist under certain (perhaps ideal) conditions.

If the $\mathbb{R}^d$-valued function $w(\cdot)$ is absolutely continuous, it can be shown that the solution to SP is also absolutely continuous; see Theorem 4.2. This suggests that derivatives of $y_i(\cdot)$ can be treated as controls. In this context the dynamic game is a d-person differential game and becomes more amenable to known techniques of optimization theory. This situation is considered in Sections 3 - 5.

As indicated in Section 3, it is convenient to treat $y(\cdot), z(\cdot)$ as state of the system satisfying (3.1), (3.2). Since $(y(\cdot), z(\cdot))$ takes values in $\bar{G} \times \bar{G}$, the problem is an optimization problem with state-space constraints. The systems of HJB equations are exhibited in Section 3, for Nash as well as utopian equilibria; the implicit "boundary conditions" are also set forth. [So] has been the first to consider control problems with state space contraints; see also [FS], [BC], and the references therein. In the case of utopian equilibrium, there are d control problems each with a d-dimensional control set; should all the d problems attain their minima at the *same* control we have a utopian equilibrium. In the case of Nash equilibrium there are d *interlinked* control problems with one dimensional control sets.

We begin Section 4 by establishing a few useful results concerning SP in an orthant, including Lipschitz continuity of the solution as a function of the initial values. A priori bounds arising in SP provide a convenient compact set in which the controls may be stipulated to take values. It is then established that under suitable conditions, the derivatives of the y-part of the solution to SP provide a Nash equilibrium to the d-person differential game. We also give conditions under which any Nash equilibrium, serving for all $t \leq T$, must be a solution to SP. Thus Skorokhod problem provides a new technique for solving d-person games with state constraints.

Since the work of [So], it is known that the right framework for studying control problems with state-space constraints is through the so called constrained viscosity solution to the HJB equation. The last section investigates the value functions as constrained viscosity solutions to the system of HJB equations. In the case of utopian equilibrium, under some conditions, the value function for the ith player is shown to be the unique bounded uniformly continuous constrained viscosity solution to the appropriate HJB equation. In the case of Nash equilibrium the Hamiltonian can be discontinuous (in the time variable) in general. We show that the value function is a constrained viscosity solution to the discontinuous HJB equation in an appropriate sense, involving the semicontinuous envelopes of the Hamiltonian. It is also shown that uniqueness does not hold in general.

If $u(\cdot) = (u_1(\cdot), \ldots, u_d(\cdot))$ denotes the control and $y_i(t) - y_i = \int_s^t u_i(r)\, dr$ is the corresponding cost function for the ith player, and if the solution to SP provides Nash (resp. utopian) equilibrium, note that the y-part of the solution to SP becomes the respective value function for each i. Thus the pushing part of the solution to SP can be interpreted as a constrained viscosity solution to the appropriate system of HJB equations.

We now indicate some connections with previous works. HJB equations with state constraints have been considered by [CL]. There have been quite a few papers where SP, determinstic as well as stochastic, has played a major role in control and 2-person zero-sum differential game problems. In many of these, the dynamics of the system is governed by the z-part of the solution to SP; often the so called Skorokhod map is assumed to be Lipschitz continuous on the function space. Moreover the reflection terms are essentially taken to be constants. Existence and uniqueness of the value function as viscosity solution to appropriate PDE are often studied. Costs corresponding to singular controls (which are similar to control costs considered here) and ergodic controls are also investigated. To get a flavour of these one may see [ADS], [AB], [BB] and the references therein.

We now fix some notations.

For $0 \leq s \leq t \leq \infty,\ d \geq 1,\ C([s,t] : I\!R^d)$ denotes the space of $I\!R^d$-valued continuous function on $[s,t]$.

$$\begin{aligned} C_{\uparrow}([s,t] : d) &= \{y(\cdot) = (y_1(\cdot), \ldots, y_d(\cdot)) \in C([s,t] : I\!R^d) : \\ y_i(\cdot) &\geq 0, y_i(\cdot) \text{ nondecreasing for all } 1 \leq i \leq d\}. \end{aligned}$$

$L^1_+([s,t] : I\!R^d) = \{u(\cdot) = u_1(\cdot), \ldots, u_d(\cdot)) : u_i(\cdot) \geq 0, u_i(\cdot) \text{ integrable on } [s,t]\}$. $\pi_i^{(s,t)}, \pi_{-i}^{(s,t)}$ are the projections on the function spaces given by

$$(\pi_i^{(s,t)} g)(\cdot) = g_i(\cdot), (\pi_{-i}^{(s,t)} g)(\cdot) = (g_1(\cdot), \ldots, g_{i-1}(\cdot), g_{i+1}(\cdot), \ldots g_d(\cdot)) = g_{-i}(\cdot)$$

where $g(\cdot) = (g_1(\cdot), \ldots, g_d(\cdot))$. We shall often identify $g(\cdot) = (g_i(\cdot), g_{-i}(\cdot))$.

For a function $(r,y,z) \mapsto f(r,y,z)$ on $[0,\infty) \times I\!R^d \times I\!R^d$ denote $\partial_0 f(r,y,z) = \frac{\partial f}{\partial r}(r,y,z)$,

$$\nabla_y f(r,y,z) = \left(\frac{\partial f}{\partial y_1}(r,y,z), \ldots, \frac{\partial f}{\partial y_d}(r,y,z) \right),$$

$$\nabla_z f(r,y,z) = \left(\frac{\partial f}{\partial z_1}(r,y,z), \ldots, \frac{\partial f}{\partial z_d}(r,y,z)\right),$$

$D_{(y,z)}f = (\nabla_y f, \nabla_z f) =$ gradient of f in (y,z)-variables. We may also write $x = (y,z)$ and $D_x f = D_{(y,z)} f$. $\mathbb{M}_d(\mathbb{R})$ denotes the space of all $d \times d$ matrices with real entries. $\mathrm{SP}(b, \theta; F, s, y, w(\cdot))$ or $\mathrm{SP}(b, \theta; F, s, y, z)$ shall denote the Skorokhod problem in the domain F with drift b, reflection matrix/vector θ, and initial data $s, y, w(\cdot)$ or s, y, z respectively.

2 General set up and dynamic programming principle

In this section we describe a constrained d-person dynamic game in a general frame work; two types of optimality are discussed; dynamic programming principle is established in each case for the finite time horizon problem.

$G := \{x \in \mathbb{R}^d : x_i > 0,\ 1 \le i \le d\}$ denotes the d-dimensional positive orthant. We have two functions $b : [0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d, R : [0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{M}_d(\mathbb{R})$ called respectively the *drift* and the *reflection field*; denote $b(s,y,z) = (b_1(s,y,z), \ldots, b_d(s,y,z))$ and $R(s,y,z) = ((R_{ij}(s,y,z)))_{1 \le i,j \le d}$. We make the following assumptions:

(A1): For $1 \le i \le d, b_i$ are bounded measurable; also $(y,z) \mapsto b_i(t,y,z)$ are Lipschitz continuous, uniformly in t; let $|b_i(t,y,z)| \le \beta_i, 1 \le i \le d, \beta = (\beta_1, \ldots, \beta_d)$.

(A2): For $1 \le i,j \le d$, R_{ij} are bounded measurable; also $(y,z) \mapsto R_{ij}(t,y,z)$ are Lipschitz continuous, uniformly in t. Moreover $R_{ii} \equiv 1$ for all i; (this is a suitable normalization.)

(A3): For $i \ne j$ there exist constants W_{ij} such that $|R_{ij}(t,y,z)| \le W_{ij}$. Set $W = ((W_{ij}))$ with $W_{ii} \equiv 0$; we assume that $\sigma(W) < 1$, where $\sigma(W)$ denotes the spectral radius of W.

Under the hypotheses (A1) - (A3) it is known that the Skorokhod problem with generalized space-time dependencies is well posed; see [Ra].

For $s \ge 0, w_0 \in \bar{G}, y_0 \in \bar{G}, w \in C([s,\infty) : \mathbb{R}^d)$ with $w(s) = w_0, y \in C_\uparrow([s,\infty) : d)$ with $y(s) = y_0$ consider the integral equation, called the *state equation*, for $t \ge s$

$$
\begin{aligned}
& z(t; s, y_0, w_0, w(\cdot), y(\cdot)) \equiv z(t) \\
= \; & w(t) + \int_s^t b(r, y(r), z(r)) \, dr + \int_s^t R(r, y(r), z(r)) \, dy(r).
\end{aligned} \qquad (2.1)
$$

That is, for $i = 1, 2, \ldots, d, t \geq s$

$$
\begin{aligned}
z_i(t) \; = \; & w_i(t) + \int_s^t b_i(r, y(r), z(r)) \, dr + y_i(t) - y_i(s) \\
& + \sum_{j \neq i} \int_s^t R_{ij}(r, y(r), z(r)) \, dy_j(r).
\end{aligned} \qquad (2.2)
$$

For a given $y(\cdot)$, to show that (2.1) has a unique solution one can argue as follows. Put

$$
(Sz)(t) = w(t) + \int_s^t b(r, y(r), z(r)) \, dr + \int_s^t R(r, y(r), z(r)) \, dy(r).
$$

By Lipschitz continuity in z-variables,

$$
\sup_{s \leq r \leq t} |(Sz)(r) - (S\hat{z})(r)| \leq K \left[|t - s| + \sum_{j=1}^{d} |y_j(t) - y_j(s)| \right] \left[\sup_{s \leq r \leq t} |z(r) - \hat{z}(r)| \right]
$$

for any $z, \hat{z} \in C([s, \infty) : \mathbb{R}^d)$. As y_j are continuous, by contraction mapping principle applied repeatedly one gets $s < t_1 < t_2 < \ldots < t_n < \ldots$ such that (2.1) is well posed on $[s, t_n]$ for each n. Now apply an argument as in Step 7 of Section 3 of [Ra].

We consider a d-person game whose state equation is given by (2.1). The nondecreasing function $y_i(\cdot)$ represents the control for i-th player, $i = 1, 2, \ldots, d$. Moreover we consider only those controls which ensure that $z(\cdot)$ lives in $\bar{G}$; so it is a *d-person dynamic game with state constraints*. It turns out to be suitable to consider $(y(\cdot), z(\cdot))$ as the state of the system.

Given $s \geq 0, w_0, y_0 \in \bar{G}, w(\cdot)$ as above, a function $y(\cdot) \in C_{\uparrow}([s, \infty) : d)$ is said to be a *feasible control* if $y(s) = y_0$ and the solution $z(\cdot)$ of (2.1) satisfies $z(\cdot) \geq 0$ on $[s, \infty)$; that is, $z_i(t; s, y_0, w_0, w(\cdot), y(\cdot)) \geq 0$ for all $t \geq s, 1 \leq i \leq d$. Let $\mathcal{A}(s, y_0, w_0, w(\cdot))$ denote the set of feasible controls; under (A1) - (A3), $\mathcal{A}(s, y_0, w_0, w(\cdot)) \neq \phi$ as the Skorokhod problem is well posed. Denote $\mathcal{A}(s, y_0, w_0, w(\cdot); T) = \{y \mid_{[s,T]}: y \in \mathcal{A}(s, y_0, w_0, w(\cdot))\}$. For $i = 1, 2, \ldots, d$ set $\mathcal{A}^{(i)}(s, y_0, w_0, w(\cdot)) \equiv \pi_i^{(s,\infty)} \mathcal{A}(s, y_0, w_0, w(\cdot)) := \{y_i \in C_{\uparrow}([s, \infty) : 1) : \exists y \in \mathcal{A}(s, y_0, w_0, w(\cdot))$ such that $y_i = \pi_i^{(s,\infty)} y\}$. Clearly $\mathcal{A}^i(\cdots)$ is the set of controls available to i-th player.

For each player i, initial time s, terminal time T, given evolution $w(\cdot)$ and feasible control $y(\cdot) = (y_1(\cdot), \ldots, y_d(\cdot))$ the cost J_i is given by

$$
\begin{aligned}
& J_i(s, y_0, w_0, w(\cdot); T, y(\cdot)) = g_i(T, y(T), z(T)) \\
& + \int_s^T L_i(r, y(r), z(r)) \, dr + \int_s^T M_i(r, y(r), z(r)) \, dy_i(r)
\end{aligned} \qquad (2.3)
$$

where $z(\cdot) \equiv z(\cdot; s, y_0, w_0, w(\cdot), y(\cdot))$. The three components are generally nonnegative. The first two components are respectively the *terminal cost* and the *running cost*; these are standard.

The third component on the r.h.s. of (2.3) may be called the *control cost* (or the *subsidy cost*); this is the cost of exercising the control (or mobilizing the the subsidy) $y_i(\cdot)$. We shall see later that this leads to an interesting mathematical interpretation. Besides, this is not an unreasonable extension, and we shall indicate now in terms of the subsidy-surplus model developed in [Ra]. For example, if the subsidy is in the form of food-aid to a drought hit country/region, temporary transport/harbour facilities, personnel employed on short term basis, etc. may be needed just for the duration of the aid program; often shipping costs may have to be borne by the recipient. When prices crash and a governmental agency comes up with a scheme of support prices, especially for agricultural/dairy products, the recipient may have to incur the cost of transporting the produce; such situations are not unknown even in developed countries. As another illustration one may mention welfare sectors like public health, education, research and development, etc.; these sectors can be non-surplus sectors, but having a positive effect on the economy as a whole; in this case the subsidy cost can be interpreted as the cost of running such a department.

Now we describe two notions of equilibrium.

Let $0 \leq s \leq T, y_0, w_0 \in \bar{G}, w \in C([s, \infty) : \mathbb{R}^d)$ with $w(s) = w_0$. A feasible control $y^* \in \mathcal{A}(s, y_0, w_0, w(\cdot); T)$ is said to be a *utopian equilibrium* in $\mathcal{A}(s, y_0, w_0, w(\cdot); T)$ if for each $i = 1, 2, \ldots, d$

$$J_i(s, y_0, w_0, w(\cdot); T, y^*(\cdot)) \leq J_i(s, y_0, w_0, w(\cdot); T, y(\cdot)) \tag{2.4}$$

for all $y \in \mathcal{A}(s, y_0, w_0, w(\cdot); T)$; the choice of the adjective 'utopian' in this context is due to a comment in [L]. In this case note that d functions attain their respective minima at the same point $y^*(\cdot)$.

A feasible control $y^*(\cdot) = (y_1^*(\cdot), \ldots, y_d^*(\cdot)) \in \mathcal{A}(s, y_0, w_0, w(\cdot); T)$ is said to be a *Nash equilibrium* in $\mathcal{A}(s, y_0, w_0, w(\cdot); T)$ if for each $i = 1, 2, \ldots, d$

$$\begin{aligned} & J_i(s, y_0, w_0, w(\cdot); T, y^*(\cdot)) \\ = \; & \inf\{J_i(s, y_0, w_0, w(\cdot); T, y(\cdot)) : y_{-i}(\cdot) = y_{-i}^*(\cdot), y \in \mathcal{A}(s, y_0, w_0, w(\cdot); T)\}. \end{aligned} \tag{2.5}$$

In this situation each player seeks to minimize her/his cost assuming that the other players make rational decisions. It is obvious that a utopian equilibrium is also a Nash equilibrium.

We now state a lemma whose proof is an easy consequence of uniqueness of solution to (2.1); parts (b), (c) of the lemma indicate that a "switching condition" holds.

Lemma 2.1 *(a) Let $y(\cdot) \in \mathcal{A}(s, y_0, w_0, w(\cdot))$ and $z(\cdot) \equiv z(\cdot; s, y_0, w_0, w(\cdot), y(\cdot))$ be the associated solution to (2.1). Let $t \geq s$. Define $y^{(t)}(r) = y(r), w^{(t,z(t))}(r) = w(r) - w(t) + z(t)$, for $r \geq t$. Then $y^{(t)}(\cdot) \in \mathcal{A}(t, y(t), z(t), w^{(t,z(t))}(\cdot))$.*

(b) Let $\hat{y}(\cdot) \in \mathcal{A}(t, y(t), z(t), w^{(t,z(t))}(\cdot))$ and $\hat{z}(\cdot) = z(\cdot; t, y(t), z(t), w^{(t,z(t))}(\cdot), \hat{y}(\cdot))$ be the associated solution to the state equation. Define

$$\tilde{y}(r) = \begin{cases} y(r), & s \leq r \leq t \\ \hat{y}(r), & r \geq t \end{cases}.$$

Then $\tilde{y}(\cdot) \in \mathcal{A}(s, y_0, w_0, w(\cdot))$ and $\tilde{z}(\cdot) = z(\cdot; s, y_0, w_0, w(\cdot), \tilde{y}(\cdot))$ is given by

$$\tilde{z}(r) = \begin{cases} z(r), & s \leq r \leq t \\ \hat{z}(r), & r \geq t \end{cases}.$$

(c) Let $1 \leq i \leq d$. Let $f \in C_{\uparrow}([t, \infty) : 1)$ be such that

$$(f, y_{-i}^{(t)}) \equiv (y_1^{(t)}, \ldots, y_{i-1}^{(t)}, f, y_{i+1}^{(t)}, \ldots, y_d^{(t)})$$

belongs to $\mathcal{A}(t, y(t), z(t), w^{(t,z(t))}(\cdot))$. Define $\tilde{y}$ by $\tilde{y}(r) = y(r)$ for $s \leq r \leq t$, and $\tilde{y}_j(r) = y_j(r), j \neq i, \tilde{y}_i(r) = f(r)$ for $r \geq t$. Then $\tilde{y}(\cdot) \in \mathcal{A}(s, y_0, w_0, w(\cdot))$. □

Regarding dynamic programming principle we look at the case of Nash equilibrium first. Fix the terminal time $T > 0$. Let the notation be as in the preceding lemma. Denote $\mathcal{A}(t) = \mathcal{A}(t, y(t), z(t), w^{(t,z(t))}(\cdot); T)$ for $t \in [s, T]$. For $i = 1, 2, \ldots, d$ put

$$\begin{aligned} & V^{(N,i)}(s, y_0, w_0, w(\cdot); T, y_{-i}(\cdot)) \\ = \; & \inf\{J_i(s, y_0, w_0, w(\cdot); T, (f(\cdot), y_{-i}(\cdot))) : (f, y_{-i}) \in \mathcal{A}(s)\}. \end{aligned} \tag{2.6}$$

Of course, infimum over an empty set is taken to be $+\infty$. $V^{(N,i)}$ is called the *N-value function* for the i-th player; here $y_{-i}(\cdot)$ represents the controls exercised by the other $(d-1)$ players; the superscript N denotes that the context of discussion is Nash equilibrium. With $T, i, w(\cdot), y_{-i}(\cdot)$ fixed, it may be considered a control problem with domain of feasible controls given by

$$\pi_i^{(s,T)}[\mathcal{A}(s) \cap (\pi_{-i}^{(s,T)})^{-1}\{y_{-i}(\cdot)\}].$$

Note that

$$\begin{aligned} & V^{(N,i)}(t, y(t), z(t), w^{(t,z(t))}(\cdot); T, y_{-i}^{(t)}(\cdot)) \\ = \; & \inf\{J_i(t, y(t), z(t), w^{(t,z(t))}(\cdot); T, (f, y_{-i}^{(t)})) : (f, y_{-i}^{(t)}) \in \mathcal{A}(t)\} \end{aligned} \tag{2.7}$$

gives the value function evaluated along a feasible trajectory for the ith player.

In view of Lemma 2.1, the following monotonicity property along a feasible trajectory is easy to establish.

Lemma 2.2 *Notation as above. Fix i. For any feasible control $y(\cdot) \equiv (y_i(\cdot), y_{-i}(\cdot))$, associated solution $z(\cdot)$ and $s \leq t_1 \leq t_2 \leq T$,*

$$\begin{aligned} & V^{(N,i)}(t_1, y(t_1), z(t_1), w^{(t_1, z(t_1))}(\cdot); T, y_{-i}(\cdot)) \\ \leq \;& V^{(N,i)}(t_2, y(t_2), z(t_2), w^{(t_2, z(t_2))}(\cdot); T, y_{-i}(\cdot)) \\ & + \int_{t_1}^{t_2} L_i(r, y(r), z(r))\ dr + \int_{t_1}^{t_2} M_i(r, y(r), z(r))\ dy_i(r). \end{aligned} \tag{2.8}$$

□

Theorem 2.3 *(Dynamic Programming Principle)*
(i) Assume (A1), (A2). Let $T > 0, s \in [0,T], y_0 \in \bar{G}, w_0 \in \bar{G}, w(\cdot) \in C([s,T] : I\!R^d)$ with $w(s) = w_0$. Fix $1 \leq i \leq d$. Let $y_{-i}(\cdot) \in C_{\uparrow}([s,T] : d-1)$ with $y_j(s) = y_{0j} =$ j-th coordinate of y_0, for $j \neq i$. Assume that

$$\pi_i^{(s,T)}[\mathcal{A}(s) \cap (\pi_{-i}^{(s,T)})^{-1}\{y_{-i}(\cdot)\}] \neq \phi. \tag{2.9}$$

Then $y_i^*(\cdot)$ is optimal in $\pi_i^{(s,T)}[\mathcal{A}(s) \cap (\pi_{-i}^{(s,T)})^{-1}\{y_{-i}\}]$, that is,

$$V^{(N,i)}(s, y_0, w_0, w(\cdot); T, y_{-i}(\cdot)) = J_i(s, y_0, w_0, w(\cdot); T, (y_i^*, y_{-i})), \tag{2.10}$$

if and only if for any $t \in [s,T]$

$$\begin{aligned} & V^{(N,i)}(s, y_0, w_0, w(\cdot); T, y_{-i}) - V^{(N,i)}(t, \bar{y}(t), \bar{z}(t), w^{(t,\bar{z}(t))}(\cdot); T, y_{-i}) \\ = \;& \int_s^t L_i(r, \bar{y}(r), \bar{z}(r))\ dr + \int_s^t M_i(r, \bar{y}(r), \bar{z}(r))\ dy_i^*(r) \end{aligned} \tag{2.11}$$

where $\bar{y} = (y_i^*, y_{-i})$ and $\bar{z}(\cdot) = z(\cdot; s, y_0, w_0, w(\cdot), \bar{y}(\cdot))$. Moreover, if (2.11) holds then for any $t \in [s,T]$

$$\begin{aligned} & V^{(N,i)}(t, \bar{y}(t), \bar{z}(t), w^{(t,\bar{z}(t))}(\cdot); T, y_{-i}(\cdot)) \\ = \;& J_i(t, \bar{y}(t), \bar{z}(t), w^{(t,\bar{z}(t))}(\cdot); T, (y_i^*, y_{-i})) \end{aligned} \tag{2.12}$$

that is, $y_i^{*(t)}(\cdot)$ is optimal on $[t,T]$ for the data $t, \bar{y}(t), \bar{z}(t), w^{(t,\bar{z}(t))}(\cdot), y_{-i}^{(t)}(\cdot)$.

(ii) Assume (A1) - (A3). Then $y^*(\cdot) \in \mathcal{A}(s, y_0, w_0, w(\cdot); T)$ is a Nash equilibrium if and only if for $s \leq t_1 \leq t_2 \leq T, i = 1, 2, \ldots, d$

$$\begin{aligned} & V^{(N,i)}(t_1, y^*(t_1), z^*(t_1), w^{*(t_1)}(\cdot); T, y_{-i}^*(\cdot)) \\ = \;& V^{(N,i)}(t_2, y^*(t_2), z^*(t_2), w^{*(t_2)}(\cdot); T, y_{-i}^*(\cdot)) \\ & + \int_{t_1}^{t_2} L_i(r, y^*(r), z^*(r))\ dr + \int_{t_1}^{t_2} M_i(r, y^*(r), z^*(r))\ dy_i^*(r) \end{aligned} \tag{2.13}$$

where $z^*(\cdot) = z(\cdot; s, y_0, w_0, w(\cdot), y^*(\cdot)), w^{*(t)}(\cdot) = w^{(t,z^*(t))}(\cdot)$. Moreover, whenever (2.13) holds, for any $t \in [s,T]$ the restriction $y^{*(t)}(\cdot)$ of y^* is a Nash equilibrium in $\mathcal{A}(t, y^*(t), z^*(t), w^{*(t)}(\cdot); T)$.

Proof: Given $s, y_0, w_0, w(\cdot)$ and a feasible control $y(\cdot)$, for $t \geq s$, corresponding to $t, y(t), z(t), w^{(t)}(\cdot)$, any feasible control $\hat{y}(\cdot)$ need to satisfy $\hat{y}(t) = y(t), \hat{y}(\cdot)$ nondecreasing. To prove, Lemma 2.2 and Theorem 2.3, one may imitate the approach given in Chapter 1 of [FS], for example; the modifications needed are pretty clear in view of Lemma 2.1. □

Remark 2.4 In Section 4 a sufficient condition will be given to ensure (2.9) see Theorem 4.10.

In connection with utopian equilibrium, with $T, s, y_0, w_0, w(\cdot), \mathcal{A}(s)$ as before, for $i = 1, 2, \ldots, d$ define

$$\begin{aligned} & V^{(U,i)}(s, y_0, w_0, w(\cdot); T) \\ = \; & \inf\{J_i(s, y_0, w_0, w(\cdot); T, y(\cdot)) : y(\cdot) \in \mathcal{A}(s)\}. \end{aligned} \qquad (2.14)$$

The superscript U indicates that the context of discussion is utopian equilibrium; this is the *U-value function* for the i-th player. The following dynamic programming principle for utopian equilibrium can now be stated without proof.

Theorem 2.5 *Assume (A1) - (A3). Let the notations be as in the preceding theorem. Then $y^*(\cdot)$ is a utopian equilibrium in $\mathcal{A}(s, y_0, w_0, w(\cdot); T)$ if and only if for $s \leq t_1 \leq t_2 \leq T, 1 \leq i \leq d$,*

$$\begin{aligned} & V^{(U,i)}(t_1, y^*(t_1), z^*(t_1), w^{*(t_1)}(\cdot); T) \\ = \; & V^{(U,i)}(t_2, y^*(t_2), z^*(t_2), w^{*(t_2)}(\cdot); T) \\ & + \int_{t_1}^{t_2} L_i(r, y^*(r), z^*(r)) \; dr + \int_{t_1}^{t_2} M_i(r, y^*(r), z^*(r)) \; dy_i^*(r). \end{aligned} \qquad (2.15)$$

Moreover, whenever (2.15) holds, for any $t \in [s, T]$ the restriction $y^{*(t)}(\cdot)$ of y^* is a utopian equilibrium in $\mathcal{A}(t, y^*(t), z^*(t), w^{*(t)}(\cdot); T)$. □

Note the difference between the two situations. In the case of utopian equilibrium, for each i the minimization problem is over $\mathcal{A}(s, y_0, w_0, w(\cdot); T)$; for each i the variables and the parameters are the same; both $y^*(\cdot), z^*(\cdot)$ vary over $\bar{G}$. In the case of Nash equilibrium, we have d interlinked minimization problems, domain of minimization changing with i; in the ith equation in (2.13), $y^*_{-i}(\cdot)$ acts like a parameter; also while $y_i^*(\cdot)$ varies over $[0, \infty), z^*(\cdot)$ varies over $\bar{G}$.

We end this section by exhibiting a class of examples to show that utopia can exist.

Theorem 2.6 *In addition to (A1) - (A3) assume*
(i) $b_i(\cdot, \cdot)$ are functions of t, y (independent of z) such that

$$b_i(t, \hat{y}) \geq b_i(t, y)$$

whenever $\hat{y} \leq y$, for $t \geq 0, 1 \leq i \leq d$;
(ii) R_{ij} are functions of t (independent of y, z) such that $t \mapsto R_{ij}(t)$ is nondecreasing, and $R_{ij}(t) \leq 0, i \neq j, 1 \leq i, j \leq d$;
(iii) g_i, L_i are nonnegative functions of t, y (independent of z) such that

$$L_i(t, \hat{y}) \leq L_i(t, y), g_i(T, \hat{y}) \leq g_i(T, y),$$

whenever $\hat{y} \leq y$, for $t \geq 0, 1 \leq i \leq d$;
(iv) $M_i(\cdot, \cdot, \cdot) = C_i > 0$ is a constant, $1 \leq i \leq d$.
Let $0 \leq s \leq T, y \in \bar{G}$ be fixed; let $w(\cdot) \in C([s, \infty) : \mathbb{R}^d)$ with $w(s) = w_0 \in \bar{G}$. Let Yw, Zw be the solution to SP$(b, R; G, s, y, w(\cdot))$, the Skorokhod problem in the orthant corresponding to $w(\cdot)$ starting at s, drift b, reflection R, with $Yw(s) = y$. Then Yw is the unique utopian equilibrium in $\mathcal{A}(s, y, w_0, w(\cdot); T)$.

If in addition, $g_i = L_i \equiv 0, M_i \equiv 1$ for all i, then

$$V^{(U,i)}(s, y, w_0, w(\cdot); T) = (Yw)_i(T) - y_i.$$

Proof: By Theorem 5.3 of [Ra], under (A1) - (A3) and (i), (ii) above $(Yw)_i(t) - y_i \leq y_i(t) - y_i$, for all $t \in [s, T], 1 \leq i \leq d$ for any $y(\cdot) \in \mathcal{A}(s, y, w_0, w(\cdot))$. The result now is immediate. □

3 A special case and HJB equations

In this section we consider the case when $w(\cdot) \equiv$ constant; the solution to the Skorokhod problem is then known to be absolutely continuous; see Theorem 4.2. This suggests that perhaps one can look at just those controls which are absolutely continuous; in such a case we might as well treat the derivatives as controls. This enables the dynamic game introduced in the preceding section to be viewed upon as a d-person differential game. The corresponding system of HJB equations with state constraints can also be derived; of course the domain will be nonsmooth.

Note: Observe that considering the case $w(\cdot) \equiv$ constant is equivalent to considering the case when $w(\cdot)$ is absolutely continuous; because in the latter case the derivative $\dot{w}(\cdot)$ can be absorbed into the drift.

Let b, R be as in Section 2. Fix the terminal time $T > 0$. For $s \in [0, T], y, z \in \bar{G}, u(\cdot) = (u_1(\cdot), \ldots, u_d(\cdot)) \in L^1_+([s, T] : \mathbb{R}^d)$, state equations is a pair of equations,

$$y(t) = y + \int_s^t u(r)\, dr, \qquad (3.1)$$

$$
\begin{aligned}
z(t) &:= z(t; s, y, z, u(\cdot)) \\
&= z + \int_s^t b(r, y(r), z(r))\ dr + \int_s^t R(r, y(r), z(r))\ u(r)\ dr
\end{aligned} \tag{3.2}
$$

equivalently, for $1 \le i \le d, t \ge s$

$$
y_i(t) = y_i + \int_s^t u_i(r)\ dr \tag{3.3}
$$

$$
\begin{aligned}
z_i(t) &= z_i + \int_s^t b_i(r, y(r), z(r))\ dr + \int_s^t u_i(r)\ dr \\
&\quad + \sum_{j \neq i} \int_s^t R_{ij}(r, y(r), z(r))\ u_j(r)\ dr.
\end{aligned} \tag{3.4}
$$

Clearly there exists a unique pair $y(\cdot) \in C_{\uparrow}([s,T] : d), z(\cdot) \in C([s,T] : I\!R^d)$ solving (3.1), (3.2). We will treat $u(\cdot)$ as control.

Given $s \in [0,T], y, z \in \bar{G}$ let $\mathcal{U}(s,y,z;T) = \{u(\cdot) \in L^1_+([s,T] : I\!R^d) : z_i(\cdot) \ge 0\ \forall\ i\}$; any $u(\cdot)$ in $\mathcal{U}(s,y,z;T)$ may be called a *feasible control* or *feasible marginal subsidy.* The *cost function* for the ith player is given by

$$
\begin{aligned}
&J_i(s, y, z; T, u(\cdot)) = g_i(T, y(T), z(T)) \\
&+ \int_s^T L_i(r, y(r), z(r))\ dr + \int_s^T M_i(r, y(r), z(r))\ u_i(r)\ dr.
\end{aligned} \tag{3.5}
$$

The three components of the cost are interpreted as before. A control $u^*(\cdot) \in \mathcal{U}(s,y,z;T)$ is called a *utopian equilibrium* in $\mathcal{U}(s,y,z;T)$ if

$$
J_i(s,y,z;T,u^*(\cdot)) \le J_i(s,y,z;T,u(\cdot)) \tag{3.6}
$$

for all $u \in \mathcal{U}(s,y,z;T), i = 1, 2, \ldots, d$. Similarly a control $u^*(\cdot) = (u_1^*(\cdot), \ldots, u_d^*(\cdot)) \in \mathcal{U}(s,y,z;T)$ is called a *Nash equilibrium* in $\mathcal{U}(s,y,z;T)$ if for $i = 1, 2, \ldots, d$

$$
\begin{aligned}
&J_i(s,y,z;T,u^*(\cdot)) \\
= \ &\inf\{J_i(s,y,z;T,u(\cdot)) : u_{-i} = u^*_{-i}, u \in \mathcal{U}(s,y,z;T)\}.
\end{aligned} \tag{3.7}
$$

Remark 3.1 Under (A1) - (A3), by Theorem 4.2 in Section 4 below, $\mathcal{U}(s,y,z;T) \neq \phi$ for any $s \in [0,T], y, z \in \bar{G}$. Analogue of Lemma 2.1 is easy to establish; in fact, as discontinuity in control $u(\cdot)$ is allowed the results are exact counterparts of the conventional ones; for example if $u(\cdot) \in \mathcal{U}(s,y,z;T)$ then $u^{(t)}(\cdot) \in \mathcal{U}(t, y(t), z(t); T)$ for $t \ge s$ where $y(\cdot), z(\cdot)$ denote the solution to (3.1), (3.2) corresponding to $u(\cdot)$. Also, using the analogue of the switching condition, if $(s,y,z) \in [0,T) \times G \times G, c = (c_1, \ldots, c_d)$ with $c_j \ge 0\ \forall j$, it can then be shown that there exists $u \in \mathcal{U}(s,y,z;T)$ such that $\lim\limits_{r \downarrow s} u(r) = c$. □

To derive the system of HJB-equations with state constraints to be satisfied by the appropriate value functions, we consider first the case of Nash equilibrium.

Let $u_{-i}(\cdot) = (u_1(\cdot), \ldots, u_{i-1}(\cdot), u_{i+1}(\cdot), \ldots, u_d(\cdot)) \in L^1_+([0,T] : I\!R^{d-1})$; here i is fixed. Define for $s \in [0,T], y, z \in \bar{G}$

$$\begin{aligned} & V^{(N,i)}(s,y,z;T,u_{-i}(\cdot)) \\ = \quad & \inf\{J_i(s,y,z;T,v(\cdot)) : v_{-i} = u^{(s)}_{-i}, v \in \mathcal{U}(s,y,z;T)\} \end{aligned} \tag{3.8}$$

with the convention that infimum over an empty set is taken to be $+\infty$. This is the value function for the control problem with $u_{-i}(\cdot)$ acting as a parameter.

With $i, u_{-i}(\cdot)$ fixed, it is convenient to define, for $s \in [0,T], y, z \in I\!R^d, c \in [0,\infty), I\!R^{2d}$-valued vector by

$$f^{(N,i)}(s,(y,z),c) := \begin{pmatrix} \begin{pmatrix} c \\ u_{-i}(s) \end{pmatrix} \\ b(s,y,z) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & R(s,y,z) \end{pmatrix} + \begin{pmatrix} 0 \\ \begin{pmatrix} c \\ u_{-i}(s) \end{pmatrix} \end{pmatrix} \tag{3.9}$$

where $(c, u_{-i}(s)) := (u_1(s), \ldots, u_{-i}(s), c, u_{i+1}(s), \ldots, u_d(s))$, the square matrix on r.h.s. is of order $2d$, and the scalar by

$$C^{(N,i)}(s,(y,z),c) = L_i(s,y,z) + M_i(s,y,z)c. \tag{3.10}$$

(It is to be kept in mind that $u_{-i}(\cdot)$ acts as a parameter.) In this notation state equations (3.1), (3.2) can be written as

$$d\begin{pmatrix} y(t) \\ z(t) \end{pmatrix} = f^{(N,i)}(t,(y(t),z(t)),u_i(t))\, dt, \ t > s \tag{3.11}$$

with initial value $(y(s), z(s)) = (y,z)$, and the cost function for i-th player as

$$\begin{aligned} & J_i(s,y,z;T,(u_i(\cdot),u_{-i}(\cdot)) \\ = \quad & g_i(T,y(T),z(T)) + \int_s^T C^{(N,i)}(r,(y(r),z(r)),u_i(r))\, dr. \end{aligned} \tag{3.12}$$

Next the *Hamiltonian* $H^{(N,i)}$ (for ith player in case of Nash equilibrium) is defined by

$$\begin{aligned} & H^{(N,i)}(s,y,z),p) \\ = \quad & \sup\{[-\langle p, f^{(N,i)}(s,(y,z),c)\rangle - C^{(N,i)}(s,(y,z),c)] : 0 \le c < \infty\} \end{aligned} \tag{3.13}$$

for $s \ge 0, y, z \in I\!R^d, p \in I\!R^{2d}$.

Assume now that

$$\begin{aligned}\mathcal{U}(s,y,z;T,u_{-i}(\cdot)) &:= \{u_i(\cdot)\in L^1_+([s,T]:\mathbb{R}) : (u_i(\cdot),u_{-i}(\cdot))\in \mathcal{U}(s,y,z;T)\} \\ &\neq \phi. \qquad (3.14)\end{aligned}$$

In view of Remark 3.1, analogues of Lemma 2.2 and Theorem 2.3 are easy to obtain. Assume also that $u_{-i}(\cdot)$ is right continuous, and that $V^{(N,i)}$ is continuously differentiable. Assume that b, R are continuous in t.

Let $s\in[0,T), y,z\in G$ (that is, interior point). Then there is $u_i(\cdot)\in\mathcal{U}(s,y,z;T,u_{-i}(\cdot))$ which is right continuous such that $\lim_{r\downarrow s} u_i(r) = c$. Then by the analogue of Lemma 2.2 it is not difficult to see, denoting $V(s,y,z) = V^{(N,i)}(s,y,z;T,u_{-i}(\cdot)), f(s,(y,z),c) = f^{(N,i)}(s,(y,z),c), C(s,(y,z),c) = C^{(N,i)}(s,(y,z),c)$, that

$$\partial_0 V(s,y,z) + \langle D_{y,z}V(s,y,z), f(s,(y,z),c)\rangle + C(s,(y,z),c) \geq 0 \qquad (3.15)$$

where $\partial_0 = \frac{\partial}{\partial s}$ and $D_{y,z}$ = gradient in the (y,z)-variables. As $c\in[0,\infty)$ is arbitrary, we get

$$\begin{aligned}\inf\ &\{[\partial_0 V(s,y,z) + \langle D_{y,z}V(s,y,z), f(s,(y,z),c)\rangle + C(s,(y,z),c)] : c\geq 0\} \\ \geq\ & 0. \qquad (3.16)\end{aligned}$$

If $u_i(\cdot)\in\mathcal{U}(s,y,z;T,u_{-i}(\cdot))$ is optimal, and right continuous at s, then by the dynamic programming principle (analogue of Theorem 2.3), note that equality holds in (3.15) with c replaced by $u_i(s)$. Therefore V satisfies

$$\begin{aligned}\inf\ &\{[\partial_0 V(s,y,z) + \langle D_{y,z}V(s,y,z), f(s,(y,z),c)\rangle + C(s,(y,z),c)] : c\in[0,\infty)\} \\ =\ & 0. \qquad (3.17)\end{aligned}$$

With the Hamiltonian introduced in (3.13), we write (3.17) in the conventional (but equivalent) form as

$$-\partial_0 V(s,y,z) + H^{(N,i)}(s,(y,z),D_{y,z}V(s,y,z)) = 0, \qquad (3.18)$$

for $(s,y,z)\in[0,T)\times G\times G$. It is to be remembered that $u_{-i}(\cdot)$ acts as a parameter in (3.18). Equation (3.18) is a *Hamilton-Jacobi-Bellman equation.*

Recall that we admit only those controls such that $z_k(\cdot)\geq 0$ for all k. This restriction implies a condition at the boundary, and leads to what is referred to as a "problem with state constraints" in the literature. [So] was the first to consider such problems; one may consult [BC] or [FS] for detailed discussions. We describe below the state constraint in our context.

Since $y(t)\in\bar{G}$ automatically note that $u_i(\cdot)$ is feasible if and only if $(y(t),z(t))\in\bar{G}\times\bar{G}$ for all t; hence $\bar{G}\times\bar{G}$ is taken at the state space. For $y,z\in\mathbb{R}^d$ denote $I_y = \{j : y_j = 0\}, I_z =$

$\{k : z_k = 0\}$. Clearly $(y,z) \in \partial(G \times G) \Leftrightarrow I_y \cup I_z \neq \phi$. For $(y,z) \in \partial(G \times G)$ let $\mathcal{N}_{(y,z)}$ = set of all unit inward normals at (y,z). It is not difficult to see that $n \in \mathcal{N}_{(y,z)} \Leftrightarrow n$ is a convex combination of e_j, e_{d+k} with $j \in I_y, k \in I_z$ where e_ℓ's denote unit vectors in $\mathbb{R}^{2d}$.

Let $(y,z) \in \partial(G \times G)$. Suppose there is a feasible control $u_i(\cdot)$ with some initial data and t such that $(y(t), z(t)) = (y,z)$ where $(y(\cdot), z(\cdot))$ is the solution to the state equation (3.11) corresponding to the control $u_i(\cdot)$. Since $(y(t), z(t))$ is a boundary point note that $(y(r), z(r)) \in \bar{G} \times \bar{G}$ for all $r \geq t \Leftrightarrow \left\langle \begin{pmatrix} y(r) - y(t) \\ z(r) - z(t) \end{pmatrix}, n \right\rangle \geq 0$ for all $n \in \mathcal{N}_{(y,z)}, r \geq t$. Hence $\left\langle d \begin{pmatrix} y(t) \\ z(t) \end{pmatrix}, n \right\rangle \geq 0$ for all $n \in \mathcal{N}_{(y,z)}$. So by (3.11) we get

$$\left\langle f^{(N,i)}(t, (y,z), u_i(t)), n \right\rangle \geq 0, \ \forall n \in \mathcal{N}_{(y,z)}. \qquad (3.19)$$

Note that $u_i(\cdot), u_j(\cdot), j \neq i$ are always nonnegative (even for $i, j \notin I_y$). So by (3.9), the "boundary condition" (3.19) is essentially

$$b_i(t,y,z) + u_i(t) + \sum_{\ell \neq i} R_{i\ell}(t,y,z) \ u_\ell(t) \geq 0 \qquad (3.20)$$

if $i \in I_z$, and

$$b_k(t,y,z) + R_{ki}(t,y,z) u_i(t) + u_k(t) + \sum_{\ell \neq i,k} R_{k\ell}(t,y,z) \ u_\ell(t) \geq 0, \ \ k \neq i, \ \ k \in I_z \qquad (3.21)$$

again remembering that $i, u_j(\cdot), j \neq i$ are fixed.

Let $(y,z) \in \partial(G \times G)$. Let $u_i(\cdot) \in \mathcal{U}(s,y,z;T,u_{-i}(\cdot))$ be optimal and right continuous. Denoting $V^{(N,i)}, f^{(N,i)}, C^{(N,i)}$ respectively by V, f, C, as $V \in C^1([0,T] \times \bar{G} \times \bar{G}), f$ is continuous, by the dynamic programming principle we get

$$\partial_0 V(s,y,z) + \langle D_{y,z} V(s,y,z), f(s,(y,z),u_i(s)) \rangle + C(s,(y,z),u_i(s)) = 0. \qquad (3.22)$$

Assume that the Hamiltonian is continuous. As (3.18) holds on $[0,T) \times G \times G$, it is now clear that it is true on $[0,T] \times \bar{G} \times \bar{G}$ as well. Consequently by (3.22) we now get

$$\begin{aligned} & H(s,(y,z), D_{(y,z)} V(s,y,z)) \\ &= -\langle D_{(y,z)} V(s,y,z), f(s,(y,z),u_i(s)) \rangle - C(s,(y,z),u_i(s)). \end{aligned} \qquad (3.23)$$

Now by the definition of H with $p = D_{(y,z)} V(s,y,z) - \gamma n$, for any $\gamma \geq 0, n \in \mathcal{N}_{(y,z)}$ by (3.19), (3.23) we get

$$H(s,(y,z), D_{(y,z)} V(s,y,z) - \gamma n) \geq H(s,(y,z), D_{(y,z)} V(s,y,z)). \qquad (3.24)$$

Thus the state constraint (3.19), which is essentially (3.20), (3.21), implies that the implicit inequality (3.24) has to be satisfied by $D_{(y,z)} V^{(N,i)}$ at a boundary point. The heuristics above on state constraints have been influenced by the discussion on pp. 102-103 of [FS].

It is well known that "viscosity solutions" is the appropriate framework to treat Hamilton-Jacobi-Bellman equations; in particular "constrained viscosity solutions" form the proper context to take care of problems with state constraints.

Lemma 3.2 *Let $D \subseteq \mathbb{R}^k$ be a convex set (not necessarily smooth or bounded). For $\xi \in \partial D$ let $\mathcal{N}_\xi$ denote the set of unit inward normal vectors at ξ. Let g be a C^1-function and $x \in \partial D$ such that $g(x) = \min\{g(x') : x' \in \bar{D}\}$. Then $\nabla g(x) = \gamma n$ for some $n \in \mathcal{N}_x, \gamma \geq 0$.*

Proof: We may assume that $|\nabla g(x)| \neq 0$. Suppose the result is not true. Then there is $x' \in \bar{D}$ such that $\left\langle \frac{\nabla g(x)}{|\nabla g(x)|}, (x' - x) \right\rangle < 0$; (since D is convex, $x \in \partial D$ we have $n \in \mathcal{N}_x \Leftrightarrow |n| = 1$ and $\langle \xi - x, n\rangle \geq 0 \; \forall \; \xi \in \bar{D}$). Clearly $x' \neq x$; put $\ell = \frac{x'-x}{|x'-x|}$. Then $\langle \nabla g(x), \ell\rangle < 0$.

Now, by the mean value theorem, for any $0 < r < 1$ there is $r' \in (0, r)$ such that

$$g(x + r\ell) = g(x) + r\langle \nabla g(x + r'\ell), \ell\rangle.$$

Since g attains minimum at x (over $\bar{D}$) it follows then that $\langle \nabla g(x + r'\ell), \ell\rangle \geq 0$. Letting $r \downarrow 0$, as g is C^1, we get $\langle \nabla g(x), \ell\rangle \geq 0$ which is a contradiction. This proves the lemma. □

Proposition 3.3 *Fix i; let b, R be continuous in t, y, z; let $u_{-i}(\cdot)$ be continuous and the Hamiltonian $H^{(N,i)}$ given by (3.13) be continuous. Assume (3.14). Assume that the value function $V^{(N,i)}(\cdot, \cdot, \cdot; T, u_{-i}(\cdot)) \in C^1([0,T] \times \bar{G} \times \bar{G})$, and that there is a right continuous optimal control $u_i(\cdot)$ in $\mathcal{U}(s, y, z; T, u_{-i}(\cdot))$ for any s, y, z. Then $V^{(N,i)}$ is a constrained viscosity solution to HJB equation (3.18) on $[0,T] \times \bar{G} \times \bar{G}$ with terminal value $g_i(T, \cdot, \cdot)$.*

Proof: We need to show that $V \equiv V^{(N,i)}$ is a viscosity subsolution on $[0, T) \times G \times G$, and is a viscosity supersolution on $[0, T] \times \bar{G} \times \bar{G}$.

Let $(s, y, z) \in [0, T) \times G \times G$. Let w be a C^1-function such that $(V - w)$ has a local maximum at (s, y, z). Then $\partial_0 w(s, y, z) = \partial_0 V(s, y, z)$ if $s > 0, -\partial_0 w(s, y, z) \leq -\partial_0 V(s, y, z)$ if $s = 0$. Since (y, z) is an interior point, we have $D_{(y,z)} w(s, y, z) = D_{(y,z)} V(s, y, z)$. Therefore, as V satisfies (3.18)

$$\begin{aligned} & -\partial_0 w(s, y, z) + H^{(N,i)}(s, (y, z), D_{(y,z)} w(s, y, z)) \\ \leq \; & -\partial_0 V(s, y, z) + H^{(N,i)}(s, (y, z), D_{(y,z)} V(s, y, z) = 0. \end{aligned}$$

Thus V is a viscosity subsolution on $[0, T] \times G \times G$. In a similar way it can be shown that it is a viscosity supersolution on $[0, T) \times G \times G$.

Remains to consider the case when $s \in [0, T], (y, z) \in \partial(G \times G)$. Let w be a C^1-function such that $(V - w)$ has a local minimum (in $[0, T] \times \bar{G} \times \bar{G}$) at (s, y, z). It is then easily verified that

$-\partial_0 w(s,y,z) \geq -\partial_0 V(s,y,z)$. Also there is a ball B around (y,z) such that

$$V(s,y,z) - w(s,y,z) = \min\{V(s,y',z') - w(s,y',z') : (y',z') \in B \cap (\bar{G} \times \bar{G})\}.$$

As $B \cap (\bar{G} \times \bar{G})$ is a convex set in $I\!R^{2d}$, by the preceding lemma

$$D_{(y,z)}[V(s,y,z) - w(s,y,z)] = \gamma n, \text{ for some } n \in \mathcal{N}_{(y,z)}, \gamma \geq 0.$$

As (y,z) is an interior point of B, note that n is an inward normal to $(G \times G)$ at (y,z). Hence

$$D_{(y,z)} w(s,y,z) = D_{(y,z)} V(s,y,z) - \gamma n.$$

Consequently by (3.24) and (3.18) we now obtain

$$\begin{aligned} & -\partial_0 w(s,y,z) + H^{(N,i)}(s,(y,z), D_{(y,z)} w(s,y,z)) \\ \geq \quad & -\partial_0 V(s,y,z) + H^{(N,i)}(s,(y,z), D_{y,z} V(s,y,z)) = 0. \end{aligned}$$

Thus V is a viscosity supersolution to (3.18) on $[0,T] \times \bar{G} \times \bar{G}$, completing the proof. □

Remark 3.4 Suppose $u^*(\cdot) = (u_1^*(\cdot), \ldots, u_d^*(\cdot))$ is a Nash equilibrium. In addition assume that $u^*(\cdot)$ is continuous and $(s,y,z) \mapsto V^{(N,i)}(s,y,z;T,u_{-i}^*(\cdot)) = J_i(s,y,z;T,u^*(\cdot))$ is in $C^1([0,T] \times \bar{G} \times \bar{G})$ for each $i = 1,2,\ldots,d$. Then by the preceding proposition $(s,y,z) \mapsto V^{(N,i)}(s,y,z;T,u_{-i}^*(\cdot)$, is a constrained viscosity solution to the HJB equation (3.18) with $u_{-i}(\cdot)$ replaced by $u_{-i}^*(\cdot)$, for each $1 \leq i \leq d$. So we will have a system of interrelated HJB equations with state constraints. However, in general the value function will not be smooth; nor can one hope to have a continuous Nash equilibrium. □

We now briefly indicate the HJB equations in the case of utopian equilibrium; note that $\bar{G}$ can be taken as the control set; as before $\bar{G} \times \bar{G}$ is the state space. For $s \in [0,T], y,z \in I\!R^d, u \in I\!R^d$ define $I\!R^{2d}$-valued vector by

$$f^{(U)}(s,(y,z),u) = \begin{pmatrix} u \\ b(s,y,z) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & R(s,y,z) \end{pmatrix} \begin{pmatrix} 0 \\ u \end{pmatrix}. \qquad (3.25)$$

So the state equations (3.1), (3.2) become

$$d \begin{pmatrix} y(t) \\ z(t) \end{pmatrix} = f^{(U)}(t,(y(t),z(t)),u(t))\ dt \qquad (3.26)$$

with initial value $(y(s),z(s)) = (y,z)$ corresponding to the control $u(\cdot)$. Define a scalar function for $i = 1,\ldots,d$ by

$$C^{(U,i)}(s,(y,z),u) = L_i(s,y,z) + M_i(s,y,z)u_i. \qquad (3.27)$$

Note that $C^{(N,i)}$ given by (3.10) and $C^{(U,i)}$ above differ in their domain of definition; so $C^{(U,i)}(s,(y,z),u) = C^{(N,i)}(s,(y,z),u_i)$. Cost function J_i for the i-th player, corresponding to the control $u(\cdot)$, is the same as before, given by (3.12). The *value function* for the i-th player is

$$V^{(U,i)}(s,y,z;T) = \inf\{J_i(s,y,z;T,u(\cdot)) : u(\cdot) \in \mathcal{U}(s,y,z;T)\}. \qquad (3.28)$$

The *Hamiltonian* is, for $s \in [0,T], y,z \in \mathbb{R}^d, p \in \mathbb{R}^{2d}$, given by

$$H^{(U,i)}(s,(y,z),p) = \sup\{[-\langle p, f^{(U)}(s,(y,z),u)\rangle - C^{(U,i)}(s,(y,z),u)] : u \in \bar{G}\}. \qquad (3.29)$$

In a manner analogues to the earlier discussion, the HJB equation in this context is seen to be

$$-\partial_0 v(s,y,z) + H^{(U,i)}(s,(y,z),D_{(y,z)}v(s,y,z)) = 0. \qquad (3.30)$$

In case $V^{(U,i)} \in C^1([0,T] \times \bar{G} \times \bar{G})$ and the optimal control is sufficiently regular, it can be shown that $V^{(U,i)}$ is a constrained viscosity solution to (3.30).

Remark 3.5 In the following sections we will be dealing with controls that are bounded a priori. For example, let $\tilde{\beta} = (\tilde{\beta}_1, \ldots \tilde{\beta}_d)$ be such that $\tilde{\beta}_j \geq 0, \ \forall\ 1 \leq j \leq d$. Let $1 \leq i \leq d, u_{-i}(\cdot) \in L^1_+([0,T] : d-1)$ be fixed such that $0 \leq u_j(\cdot) \leq \tilde{\beta}_j, j \neq i$. Put

$$\mathcal{U}_{\tilde{\beta}}(s,y,z;T,u_{-i}(\cdot)) = \{u_i(\cdot) \in \mathcal{U}(s,y,z;T,u_{-i}(\cdot)) : 0 \leq u_i(\cdot) \leq \tilde{\beta}_i\}. \qquad (3.31)$$

The value function is

$$\begin{aligned} & V^{(N,i)}_{\tilde{\beta}}(s,y,z;T,u_{-i}(\cdot)) \\ = {} & \inf\{J_i(s,y,z;T,(u_i(\cdot),u_{-i}(\cdot))) : u_i(\cdot) \in \mathcal{U}_{\tilde{\beta}}(s,y,z;T,u_{-i}(\cdot))\}. \end{aligned} \qquad (3.32)$$

In such a case the Hamiltonian is

$$\begin{aligned} & H^{(N,i)}_{\tilde{\beta}}(s,(y,z),p) \\ = {} & \sup\{[-\langle p, f^{(N,i)}(s,(y,z),c)\rangle - C^{(N,i)}(s,(y,z),c)] : 0 \leq c \leq \tilde{\beta}_i.\}. \end{aligned} \qquad (3.33)$$

So the HJB equation is the analogue of (3.18) with $H^{(N,i)}$ replaced by $H^{(N,i)}_{\tilde{\beta}}$. Similar comments apply in the case of utopian equilibrium.

4 Nash equilibrium and Skorokhod problem

In this section we investigate some conditions under which the solution to the Skorokhod problem gives a Nash equilibrium. First we put together some results concerning the Skorokhod problem.

We begin with a result which may be known to experts. As we have not seen an easily accessible proof we include it for the sake of completeness.

Proposition 4.1 *Let* $w \in C([0,\infty) : \mathbb{R})$ *be absolutely continuous with derivative* $\dot{w}(\cdot)$*; assume* $w(0) \geq 0$*. Let* $y^w(\cdot), z^w(\cdot)$ *be the solution to the one dimensional Skorokhod problem for* $w(\cdot)$*. Then* $y^w(\cdot), z^w(\cdot)$ *are also absolutely continuous and* $0 \leq \dot{y}^w(\cdot) \leq |\dot{w}(\cdot)|, |\dot{z}^w(\cdot)| \leq 2|\dot{w}(\cdot)|$ *a.s.*

Proof: Let $s \geq 0$. Put $\hat{w}(t) = z^w(s) + \int_s^t \dot{w}(r)\ dr, \hat{y}(t) = y^w(t) - y^w(s), \hat{z}(t) = z^w(s),\ t \geq s$. Then $\hat{y}(\cdot), \hat{z}(\cdot)$ is the unique solution to the Skorokhod problem for $\hat{w}(\cdot)$ starting at time s. Also put $\tilde{w}(t) \equiv z^w(s), \tilde{y}(t) \equiv 0, \tilde{z}(t) \equiv z^w(t), t \geq s$. Then $\tilde{y}(\cdot), \tilde{z}(\cdot)$ is the unique solution to the Skorokhod problem for $\tilde{w}(\cdot)$ starting at time s.

Note that $\hat{w} - \tilde{w}$ is of bounded variation over $[s,t]$ for any $t \geq s$. So by the Lemma of variational distance between maximal functions in Section 2 of [Sh], for any $t \geq s$

$$\text{Var}\ (y^w : [s,t]) = \text{Var}\ (\hat{y} - \tilde{y} : [s,t]) \leq\ \text{Var}\ (\hat{w} - \tilde{w} : [s,t]) = \int_s^t |\dot{w}(r)|\ dr \tag{4.1}$$

where $\text{Var}(g : [a,b])$ denotes the total variation of g over $[s,t]$.

As (4.1) holds for every $0 \leq s \leq t$, it follows that $\text{Var}(y^w : d\alpha) = dy^w(\cdot)$ is absolutely continuous. The other assertions are now easy to obtain. □

Now we consider the Skorokhod problem in an orthant with $w(\cdot) \equiv$ constant, and general time-space dependent drift and reflection, and show that absolute continuous solution exists. In case $w(\cdot)$ is an absolutely continuous function, note that $w(\cdot) - w(0)$ can be incorporated as part of time dependent drift; so it is enough to consider $w(\cdot) \equiv$ constant.

Given $s \geq 0, y \in \bar{G}, z \in \bar{G}$ consider the problem: Find functions $P(\cdot; s, y, z) \equiv P(\cdot) = (P_1(\cdot), \ldots, P_d(\cdot)), Q(\cdot; s, y, z) \equiv Q(\cdot) = (Q_1(\cdot), \ldots, Q_d(\cdot))$ on $[s, \infty)$ satisfying the following:

1. $P_i(t) \geq 0$ a.a. $t \geq s, 1 \leq i \leq d$;

2. $Q_i(\cdot)$ integrable over every compact interval;

3. $Y(\cdot; s, y, z) \equiv Y(\cdot) = (Y_1(\cdot), \ldots, Y_d(\cdot))$ with

$$Y_i(t) = y_i + \int_s^t P_i(r)\ dr, t \geq s, 1 \leq i \leq d; \tag{4.2}$$

   so $Y_i(\cdot) \geq 0$ and nondecreasing;

4. $Z(\cdot; s, y, z) \equiv Z(\cdot) = (Z_1(\cdot), \ldots, Z_d(\cdot))$ with $Z(\cdot) \in \bar{G}$ and

$$Z_i(t) = z_i + \int_s^t Q_i(r)\ dr, t \geq s, 1 \leq i \leq d \tag{4.3}$$

5. $Z(\cdot)$ satisfies the Skorokhod equation, viz. for $i = 1, 2, \ldots, d, t \geq s$

$$\begin{aligned} Z_i(t) &= z_i + \int_s^t b_i(r, Y(r), Z(r))\ dr + Y_i(t) - y_i \\ &\quad + \sum_{j \neq i} \int_s^t R_{ij}(r, Y(r), Z(r))\ P_j(r)\ dr \end{aligned} \tag{4.4}$$

6. $Y_i(\cdot)$ can increase only when $Z_i(\cdot) = 0$, that is, $Z_i(t)\ P_i(t) = 0$ a.a.t, $1 \leq i \leq d$.

In such a case we say P, Q (or equivalently Y, Z) solves SP$(b, R; G, s, y, z)$.

Theorem 4.2 *Assume (A1) - (A3). Then the above Skorokhod problem is well posed. Moreover*

$$0 \leq P_i(r) \leq \tilde{\beta}_i, |Q_i(r)| \leq 2\tilde{\beta}_i, \ a.e.\ r, 1 \leq i \leq d \tag{4.5}$$

where $\tilde{\beta} = (\tilde{\beta}_1, \ldots, \tilde{\beta}_d) = (I - W)^{-1}\beta$ *with* β, W *as in (A1), (A3).*

Proof: In view of the preceding proposition the a priori bounds (4.5) can be established as in Proposition 3.2 of [Ra]. So it is enough to consider $P(\cdot)$ satisfying (4.5). For simplicity of notation take $s = 0$.

For $t > 0$ put

$$\begin{aligned} \mathcal{H}_t = \ & \{(P(\cdot), Q(\cdot)) : P(\cdot) = (P_1(\cdot), \ldots, P_d(\cdot)), Q(\cdot) = (Q_1(\cdot), \ldots, Q_d(\cdot)), P_i, Q_i \\ & \text{satisfy (4.5) on } [0, t], 1 \leq i \leq d\}. \end{aligned}$$

As (A3) holds note that there exist $a_i > 0, 1 \leq i \leq d, 0 < \alpha < 1$ such that (see [DI1] for a proof)

$$\sum_{j \neq i} a_j |R_{ji}(t, y, z)| \leq \sum_{j \neq i} a_j W_{ji} \leq \alpha a_i, \tag{4.6}$$

for $1 \leq i \leq d, t \geq 0, y, z \in I\!R^d$. Take $\zeta > \frac{1+2\alpha}{1-\alpha}$. For $(P^{(k)}, Q^{(k)}) \in \mathcal{H}_t, k = 1, 2$ define

$$\begin{aligned} d_t((P^{(1)}, Q^{(1)}), (P^{(2)}, Q^{(2)})) &= \zeta \sum_{i=1}^d a_i \int_0^t |P_i^{(1)}(r) - P_i^{(2)}(r)|\ dr \\ &+ \sum_{i=1}^d a_i \int_0^t |Q_i^{(1)}(r) - Q_i^{(2)}(r)|\ dr. \end{aligned} \tag{4.7}$$

Note that $(\mathcal{H}_t, d_t)$ is a complete separable metric space.

Define a map $S : \mathcal{H}_t \to \mathcal{H}_t$ as follows. Let $(P, Q) \in \mathcal{H}_t$. Put $Y_i(r) = y_i + \int_0^r P_i(\theta)\ d\theta, Z_i(r) = z_i + \int_0^r Q_i(\theta) d\theta, 0 \le r \le t, 1 \le i \le d$. Let $\hat{P}(\cdot) = (\hat{P}_1(\cdot), \ldots, \hat{P}_d(\cdot)), \hat{Q}(\cdot) = (\hat{Q}_1(\cdot), \ldots, \hat{Q}_d(\cdot))$ be such that if we define $\hat{Y}_k(r) = y_k + \int_0^r \hat{P}_k(\theta)\ d\theta, \hat{Z}_k(r) = z_k + \int_0^r \hat{Q}_k(\theta)\ d\theta, 1 \le k \le d$, then for each $i = 1, 2, \ldots, d$, the functions $(\hat{Y}_i(\cdot), \hat{Z}_i(\cdot))$ solve the one dimensional Skorokhod problem on $[0, \infty)$ corresponding to the function

$$r \mapsto z_i + \int_0^r b_i(\theta, Y(\theta), Z(\theta))\ d\theta + \sum_{j \ne i} \int_0^r R_{ij}(\theta, Y(\theta), Z(\theta)) P_j(\theta)\ d\theta.$$

Put $S(P, Q) = (\hat{P}, \hat{Q})$.

Let $(\hat{P}^{(\ell)}, \hat{Q}^{(\ell)}) = S(P^{(\ell)}, Q^{(\ell)})$ for $(P^{(\ell)}, Q^{(\ell)}) \in \mathcal{H}_t, \ell = 1, 2.$ Then $\int_0^t |\hat{P}_i^{(1)}(r) - \hat{P}_i^{(2)}(r)|\ dr = \text{Var}\ (\hat{Y}_i^{(1)} - \hat{Y}_i^{(2)} : [0, t])$ and by the preceding proposition and Skorokhod equation

$$\begin{aligned}
&\int_0^t |\hat{Q}_i^{(1)}(r) - \hat{Q}_i^{(2)}(r)|\ dr = \text{Var}\ (\hat{Z}_i^{(1)} - \hat{Z}_i^{(2)} : [0, t]) \\
\le\ & \text{Var}\ (\hat{Y}_i^{(1)} - \hat{Y}_i^{(2)} : [0, t]) + \int_0^t |b_i(r, Y^{(1)}(r), Z^{(1)}(r)) - b_i(r, Y^{(2)}(r), Z^{(2)}(r))|\ dr \\
+\ & \sum_{j \ne i} \int_0^t |R_{ij}(r, Y^{(1)}(r), Z^{(1)}(r)) P_j^{(1)}(r) - R_{ij}(r, Y^{(2)}(r), Z^{(2)}(r)) P_j^{(2)}(r)| dr. \qquad (4.8)
\end{aligned}$$

So proceeding as in the proofs of Proposition 3.6 and Theorem 3.7 of [Ra] it can be shown that $(\hat{P}, \hat{Q}) \in \mathcal{H}_t$ for $(P, Q) \in \mathcal{H}_t$ and that S is a contraction on $\mathcal{H}_{t_0}$ if $t_0 = \frac{\alpha}{K(\zeta - 1)}$ where K is a constant depending on the Lipschitz constant of b, R and $d, \tilde{\beta}, \{a_i\}$. In fact the details are somewhat simpler here. Moreover in this case one can proceed in steps of t_0 to complete the proof. □

Note: By Theorem 3.7 of [Ra], uniqueness of (Y, Z) can be asserted in the larger class $C_{\uparrow}([0, \infty) : \bar{G}) \times C([0, \infty) \times \bar{G})$. However the above proof (in the absolutely continuous case) gives L^1-approximation of the derivatives of Y, Z; so the convergence in the case of the Z-part of the solution is in the variational norm which is stronger than the supremum norm. □

Remark 4.3 Let $s, \hat{s} \in [0, T]$. Let f be defined over $[s, T]$ and g over $[\hat{s}, T]$. Suppose $\hat{s} \ge s$. Then we take

$$\text{Var}\ (f - g : [s, T]) = \text{Var}\ (f - g : [\hat{s}, T]) + \text{Var}\ (f : [s, \hat{s}]). \qquad (4.9)$$

This amounts to taking $g(r) = g(\hat{s}), s \le r \le \hat{s}$. In case f, g are absolutely continuous, then (4.9) means

$$\int\limits_{[s,T]} |\dot{f} - \dot{g}| = \int\limits_{[s,T]} |\dot{f} - \dot{g}| + \int\limits_{[s,\hat{s}]} |\dot{f}|.$$

Theorem 4.4 *Assume (A1) - (A3). Let $T > 0$ be fixed. Let $(s, y, z), (\hat{s}, \hat{y}, \hat{z}) \in [0, T] \times \bar{G} \times \bar{G}$; let P, Q (or equivalently Y, Z) solve $SP(b, R; G, s, y, z)$ and $\hat{P}, \hat{Q}$ (or equivalently $\hat{Y}, \hat{Z}$) solve $SP(b, R; G, \hat{s}, \hat{y}, \hat{z})$. Then there exists a constant K_0 depending only on d, W, T, Lipshitz constants and bounds of b, R such that*

$$\begin{aligned} Var\ (\hat{Y}_i - Y_i : [s,T]) &= \|\hat{P}_i - P_i\|_{L^1[s,T]} \\ &\le K_0[|\hat{s} - s| + |\hat{y} - y| + |\hat{z} - z|] \end{aligned} \tag{4.10}$$

$$\begin{aligned} Var\ (\hat{Z}_i - Z_i : [s,T]) &= \|\hat{Q}_i - Q_i\|_{L^1[s,T]} \\ &\le K_0[|\hat{s} - s| + |\hat{y} - y| + |\hat{z} - z|] \end{aligned} \tag{4.11}$$

for $1 \le i \le d, 0 \le s, \hat{s} \le T, y, z, \hat{y}, \hat{z} \in \bar{G}$. In particular

$$|\hat{Y}_i(T) - Y_i(T)| \le K_0[|\hat{s} - s| + |\hat{y} - y| + |\hat{z} - z|]. \tag{4.12}$$

Proof: Let $\{a_i\}$ be as in (4.6). As $a_i > 0\ \forall\ i$ note that $|\xi| = \sum_{i=1}^{d} a_i|\xi_i|$ is a norm on $I\!R^d$. Without loss of generality we may assume that Lipschitz continuity is w.r.t. this metric/norm. For the duration of this proof, $|\cdot|$ on $I\!R^d$ shall denote this norm.

Suppose $\hat{s} \ge s$. For $t \in [\hat{s}, T]$, put

$$g(t) = \sum_i a_i\{\text{Var } (\hat{Y}_i - Y_i : [\hat{s}, t]) + \text{Var } (\hat{Z}_i - Z_i : [\hat{s}, t])\}.$$

For $r \in [\hat{s}, T]$ it is easy to see that

$$|\hat{Y}(r) - Y(r)| + |\hat{Z}(r) - Z(r)| \le g(r) + K[|\hat{s} - s| + |\hat{y} - y| + |\hat{z} - z|]. \tag{4.13}$$

Now using the lemma (see [Sh]) on the variational distance between maximal functions of Skorokhod problems starting from $\hat{s}$, we get for $i = 1, \ldots, d$,

$$\begin{aligned} &\text{Var } ((\hat{Y}_i(\cdot) - \hat{y}_i) - (Y_i(\cdot) - Y_i(\hat{s})) : [\hat{s}, t]) \\ \le\ & |\hat{z}_i - Z_i(\hat{s})| + K \int_{\hat{s}}^{t} (|\hat{Y}(r) - Y(r)| + |\hat{Z}(r) - Z(r)|)\, dr \end{aligned}$$

$$
\begin{aligned}
&+\sum_{j\neq i} W_{ij}\int_{\hat{s}}^{t} |\hat{P}_j(r)-P_j(r)|\ dr \\
\leq\ & K[|\hat{s}-s|+|\hat{z}_i-z_i|]+K[|\hat{s}-s|+|\hat{y}-y|+|\hat{z}-z|](t-\hat{s}) \\
&+K\int_{\hat{s}}^{t} g(r)\ dr+\sum_{j\neq i} W_{ij}\mathrm{Var}\ ((\hat{Y}_j(\cdot)-\hat{y}_j)-(Y_j(\cdot)-Y_j(\hat{s})) : [\hat{s},t]).
\end{aligned}
$$

Multiplying both sides by a_i, summing over i, using (4.6) we get

$$
\begin{aligned}
&(1-\alpha)\sum_{i} a_i \mathrm{Var}\ ((\hat{Y}_i(\cdot)-\hat{y}_i)-(Y_i(\cdot)-Y_i(\hat{s})) : [\hat{s},t]) \\
\leq\ & K[|\hat{s}-s|+|\hat{z}-z|]+K[|\hat{s}-s|+|\hat{y}-y|+|\hat{z}-z|](t-\hat{s}) \\
&+K\int_{\hat{s}}^{t} g(r)\ dr. \qquad (4.14)
\end{aligned}
$$

In the above K denotes a generic constant. Using the Skorokhod equation (4.4) and (4.13) it is easy to see

$$
\begin{aligned}
&\mathrm{Var}\ ((\hat{Z}_i(\cdot)-\hat{z}_i)-(Z_i(\cdot)-Z_i(\hat{s})) : [\hat{s},t]) \\
\leq\ & \mathrm{Var}\ ((\hat{Y}_i(\cdot)-\hat{y}_i)-(Y_i(\cdot)-Y_i(\hat{s})) : [\hat{s},t]) \\
&+\sum_{j\neq i} W_{ij}\mathrm{Var}\ ((\hat{Y}_j(\cdot)-\hat{y}_j)-(Y_j(\cdot)-Y_j(\hat{s})) : [\hat{s},t]) \\
&+K\int_{\hat{s}}^{t} g(r)\ dr+K[|\hat{s}-s|+|\hat{y}-y|+|\hat{z}-z|](t-\hat{s}). \qquad (4.15)
\end{aligned}
$$

Using (4.14), (4.15), (4.6) it can now be shown that

$$
\begin{aligned}
g(t)\ \leq\ & \frac{3}{(1-\alpha)}[K\int_{\hat{s}}^{t} g(r)\ dr+K(|\hat{s}-s|+|\hat{z}-z|) \\
&+K(|\hat{s}-s|+|\hat{y}-y|+|\hat{z}-z|)(t-\hat{s})]. \qquad (4.16)
\end{aligned}
$$

Gronwall inequality now implies, for $t\in[\hat{s},T]$,

$$
g(t)\leq K_0[|\hat{s}-s|+|\hat{y}-y|+|\hat{z}-z|]. \qquad (4.17)
$$

From (4.17) assertions (4.10), (4.11) are now clear. □

To investigate the connection between Nash equilibrium and the Skorokhod problem in an orthant, we need to consider a control problem in a half space; to motivate the set up we recall the deterministic Skorokhod problem below.

Let $H = H_1 := \{x\in\mathbb{R}^d : x_1 > 0\}$ be the half space. The drift function $b : [0,\infty)\times\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}^d$, and the reflection field $\gamma : [0,\infty)\times\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}^d$ are given. (In the context of reflected processes, γ may be specified only on $[0,\infty) \times I\!R \times \partial H$; but for our purpose of defining the control problem we take it to be a function on $[0,\infty) \times I\!R \times I\!R^d$). We assume

(B1): b, γ are bounded measurable functions; there exists a constant $K > 0$ such that

$$|f(t,\xi,z) - f(t,\xi',z')| \leq K[|\xi - \xi'| + |z - z'|] \tag{4.18}$$

for all $\xi, \xi' \in I\!R, z, z' \in I\!R^d, t \geq 0$, for $f = b_i, \gamma_j, 1 \leq i,j \leq d$; moreover $\gamma_1(\cdots) \equiv 1$.

Given $s \geq 0, y_1 \in [0,\infty), z \in \bar{H}$ the Skorokhod problem $\mathrm{SP}(b,\gamma; H, s, y_1, z)$ consists in finding functions

$$\varphi(\cdot; s, y_1, z) \equiv \varphi(\cdot), Q(\cdot; s, y_1, z) \equiv Q(\cdot) = (Q_1(\cdot), \ldots, Q_d(\cdot))$$

such that

$(1)'$ $\varphi(t) \geq 0, a.a.t \geq s$;
$(2)'$ $Q_i(\cdot)$ is integrable over every bounded interval;
$(3)'$ $\Phi(\cdot; s, y_1, z) \equiv \Phi(\cdot)$ with

$$\Phi(t) = y_1 + \int_s^t \varphi(r)\ dr, t \geq s; \tag{4.19}$$

so $\Phi(\cdot) \geq 0$ and nondecreasing
$(4)'$ $Z(\cdot; s, y_1, z) \equiv Z(\cdot) = (Z_1(\cdot), \ldots, Z_d(\cdot))$ with $Z(t) \in \bar{H}$ and

$$Z_i(t) = z_i + \int_s^t Q_i(r)\ dr \tag{4.20}$$

$(5)'$ Skorokhod equation holds, viz.

$$Z(t) = z + \int_s^t b(r,\Phi(r),Z(r))dr + \int_s^t \gamma(r,\Phi(r),Z(r))\ \varphi(r)\ dr \tag{4.21}$$

that is,

$$Z_1(t) \quad = \quad z_1 + \int_s^t b_1(r,\Phi(r),Z(r))\ dr + \Phi(t) - y_1 \tag{4.22}$$

$$Z_i(t) \quad = \quad z_i + \int_s^t b_i(r,\Phi(r),Z(r))dr + \int_s^t \gamma_i(r,\Phi(r),Z(r))\ \varphi(r)\ dr \tag{4.23}$$

for $i \neq 1$.
$(6)'$ $\Phi(\cdot)$ can increase only when $Z_1(\cdot) = 0$; that is $Z_1(t)\ \varphi(t) = 0$ for $\ a \cdot a \cdot t$.

In such a case we say that φ, Q (or equivalently Φ, Z) solves $\mathrm{SP}(b,\gamma;H,s,y_1,z)$.

By arguments entirely analogous to those in Theorem 4.2, with obvious modifications, it can be shown that under (B1) the above Skorokhod problem is well posed. Moreover

$$0 \le \varphi(\cdot) \le \beta_1 \tag{4.24}$$

$$|Q_1(\cdot)| \le 2\beta_1 \tag{4.25}$$

$$|Q_i(\cdot)| \le \beta_i + \beta_1\|\gamma_i\|_\infty \tag{4.26}$$

where $\beta_i \ge \|b_i\|_\infty, 1 \le i \le d$.

We can now describe the control problem in the half space. Let b, γ be given as above satisfying (B1). In view of the a priori bound (4.24), it is convenient to consider controls that are bounded. Let $\beta_1 \ge \|b_1\|_\infty$.

For $s \ge 0, y_1 \in [0,\infty), z \in \bar{H}_1$, measurable function $v(\cdot)$ with $0 \le v(t) \le \beta_1$ for $a \cdot a \cdot t \ge s$, consider the *state equations*

$$\psi(t) = y_1 + \int_s^t v(r)\, dr \tag{4.27}$$

$$z(t) = z + \int_s^t b(r,\psi(r),z(r))dr \int_s^t \gamma(r,\psi(r),z(r))\; v(r)\; dr. \tag{4.28}$$

Let $T > 0$ be fixed. Denote

$$\begin{aligned} \mathcal{V} &\equiv \mathcal{V}_\beta^{(1)}(s,y_1,z;T) \\ &= \{v : [s,T] \to I\!R : 0 \le v(\cdot) \le \beta_1 \text{ a.s and } z(t) \in \bar{H}_1\ \forall\ s \le t \le T\}. \end{aligned} \tag{4.29}$$

Any $v \in \mathcal{V}$ is called a *feasible control.* The cost function is

$$J(s,y_1,z;T,v(\cdot)) = \int_s^T v(r)\, dr \tag{4.30}$$

and the *value function* is

$$V_\beta^{(1)}(s,y_1,z;T) = \inf\left\{\int_s^T v(r)\, dr : v \in \mathcal{V}_\beta^{(1)}(s,y_1,z;T)\right\}. \tag{4.31}$$

Lemma 4.5 *Assume (B1). Then for any $s \in [0,T], y_1 \in [0,\infty), z \in \bar{H}_1$ the set $\mathcal{V}_\beta^{(1)}(s,y_1,z;T)$ is a nonempty weakly compact set in $L^2[s,T]$.*

Proof: $\mathcal{V} \neq \phi$ because $v(\cdot) \equiv \beta_1$ as well as $v(\cdot) = \phi(\cdot)$ are feasible controls. Clearly $\mathcal{V}$ is bounded; so it is enough to show that $\mathcal{V}$ is closed in the weak L^2-topology. Let $v^{(n)}(\cdot) \in \mathcal{V}, n = 1, 2, \ldots$ and $v^{(n)} \to v$ in the weak topology. Let $\bar{A} = \{t \in [s,T] : v(t) > \beta_1\}$. Suppose $m(\bar{A}) > 0$ where m denotes one dimensional Lebesgue measure. Then

$$\begin{aligned} 0 &< \int_{\bar{A}} (v(t) - \beta_1)dt = \langle v, I_{\bar{A}}\rangle - \beta_1 m(\bar{A}) \\ &= \lim_{n\to\infty} \langle v^{(n)}, I_{\bar{A}}\rangle - \beta_1 m(\bar{A}) \leq 0 \end{aligned}$$

which is a contradiction. So $v(\cdot) \leq \beta_1$ a.s. Similarly $v(\cdot) \geq 0$ a.s.

Denote by $\psi^{(n)}, z^{(n)}$ the solution to the state equations (4.27), (4.28) corresponding to the control $v^{(n)}$; similarly let ψ, z correspond to the control v. Clearly $\psi^{(n)}(t) \to \psi(t)$ for all t, and they are uniformly bounded. To complete the proof it is enough to prove that $z^{(n)}(t) \to z(t)\ \forall\ t$, as it would show that $z(\cdot)$ lives in $\bar{H}_1$. Using Lipschitz continuity of b, γ and uniform boundedness of $v^{(n)}$, we can show that

$$|z^{(n)}(t) - z(t)| \leq K_1\theta^{(n)} + \sigma^{(n)}(t) + K_1 \int_s^t |z^{(n)}(r) - z(r)|dr \tag{4.32}$$

where $K_1 > 0$ is a constant,

$$\begin{aligned} \theta^{(n)} &= \int_s^T \Big[|b(r, \psi^{(n)}(r), z(r)) - b(r, \psi(r), z(r))| \\ &\quad + |\gamma(r, \psi^{(n)}(r), z(r)) - \gamma(r, \psi(r), z(r))|\Big]\, dr \\ \sigma^{(n)}(t) &= \left| \int_s^t \gamma(r, \psi(r), z(r))[v^{(n)}(r) - v(r)]dr \right| \end{aligned}$$

for $t \in [s,T]$, for each n. Clearly $\theta^{(n)} \to 0$ as $n \to \infty$. As $v^n \to v$ weakly in $L^2[s,T]$ and $r \mapsto 1_{[s,t]}(r)\gamma(r, \psi(r), z(r))$ is in $L^2[s,T]$ it follows that $\sigma^{(n)}(t) \to 0$ as $n \to \infty$ for each t. Uniform boundedness of $\sigma^{(n)}(\cdot)$, (4.32) and Gronwall inequality now imply $z^{(n)}(t) \to z(t)$ for each t.

Lemma 4.6 *Assume (B1). Suppose b_1 is a function only of the time variable t. Then*

$$\int_s^t \varphi(r)\ dr \leq \int_s^t v(r)\ dr,\ \forall\ t \in [s,T], v \in \mathcal{V}.$$

In particular

$$V_\beta^{(1)}(s, y_1, z; T) = J(s, y_1, z; T, \varphi(\cdot)) = \Phi(T) - y_1.$$

Proof: Note that $\Phi(\cdot)-y_1, Z_1(\cdot)$ solve the one dimensional Skorokhod problem for the function $z_1 + \int_s^t b_1(r)\ dr$. Suppose $v \in \mathcal{V}$. Then, as b_1 is independent of the space variables we get

$$z_1(t) = z_1 + \int_s^t b_1(r)\ dr + \int_s^t v(r)\ dr \geq 0, t \geq s.$$

It is now easily seen that

$$\begin{aligned} \int_s^t v(r)\ dr &\geq \sup_{s\leq r\leq t} \max\left\{0, -\left[z_1 + \int_s^r b_1(r')\ dr'\right]\right\} \\ &= \Phi(t) - y_1 = \int_s^t \varphi(r)\ dr. \end{aligned}$$

□

Theorem 4.7 *In addition to (B1) assume the following:*
(B2): $b_1(t,\xi,z) = b_1(t,\xi,z_1)$ *for* $t \geq 0, \xi \in I\!R, z = (z_1, z_2, \ldots, z_d) \in I\!R^d$; *also* $\xi \leq \tilde{\xi}$ *implies* $b_1(t,\xi,z_1) \geq b_1(t,\tilde{\xi},z_1)$ *for all* t, z_1; *moreover* b_1 *is differentiable w.r.t.* z_1 *and*

$$\frac{\partial}{\partial z_1} b_1(t,\xi,z_1) \leq 0,\ \forall\ t,\xi,z_1.$$

Let φ, Q *be the solution to* $SP(b,\gamma;H_1,s,y_1,z)$. *Then*

$$\int_s^t \varphi(r)dr \leq \int_s^t v(r)\ dr,\ \forall\ t \in [s,T], v \in \mathcal{V}. \tag{4.33}$$

In particular

$$V_\beta^{(1)}(s,y_1,z;T) = \int_s^T \varphi(r)\ dr = \Phi(T) - y_1. \tag{4.34}$$

Proof: Since $\mathcal{V} \equiv \mathcal{V}_\beta^{(1)}(s,y_1,z;T)$ is weakly compact in $L^2[s,T]$ and $v(\cdot) \mapsto \int_s^T v(r)\ dr$ is weakly continuous, there is $v^{(0)}(\cdot) \in \mathcal{V}$ such that

$$\int_s^T v^{(0)}(r)\ dr = \inf\left\{\int_s^T v(r)\ dr : v(\cdot) \in \mathcal{V}\right\}. \tag{4.35}$$

Let $\psi^{(0)}(t) = y_1 + \int_s^t v^{(0)}(r)\ dr,$ and $z^{(0)}(\cdot)$ denote the solution to the integral equation

$$z^{(0)}(t) = z + \int_s^t b(r,\psi^{(0)}(r),z^{(0)}(r))\ dr + \int_s^t \gamma(r,\psi^{(0)}(r),z^{(0)}(r))\ v^{(0)}(r)\ dr.$$

As $v^{(0)}$ is feasible, note that $z_1^{(0)}(\cdot) \geq 0$. Put $\tilde{z}^{(0)}(\cdot) = z^{(0)}(\cdot)$.

For $k = 1, 2, \ldots$ define $v^{(k)}(\cdot), \psi^{(k)}(\cdot), z^{(k)}(\cdot), \tilde{z}^{(k)}(\cdot)$ inductively as follows: Set $b^{(k-1)}(r) = b(r, \psi^{(k-1)}(r), \tilde{z}^{(k-1)}(r))$ and

$$\gamma^{(k-1)}(r) = (1, \gamma_2(r, \psi^{(k-1)}(r), \tilde{z}^{(k-1)}(r)), \ldots, \gamma_d(r, \psi^{(k-1)}(r), \tilde{z}^{(k-1)}(r))).$$

Note that these are, in principle, known functions only of r. Let $\psi^{(k)}(\cdot), z^{(k)}(\cdot)$ be the solution to $\mathrm{SP}(b^{(k-1)}, \gamma^{(k-1)}; H_1, s, y_1, z)$ so that $\psi^{(k)}(t) = y_1 + \int_s^t v^{(k)}(r)\ dr$, for some function $v^{(k)}(\cdot)$ with $0 \leq v^{(k)}(t) \leq \beta_1, z_1^{(k)}(t) \geq 0, z_1^{(k)}(\cdot)v^{(k)}(\cdot) = 0$ a.s. and the Skorokhod equation holds, viz.

$$\begin{aligned} z^{(k)}(t) &= z + \int_s^t b(r, \psi^{(k-1)}(r), \tilde{z}^{(k-1)}(r))\ dr \\ &\quad + \int_s^t \gamma(r, \psi^{(k-1)}(r), \tilde{z}^{(k-1)}(r))\ v^{(k)}(r)\ dr. \end{aligned} \qquad (4.36)$$

Next let $\tilde{z}^{(k)}(\cdot)$ be the solution to the integral equation

$$\begin{aligned} \tilde{z}^{(k)}(t) &= z + \int_s^t b(r, \psi^{(k)}(r), \tilde{z}^{(k)}(r))\ dr \\ &\quad + \int_s^t \gamma(r, \psi^{(k)}(r), \tilde{z}^{(k)}(r))\ v^{(k)}(r)\ dr. \end{aligned} \qquad (4.37)$$

We claim that $\tilde{z}_1^{(k)}(t) \geq 0, t \in [s, T], k = 0, 1, 2, \ldots$ This is proved by induction on k. For $k = 0$ it is already done. Assume it for $k \leq n - 1$. As $b^{(n-1)}, \gamma^{(n-1)}$ are functions only of r, by the preceding lemma

$$\int_s^t v^{(n)}(r) dr \leq \int_s^t v(r)\ dr, t \in [s, T]$$

for any $v(\cdot)$ such that $z_1 + \int_s^t b_1^{(n-1)}(r) dr + \int_s^t v(r)\ dr \geq 0, t \in [s, T]$. By induction hypothesis note that

$$z_1 + \int_s^t b_1^{(n-1)}(r)\ dr + \int_s^t v^{(n-1)}(r)\ dr = \tilde{z}_1^{(n-1)}(t) \geq 0, t \in [s, T].$$

Therefore $\int_s^t v^{(n)}(r)\ dr \leq \int_s^t v^{(n-1)}(r)\ dr$ and hence $\psi^{(n)}(t) \leq \psi^{(n-1)}(t)$; consequently by our hypothesis (B2), $b_1(r, \psi^{(n)}(r), \tilde{z}_1^{(n)}(r)) \geq b_1(r, \psi^{(n-1)}(r), \tilde{z}_1^{(n)}(r))$ for all r. Moreover $z_1^{(n)}(t) - \tilde{z}_1^{(n-1)}(t) = \int_s^t v^{(n)}(r)\ dr - \int_s^t v^{(n-1)}(r)\ dr \leq 0$ for all t. Now denote $b_{13}(\cdots) = \frac{\partial}{\partial z_1} b_1(\cdots)$. By

(4.36), (4.37) and the mean value theorem

$$\tilde{z}_1^{(n)}(t) - z_1^{(n)}(t) = \int_s^t \alpha^{(n)}(r)\ dr$$
$$+ \int_s^t b_{13}(r, \psi^{(n-1)}(r), \theta(r))[\tilde{z}_1^{(n)}(r) - z_1^{(n)}(r)]dr \qquad (4.38)$$

where $\theta(r)$ is a point between $\tilde{z}_1^{(n)}(r)$ and $\tilde{z}_1^{(n-1)}(r)$, and

$$\begin{aligned}\alpha^{(n)}(r) &= b_1(r, \psi^{(n)}(r), \tilde{z}_1^{(n)}(r)) - b_1(r, \psi^{(n-1)}(r), \tilde{z}_1^{(n)}(r)) \\ &\quad + b_{13}(r, \psi^{(n-1)}(r), \theta(r))[z_1^{(n)}(r) - \tilde{z}_1^{(n-1)}(r)].\end{aligned}$$

By our hypothesis (B2) and the preceding paragraph note that $\alpha^{(n)}(\cdot) \geq 0$. Therefore (4.38) now implies

$$\tilde{z}_1^{(n)}(t) - z_1^{(n)}(t) = \int_s^t \alpha^{(n)}(r) \exp\left(\int_r^t b_{13}(r'; \psi^{(n-1)}(r'), \theta(r'))dr'\right) dr$$

and hence $\tilde{z}_1^{(n)}(t) - z_1^{(n)}(t) \geq 0$. As $z_1^{(n)}(\cdot) \geq 0$ being the solution to Skorokhod problem we now get $\tilde{z}_1^{(n)}(t) \geq 0$ for all $t \in [s, T]$, proving our claim.

Thus $v^{(n)}(\cdot) \in \mathcal{V}$ for each n. Therefore our proof and (4.35) now imply

$$\begin{aligned}\int_s^T v^{(0)}(r)\ dr &\leq \int_s^T v^{(n)}(r)\ dr \leq \int_s^T v^{(n-1)}(r)\ dr \\ &\leq \ldots \leq \int_s^T v^{(0)}(r)\ dr.\end{aligned}$$

So $\int_s^T v^{(n)}(r)\ dr = \int_s^T v^{(0)}(r)\ dr$ for each n. The proof will now be complete once the following lemma is established. □

Lemma 4.8 *Assume (B1); (note that (B2) is not assumed in this lemma). Let* $s \in [0, T], y_1 \in [0, \infty), z \in \bar{H}_1$ *be fixed. Let* $v^{(0)}(\cdot)$ *be such that* $0 \leq v^{(0)}(t) \leq \beta_1$ *for* $a \cdot a \cdot t \in [s, T]$*; let* $z(\cdot)$ *be a continuous* $\bar{H}_1$*-valued function with* $z(s) = z$*; put* $\psi^{(0)}(t) = y_1 + \int_s^t v^{(0)}(r)\ dr$ *and* $z^{(0)}(\cdot) \equiv \tilde{z}^{(0)}(\cdot) \equiv z(\cdot)$*. For* $k = 1, 2, \ldots$ *define* $v^{(k)}(\cdot), \psi^{(k)}(\cdot), z^{(k)}(\cdot), \tilde{z}^{(k)}(\cdot)$ *inductively analogous to (4.36), (4.37). Then* $v^{(k)}(\cdot) \to \varphi(\cdot)$ *in* $L^1[s, T], z^{(k)}(\cdot) \to Z(\cdot), \tilde{z}^{(k)}(\cdot) \to Z(\cdot)$ *both in* $C[s, T]$ *as* $k \to \infty$*.*

Proof: Note that $z^{(k)}(t) \in \bar{H}_1$ for all $t \in [s, T]$, but $\tilde{z}^{(k)}(\cdot)$ need not be $\bar{H}_1$-valued. For $n \geq 1$ denote

$$g^{(n)}(t) = \sup_{s \leq r \leq t} |\tilde{z}^{(n)}(r) - \tilde{z}^{(n-1)}(r)| + \int_s^t |v^{(n)}(r) - v^{(n-1)}(r)|\ dr. \qquad (4.39)$$

For $n \geq 2, v^{(n)}(\cdot), z^{(n)}(\cdot)$ arise by solving a Skorokhod problem with drift and reflection fields depending only on time variable; and $\psi^{(n)}(\cdot) - y_1$ is the corresponding maximal function. So again using the estimate on the variational distance between maximal functions [Sh], and Lipschitz continuity in space variables, it can be seen that

$$\int_s^t |v^{(n)}(r) - v^{(n-1)}(r)| \, dr \leq K \int_s^t g^{(n-1)}(r) \, dr. \tag{4.40}$$

Note that $0 \leq v^{(n)}(\cdot) \leq \beta_1$ for all n. Therefore

$$\begin{aligned} &\sup_{s \leq r \leq t} |\tilde{z}^{(n)}(r) - \tilde{z}^{(n-1)}(r)| \\ \leq\ & K_1 \int_s^t \left[\sup_{s \leq q \leq r} |\tilde{z}^{(n)}(q) - \tilde{z}^{(n-1)}(q)| \right] dr + K_2 \int_s^t |v^{(n)}(r) - v^{(n-1)}(r)| \, dr \end{aligned}$$

and hence by Gronwall inequality

$$\begin{aligned} &\sup_{s \leq r \leq t} |\tilde{z}^{(n)}(r) - \tilde{z}^{(n-1)}(r)| \\ \leq\ & K_2 e^{K_1(t-s)} \int_s^t |v^{(n)}(r) - v^{(n-1)}(r)| \, dr. \end{aligned} \tag{4.41}$$

Now (4.40), (4.41) imply

$$g^{(n)}(t) \leq C \int_s^t g^{(n-1)}(r) \, dr. \tag{4.42}$$

As $g^{(1)}(r) \leq C(r-s)$, iterating (4.42)

$$g^{(n)}(t) \leq \frac{C^n (t-s)^n}{(n-1)!} \to 0, \text{ as } n \to \infty.$$

So $\{v^{(n)}(\cdot)\}$ is a Cauchy sequence in $L^1[s,T]$, and $\{\tilde{z}^{(n)}(\cdot)\}$ a Cauchy sequence in $C([s,T] : \mathbb{R}^d)$. Let $\bar{v}(\cdot) = \lim_n v^{(n)}(\cdot), \bar{z}(\cdot) = \lim_n \tilde{z}^{(n)}(\cdot)$. As b, γ are continuous in ξ, z variables, and bounded it is now easily seen that

$$\bar{z}(t) = z + \int_s^t b(r, \bar{\psi}(r), \bar{z}(r)) \, dr + \int_s^t (\gamma(r, \bar{\psi}(r), \bar{z}(r)) \; \bar{v}(r) \, dr \tag{4.43}$$

where $\bar{\psi}(t) = y_1 + \int_s^t \bar{v}(r) \, dr$; similarly

$$\lim_{k \to \infty} z^{(k)}(t) = \bar{z}(t). \tag{4.44}$$

So (4.44) implies $\bar{z}(\cdot)$ is $\bar{H}_1$-valued. By going to a subsequence if necessary, $v^{(k)}(\cdot) \to \bar{v}(\cdot)$ a.s. Hence it follows that $\bar{z}_1(\cdot)\bar{v}(\cdot) = \lim_k z_1^{(k)}(\cdot) v^{(k)}(\cdot) = 0$ a.s. So $\bar{v}(\cdot) = \varphi(\cdot), \bar{z}(\cdot) = Z(\cdot)$, proving the lemma. □

We now go back to the situation in an orthant G; so $b : [0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d, R : [0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{M}_d(\mathbb{R})$ are as in Sections 2 and 3.

Theorem 4.9 *In addition to (A1) - (A3), let b and R satisfy the following conditions.*
(C1): For $1 \leq i \leq d, b_i, R_{ij}$ are independent of $z_\ell, \ell \neq i$; that is, $b_i(t,y,z) = b_i(t,y,z_i), R_{ij}(t,y,z) = R_{ij}(t,y,z_i)$.
(C2): For fixed $1 \leq i \leq d, y_{-i} = (y_1,\ldots,y_{i-1},y_{i+1},\ldots,y_d), t \geq 0, z \in \mathbb{R}^d$

$$\begin{aligned} b_i(t,(\xi,y_{-i}),z) &\geq b_i(t,(\tilde{\xi},y_{-i}),z) \\ R_{ij}(t,(\xi,y_{-i}),z) &\geq R_{ij}(t,(\tilde{\xi},y_{-i}),z), 1 \leq j \leq d \end{aligned}$$

whenever $\xi \leq \tilde{\xi}$; that is b_i, R_{ij} are nonincreasing in y_i.
(C3): The functions $z_i \mapsto b_i(t,y,z_i) = b_i(t,y,z), z_i \mapsto R_{ij}(t,y,z_i) = R_{ij}(t,y,z)$ are differentiable and

$$\frac{\partial}{\partial z_i} b_i(t,y,z) \leq 0, \; \frac{\partial}{\partial z_i} R_{ij}(t,y,z) \leq 0, 1 \leq i,j \leq d.$$

Let $P(\cdot) = (P_1(\cdot),\ldots,P_d(\cdot)), Q(\cdot) = (Q_1(\cdot),\ldots,Q_d(\cdot))$ be the solution to $SP(b,R;\bar{G},s,y,z)$. Consider the d-person dynamic game as in Section 3. Let $\tilde{\beta} = (\tilde{\beta}_1,\ldots,\tilde{\beta}_d)$ be such that $\tilde{\beta}_j \geq ((I-W)^{-1}\beta)_j, 1 \leq j \leq d$. Let the cost function for the ith player corresponding to the control $u_i(\cdot)$ be $\int_s^T u_i(r)\,dr$; that is, $g_i \equiv 0, L_i \equiv 0, M_i \equiv 1$. Then $P(\cdot)$ is a Nash equilibrium in $\mathcal{U}_{\tilde{\beta}}(s,y,z;T)$. Moreover $P(\cdot)$ is a Nash equilibrium for the game in $\mathcal{U}_{\tilde{\beta}}(s,y,z;t)$ for any $t \in [s,T]$.

Proof: Note that under (A1) - (A3), $\mathcal{U}_{\tilde{\beta}}(s,y,z;T) \neq \phi$. Fix s,y,z. Fix i; so $P_{-i}(\cdot) = (P_1(\cdot),\ldots,P_{i-1}(\cdot),P_{i+1}(\cdot),\ldots,P_d(\cdot))$. Denote $\mathcal{U}^{(i)}_{\tilde{\beta}} = \mathcal{U}_{\tilde{\beta}}(s,y,z;T,P_{-i}(\cdot))$. Note that $\mathcal{U}^{(i)}_{\tilde{\beta}} \neq \phi$ as $P_i(\cdot) \in \mathcal{U}^{(i)}_{\tilde{\beta}}$. Put $Y_{-i}(t) = y_{-i} + \int_s^t P_{-i}(r)\,dr$. Define $\tilde{b}^{(i)} : [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ and $\tilde{\gamma}^{(i)} : [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ by

$$\begin{aligned} \tilde{b}^{(i)}_i(t,\xi,z) &= b_i(t,(\xi,Y_{-i}(t)),z) + \sum_{j\neq i} R_{ij}(t,(\xi,Y_{-i}(t)),z)P_j(t), \\ \tilde{b}^{(i)}_k(t,\xi,z) &= b_k(t,(\xi,Y_{-i}(t)),z) + \sum_{j\neq i} R_{kj}(t,(\xi,Y_{-i}(t)),z)P_j(t), \text{ for } k \neq i \\ \tilde{\gamma}^{(i)}_i(\ldots) &\equiv R_{ii}(\ldots) \equiv 1, \\ \tilde{\gamma}^{(i)}_k(t,\xi,z) &= R_{ki}(t,(\xi,Y_{-i}(t)),z), \text{ for } k \neq i. \end{aligned}$$

Note that $\tilde{b}^{(i)}, \tilde{\gamma}^{(i)}$ satisfy the hypotheses of Theorem 4.7, with $i, \bar{H}_i$ in the place of 1, $\bar{H}_1$ respectively. For any $0 \leq v(\cdot) \leq \tilde{\beta}_i$ put $\psi^{(i)}(t) = y_i + \int_s^t v(r)\,dr$ and consider the integral equation

$$z(t) = z + \int_s^t \tilde{b}^{(i)}(r, \psi^{(i)}(r), z(r))\ dr + \int_s^t \tilde{\gamma}^{(i)}(r, \psi^{(i)}(r), z(r))\ v(r)\ dr. \tag{4.45}$$

This is the analogue of (4.28). Denote

$$\mathcal{V}^{(i)}_{\tilde{\beta}} = \left\{ \begin{array}{l} v(\cdot) : [s,T] \to I\!R : 0 \le v(\cdot) \le \tilde{\beta}_i, \text{ and the} \\ \text{solution } z(\cdot) \text{ of (4.45) is } \bar{H}_i - \text{valued} \end{array} \right\}.$$

Then by Theorem 4.7

$$\int_s^t \varphi(r)\ dr \le \int_s^t v(r)\ dr,\ \forall\ t \in [s,T],\ \forall\ v \in \mathcal{V}^{(i)}_{\tilde{\beta}} \tag{4.46}$$

where $\varphi(\cdot)$, along with Q solves the Skorokhod problem $\mathrm{SP}(\tilde{b}^{(i)}, \tilde{\gamma}^{(i)}; H_i, s, y, z)$; so $\varphi(\cdot)$ is optimal in $\mathcal{V}^{(i)}_{\tilde{\beta}}$ for the cost function $\int_s^T v(r)\ dr$. Since $\bar{G} \subset \bar{H}_i$ and the solution to the Skorokhod problem in the half space is unique it now follows that $\varphi(\cdot) = P_i(\cdot)$. It is clear that $\phi \neq \mathcal{U}^{(i)}_{\tilde{\beta}} \subseteq \mathcal{V}^{(i)}_{\tilde{\beta}}$. Hence (4.46) now implies that

$$\int_s^t P_i(r)\ dr \le \int_s^t u_i(r) dr,\ \ s \le t \le T,\ \forall\ u_i(\cdot) \in \mathcal{U}^{(i)}_{\tilde{\beta}}(s, y, z; t, P_{-i}(\cdot)). \tag{4.47}$$

The theorem is now clear. □

We now consider a possible converse to the preceding theorem. For this we need a preliminary result which incidentally gives a sufficient condition for (3.14) (and (2.9) as well) to hold.

Theorem 4.10 *Assume (A1) - (A3); moreover assume that $b, I\!R$ are continuous in the time variable as well. In addition let the following hypothesis hold:*
(A4): For $i = 1, 2, \ldots, d, s \in [0,T], y \in \bar{G}, z \in \partial G, u_j \in [0, \tilde{\beta}_j], j \neq i$ there exists $u_i \in [0, \tilde{\beta}_i]$ with

$$\left[b_k(s,y,z) + \sum_{\ell=1}^{d} R_{k\ell}(s,y,z) u_\ell \right] > 0 \tag{4.48}$$

for any $k \in I(z)$. Here $\tilde{\beta} = (\tilde{\beta}_1, \ldots, \tilde{\beta}_d)$ is as in Theorem 4.9.
Then $\mathcal{U}_{\tilde{\beta}}(s, y, z; T, u_{-i}(\cdot))$ is a nonempty weakly compact subset of $L^2[s,T]$ for any $s \in [0,T], y, z \in \bar{G}, 0 \le u_j(\cdot) \le \tilde{\beta}_j, j \neq i$.

Note: Compare (4.48) with (3.20), (3.21). We know that (3.20), (3.21) give a necessary boundary condition for $u(\cdot)$ to be feasible. See also Section 5, Chapter 4 of [BC].

We first prove the following lemma

Lemma 4.11 *Under the above hypotheses* $\mathcal{U}_{\tilde{\beta}}(s, y, z; T, u_{-i}(\cdot)) \neq \phi$ *if* $u_{-i}(\cdot)$ *is right continuous. Moreover* $u_i(\cdot)$ *can also chosen to be right continuous in this case.*

Proof: Fix $i, s, y, z, u_{-i}(\cdot)$. We first claim that there exists a right continuous $u_i(\cdot)$ taking values in $[0, \tilde{\beta}_i]$ such that

$$\tau_0 = \inf\{t > s : z(t) \notin G\} > s$$

where $y(\cdot), z(\cdot)$ is the solution to the state equation corresponding to the control $u(\cdot) = (u_i(\cdot), u_{-i}(\cdot))$ with $y(s) = y, z(s) = z$.

Indeed, if $z \in G$ then $u_i(\cdot)$ can be taken to be any right continuous function on $[s, T]$ taking values in $[0, \tilde{\beta}_i]$. So let $z \in \partial G$; note that $I(z) \neq \phi$. Taking $u_j = u_j(s), j \neq i$, by assumption (A4) choose u_i so that (4.48) holds for all $k \in I(z)$. Set $u(t) = (u_i, u_{-i}(t)), t \geq s$; note that the ith component of $u(\cdot)$ is constant; let $y(\cdot), z(\cdot)$ denote the corresponding solution to the state equation. By continuity of $b_k, R_{kj}, y(\cdot), z(\cdot)$ and right continuity of $u(\cdot)$ at s, note that there exist $\epsilon_0 > 0, \eta > 0$ such that for all $t \in [s, s + \epsilon_0]$ we have $I(z(t)) \subseteq I(z)$ and

$$b_k(t, y(t), z(t)) + \sum_{\ell} R_{k\ell}(t, y(t), z(t)) \; u_\ell(t) \geq \eta$$

for all $k \in I(z)$. So $z_k(t) \geq \eta(t - s) > 0$ for $k \in I(z)$ and $z_\ell(t) > 0$ for $\ell \notin I(z)$, for all $s < t \leq s + \epsilon_0$. The claim now follows.

Now put $\tau := \sup\{t \in [s, T]$: there exists a right continuous $u_i(\cdot)$ taking values in $[0, \tilde{\beta}_i]$ and $z(r) \in \bar{G}$ for all $s \leq r \leq t\}$; here $y(\cdot), z(\cdot)$ denote the solution to the state equation corresponding to $(u_i(\cdot), u_{-i}(\cdot))$. If $\tau < T$ then apply the claim above with $\tau_0, y(\tau_0), z(\tau_0)$ respectively replacing s, y, z; we now get a contradiction to the definition of τ. So $\tau = T$ and hence $z(t) \in \bar{G}$ for all $s \leq t \leq T$. □

Proof of Theorem 4.10: Fix i, s, y, z; let $u_j(\cdot), j \neq i$ be as in the theorem. For $n \geq 1$ choose $u_{-i}^{(n)}(\cdot) = (u_1^{(n)}(\cdot), \ldots, u_{i-1}^{(n)}(\cdot), u_{i+1}^{(n)}(\cdot), \ldots, u_d^{(n)}(\cdot))$ such that $u_j^{(n)}(\cdot)$ is right continuous, $0 \leq u_j^{(n)}(\cdot) \leq \tilde{\beta}_j$, and $u_j^{(n)}(\cdot) \to u_j(\cdot)$ in $L^2[s, T]$ as $n \to \infty$, for $j \neq i$. By the preceding lemma, for $n \geq 1$, there exists $u_i^{(n)}(\cdot)$, right continuous on $[s, T]$, taking values in $[0, \tilde{\beta}_i]$, and if $(y^{(n)}(\cdot), z^{(n)}(\cdot))$ denotes the solution to the state equation corresponding to the control $u^{(n)}(\cdot) = (u_i^{(n)}(\cdot), u_{-i}^{(n)}(\cdot))$ with $y^{(n)}(s) = y, z^{(n)}(s) = z$ then $y^{(n)}(t) \geq 0, z^{(n)}(t) \geq 0$ for all $t \in [s, T]$.

As $\{u_i^{(n)}(\cdot) : n \geq 1\}$ is bounded, by Banach-Alaoglu theorem there exists $u_i(\cdot) \in L^2[s,T]$ such that $u_i^{(n)} \to u_i(\cdot)$ weakly. Put $u(\cdot) = (u_i(\cdot), u_{-i}(\cdot))$; let $y(\cdot), z(\cdot)$ denote the solution to the state equation corresponding to $u(\cdot)$ starting at s, y, z. It can be proved as in Lemma 4.5 that $0 \leq u_i(\cdot) \leq \tilde{\beta}_i$. Clearly $y_i(\cdot) \geq 0$ and nondecreasing. Again as in Lemma 4.5 using Lipschitz continuity of b, R, uniform boundedness of $u^n(\cdot), u(\cdot)$ and Gronwall inequality it can be shown that $z(\cdot)$ is $\bar{G}$-valued. Thus $\mathcal{U}_{\tilde{\beta}}(s, y, z; T, u_{-i}(\cdot)) \neq \phi$. The same argument implies weak compactness as well. □

Theorem 4.12 *Let $g_i \equiv 0, L_i \equiv 0, M_i \equiv 1$ for $1 \leq i \leq d$. In addition to the hypotheses of Theorem 4.10, assume that $R_{k\ell}(\cdots) \leq 0$ for $k \neq \ell$. Fix $s \in [0,T], y \in \bar{G}, z \in \bar{G}$. Let $\hat{u}(\cdot) = (\hat{u}_1(\cdot), \ldots, \hat{u}_d(\cdot))$ be such that for any $t \in [s,T]$, the restriction of $\hat{u}(\cdot)$ to $[s,t]$ is a Nash equilibrium in $\mathcal{U}_{\tilde{\beta}}(s,y,z;t)$. Then $\hat{u}(\cdot) = P(\cdot)$, where $P(\cdot)$, along with $Q(\cdot)$, solves the Skorokhod problem SP$(b, R; \bar{G}, s, y, z)$.*

Proof: Let $\hat{y}(\cdot), \hat{z}(\cdot)$ denote the solution to the state equation corresponding to the control $\hat{u}(\cdot)$. As $\hat{u}(\cdot)$ is feasible it is clear that $\hat{z}(\cdot)$ is $\bar{G}$-valued. So we just need to prove that $\hat{z}_i(\cdot)\ \hat{u}_i(\cdot) = 0$ a.s. for each i. Put $\hat{h}_i(t) = \int_s^t \hat{z}_i(r)\ \hat{u}_i(r)\ dr, t \geq s$. We need to prove $\hat{h}_i(\cdot) \equiv 0, 1 \leq i \leq d$. Suppose not. Then $\hat{h}_i(t) > 0$ for some $t > s, 1 \leq i \leq d$. Hence, by continuity of $\hat{z}_i(\cdot)$, there exist $x > 0, s \leq t_0 < \tilde{t}$ such that $\hat{z}_i(t) \geq x > 0$ for all $t_0 \leq t \leq \tilde{t}$ and

$$m(\{r : \hat{u}_i(r) > 0 \text{ for } r \in [t_0, t]\}) > 0, \ \forall\ t_0 \leq t \leq \tilde{t}. \qquad (4.49)$$

We now make a remark concerning (A4). Let $i, u_j, j \neq i, s, y$ be as in (A4). Let $z \in \partial G$ be such that $z_i > 0$; so $R_{ki}(\ldots) \leq 0$ for any $k \in I(z)$. Therefore one can take $u_i = 0$ so that (4.48) holds.

Now define $\tilde{u}(\cdot) = (\tilde{u}_i(\cdot), \tilde{u}_{-i}(\cdot))$ by $\tilde{u}_j(\cdot) = \hat{u}_j(\cdot)$, for $j \neq i, \tilde{u}_i(r) = \hat{u}_i(r), s \leq r < t_0$ and $\tilde{u}_i(r) = 0, r \geq t_0$. Let $\tilde{y}(\cdot), \tilde{z}(\cdot)$ denote the solution to the state equation corresponding to $\tilde{u}(\cdot)$. Clearly $\tilde{z}(r) = \hat{z}(r), s \leq r \leq t_0$. In particular $\tilde{z}_i(t_0) \geq x > 0$. Put $t_1 = \inf\{t \geq t_0 : \tilde{z}_i(t) = 0\}$. Clearly $t_1 > t_0$. Now Theorem 4.10 and the remark above give that $\mathcal{U}_{\tilde{\beta}}(s, y, z; t, \hat{u}_{-i}(\cdot)) \neq \phi$ and in fact $\tilde{u}_i(\cdot) \in \mathcal{U}_{\tilde{\beta}}(s, y, z; t, \hat{u}_i(\cdot))$ for any $t \in [t_0, t_1]$. In view of (4.49) it is now clear that

$$\int_s^t \tilde{u}_i(r)\ dr < \int_s^t \hat{u}_i(r)\ dr, t_0 < t \leq (t_1 \wedge T). \qquad (4.50)$$

As (4.50) contradicts our hypothesis that $\hat{u}(\cdot)$ is a Nash equilibrium in $\mathcal{U}_{\tilde{\beta}}(s, y, z; t)$ for $t \in [s, T]$, the result now follows. □

Remark 4.13 Under the hypotheses of Theorem 4.12, as the Skorokhod problem is well posed, it follows that there can be at most one Nash equilibrium serving for all t. It is also now clear that under the combined hypotheses of Theorems 4.9, 4.10, 4.12, the solution to the Skorokhod problem gives the unique Nash equilibrium serving for all t. All these conclusions are valid even if one considers more general cost functions with M_i = positive constant, g_i, L_i being independent of z, and satisfying

$$\begin{aligned} g_i(T, (\tilde{\xi}, y_{-i})) &\leq g_i(T, (\xi, y_{-i})) \\ L_i(t, (\tilde{\xi}, y_{-i})) &\leq L_i(t, (\xi, y_{-i})) \end{aligned}$$

whenever $\tilde{\xi} \leq \xi, \ \forall\, t, y_{-i}, 1 \leq i \leq d.$ □

We conclude this section with the following example.

Example 4.14 Let $d = 2, b(\ldots) \equiv (1, -1), R_{12}(\ldots) \equiv R_{12} > 0, R_{21}(\ldots) \equiv R_{21} > 0$ are constants such that $R_{12}R_{21} < 1$. Take $s = 0, y = 0, z = 0$. The state equation for the z-part, viz. analogue of (3.4), is given by

$$\begin{aligned} z_1(t) &= t + y_1(t) + R_{12}y_2(t) \\ z_2(t) &= -t + y_2(t) + R_{21}y_1(t). \end{aligned}$$

Solution to the corresponding Skorokhod problem is given by $Y_1(t) \equiv 0, Y_2(t) = t, Z_1(t) = (1 + R_{12})t, Z_2(t) \equiv 0$. By Theorem 4.9 the solution to the Skorokhod problem gives a Nash equilibrium; this also follows from the argument given below.

Let $\lambda_1 \geq 0, 0 \leq \lambda_2 \leq 1$ be such that $\lambda_2 + R_{21}\lambda_1 = 1$. Put $\hat{y}_1(t) = \lambda_1(t), \hat{y}_2(t) = \lambda_2 t$. Fix $\hat{y}_1(\cdot)$. Let $0 \leq y_2(t) < \lambda_2 t$ for some t. Corresponding to $\hat{y}_1(\cdot), y_2(\cdot)$ note that $z_1(t) \geq 0$ but $z_2(t) = -t + y_2(t) + R_{21}\hat{y}_1(t) < 0$.

So with $\hat{y}_1(\cdot)$ fixed, $y_2(\cdot)$ cannot be feasible unless $y_2(t) \geq \lambda_2 t, \ \forall\, t$. In an entirely analogous manner with $\hat{y}_2(\cdot)$ fixed, $y_1(\cdot)$ cannot be feasible unless $y_1(t) \geq \lambda_1 t$ for all t. Therefore it follows that for any λ_1, λ_2 as above $(\hat{u}_1(\cdot), \hat{u}_2(\cdot)) \equiv (\lambda_1, \lambda_2)$ gives a Nash equilibrium for each $t \geq 0$. So even Nash equilibrium serving for all t need not be unique. Next note that $(\lambda_1, \lambda_2) = (0, 1)$ as well as $(\lambda_1, \lambda_2) = \left(\frac{1}{R_{21}}, 0\right)$ give feasible controls (in fact both are Nash equilibria). So the only possible candidate for utopian equilibrium is (0,0). But (0,0) cannot be a feasible control. Hence there is no utopian equilibrium even for a single $t > 0$. □

5 Viscosity solutions

As mentioned earlier, an appropriate framework to discuss the value function is in terms of viscosity solutions to HJB equations. We take up the case of utopian equilibrium first. In this case there are d control problems, with a d-dimensional control set for each problem. We will consider only controls taking values in $\bar{G}_{\tilde{\beta}} = \prod_{i=1}^{d} [0, \tilde{\beta}_i]$ with $\tilde{\beta}_i \geq ((I-W)^{-1}\beta)_i, 1 \leq i \leq d$.

The value function for the ith player is

$$V_{\tilde{\beta}}^{(U,i)}(s, y, z; T) = \inf\{J_i(s, y, z; T, u(\cdot)) : u(\cdot) \in \mathcal{U}_{\tilde{\beta}}(s, y, z; T)\} \tag{5.1}$$

and the HJB equation is

$$-\partial_0 v(s, y, z) + H_{\tilde{\beta}}^{(U,i)}(s, (y, z), D_{(y,z)} v(s, y, z) = 0 \tag{5.2}$$

where the Hamiltonian is given by

$$\begin{aligned} &H_{\tilde{\beta}}^{(U,i)}(s, (y, z), p) \\ &= \sup\{[-\langle p, f^{(U)}(s, (y, z), u)\rangle - C^{(U,i)}(s, (y, z), u)] : u \in \bar{G}_{\tilde{\beta}}\} \end{aligned} \tag{5.3}$$

and $J_i, f^{(U)}, C^{(U,i)}$ are given respectively by (3.12), (3.25), (3.27). Recall (see [BC] or [FS]) that a locally bounded function v is said to be a *viscosity subsolution* to (5.2) on $[0, T) \times G \times G$ if for any $(s, y, z) \in [0, T) \times G \times G$, any C^1-function w such that $(v - w)$ has a local maximum at (s, y, z) one has

$$-\partial_0 w(s, y, z) + H_{\tilde{\beta}}^{(U,i)}(s, (y, z), D_{(y,z)} w(s, y, z)) \leq 0. \tag{5.4}$$

Similarly a locally bounded function v is said to be a *viscosity supersolution* to (5.2) on $[0, T] \times \bar{G} \times \bar{G}$ if for any $(s, y, z) \in [0, T] \times \bar{G} \times \bar{G}$ any C^1-function w such that $(v - w)$ has a local minimum in $[0, T] \times \bar{G} \times \bar{G}$ at (s, y, z) one has

$$-\partial_0 w(s, y, z) + H_{\tilde{\beta}}^{(U,i)}(s, (y, z), D_{(y,z)} w(s, y, z)) \geq 0. \tag{5.5}$$

If v is a viscosity subsolution on $[0, T) \times G \times G$ and a viscosity supersolution on $[0, T] \times \bar{G} \times \bar{G}$, then v is said to be a *constrained viscosity solution* to (5.2) on $[0, T] \times \bar{G} \times \bar{G}$.

Theorem 5.1 *In addition to (A1) - (A3) assume that b, R are continuous in the time variable as well. Let g_i, L_i, M_i be bounded continuous. Assume that $(s, y, z) \mapsto V_{\tilde{\beta}}^{(U,i)}(s, y, z; T)$ is a bounded continuous function on $[0, T] \times \bar{G} \times \bar{G}$. Then $V_{\tilde{\beta}}^{(U,i)}$ is a constrained viscosity solution to (5.2) on $[0, T] \times \bar{G} \times \bar{G}$.*

Proof: Refer to the proof of Theorem 5.9 below. Because of continuity of $f^{(U)}$, the proof is much simpler. See also [BC]. □

We now address the question of uniqueness. It is well known that this involves proving a comparison result. Our approach below is inspired by the proofs of Theorem III.3.7 and IV.5.8 of [BC], of course, with some crucial deviations/modifications.

We denote $x = (y,z) \in I\!R^d \times I\!R^d$ and $\langle x \rangle = \langle (y,z) \rangle = \left[1 + \sum_{i=1}^d y_i^2 + \sum_{j=1}^d z_j^2\right]^{1/2}$; also $e_{2d} = (1,1,\ldots,1,1) \in I\!R^{2d}$ with all the coordinates equal to 1. The following lemma is the analogue of Lemma III.2.11 of [BC], and the proof is similar.

Lemma 5.2 *Let b, R, L_i, M_i be bounded and Lipschitz continuous in all the variables (including the time variable). Then for $s, s' \in [0,T], x, x', \xi \in \bar{G} \times \bar{G}, \sigma > 0, \theta > 0, \theta' > 0$*

$$\begin{aligned} & \Big| H_{\tilde{\beta}}^{(U,i)}(s, x, \{\sigma[x - x' + \xi] + \theta x\}) \\ & - H_{\tilde{\beta}}^{(U,i)}(s', x', \{\sigma[x - x' + \xi] - \theta' x'\}) \Big| \\ \leq\ & K\sigma|x - x' + \xi| \cdot \{|s - s'| + |x - x'|\} \\ & + K\theta\langle x\rangle^2 + K\theta'\langle x'\rangle^2 + K\{|s - s'| + |x - x'|\} \end{aligned} \tag{5.6}$$

where K depends only on bounds and Lipschitz constants of b, R, L_i, M_i. □

Theorem 5.3 *Assume (A3) and that b, R, L_i, M_i are bounded Lipschitz continuous functions in all the variables (including the time variable). Let v_1, v_2 be functions on $[0,T] \times \bar{G} \times \bar{G}$ such that*
(a) v_1, v_2 are bounded uniformly continuous functions on $[0,T] \times \bar{G} \times \bar{G}$;
(b) v_1 is a viscosity subsolution to (5.2) on $[0,T) \times G \times G$;
(c) v_2 is a viscosity supersolution to (5.2) on $[0,T] \times \bar{G} \times \bar{G}$;
(d) $v_1(T,\cdot,\cdot) = v_2(T,\cdot,\cdot)$ on $\bar{G} \times \bar{G}$.
Then $v_1 \leq v_2$ on $[0,T] \times \bar{G} \times \bar{G}$. In particular, under the above hypotheses, if the value function $V_{\tilde{\beta}}^{(U,i)}$ given in the preceding theorem is bounded uniformly continuous on $[0,T] \times \bar{G} \times \bar{G}$, then it is the unique constrained viscosity solution to the HJB equation (5.2) in the class of bounded uniformly continuous functions with terminal value $g_i(T,\cdot,\cdot)$.

Proof: Suppose $M \equiv \sup\{v_1(s,y,z) - v_2(s,y,z) : s \in [0,T], y, z \in \bar{G}\} > 0$. For $\delta \in (0, M)$ note that there exists $(\tilde{s}, \tilde{x}) = (\tilde{s}, \tilde{y}, \tilde{z}) \in [0,T] \times \bar{G} \times \bar{G}$ such that $v_1(\tilde{s}, \tilde{x}) - v_2(\tilde{s}, \tilde{x}) = \delta$. Clearly $s < T$ by (d). Choose $\lambda > 0, \eta > 0, \mu > 0, \nu > 0$ such that

$$2\lambda\langle\tilde{x}\rangle + 2\eta(T - \tilde{s}) + \nu + 2d\mu \leq \frac{1}{2}\delta$$

ensuring that

$$2\lambda\langle\tilde{x}\rangle^m + 2\eta(T - \tilde{s}) + \nu + 2d\mu \leq \frac{1}{2}\delta, \text{ for all } 0 < m \leq 1. \tag{5.7}$$

For $(s, x; s', x') \in ([0,T] \times \bar{G} \times \bar{G})^2$ define

$$\Psi_\epsilon(s, x; s', x') = v_1(s,x) - v_2(s', x') - \mu\left|\frac{x - x'}{\epsilon} - e_{2d}\right|^2$$
$$- \nu\left|\frac{s - s'}{\epsilon} + 1\right|^2 - \lambda(\langle x\rangle^m + \langle x'\rangle^m) - \eta[(T - s) + (T - s')]. \tag{5.8}$$

We will choose $\epsilon > 0, 0 < m \leq 1$ suitably later; m will be chosen appropriately and fixed, whereas ϵ will be treated as a parameter; so only dependence on ϵ is highlighted in Ψ_ϵ. Since $\Psi_\epsilon \to -\infty$ as $|x| + |x'| \to \infty$ and Ψ_ϵ is continuous, it follows that there exists $(s_\epsilon, x_\epsilon; s'_\epsilon, x'_\epsilon)$ such that, by (5.7),

$$\begin{aligned}\Psi_\epsilon(s_\epsilon, x_\epsilon; s'_\epsilon, x'_\epsilon) &= \sup\{\Psi_\epsilon(s, x; s', x') : s, s' \in [0, T], x, x' \in \bar{G}\} \\ &\geq \Psi_\epsilon(\tilde{s}, \tilde{x}; \tilde{s}, \tilde{x}) \geq \frac{1}{2}\delta.\end{aligned}$$

Consequently

$$\begin{aligned}&\lambda[\langle x_\epsilon\rangle^m + \langle x'_\epsilon\rangle^m] + \eta[(T - s_\epsilon) + (T - s'_\epsilon)] \\ &+\mu\left|\frac{x_\epsilon - x'_\epsilon}{\epsilon} - e_{2d}\right|^2 + \nu\left|\frac{s_\epsilon - s'_\epsilon}{\epsilon} + 1\right|^2 \\ \leq\ & \sup v_1 - \inf v_2 \equiv C_0 \qquad (5.9)\end{aligned}$$

for all $\epsilon > 0, 0 < m \leq 1$. (Note that $C_0 > 0$, otherwise $M = 0$). It follows from (5.9) that $x_\epsilon, x'_\epsilon \in \overline{B(0 : (C_0/\lambda)^{1/m})}$ for all $\epsilon > 0$.

As Ψ_ϵ has its maximum at $(s_\epsilon, x_\epsilon; s'_\epsilon, x'_\epsilon)$

$$\Psi_\epsilon(s_\epsilon, x_\epsilon; s_\epsilon + \epsilon, x_\epsilon - \epsilon e_{2d}) + \Psi_\epsilon(s'_\epsilon - \epsilon, x'_\epsilon + \epsilon e_{2d}; s'_\epsilon, x'_\epsilon) \leq 2\Psi_\epsilon(s_\epsilon, x_\epsilon; s'_\epsilon, x'_\epsilon). \qquad (5.10)$$

By mean value theorem $\langle x_\epsilon - \epsilon e_{2d}\rangle^m = \langle x\rangle^m + O(\epsilon)$ as x_ϵ varies over a bounded set. Therefore

$$\begin{aligned}\Psi_\epsilon(s_\epsilon, x_\epsilon; s_\epsilon + \epsilon, x_\epsilon - \epsilon e_{2d}) &= v_1(s_\epsilon, x_\epsilon) - v_2(s_\epsilon + \epsilon, x_\epsilon - \epsilon e_{2d}) - 2\lambda\langle x_\epsilon\rangle^m \\ &\quad -2\eta(T - s_\epsilon) + \eta\epsilon + O(\epsilon) \qquad (5.11)\end{aligned}$$

and similarly

$$\begin{aligned}&\Psi_\epsilon(s'_\epsilon - \epsilon, x'_\epsilon + \epsilon e_{2d}; s'_\epsilon, x'_\epsilon) = v_1(s'_\epsilon - \epsilon, x'_\epsilon + \epsilon e_{2d}) \\ &-v_2(s'_\epsilon, x'_\epsilon) - 2\lambda\langle x'_\epsilon\rangle^m - 2\eta(T - s'_\epsilon) - \eta\epsilon + O(\epsilon). \qquad (5.12)\end{aligned}$$

Denote by ω the common modulus of continuity of v_1 and v_2; note that ω can be taken to be bounded as v_1, v_2 are bounded uniformly continuous. Now (5.10) - (5.12) imply

$$\begin{aligned}&\mu\left|\frac{x_\epsilon - x'_\epsilon}{\epsilon} - e_{2d}\right|^2 + \nu\left|\frac{s_\epsilon - s'_\epsilon}{\epsilon} + 1\right|^2 \\ \leq\ & \omega(|s_\epsilon - s'_\epsilon + \epsilon| + |x_\epsilon - x'_\epsilon - \epsilon e_{2d}|) + O(\epsilon). \qquad (5.13)\end{aligned}$$

If $\epsilon \leq 1$, then r.h.s. of (5.13) is bounded by a constant independent of ϵ. So from (5.13) we get

$$|x_\epsilon - x'_\epsilon| + |s_\epsilon - s'_\epsilon| \leq K_1\epsilon. \qquad (5.14)$$

Plugging (5.14) back into (5.13)

$$\left|\frac{x_\epsilon - x'_\epsilon}{\epsilon} - e_{2d}\right|^2 + \left|\frac{s_\epsilon - s'_\epsilon}{\epsilon} + 1\right|^2 \leq \omega(K_2\epsilon) + O(\epsilon) \qquad (5.15)$$

as $\epsilon \downarrow 0$, where K_2 is constant independent of ϵ.

Clearly the ball $B(x + \epsilon e_{2d} : \epsilon) \subset G \times G$ for any $x \in \bar{G} \times \bar{G}$. So by (5.15) it is now easily seen that $(s_\epsilon, x_\epsilon) \in [0, T) \times G \times G$ for any $\epsilon > 0$ such that r.h.s. of (5.15) < 1.

With $0 < \epsilon \leq 1$ as above, for $(s, x) = (s, y, z) \in [0, T] \times \bar{G} \times \bar{G}$ define

$$\begin{aligned} w_{1\epsilon}(s, x) &= v_2(s'_\epsilon, x'_\epsilon) + \mu \left| \frac{x - x'_\epsilon}{\epsilon} - e_{2d} \right|^2 \\ &+ \nu \left| \frac{s - s'_\epsilon}{\epsilon} + 1 \right|^2 + \lambda(\langle x \rangle^m + \langle x'_\epsilon \rangle^m) + \eta[(T - s) + (T - s'_\epsilon)]. \end{aligned}$$

Note that $(v_1 - w_{1\epsilon})$ has a maximum at $(s_\epsilon, y_\epsilon, z_\epsilon)$ and that $w_{1\epsilon}$ is C^1. Similarly, for $(s', x') = (s', y', z') \in [0, T] \times \bar{G} \times \bar{G}$ define

$$\begin{aligned} w_{2\epsilon}(s', x') &= v_1(s_\epsilon, x_\epsilon) - \mu \left| \frac{x_\epsilon - x'}{\epsilon} - e_{2d} \right|^2 \\ &- \nu \left| \frac{s_\epsilon - s'}{\epsilon} + 1 \right|^2 - \lambda[\langle x_\epsilon \rangle^m + \langle x' \rangle^m] - \eta[(T - s_\epsilon) + (T - s')]. \end{aligned}$$

Clearly $w_{2\epsilon}$ is C^1 and $(v_2 - w_{2\epsilon})$ has a minimum in $[0, T] \times \bar{G} \times \bar{G}$ at $(s'_\epsilon, y'_\epsilon, z'_\epsilon)$. Since v_1 is a viscosity subsolution in the interior and v_2 is a viscosity supersolution to (5.2) in the closure, it can be seen that

$$\begin{aligned} &2\eta + H\left(s_\epsilon, x_\epsilon, \left\{ \frac{2\mu}{\epsilon^2}[x_\epsilon - x'_\epsilon - \epsilon e_{2d}] + m\lambda \langle x_\epsilon \rangle^{m-2} x_\epsilon \right\}\right) \\ &- H\left(s'_\epsilon, x'_\epsilon, \left\{ \frac{2\mu}{\epsilon^2}[x_\epsilon - x'_\epsilon - \epsilon e_{2d}] - m\lambda \langle x'_\epsilon \rangle^{m-2} x'_\epsilon \right\}\right) \\ &\leq 0 \end{aligned} \qquad (5.16)$$

where $H = H^{(U,i)}_{\tilde{\beta}}$. Now by Lemma 5.2, (5.9), (5.14), (5.15)

$$\begin{aligned} &\left| H\left(s_\epsilon, x_\epsilon, \left\{ \frac{2\mu}{\epsilon^2}[x_\epsilon - x'_\epsilon - \epsilon e_{2d}] + m\lambda \langle x_\epsilon \rangle^{m-2} x_\epsilon \right\}\right) \right. \\ &\left. - H\left(s'_\epsilon, x'_\epsilon, \left\{ \frac{2\mu}{\epsilon^2}[x_\epsilon - x'_\epsilon - \epsilon e_{2d}] - m\lambda \langle x'_\epsilon \rangle^{m-2} x'_\epsilon \right\}\right) \right| \\ \leq\ & K \frac{2\mu}{\epsilon} \left| \frac{x_\epsilon - x'_\epsilon}{\epsilon} - e_{2d} \right| [|s_\epsilon - s'_\epsilon| + |x_\epsilon - x'_\epsilon|] \\ &+ K\ m\lambda[\langle x_\epsilon \rangle^m + \langle x'_\epsilon \rangle^m] + K[|s_\epsilon - s'_\epsilon| + |x_\epsilon - x'_\epsilon|] \\ \leq\ & 2KK_1\mu[\omega(K_2\epsilon) + O(\epsilon)]^{1/2} + m\ K\ C_0 + KK_1\epsilon \end{aligned} \qquad (5.17)$$

Now choose $m \in (0, 1)$ such that $m < \frac{\eta}{2KC_0}$. Then (5.16), (5.17) imply

$$\frac{3}{2}\eta - KK_1\epsilon - 2KK_1\mu[\omega(K_2\epsilon) + O(\epsilon)]^{1/2} \leq 0.$$

But this would contradict $\eta > 0$ for $\epsilon \downarrow 0$. Thus $M = 0$ and hence $v_1 \leq v_2$ on $[0, T] \times \bar{G} \times \bar{G}$. □

The following result gives another interpretation for the "pushing part" of the solution to the Skorokhod problem.

Theorem 5.4 *Assume (A1) - (A3), (i), (ii) of Theorem 2.6; assume that b, R are Lipschitz continuous in the time variable as well. Let $T > 0$ be fixed. Let $Y(\cdot; s, y, z)$, along with $Z(\cdot; s, y, z)$ solve SP$(b, R; G, s, y, z)$. Then for $1 \le i \le d$, the function $(s, y, z) \mapsto Y_i(T; s, y, z) - y_i$ is the unique constrained viscosity solution to the HJB equation*

$$-\partial_0 v(s, y, z) + \hat{H}^{(i)}_{\tilde{\beta}}(s, (y, z), D_{y,z} v(s, y, z)) = 0 \qquad (5.18)$$

with terminal value $v(T, \cdot, \cdot) \equiv 0$ in the class of bounded uniformly continuous functions on $[0, T] \times \bar{G} \times \bar{G}$, where the Hamiltonian $\hat{H}^{(i)}_{\tilde{\beta}}$ is given by

$$\hat{H}^{(i)}_{\tilde{\beta}}(s, (y, z), p) = \sup\{[-\langle p, f^{(U)}(s, (y, z), u)\rangle - u_i] : u \in \bar{G}_{\tilde{\beta}}\}. \qquad (5.19)$$

Proof: Consider the d-person game as in Section 3 with cost function $\int_s^T u_i(r)\, dr$ for the ith player; that is $g_i \equiv L_i \equiv 0, M_i \equiv 1$. By Theorem 2.6, the utopian equilibrium is given by the solution to the Skorokhod problem, and the value function is $Y_i(T; s, y, z) - y_i$, for the ith player. By Theorem 4.4, Y_i is a bounded Lipschitz continuous function of s, y, z. So the result now follows by Theorems 5.1 and 5.3. □

Remark 5.5 Though the function $v(\ldots) \equiv 0$ satisfies the equation (5.18) in the classical sense even beyond the orthant, it is not a constrained viscosity solution to (5.18) on $[0, T] \times \bar{G} \times \bar{G}$ in general. To see this take $d = 1, R_{11} = 1, b < 0$, a constant. Take $w(s, y, z) = -z$. Then $(v - w)$ has a local minimum in $[0, T] \times [0, \infty) \times [0, \infty)$ at $(0, 0, 0)$, or even at $(s, y, 0)$ for $s \in [0, T], y \ge 0$. But

$$-\partial_0 w(s, y, 0) + \hat{H}^{(1)}_{\tilde{\beta}}(s, (y, 0), D_{y,z} w(s, y, 0)) = b < 0,$$

and hence v is not a viscosity supersolution on $[0, T] \times [0, \infty) \times [0, \infty)$. The reason for this is, of course, that the control $u(\cdot) \equiv 0$, which corresponds to the cost $v(\cdot) \equiv 0$, is not feasible when $z = 0, b < 0$; this is also reflected in the fact that the boundary condition (3.19) or (3.20) fails to hold when $u(\cdot) \equiv 0, b < 0$ at $z = 0$; (cf. see also p.277 of [BC]).

Remark 5.6 Let the hypotheses of Theorems 2.6 and 5.4 hold; in addition assume that g_i, L_i are bounded Lipschitz continuous in all the variables. Consider the d-person game as in Section 3. It can be proved that utopian equilibrium is achieved at the solution to the Skorokhod problem, the value functions are Lipschitz continuous and form the unique constrained viscosity solution to the respective HJB equations in the class of bounded uniformly continuous functions. □

We now consider the case of Nash equilibrium. Once again we will consider only controls taking values in a bounded set. As indicated in Remark 3.5, fix $i = 1, 2, \ldots, d$ and $u_{-i}(\cdot)$ such that

$0 \le u_j(\cdot) \le \tilde{\beta}_j, j \neq i$. We will assume that the set of feasible controls

$$\mathcal{U}_{\tilde{\beta}}(s, y, z; T, u_{-i}(\cdot)) \neq \phi, \ \forall \ (s, y, z) \in [0, T] \times \bar{G} \times \bar{G} \tag{5.20}$$

where $\mathcal{U}_{\tilde{\beta}}(\ldots)$ is given by (3.31); note that Theorem 4.10 gives sufficient conditions for (5.20) to hold. The HJB equation in this case is

$$-\partial_0 v(s, y, z) + H_{\tilde{\beta}}^{(N,i)}(s, (y, z), D_{y,z} v(s, y, z)) = 0, \tag{5.21}$$

where the Hamiltonian $H_{\tilde{\beta}}^{(N,i)}$ is given by (3.33). In general $u_{-i}(\cdot)$ will not be continuous or even right continuous. So the Hamiltonian $H_{\tilde{\beta}}^{(N,i)}$ will not be continuous in the time variable. So we need to define the semicontinuous envelopes.

For notational convenience write $H = H_{\tilde{\beta}}^{(N,i)}$. For $t \in [0, T], y, z \in \bar{G}, p \in \mathbb{R}^{2d}$ set

$$\begin{aligned} & H_*(t, (y, z), p) \\ &= \liminf\{H(t', (y', z'), p') : (t', (y', z'), p') \to (t, (y, z), p) \text{ in } [0, T] \times \bar{G} \times \bar{G} \times \mathbb{R}^{2d}\} \\ &= \lim_{\theta \downarrow 0} \inf\{H(t', (y', z'), p') : |(t', (y', z'), p') - (t, (y, z), p)| \le \theta, 0 \le t' \le T, y', z' \in \bar{G}\} \end{aligned} \tag{5.22}$$

which is the *lower semicontinuous evelope*, and

$$\begin{aligned} & H^*(t, (y, z), p) \\ &= \limsup\{H(t', (y', z'), p') : (t', (y', z'), p') \to (t, (y, z), p) \text{ in } [0, T] \times \bar{G} \times \bar{G} \times \mathbb{R}^{2d}\} \\ &= \lim_{\theta \downarrow 0} \sup\{H(t', (y', z'), p') : |(t', (y', z'), p') - (t, (y, z), p)| \le \theta, 0 \le t' \le T, y', z' \in \bar{G}\} \end{aligned} \tag{5.23}$$

which is the *upper semicontinuous envelope.*

Lemma 5.7 *Let b, R, L_i, M_i be bounded and continuous; let $f^{(N,i)}, C^{(N,i)}$ be defined by (3.9), (3.10) respectively. For $0 \le t \le T, y, z \in \bar{G}, p \in \mathbb{R}^{2d}, 0 \le c \le \tilde{\beta}_i$ denote*

$$h(t, (y, z), p; c) = -\langle p, f^{(N,i)}(t, (y, z), c)\rangle - C^{(N,i)}(t, (y, z), c)$$

and

$$\begin{aligned} h_*(t, (y, z), p; c) &= \liminf\{h(t', (y', z'), p'; c) : (t', (y', z'), p') \to (t, (y, z), p)\} \\ h^*(t, (y, z), p; c) &= \limsup\{h(t', (y', z'), p'; c) : (t', (y', z'), p') \to (t, (y, z, p)\}. \end{aligned}$$

Here $c \in [0, \tilde{\beta}_i]$ acts as a parameter. Then

$$\begin{aligned} H_*(t, (y, z), p) &= \max\{h_*(t, (y, z), p; 0), h_*(t, (y, z), p; \tilde{\beta}_i)\} \\ H^*(t, (y, z), p) &= \max\{h^*(t, (y, z), p; 0), h^*(t, (y, z), p; \tilde{\beta}_i)\} \end{aligned}$$

Proof: Observe that $h(t,(y,z),p;c) = \Psi(t,(y,z),p)c + \Phi(t,(y,z),p)$, where Ψ is a bounded continuous function and Φ is bounded measurable. By linearity in c

$$H(t(y,z),p) = \Phi(t,(y,z),p) + \max\{0, \Psi(t,(y,z),p)\tilde{\beta}_i\}.$$

The required conclusions are now easy to obtain. □

Definition 5.8 (a) A locally bounded function v is said to be a *viscosity subsolution* to the discontinuous HJB equation (5.21) on $[0,T) \times G \times G$ if for any $s \in [0,T), y \in G, z \in G$, any C^1-function w such that $(v-w)$ has a local maximum at (s,y,z) one has

$$-\partial_0 w(s,y,z) + H_*(s,(y,z), D_{y,z}w(s,y,z)) \leq 0. \tag{5.24}$$

(b) A locally bounded function v is said to be a *viscosity supersolution* to (5.21) on $[0,T]\times\bar{G}\times\bar{G}$ if for any $s \in [0,T], y \in \bar{G}, z \in \bar{G}$, any C^1-function w such that $(v-w)$ has a local minimum in $[0,T] \times \bar{G} \times \bar{G}$ at (s,y,z) one has

$$-\partial_0 w(s,y,z) + H^*(s,(y,z), D_{y,z}w(s,y,z)) \geq 0. \tag{5.25}$$

(c) If v satisfies both (a), (b) above then it is called a *constrained viscosity solution* to (5.21) on $[0,T] \times \bar{G} \times \bar{G}$.

Note: [BC] (see Remark V.4.2 and Exercise V.4.1) very briefly discusses viscosity solution of a discontinuous HJB equation. However we do not know of any other instance of a discontinuous HJB equation with state constraints.

Theorem 5.9 *Let $i, u_{-i}(\cdot)$ be fixed such that $0 \leq u_j(\cdot) \leq \tilde{\beta}_j, j \neq i$. Assume that b, R, L_i, M_i, g_i are bounded continuous. Assume that (5.20) holds. Suppose the value function $V^{(N,i)}_{\tilde{\beta}_i}$, defined by (3.32), is a bounded continuous function on $[0,T]\times\bar{G}\times\bar{G}$. Then $(s,y,z) \mapsto V^{(N,i)}_{\tilde{\beta}}(s,y,z;T,u_{-i}(\cdot))$ is a constrained viscosity solution to the discontinuous HJB equation (5.21) on $[0,T] \times \bar{G} \times \bar{G}$ with terminal value $g_i(T,\cdot,\cdot)$.*

Proof: For simplicity of notation we shall drop the superscripts N, i and the subscript $\tilde{\beta}$.

Subsolution: Let $(s,y,z) \in [0,T)\times G\times G$. Let w be a C^1-function such that $(V-w)$ has a local maximum at (s,y,z). Taking $u_i(\cdot) \equiv c \in [0,\tilde{\beta}_i]$ arbitrarily, denoting by $y(\cdot), z(\cdot)$ the solution to the state equation corresponding to the control $u(\cdot) = (u_i(\cdot), u_{-i}(\cdot))$ with $y(s) = y, z(s) = z$, using dynamic programming principle, the state equation and usual arguments as in [BC], one can obtain

$$\begin{aligned} & - \int_s^{s'} [\partial_0 w(r,y(r),z(r)) + \langle D_{y,z}w(r,y(r),z(r)), f(r,(y(r),z(r)),c)\rangle \\ & \quad + C(r,(y(r),z(r)),c)]dr \\ & \leq \ 0 \end{aligned}$$

for all s' sufficiently close to s with $s' > s$. Hence

$$\liminf_{r \downarrow s} [-\partial_0 w(r, y(r), z(r)) + h(r, (y(r), z(r)), D_{y,z} w(r, y(r), z(r)); c)] \leq 0$$

where $h(\cdots)$ is an in Lemma 5.7. As $y(\cdot), z(\cdot), \partial_0 w, D_{y,z} w$ are continuous it now follows that

$$-\partial_0 w(s, y, z) + h_*(s, (y, z), D_{y,z} w(s, y, z); c) \leq 0 \qquad (5.26)$$

for any $0 \leq c \leq \tilde{\beta}_i$, where $h_*(\cdots)$ is as in Lemma 5.7. Now use (5.26) and Lemma 5.7 to get the required conclusion (5.24).

Supersolution: Let $(s, y, z) \in [0, T] \times \bar{G} \times \bar{G}$; (it could be a boundary point). Let w be a C^1-function such that $(V - w)$ has a local minimum in $[0, T] \times \bar{G} \times \bar{G}$ at (s, y, z). Using the definition of V, dynamic programming principle, state equation and arguments as in [BC] it can be shown that for any $\epsilon > 0, s' \in (s, T]$ sufficiently close to s, there exists $\bar{u}_i(\cdot)$, possibly depending on ϵ, s', such that $(\bar{u}_i(\cdot), u_{-i}(\cdot)) \in \mathcal{U}_{\tilde{\beta}}(s, y, z; T)$ and

$$\int_s^{s'} [-\partial_0 w(r, \bar{y}(r), \bar{z}(r)) + h(r, (\bar{y}(r), \bar{z}(r)), D_{y,z} w(r, \bar{y}(r), \bar{z}(r)); \bar{u}_i(r))] dr$$
$$\geq -\epsilon(s' - s)$$

where $\bar{y}(\cdot), \bar{z}(\cdot)$ denotes the solution to the state equation corresponding to the control $(\bar{u}_i(\cdot), u_{-i}(\cdot))$. Therefore by the definition of the Hamiltonian

$$\int_s^{s'} [-\partial_0 w(r, \bar{y}(r), \bar{z}(r)) + H(r, (\bar{y}(r), \bar{z}(r)), D_{y,z} w(r, \bar{y}(r), \bar{z}(r)))] dr$$
$$\geq -\epsilon(s' - s').$$

So, with $\epsilon > 0$ fixed, for any $s' \in (s, T]$ sufficiently close to s, there is a feasible control $\bar{u}_i(\cdot)$ such that

$$\sup_{s \leq r \leq s'} \{-\partial_0 w(r, \bar{y}(r), \bar{z}(r)) + H(r, (\bar{y}(r), \bar{z}(r)), D_{y,z} w(r, \bar{y}(r), \bar{z}(r)))\}$$
$$\geq -\epsilon. \qquad (5.27)$$

Note that the solution to the state equation is Lipschitz continuous in r, with the Lipschitz constant independent of the control. Hence given any small neighbourhood N of (s, y, z), there exists $s' \in (s, T]$ sufficiently close to s such that $(r, y(r), z(r)) \in N \cap ([s, T] \times \bar{G} \times \bar{G})$ for any $r \in [s, s']$ and any feasible control $u_i(\cdot)$. Consequently (5.27) now implies that

$$\sup\{-\partial_0 w(r', y', z') + H(r', (y', z'), D_{y,z} w(r', y', z')) :$$
$$(r', y', z') \in N \cap ([s, T] \times \bar{G} \times \bar{G})\}$$
$$\geq -\epsilon$$

whence it follows that

$$-\partial_0 w(s,y,z) + H^*(s,(y,z),D_{y,z}w(s,y,z)) \geq -\epsilon. \qquad (5.28)$$

As $\epsilon > 0$ is arbitrary (5.28) implies (5.25), completing the proof. □

Uniqueness cannot be expected to hold in general as the following counterexample indicates.

Example 5.10 Take $d = 2$. Let $A, B \subset [0,T]$ be subsets such that
(i) $A \cup B = [0,T], A \cap B = \phi$; (ii) both A and B are dense in $[0,T]$;
(iii) $m(A) > 0, m(B) > 0$ where m is the one dimensional Lebesgue measure. Let $K_1 < K_2$ be constants. Define $u_2(s) = K_1 I_A(s) + K_2 I_B(s), 0 \leq s \leq T$. Let $b \equiv (0,0), R_{12} \equiv R_{21} \equiv 0, R_{11} \equiv R_{22} \equiv 1, M_1 \equiv 1, L_1 \equiv 0$. So f is independent of (y,z) and is given by $f(s,c) = (c, u_2(s), c, u_2(s)) \in \mathbb{R}^2 \times \mathbb{R}^2$ and the Hamiltonian, for $s \in [0,T], p = (p_1,p_2,p_3,p_4) \in \mathbb{R}^4$ by

$$\begin{aligned} H(s,p) &= \sup\{-\langle p, f(s,c)\rangle - c : 0 \leq c \leq \tilde{\beta}_1\} \\ &= \sup\{-u_2(s)(p_2+p_4) - c(1+p_1+p_3) : 0 \leq c \leq \tilde{\beta}_1\}. \end{aligned}$$

If $1 + p_1 + p_3 \geq 0$ then clearly

$$H(s,p) = \begin{cases} -K_1(p_2+p_4), & \text{if } s \in A, p \in \mathbb{R}^4 \\ -K_2(p_2+p_4), & \text{if } s \in B, p \in \mathbb{R}^4. \end{cases}$$

Consequently, as A, B are dense in $[0,T]$,

$$H_*(s,p) = -K_2(p_2+p_4), \text{if } p_2+p_4 > 0, \qquad (5.29)$$

$$H^*(s,p) = -K_1(p_2+p_4), \text{if } p_2+p_4 > 0 \qquad (5.30)$$

for all s.

For a smooth function $(s,y,z) \mapsto v(s,y,z)$ such that $1 + \frac{\partial v}{\partial y_1} + \frac{\partial v}{\partial z_1} \geq 0$ and $\frac{\partial v}{\partial y_2} + \frac{\partial v}{\partial z_2} > 0$ on $[0,T] \times \bar{G} \times \bar{G}$, note that

$$\begin{aligned} &-\partial_0 v(s,y,z) + H_*(s, D_{y,z}v(s,y,z)) \\ &= -\frac{\partial v}{\partial s}(s,y,z) - K_2\frac{\partial v}{\partial y_2}(s,y,z) - K_2\frac{\partial v}{\partial z_2}(s,y,z) \end{aligned} \qquad (5.31)$$

and

$$\begin{aligned} &-\partial_0 v(s,y,z) + H^*(s, D_{y,z}v(s,y,z)) \\ &= -\frac{\partial v}{\partial s}(s,y,z) - K_1\frac{\partial v}{\partial y_2}(s,y,z) - K_1\frac{\partial v}{\partial z_2}(s,y,z). \end{aligned} \qquad (5.32)$$

For the linear first order p.d.e.

$$\left(-\frac{\partial v}{\partial s} - K\frac{\partial v}{\partial y_2} - K\frac{\partial v}{\partial z_2}\right)(s,y,z) = 0, 0 < s < T, y, z \in G, \qquad (5.33)$$

with terminal value

$$v(T,y,z) = 1 - e^{-y_2}e^{-z_2}, y,z \in \bar{G} \times \bar{G} \tag{5.34}$$

the solution is given by

$$v(s,y,z) = 1 - e^{-2K(T-s)}e^{-y_2}e^{-z_2}, 0 \le s \le T, y,z \in \bar{G}. \tag{5.35}$$

(Note that the general solution to (5.33) can be written in the form $\varphi(Ks - y_2, y_2 - z_2)$ where φ is an arbitrary C^1-function). Observe that $\frac{\partial v}{\partial y_2} + \frac{\partial v}{\partial z_2} > 0$. In view of this define two functions

$$v_1(s,y,z) = 1 - e^{-2K_1(T-s)}e^{-y_2}e^{-z_2} \tag{5.36}$$

$$v_2(s,y,z) = 1 - e^{-2K_2(T-s)}e^{-y_2}e^{-z_2} \tag{5.37}$$

for $0 \le s \le T, y,z \in \bar{G}$. Note that $v_1(T,y,z) = v_2(T,y,z) = 1 - e^{-y_2}e^{-z_2}$. By (5.31) - (5.37) it is clear that

$$-\partial_0 v_1(s,y,z) + H^*(s, D_{y,z}v_1(s,y,z)) = 0 \tag{5.38}$$

$$-\partial_0 v_2(s,y,z) + H_*(s, D_{y,z}v_2(s,y,z)) = 0. \tag{5.39}$$

Since $K_1 < K_2$, (5.31), (5.32), (5.36), (5.37) imply

$$-\partial_0 v_1(s,y,z) + H_*(s, D_{y,z}v_1(s,y,z)) < 0 \tag{5.40}$$

$$-\partial_0 v_2(s,y,z) + H^*(s, D_{y,z}v_2(s,y,z)) > 0 \tag{5.41}$$

on $[0,T] \times \bar{G} \times \bar{G}$. Now using (5.38), (5.40) arguing as in Proposition 3.3 it can be shown that v_1 is a constrained viscosity solution to (5.21) on $[0,T] \times \bar{G} \times \bar{G}$. Similarly (5.39), (5.41) lead to showing that v_2 is also a constrained viscosity solution to (5.21) on $[0,T] \times \bar{G} \times \bar{G}$. Clearly v_1, v_2 are both bounded and Lipschitz continuous.

In the above example, take $A = A_0 \cup A_1$ where A_0 is a Cantor set of positive Lebesgue measure and A_1 a countable dense set in $[0,T]$. Then there does not exist any function $\hat{u}(\cdot)$ on $[0,T]$ such that $u_2(\cdot) = \hat{u}(\cdot)$ a.s. and $m(\hat{D}) = 0$ where $\hat{D}$ is the set of discontinuities of $\hat{u}(\cdot)$. Because, if so, then for $a \cdot a \cdot s \in A_0$ there exist $s_n \in B$ with $s_n \to s$ and $K_1 = u_2(s) = \hat{u}(s) = \lim_n \hat{u}(s_n) = \lim_n u(s_n) = \lim_n K_2 = K_2$ which is a contradiction. □

The following is the analogue of Theorem 5.4 for Nash equilibrium, modulo uniqueness.

Theorem 5.11 *In addition to the hypotheses of Theorem 4.9 assume that b, R are Lipschitz continuous. Let $P(\cdot), Q(\cdot)$, or equivalently $Y(\cdot), Z(\cdot)$, solve $SP(b,R;\bar{G},s,y,z)$. Then for $1 \le i \le d, Y_i(T;s,y,z) - y_i$, as a function of s,y,z, is a constrained viscosity solution to the discontinuous HJB equation (5.21) on $[0,T] \times \bar{G} \times \bar{G}$ with $u_{-i}(\cdot) = P_{-i}(\cdot), L_i \equiv 0, M_i \equiv 0$ and terminal value $g_i \equiv 0$. That is, $Y_i(T_i;\cdot,\cdot,\cdot) - y_i, 1 \le i \le d$ form a system of bounded Lipschitz continuous constrained viscosity solutions on $[0,T] \times \bar{G} \times \bar{G}$ to the interlinked system of discontinuous HJB equations (5.21) with $u_{-i}(\cdot) = P_{-i}(\cdot), L_i \equiv 0, M_i \equiv 1, g_i \equiv 0, 1 \le i \le d$.*

Proof: As the Skorokhod problem is well posed note that (5.20) holds with $u_{-i}(\cdot) = P_{-i}(\cdot)$. By Theorem 4.9, $P(\cdot) = (P_1(\cdot), \dots, P_d(\cdot))$ is a Nash equilibrium in $\mathcal{U}_{\tilde{\beta}}(s, y, z; T)$ with $g_i \equiv 0, L_i \equiv 0, M_i \equiv 1, 1 \leq i \leq d$; hence $V_{\tilde{\beta}}^{(N,i)}(s, y, z; T, P_{-i}(\cdot)) = Y_i(T; s, y, z) - y_i$, for each i. By Theorem 4.4, $(s, y, z) \mapsto Y_i(T; s, y, z)$ is bounded Lipschitz continuous. The result now follows by Theorem 5.9. □

Finally we list some interesting questions.

1. Obtain an interesting set of conditions to guarantee existence of a unique constrained viscosity solution to (5.21) in the class of bounded uniformly continuous functions, and existence of a Nash equilibrium.

2. Does uniqueness in 1) imply that the set of discontinuities of the Hamiltonian should be a set of measure zero ?

3. Does the existence of a unique Nash equilibrium (serving for all $t \in [0, T]$) imply that it should be utopian ?

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