isibang/ms/2005/18
February 28th,2005
http://www.isibang.ac.in/~statmath/eprints

Riesz transform and Riesz potentials for Dunkl transform

Sundaram Thangavelu and Yuan Xu

Indian Statistical Institute, Bangalore Centre
8th Mile Mysore Road, Bangalore, 560059 India

RIESZ TRANSFORM AND RIESZ POTENTIALS FOR DUNKL TRANSFORM

SUNDARAM THANGAVELU AND YUAN XU

ABSTRACT. Analogues of Riesz potentials and Riesz transforms are defined and studied for the Dunkl transform associated with a family of weight functions that are invariant under a reflection group. The L^p boundedness of these operators is established in certain cases.

1. INTRODUCTION

For a family of weight functions, h_κ, invariant under a finite reflection group, Dunkl transform is a generalisation of the Fourier transform that defines an isometry of $L^2(\mathbb{R}^d, h_\kappa^2)$ onto itself. The basic properties of the Dunkl transform have been studied by several authors, see [2, 4, 6, 7, 8, 11, 12] and the references therein. Given the important role of Fourier transform in analysis, one naturally asks if it is possible to extend results established for the Fourier transform to the Dunkl transform.

In analogy to the ordinary Fourier analysis, one can define a convolution operator and study various summabilities of the inverse Dunkl transforms. The convolution is defined through a generalized translation operator, τ_y, which plays the role of $f \mapsto f(\cdot - y)$ but is defined on the Dunkl transform side. The explicit expression of $\tau_y f$ is known only in some special cases and it is not a positive operator in general. In fact, even the L^p boundedness of τ_y is not established in general. This is the main reason that only part of the results for the Fourier transforms has been extended to the Dunkl transform at the moment.

Recently, in [11], the L^p theory for convolution operators was studied. In particular, the L^p boundedness of the convolution operator is established in the case where the kernel is a suitable radial function. Furthermore, a maximal function is defined and shown to be of strong type (p,p) and weak type $(1,1)$. This provides a handy tool for extending some results from the Fourier transform to the Dunkl transform. In the present paper we study the analogues of the Riesz potentials and the Riesz transforms for the Dunkl transform. We will study the boundedness of the Riesz potentials as well as the related Bessel potentials. The Riesz transforms are examples of singular integrals. A general theory of singular integral for the Dunkl transform appears to be out of reach at the moment. We will prove the L^p boundedness of the weighted Riesz transform only in a very special case of $d = 1$ and $G = \mathbb{Z}_2$. Even in this simple case, however, the proof turns out to be rather nontrivial.

The paper is organized as follows. In the next section we collect the background materials. In Section 3 we recall the definition of the ordinary Riesz transforms and Riesz potentials, and prove a weighted L^p boundedness for the Riesz potentials that will be used later in the paper. The weighted Riesz potentials and the Bessel potentials for the Dunkl transform will be studied in Section 4. The weighted Riesz transform is discussed in Section 5.

Throughout this paper we use the convention that c denotes a generic constant, depending on d, p, κ or other fixed parameters, its value may change from line to line.

Date: April 6, 2005.

1991 *Mathematics Subject Classification.* 42A38, 42B08, 42B15.

Key words and phrases. Dunkl transforms, reflection invariance, Riesz transform, singular integrals.

The work of YX was supported in part by the National Science Foundation under Grant DMS-0201669.

2. Preliminaries

2.1. **Dunkl Transform.** The Dunkl transform is associated to a weight function that is invariant under a reflection group. Let G be a finite reflection group on $\mathbb{R}^d$ with a fixed positive root system R_+, normalized so that $\langle v, v\rangle = 2$ for all $v \in R_+$, where $\langle x, y\rangle$ denotes the usual Euclidean inner product. Let κ be a nonnegative multiplicity function $v \mapsto \kappa_v$ defined on R_+ with the property that $\kappa_u = \kappa_v$ whenever σ_u is conjugate to σ_v in G; then $v \mapsto \kappa_v$ is a G-invariant function. The weight function h_κ is defined by

$$h_\kappa(x) = \prod_{v\in R_+} |\langle x, v\rangle|^{\kappa_v}, \qquad x \in \mathbb{R}^d. \tag{2.1}$$

This is a positive homogeneous function of degree $\gamma_\kappa := \sum_{v\in R_+} \kappa_v$, and it is invariant under the reflection group G.

To define the Dunkl transform we will also need the intertwining operator V_κ. Let $\mathcal{D}_j$ denote Dunkl's differential-difference operators defined by [1]

$$\mathcal{D}_j f(x) = \partial_j f(x) + \sum_{v\in R_+} k_v \frac{f(x) - f(x\sigma_v)}{\langle x, v\rangle}\langle v, \varepsilon_j\rangle, \qquad 1 \le j \le d,$$

where $\varepsilon_1, \ldots, \varepsilon_d$ are the standard unit vectors of $\mathbb{R}^d$ and σ_v denote the reflection with respect to the hyperplane perpendicular to v, $x\sigma_v := x - 2(\langle x, v\rangle/\|v\|^2)v$, $x \in \mathbb{R}^d$. The operators $\mathcal{D}_j$, $1 \le j \le d$, map $\mathcal{P}_n^d$ to $\mathcal{P}_{n-1}^d$, where $\mathcal{P}_n^d$ denotes the space of homogeneous polynomials of degree n in d variables, and they mutually commute; that is, $\mathcal{D}_i\mathcal{D}_j = \mathcal{D}_j\mathcal{D}_i$, $1 \le i, j \le d$. The intertwining operator V_κ is a linear operator determined uniquely by

$$V_\kappa \mathcal{P}_n \subset \mathcal{P}_n, \quad V_\kappa 1 = 1, \quad \mathcal{D}_i V_\kappa = V_\kappa \partial_i, \quad 1 \le i \le d.$$

The explicit formula of V_κ is not known in general. For the group $G = \mathbb{Z}_2^d$, $h_\kappa(x) = \prod_{i=1}^d |x_i|^{\kappa_i}$, it is an integral transform

$$V_\kappa f(x) = b_\kappa \int_{[-1,1]^d} f(x_1t_1, \ldots, x_dt_d) \prod_{i=1}^d (1 + t_i)(1 - t_i^2)^{\kappa_i - 1} dt. \tag{2.2}$$

It is known that V_κ is a positive operator [6]; that is, $p \ge 0$ implies $V_\kappa p \ge 0$.

Let $E(x, iy) = V_\kappa^{(x)}\left[e^{i\langle x, y\rangle}\right]$, $x, y \in \mathbb{R}^d$, where the superscript means that V_κ is applied to the x variable. For $f \in L^1(\mathbb{R}^d, h_\kappa^2)$, the Dunkl transform is defined by

$$\widehat{f}(y) = c_h \int_{\mathbb{R}^d} f(x) E(x, -iy) h_\kappa^2(x) dx \tag{2.3}$$

where c_h is the constant defined by $c_h^{-1} = \int_{\mathbb{R}^d} h_\kappa^2(x) e^{-\|x\|^2/2} dx$. If $\kappa = 0$ then $V_\kappa = id$ and the Dunkl transform coincides with the usual Fourier transform. If $d = 1$ and $G = \mathbb{Z}_2$, then the Dunkl transform is related closely to the Hankel transform on the real line.

Some of the properties of the Dunkl transform is collected below ([2, 4]).

Proposition 2.1. (1) *For $f \in L^1(\mathbb{R}^d, h_\kappa^2)$, $\widehat{f}$ is in $C_0(\mathbb{R}^d)$.*

(2) *When both f and $\widehat{f}$ are in $L^1(\mathbb{R}^d, h_\kappa^2)$ we have the inversion formula*

$$f(x) = \int_{\mathbb{R}^d} E(ix, y) \widehat{f}(y) h_\kappa^2(y) dy.$$

(3) *The Dunkl transform extends to an isometry of $L^2(\mathbb{R}^d, h_\kappa^2)$.*

2.2. **Generalized translation operator.** Let $y \in \mathbb{R}^d$ be given. The generalized translation operator $f \mapsto \tau_y f$ is defined on $L^2(\mathbb{R}^d, h_\kappa^2)$ by the equation

$$\widehat{\tau_y f}(x) = E(y, -ix)\widehat{f}(x), \qquad x \in \mathbb{R}^d. \tag{2.4}$$

It plays the role of the ordinary translation $\tau_y f = f(\cdot - y)$ of $\mathbb{R}^d$, since the Fourier transform of τ_y is given by $\widehat{\tau_y f}(x) = e^{-i\langle x,y\rangle}\widehat{f}(x)$.

The generalized translation operator has been studied in [6, 7, 11, 12]. The definition gives $\tau_y f$ as an L^2 function. Let us define

$$\mathcal{A}_\kappa(\mathbb{R}^d) = \{f \in L^1(\mathbb{R}^d; h_\kappa^2) : \widehat{f} \in L^1(\mathbb{R}^d; h_\kappa^2)\}.$$

Then (2.4) holds pointwise. Note that $\mathcal{A}_\kappa(\mathbb{R}^d)$ is contained in the intersection of $L^1(\mathbb{R}^d; h_\kappa^2)$ and L^∞ and hence is a subspace of $L^2(\mathbb{R}^d; h_\kappa^2)$. The operator τ_y satisfies the following properties:

Proposition 2.2. *Assume that $f \in \mathcal{A}_\kappa(\mathbb{R}^d)$ and $g \in L^1(\mathbb{R}^d; h_\kappa^2)$ is bounded. Then*

(1) $\displaystyle\int_{\mathbb{R}^d} \tau_y f(\xi) g(\xi) h_\kappa^2(\xi) d\xi = \int_{\mathbb{R}^d} f(\xi)\tau_{-y} g(\xi) h_\kappa^2(\xi) d\xi.$

(2) $\tau_y f(x) = \tau_{-x} f(-y)$.

A formula of $\tau_y f$ is known, at the moment, only in two cases. One is in the case of $G = \mathbb{Z}_2$ and $h_\kappa(x) = |x|^\kappa$ on $\mathbb{R}$ ([5])

$$\begin{aligned}\tau_y f(x) =& \frac{1}{2}\int_{-1}^{1} f\left(\sqrt{x^2+y^2-2xyt}\right)\left(1+\frac{x-y}{\sqrt{x^2+y^2-2xyt}}\right)\Phi_\kappa(t)dt \\ &+ \frac{1}{2}\int_{-1}^{1} f\left(-\sqrt{x^2+y^2-2xyt}\right)\left(1-\frac{x-y}{\sqrt{x^2+y^2-2xyt}}\right)\Phi_\kappa(t)dt,\end{aligned} \tag{2.5}$$

where $\Phi_\kappa(t) = b_\kappa(1+t)(1-t^2)^{\kappa-1}$, from which also follows a formula of $\tau_y f$ in the case of $G = \mathbb{Z}_2^d$. The explicit formula implies the boundendess of $\tau_y f$. Let $\|\cdot\|_{\kappa,p}$ denote the norm of $L^p(\mathbb{R}^d, h_\kappa^2)$.

Proposition 2.3. *Let $G = \mathbb{Z}_2^d$. For $f \in L^p(\mathbb{R}^d, h_\kappa^2)$, $1 \le p \le \infty$,*

$$\|\tau_y f\|_{\kappa,p} \le c\|f\|_{\kappa,p}.$$

Another case where a formula for $\tau_y f$ is known is when f are radial functions, $f(x) = f_0(\|x\|)$, and G being any reflection group ([7]),

$$\tau_y f(x) = V_\kappa\left[f_0\left(\sqrt{\|x\|^2+\|y\|^2-2\|x\|\,\|y\|\langle x',\cdot\rangle}\right)\right](y'), \tag{2.6}$$

from which it follows that $\tau_y f(x) \ge 0$ for all $y \in \mathbb{R}^d$ if $f(x) = f_0(\|x\|) \ge 0$.

Several essential properties of $\tau_y f$ are established for radial functions. This is collected in the following proposition ([11]). Let $L^p_{\text{rad}}(\mathbb{R}^d, h_\kappa^2)$ stand for the subspace of radial functions in $L^p(\mathbb{R}^d, h_\kappa^2)$.

Proposition 2.4. (1) *For every $f \in L^1_{\text{rad}}(\mathbb{R}^d; h_\kappa^2)$,*

$$\int_{\mathbb{R}^d} \tau_y f(x) h_\kappa^2(x) dx = \int_{\mathbb{R}^d} f(x) h_\kappa^2(x) dx.$$

(2) *For $1 \le p \le 2$, $\tau_y : L^p_{\text{rad}}(\mathbb{R}^d, h_\kappa^2) \to L^p(\mathbb{R}^d; h_\kappa^2)$ is a bounded operator.*

The generalized translation τ_y also satisfies the following property [11, 12]: If f is supported in $\{x : \|x\| \le B\}$ then $\tau_y f$ is supported in $\{x : \|x\| \le B + \|y\|\}$. An important consequence of this property is as follows ([11]): If $f \in C_0^\infty(\mathbb{R}^d)$ then for $1 \le p \le \infty$

$$\lim_{y\to 0} \|\tau_y f - f\|_{\kappa,p} = 0.$$

2.3. **The generalized convolution and maximal function.** The generalized translation operator can be used to define a convolution. For f, g in $L^2(\mathbb{R}^d, h_\kappa^2)$, we define

$$(f *_\kappa g)(x) = \int_{\mathbb{R}^d} f(y)\tau_x g^\vee(y) h_\kappa^2(y)dy \tag{2.7}$$

where $g^\vee(y) = g(-y)$. Since $\tau_x g^\vee \in L^2(\mathbb{R}^d, h_\kappa^2)$ the convolution is well defined.

This convolution has been considered by several authors ([7, 11, 12] and the refrences therein). It satisfies the basic properties $\widehat{f *_\kappa g} = \widehat{f} \cdot \widehat{g}$ and $f *_\kappa g = g *_\kappa f$ provided f and g are Schwartz class functions. Furthermore, if the generalized translation operator is bounded in norm, then the usual Young's inequality holds. For the general reflection group, the following result is proved in [11].

Theorem 2.5. *Let g be a bounded radial function in $L^1(\mathbb{R}^d; h_\kappa^2)$. Then $f *_\kappa g$ initially defined by (2.7) on the intersection of $L^1(\mathbb{R}^d; h_\kappa^2)$ and $L^2(\mathbb{R}^d; h_\kappa^2)$ extends to all $L^p(\mathbb{R}^d; h_\kappa^2)$, $1 \le p \le \infty$ as a bounded operator. In particular,*

$$\|f *_\kappa g\|_{\kappa,p} \le \|g\|_{\kappa,1} \|f\|_{\kappa,p}. \tag{2.8}$$

For $f \in L^2$ the maximal function $M_\kappa f$ is defined in [11] by

$$M_\kappa f(x) = \sup_{r>0} \frac{1}{d_\kappa r^{d+2\gamma_\kappa}} |f *_\kappa \chi_{B_r}(x)|,$$

where χ_{B_r} is the characteristic function of the ball B_r of radius r centered at 0 and

$$(d_\kappa)^{-1} = \int_{B_1} h_\kappa^2(y)dy = (d + 2\gamma_\kappa) \int_{S^{d-1}} h_\kappa^2(x) d\omega.$$

The maximal function can also be written as

$$M_\kappa f(x) = \sup_{r>0} \frac{\left|\int_{B_r} \tau_y f(x) h_\kappa^2(y)dy\right|}{\int_{B_r} h_\kappa^2(y)dy}.$$

Since $\tau_y \chi_{B_r}(x) \ge 0$, see [11] we have $M_\kappa f(x) \le M_\kappa |f|(x)$. Furthermore, the following theorem holds:

Theorem 2.6. *The maximal function is bounded on $L^p(\mathbb{R}^d, h_\kappa^2)$ for $1 < p \le \infty$; moreover it is of weak type $(1,1)$, that is, for $f \in L^1(\mathbb{R}^d, h_\kappa^2)$ and $a > 0$,*

$$\int_{E(a)} h_\kappa^2(x)dx \le \frac{c}{a}\|f\|_{\kappa,1}$$

where $E(a) = \{x : M_\kappa f(x) > a\}$ and c is a constant independent of a and f.

For $\phi \in L^1(\mathbb{R}^d, h_\kappa^2)$ and $\varepsilon > 0$, the dilation ϕ_ε is defined by

$$\phi_\varepsilon(x) = \varepsilon^{-(2\gamma_\kappa + d)}\phi(x/\varepsilon).$$

A change of variables shows that

$$\int_{\mathbb{R}^d} \phi_\varepsilon(x) h_\kappa^2(x)dx = \int_{\mathbb{R}^d} \phi(x) h_\kappa^2(x)dx, \qquad \text{for all } \varepsilon > 0.$$

For convolution with a radial kernel, the following result is established in [11].

Theorem 2.7. *Let $\phi \in \mathcal{A}_\kappa(\mathbb{R}^d)$ be a real valued radial function which satisfies $|\phi(x)| \le c(1+\|x\|)^{-d-2\gamma_\kappa-1}$. Then*

$$\sup_{\varepsilon>0} |f *_\kappa \phi_\varepsilon(x)| \le c M_\kappa f(x).$$

Consequently, $f *_\kappa \phi_\varepsilon(x) \to f(x)$ for almost every x as ε goes to 0 for all f in $L^p(\mathbb{R}^d; h_\kappa^2)$, $1 \le p < \infty$.

If τ_y is bounded in $L^p(\mathbb{R}^d; h_\kappa^2)$ then the conditions in the above theorem can be relaxed. At the moment this holds in the case of $G = \mathbb{Z}_2^d$ ([11]).

Theorem 2.8. *Set $G = \mathbb{Z}_2^d$. Let $\phi(x) = \phi_0(\|x\|) \in L^1(\mathbb{R}^d, h_\kappa^2)$ be a radial function. Assume that ϕ_0 is differentiable, $\lim_{r\to\infty} \phi_0(r) = 0$ and $\int_0^\infty r^{2\gamma_\kappa+d}|\phi_0(r)|dr < \infty$, then*

$$|(f *_\kappa \phi)(x)| \le cM_\kappa f(x) \int_0^\infty r^{2\gamma_\kappa+d}|\phi_0(r)|dr < \infty.$$

In particular, if $\widehat{\phi}(x) = \Phi(x)$ satisfies $\phi \in L^1(\mathbb{R}^d, h_\kappa^2)$ and $\Phi(0) = 1$, then

(1) *For $1 \le p \le \infty$, $f *_\kappa \phi_\varepsilon$ converges to f as $\varepsilon \to 0$ in $L^p(\mathbb{R}^d, h_\kappa^2)$;*

(2) *For $f \in L^1(\mathbb{R}^d, h_\kappa^2)$, $(f *_\kappa \phi_\varepsilon)(x)$ converges to $f(x)$ as $\varepsilon \to 0$ for almost all $x \in \mathbb{R}^d$.*

3. Ordinary Riesz transforms and Riesz potentials

In this section the notation $\widehat{f}$ stands fo the ordinary Fourier transform of f on $\mathbb{R}^m$.

We recall the classical definition of the Riesz transforms and Riesz potentials. The Riesz transform, $R_j f$ $(1 \le j \le m)$, for the ordinary Fourier transform on $\mathbb{R}^m$ is defined by

$$R_j f(x) = \lim_{\varepsilon\to 0} c_j \int_{\|y\|\ge\varepsilon} f(x-y)\frac{y_j}{\|y\|^{m+1}}dy, \qquad c_j = \frac{2^{m/2}}{\sqrt{\pi}}\Gamma\left(\frac{m+1}{2}\right).$$

It is a multiplier operator in the sense that

$$R_j f(x) = c\int_{\mathbb{R}^m} f(x-y)\frac{y_j}{\|y\|^{m+1}}dy = c\int_0^\infty \int_{S^{d-1}} f_0(\|x - sy'\|)y_j' d\omega(y')\frac{ds}{s} \qquad = cx_j' \int_0^\infty \int_{-1}^1 f_0\Big(\sqrt{\|x\|^2 + s^2 - 2\|$$

Let $\widetilde{R}_j f_0$ be defined by the relation

$$x_j' \widetilde{R}_j f_0(r) = R_j f(rx'), \quad \text{when} \quad f(x) = f_0(\|x\|)$$

where $x = rx', x' \in S^{m-1}$. The boundedness of $R_j f$ gives us

Proposition 3.1. *If $f(x) = f_0(\|x\|)$ and $f \in L^p(\mathbb{R}^m)$ then*

$$\left(\int_0^\infty |\widetilde{R}_j f_0(r)|^p r^{m-1} dr\right)^{1/p} \le c\left(\int_0^\infty |f_0(r)|^p r^{m-1} dr\right)^{1/p}, \quad 1 < p < \infty.$$

The Riesz potential on $\mathbb{R}^m$ is defined as the ordinary convolution of f with the kernel $K_\alpha(x) = \gamma(\alpha)^{-1}\|x\|^{\alpha-m}$, $0 < \alpha < m$,

$$f * K_\alpha(x) = \gamma(\alpha)^{-1}\int_{\mathbb{R}^m} \|x-y\|^{-m+\alpha} f(y)dy, \tag{3.1}$$

where $\gamma(\alpha)$ is a constant whose value we will not need. It is well known (see, for example, [9, p. 119]) that for $1 < p < q < \infty$ where $1/q = 1/p - \alpha/m$,

$$\|f * K_\alpha\|_q \le A_{p,q}\|f\|_p.$$

The Riesz potentials will be used in Section 5. In particular, we need the following weighted inequality in the case of $\alpha = 1$.

Proposition 3.2. *If $f(\|\cdot\|) \in L^p$, then*

$$\left(\int_{\mathbb{R}^m} |f * K_1(x)|^p dx\right)^{1/p} \le c\left(\int_{\mathbb{R}^m} |f(x)|^p \|x\|^p dx\right)^{1/p}$$

for all p satisfying $p > m/(m-1)$, $m \ge 2$.

Proof. Let Mf be the Hardy-Littlewood maximal function defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|B_r|} \int_{B_r} |f(x-y)|dy$$

where $|B_r|$ denotes the volume of the ball B_r. We will make use of the weighted norm ineqaulity

$$\int_{\mathbb{R}^m} |Mf(x)|^p w(x)dx \le c \int_{\mathbb{R}^m} |f(x)|^p w(x)dx,$$

which holds whenever w belongs to Muckenhoupt's A_p class $A_p(\mathbb{R}^m)$. It is known that $w(x) = \|x\|^\alpha \in A_p(\mathbb{R}^m)$ whenever $-m < \alpha < m(p-1)$ (ses, for example, [10, p. 218]). In particular, $\|x\|^p \in A_p(\mathbb{R}^d)$ for $p > m/(m-1)$ and $\|x\|^{-\delta p} \in A_p(\mathbb{R}^d)$ for $0 < \delta < m/p$.

To prove the weighted inequality we split $f * K_1$ as follows,

$$\begin{aligned}(f * \|\cdot\|^{1-m})(x) &= \int_{\|y\|\le 2\|x\|} + \int_{\|y\|>2\|x\|} f(x-y)\|y\|^{1-m}dy \\ &= T_1f(x) + T_2f(x).\end{aligned}$$

We estimate T_1f as follows:

$$\begin{aligned}|T_1f(x)| &\le \sum_{j=0}^{\infty} \int_{2^{-j}\|x\|\le\|y\|<2^{-j+1}\|x\|} |f(x-y)|\,\|y\|^{1-m}dy \\ &\le \sum_{j=0}^{\infty} 2^{-j(1-m)}\|x\|^{1-m} \int_{2^{-j}\|x\|\le\|y\|<2^{-j+1}\|x\|} |f(x-y)|dy \\ &\le c\sum_{j=0}^{\infty} 2^{-j}\|x\|Mf(x) \le c\|x\|Mf(x).\end{aligned}$$

Using the weighted ineqaulity of the maximal function it follows that,

$$\int_{\mathbb{R}^d} |T_1f(x)|^p dx \le c\int_{\mathbb{R}^d} |Mf(x)|^p\|x\|^p dx \le c\int_{\mathbb{R}^d} |f(x)|^p\|x\|^p dx,$$

as $\|x\|^p \in A_p(\mathbb{R}^m)$ for $p > m/(m-1)$.

To deal with T_2f we choose δ such that $0 < \delta < m/p$ and write the integral as

$$\begin{aligned}|T_2f(x)| &\le \sum_{j=1}^{\infty} \int_{2^j\|x\|\le\|y\|<2^{j+1}\|x\|} |f(x-y)|\,\|y\|^{1-m}dy \\ &= \sum_{j=1}^{\infty} \int_{2^j\|x\|\le\|y\|<2^{j+1}\|x\|} |f(x-y)|\,\|x-y\|^{1+\delta}\|x-y\|^{-1-\delta}\|y\|^{1-m}dy.\end{aligned}$$

For $2^j\|x\| \le \|y\| < 2^{j+1}\|x\|$ we have $\|y\| \ge 2\|x\|$ and hence

$$\|x-y\| \ge \|y\| - \|x\| = \|y\|/2 + (\|y\|/2 - \|x\|) \ge \|y\|/2 \ge 2^{j-1}\|x\|$$

so that

$$\begin{aligned}|T_2f(x)| &\le c\|x\|^{-\delta}\sum_{j=0}^{\infty} 2^{-j\delta}(2^j\|x\|)^{-m} \int_{\|x\|\le\|y\|<2^{j+1}\|x\|} |f(x-y)|\|x-y\|^{1+\delta}dy \\ &\le c\|x\|^{-\delta}\sum_{j=0}^{\infty} 2^{-j}Mf_\delta(x) \\ &\le c\|x\|^{-\delta}Mf_\delta(x),\end{aligned}$$

where $f_\delta(x) = f(x)\|x\|^{1+\delta}$. Then the weighted inequality of the maximal function implies, as $\|x\|^{-\delta p} \in A_p(\mathbb{R}^d)$,

$$\begin{aligned}\int_{\mathbb{R}^d} |T_2 f(x)|^p dx &\le c \int_{\mathbb{R}^d} |M f_\delta(x)|^p \|x\|^{-\delta p} dx \\ &\le c \int_{\mathbb{R}^d} |f_\delta(x)|^p \|x\|^{-\delta p} dx = c \int_{\mathbb{R}^d} |f(x)|^p \|x\|^p dx.\end{aligned}$$

This completes the proof. □

4. Weighted Riesz Potentials and Bessel Potentials

In this section the notation $\widehat{f}$ denotes the Dunkl transform of f.

4.1. **Riesz potentials.** For $0 < \alpha < 2\gamma_\kappa + d$, the weighted Riesz potential, $I_\alpha^\kappa f$, is defined on $\mathcal{S}$ by

$$I_\alpha^\kappa f(x) = (d_\kappa^\alpha)^{-1} \int_{\mathbb{R}^d} \tau_y f(x) \frac{1}{\|y\|^{2\gamma_\kappa + d - \alpha}} h_\kappa^2(y) dy, \tag{4.1}$$

where $d_\kappa^\alpha = 2^{-\gamma_\kappa - d/2 + \alpha} \Gamma(\frac{\alpha}{2})/\Gamma(\gamma_\kappa + \frac{d-\alpha}{2})$.

In order to derive the Dunkl transform of I_α^κ, we start with a lemma, which is a little more general than what is needed. A homogeneous polynomial P is called an h-harmonics if $\Delta_h P = 0$, where $\Delta_h = \mathcal{D}_1^2 + \ldots + \mathcal{D}_d^2$ is the so-called the h-Laplacian. Let $\mathcal{H}_n^d(h_\kappa^2)$ denote the space of h-harmonics of degree n. It is known that

$$\int_{S^{d-1}} P(x) q(x) h_\kappa^2(x) d\omega = 0$$

whenever $P \in \mathcal{H}_n^d(h_\kappa^2)$ and the degree q is less than n.

Lemma 4.1. *For $P \in \mathcal{H}_n^d(h_\kappa^2)$ and $0 < \Re\{\alpha\} < 2\gamma_\kappa + d$, the identity*

$$\left(\frac{P(x)}{\|x\|^{2\gamma_\kappa + d + n - \alpha}}\right)^{\widehat{}} = d_{n,\kappa} \frac{P(x)}{\|x\|^{n+\alpha}}, \qquad d_{n,\kappa}^\alpha = i^{-n} 2^{-\gamma_\kappa - d/2 + \alpha} \frac{\Gamma(\frac{n+\alpha}{2})}{\Gamma(\gamma_\kappa + \frac{n+d-\alpha}{2})},$$

holds in the sense that

$$\int_{\mathbb{R}^d} \frac{P(x)}{\|x\|^{2\gamma_\kappa + d + n - \alpha}} \widehat{\phi}(x) h_\kappa^2(x) dx = d_{n,\kappa}^\alpha \int_{\mathbb{R}^d} \frac{P(x)}{\|x\|^{n+\alpha}} \phi(x) h_\kappa^2(x) dx \tag{4.2}$$

for every ϕ which is sufficiently rapidly decreasing at ∞, and whose Dunkl transform has the same property.

Proof. If $P_n \in \mathcal{H}_n^d(h_\kappa^2)$, then Theorem 5.7.5 of [3] shows that

$$\left(P_n(x) e^{-\|x\|^2/2}\right)^{\widehat{}} = (-i)^n P_n(x) e^{-\|x\|^2/2}.$$

Since the Dunkl transform satisfies $\widehat{f(s\cdot)}(y) = s^{-2\gamma_\kappa - d} \widehat{f}(s^{-1} y)$, it follows that

$$\left(P_n(x) e^{-s\|x\|^2/2}\right)^{\widehat{}} = (-i)^n s^{-n-\gamma_\kappa - d/2} P_n(x) e^{-\|x\|^2/(2s)}.$$

Let ϕ be a function that satisfies the property in the statement of the lemma. For $s > 0$, the above formula leads to the relation

$$\begin{aligned}&\int_{\mathbb{R}^d} P_n(x) e^{-s\|x\|^2/2} \widehat{\phi}(x) h_\kappa^2(x) dx \\ &\qquad = (-i)^n s^{-n-\gamma_\kappa - d/2} \int_{\mathbb{R}^d} P_n(x) e^{-\|x\|^2/(2s)} \phi(x) h_\kappa^2(x) dx.\end{aligned} \tag{4.3}$$

We then multiply the above equation by $s^{\beta-1}$, where $\beta = \gamma_\kappa + (d+n-\alpha)/2$, and integrate the result with respect to s on $[0,\infty)$. Using

$$\int_0^\infty s^{a-1} e^{-s\|x\|^2/2} ds = 2^a \Gamma(a) \|x\|^{-2a}$$

and changing the order of the integrals, it is easy to see that this leads to

$$2^{\gamma_\kappa+(d+n-\alpha)/2}\Gamma(\gamma_\kappa + \tfrac{d+n-\alpha}{2}) \int_{\mathbb{R}^d} \frac{P_n(x)}{\|x\|^{2\gamma_\kappa+d+n-\alpha}} \widehat{\phi}(x) h_\kappa^2(x) dx$$
$$= (-i)^n 2^{(n+\alpha)/2} \Gamma(\tfrac{n+\alpha}{2}) \int_{\mathbb{R}^d} \frac{P_n(x)}{\|x\|^{n+\alpha}} \phi(x) h_\kappa^2(x) dx,$$

which simplifies to the stated equation. In the above we can assume the decay of ϕ and $\widehat{\phi}$ is in the order of

$$|\phi(x)| \le A(1+\|x\|)^{-d-2\gamma_\kappa} \quad \text{and} \quad |\widehat{\phi}(x)| \le A(1+\|x\|)^{-d-2\gamma_\kappa}$$

to ensure that the double integrals that occur above converge absolutely, so that the Fubini theorem applies. □

Proposition 4.2. *Let $0 < \alpha < 2\gamma_\kappa + d$. The identity*

$$\widehat{I_\alpha^\kappa f}(x) = \|x\|^{-\alpha} \widehat{f}(x) \tag{4.4}$$

holds in the sense that

$$\int_{\mathbb{R}^d} I_\alpha^\kappa f(x) g(x) h_\kappa^2(x) dx = \int_{\mathbb{R}^d} \widehat{f}(x) \|x\|^{-\alpha} \widehat{g}(x) h_\kappa^2(x) dx$$

whenever $f, g \in \mathcal{S}$.

Proof. Setting $n = 0$ in Lemma 4.3 shows that the Dunkl transform of $\|x\|^{-d-2\gamma_\kappa+\alpha}$ is the function $d_\kappa^\alpha \|x\|^{-\alpha}$ in the sense that

$$\int_{\mathbb{R}^d} \|y\|^{-d-2\gamma_\kappa+\alpha} \phi(y) h_\kappa^2(y) dy = d_\kappa^\alpha \int_{\mathbb{R}^d} \|y\|^{-\alpha} \widehat{\phi}(y) h_\kappa^2(y) dy,$$

where $\phi \in \mathcal{S}$. Set $\phi(y) = \tau_y f(x)$ in the above identity leads to

$$\int_{\mathbb{R}^d} \|y\|^{-d-2\gamma_\kappa+\alpha} \tau_y f(x) h_\kappa^2(y) dy = d_\kappa^\alpha \int_{\mathbb{R}^d} \|y\|^{-\alpha} E(-ix, y) \widehat{f}(y) h_\kappa^2(y) dy.$$

Multiplying this identity by $g(x)$ and integrating, we obtain the stated identity. □

Recall that $(-\Delta_h f)\widehat{\ }(x) = \|x\|^2 \widehat{f}(x)$ ([2]) for $f \in \mathcal{S}$, the identity (4.4) shows that the weighted Riesz potential can be defined as $(-\Delta_h)^{-\alpha/2} f$. The identity (4.4) also shows that

$$I_\alpha^\kappa(I_\beta^\kappa f) = I_{\alpha+\beta}^\kappa(f), \qquad f \in \mathcal{S}, \quad \alpha, \beta > 0, \quad \alpha + \beta < 2\gamma_\kappa + d.$$

$$\Delta_h(I_\alpha^\kappa f) = I_\alpha^\kappa(\Delta_h f) = -I_{\alpha-2}^\kappa(f), \qquad f \in \mathcal{S}, \quad 2\gamma_\kappa + d > \alpha \ge 2,$$

which are extensions of familiar identities for the ordinary Riesz potentials. Next we consider the boundedness of I_α^κ as an operator from $L^q(\mathbb{R}^d, h_\kappa^2)$ to $L^p(\mathbb{R}^d, h_\kappa^2)$. The following necessary condition holds:

Proposition 4.3. *If $\|I_\alpha^\kappa f\|_{\kappa,q} \le c \|f\|_{\kappa,p}$ for $f \in \mathcal{S}$, then it is necessary that*

$$\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{2\gamma_\kappa + d}. \tag{4.5}$$

Proof. Let $f_s(x) = e^{-s^2\|x\|^2}$. Using the fact $\tau_y f_s(x) = e^{-s^2(\|x\|^2+\|y\|^2)}E(2s^2x,y)$, a change of variable shows that

$$I_\alpha^\kappa(e^{-s^2\|x\|^2})(x) = s^{-\alpha} I_\alpha^\kappa(e^{-\|x\|^2})(sx).$$

Consequently, setting $f(x) = e^{-\|x\|^2}$, a changing of variables shows that

$$\|I_\alpha^\kappa f_s\|_{\kappa,q} = s^{-\alpha-(2\gamma_\kappa+d)/q}\|I_\alpha^\kappa f\|_{\kappa,q} \quad \text{and} \quad \|f_s\|_{\kappa,p} = s^{-(2\gamma_\kappa+d)/p}\|f\|_{\kappa,p}$$

for all $s > 0$. Considering $s \to 0$ and $s \to \infty$ shows that if $\|I_\alpha f\|_{\kappa,q} \le c\|f\|_{\kappa,p}$ holds, then we must have $\alpha + (2\gamma_\kappa + d)/q - (2\gamma_\kappa + d)/p = 0$, which gives (4.5). □

The main result on the weighted Riesz potential is the following theorem.

Theorem 4.4. *Let $G = \mathbb{Z}_2^d$. Let α be a real number such that $0 < \alpha < 2\gamma_\kappa + d$ and let $1 \le p < q < \infty$ satisfies* (4.5).

(1) *For $f \in L^p(\mathbb{R}^d, h_\kappa^2)$, $p > 1$,*

$$\|I_\alpha^\kappa f\|_{\kappa,q} \le c\|f\|_{\kappa,p}.$$

(2) *For $f \in L^1(\mathbb{R}^d, h_\kappa^2)$, the mapping $f \mapsto I_\alpha^\kappa f$ is of weak type $(1, q)$; that is,*

$$\int_{\{x:|I_\alpha^\kappa(x)|>\sigma\}} h_\kappa^2(x)dx \le c\left(\frac{\|f\|_{\kappa,1}}{\sigma}\right)^q.$$

Proof. Let $R > 0$ be fixed. We write the operator as a sum of two terms,

$$\begin{aligned} I_\alpha^\kappa f(x) = (d_\kappa^\alpha)^{-1} &\int_{\{x:\|x\|\le R\}} \tau_y f(x) \frac{1}{\|y\|^{2\gamma_\kappa+d-\alpha}} h_\kappa^2(y)dy \\ + (d_\kappa^\alpha)^{-1} &\int_{\{x:\|x\|\ge R\}} \tau_y f(x) \frac{1}{\|y\|^{2\gamma_\kappa+d-\alpha}} h_\kappa^2(y)dy := S_1 f(x) + S_2 f(x). \end{aligned}$$

For $S_1 f$, we use the maximal function and the Theorem 2.8 to get the estimate

$$|S_1 f(x)| \le c M_\kappa f(x) \int_0^R r^{2\gamma_\kappa+d} \frac{d}{dr}\left(r^{-2\gamma_\kappa-d+\alpha}\right) dr \le \frac{c}{\alpha} R^\alpha M_\kappa f(x).$$

To estimate $S_2 f$ we use Proposition 2.3 which states that $\|\tau_y f\|_{\kappa,p} \le c\|f\|_{\kappa,p}$. Let $p' = p/(p-1)$. Then Hölder's inequality shows that

$$\begin{aligned} |S_2 f(x)| &\le \left(\int_{y:\|y\|\ge R} \left(\|y\|^{-2\gamma_\kappa-d+\alpha}\right)^{p'} h_\kappa^2(y)dy\right)^{1/p'} \|\tau_{-x} f\|_{\kappa,p} \\ &\le cR^{-(d+2\gamma_\kappa)/p+\alpha}\|f\|_{\kappa,p} = cR^{-(d+2\gamma_\kappa)/q}\|f\|_{\kappa,p}, \end{aligned}$$

where the last step follows from (4.5). Together, the two estimates show that

$$|I_\alpha^\kappa f(x)| \le c\left(R^\alpha M_\kappa f(x) + R^{-(d+2\gamma_\kappa)/q}\|f\|_{\kappa,p}\right)$$

for all $R > 0$. Choosing $R = (M_\kappa f(x)/\|f\|_{\kappa,p})^{-p/(d+2\gamma_\kappa)}$ and using (4.5), we obtain the inequality

$$|I_\alpha^\kappa f(x)| \le c\,(M_\kappa f(x))^{p/q}\,\|f\|_{\kappa,p}^{1-p/q}.$$

Consequently, for $p > 1$, we can use the L^p boundedness of the maximal function in Theorem 2.6 to conclude that

$$\|I_\alpha^\kappa f\|_{\kappa,q} \le c\|M_\kappa f(x)\|_{\kappa,p}^{p/q}\|f\|_{\kappa,p}^{1-p/q} \le c\|f\|_{\kappa,p}.$$

For $p=1$, we use the weak type $(1,1)$ inequality of the maximal function to get

$$\int_{\{x: I_\alpha^\kappa(x)\ge\sigma\}} h_\kappa^2(x)dx \le \int_{\{x: c\|M_\kappa f(x)\|_{\kappa,p}^{p/q}\|f\|_{\kappa,p}^{1-p/q}\ge\sigma\}} h_\kappa^2(x)dx$$
$$\le c\left(\frac{\|f\|_{\kappa,1}^{1-1/q}}{\sigma}\right)^q \|f\|_{\kappa,1} = c\left(\frac{\|f\|_{\kappa,1}}{\sigma}\right)^q.$$

The proof is completed. □

The proof shows that if τ_y is a bounded operator from $L^p(\mathbb{R}^d, h_\kappa^2)$ to itself for some other reflection group, then the conclusion of the theorem will hold for that group. At the moment, the theorem holds only if $G=\mathbb{Z}_2^d$ or if f are radial functions and $1\le p\le 2$ by Propostion 2.4.

4.2. **Weighted Bessel potentials.** The Bessel potentials are closely related to the Riesz potential. The kernel functions for the Bessel potentials have essentially the same local behavior as that of the Riesz potentials as $\|x\|\to 0$, but have much better behavior for $\|x\|$ large. In analogous to the ordinary Fourier transform, the weighted Bessel potentials, $\mathcal{J}_\alpha$, can be defined by

$$\mathcal{J}_\alpha^\kappa = (I-\Delta_h)^{-\alpha/2}, \qquad \alpha>0.$$

To be more precise, we define $\mathcal{J}_\alpha^\kappa$ as a convolution operator

$$\mathcal{J}_\alpha^\kappa f = f *_\kappa G_\alpha^\kappa, \quad \text{where} \quad \widehat{G}_\alpha^\kappa(x) = (1+\|x\|^2)^{-\alpha/2}$$

for $f\in\mathcal{S}$. The following proposition gives an explicit expression for G_α^κ.

Proposition 4.5. *Let G_α^κ be defined as above. Then $G_\alpha^\kappa(x)\ge 0$ for all $x\in\mathbb{R}$, $G_\alpha^\kappa\in L^1(\mathbb{R}^d,h_\kappa^2)$, and*

$$G_\alpha^\kappa(x) = \frac{1}{\Gamma(\alpha/2)}\int_0^\infty e^{-t}e^{-\|x\|^2/(4t)}t^{-\gamma_\kappa+(\alpha-d)/2}\frac{dt}{t}. \tag{4.6}$$

Proof. This follows as in the ordinary Bessel potentials ([9, p. 132]). Let us work backward and start with (4.6). Evidently then $G_\alpha^\kappa(x)>0$. Furthermore, since

$$c_h\int_{\mathbb{R}^d} e^{-\|x\|^2/(2t)}h_\kappa^2(x)dx = (2t)^{\gamma_\kappa+d/2}c_h\int_{\mathbb{R}^d} e^{-\|u\|^2/2}h_\kappa^2(u)du = t^{\gamma_\kappa+d/2}$$

the Fubini's theorem applied to (4.6), which shows

$$c_h\int_{\mathbb{R}^d} G_\alpha^\kappa(x)h_\kappa^2(x)dx = \frac{1}{\Gamma(\alpha/2)}\int_0^\infty t^{\alpha/2-1}e^{-t}dt = 1.$$

Using the fact that $\left((4)^{-\gamma_\kappa-\frac{d}{2}}e^{-\|x\|^2/(4t)}\right)^\wedge = e^{-t\|x\|^2}$, it follows that

$$\widehat{G}_\alpha^\kappa(x) = \frac{1}{\Gamma(\alpha/2)}\int_0^\infty e^{-t}e^{-t\|x\|^2)}t^{\alpha/2}\frac{dt}{t} = (1+\|x\|^2)^{-\alpha/2},$$

where the interchange of integrals can be easily justified by Fubini's theorem. □

The behavior of G_α^κ is described in the following lemma.

Lemma 4.6. *For $\alpha>0$,*

$$G_\alpha^\kappa(x)\le c\left(1+\|x\|^{-2\gamma_\kappa-d+\alpha}\right)e^{-\|x\|/2}, \qquad \|x\|>0.$$

Proof. The elementary inequality $t + r^2/(4t) \ge 2\sqrt{t}\sqrt{r^2/4t} = r$ leads to

$$G_\alpha^\kappa(x) \le \frac{1}{\Gamma(\alpha/2)} e^{-\|x\|/2} \int_0^\infty e^{-\frac{1}{2}(t+\frac{\|x\|^2}{4t})} t^{-\gamma_\kappa+(\alpha-d)/2} \frac{dt}{t}.$$

To estimate the integral we split it into two parts. Changing variable gives

$$\int_0^{\|x\|^2} e^{-\frac{1}{2}(t+\frac{\|x\|^2}{4t})} t^{-\gamma_\kappa+(\alpha-d)/2} \frac{dt}{t} \le \int_0^{\|x\|^2} e^{-\frac{\|x\|^2}{8t}} t^{-\gamma_\kappa+\frac{\alpha-d}{2}} \frac{dt}{t}$$
$$= \|x\|^{-2\gamma_\kappa+\alpha-d} \int_0^1 e^{-\frac{1}{8u}} u^{-\gamma_\kappa+\frac{\alpha-d}{2}} \frac{du}{u} \le c\|x\|^{-2\gamma_\kappa+\alpha-d}.$$

Furthermore, if $\|x\| \ge 1$, then

$$\int_{\|x\|^2}^\infty e^{-\frac{1}{2}(t+\frac{\|x\|^2}{4t})} t^{-\gamma_\kappa+(\alpha-d)/2} \frac{dt}{t} \le \int_{\|x\|^2}^\infty e^{-\frac{t}{2}} t^{-\gamma_\kappa+\frac{\alpha-d}{2}} \frac{dt}{t} \le c$$

and if $\|x\| \le 1$, then

$$\int_{\|x\|^2}^\infty e^{-\frac{1}{2}(t+\frac{\|x\|^2}{4t})} t^{-\gamma_\kappa+(\alpha-d)/2} \frac{dt}{t} \le \int_{\|x\|^2}^\infty e^{-\frac{t}{2}} t^{-\gamma_\kappa+(\alpha-d)/2} \frac{dt}{t}$$
$$\le c\left(1 + \int_{\|x\|^2}^1 t^{-\gamma_\kappa+(\alpha-d)/2} \frac{dt}{t}\right) \le c\left(1 + \|x\|^{-2\gamma_\kappa+\alpha-d}\right).$$

Putting these estimates together proves the stated inequality. □

This shows, in particular, that G_α^κ behaves as $\|x\|^{-2\gamma_\kappa-d+\alpha}$ for $\|x\| \to 0$, same as the kernel for the Riesz potentials.

Theorem 4.7. *Under the same assumption, the conclusion of Theorem 4.4 holds for Bessel potentials.*

The proof is essentially the same. We will not repeat it. Instead, we state the following theorem which holds for all reflection groups.

Theorem 4.8. *Let* $\alpha > 0$.

(1) *The Bessel potentials are bounded operators from* $L^p(\mathbb{R}^d, h_\kappa^2)$ *into itself for* $1 \le p \le \infty$.
(2) *For* $f \in L^1(\mathbb{R}^d, h_\kappa^2)$,

$$|\mathcal{J}_\alpha^\kappa f(x)| \le c M_\kappa f(x), \qquad x \in \mathbb{R}^d.$$

Proof. Since $G_\alpha^\kappa(x)$ is a radial function, let us write $G_\alpha^\kappa(r)$ for the function defined on $\mathbb{R}_+$. The estimate in Lemma 4.6 shows that $G_\alpha^\kappa \in L^1(\mathbb{R}^d, h_\kappa^2)$ and it has integral 1, so that the first part of the theorem follows from Theorem 2.8.

For $\alpha > 0$, the estimate in Lemma 4.6 shows that the conditions on ϕ of Theorem 2.7 are satisfied which gives the second part. □

Such a theorem does not hold for the Riesz potentials. The definition of the Bessel potentials also shows that $\mathcal{J}_\alpha^\kappa$ satisfies

$$\mathcal{J}_\alpha^\kappa \cdot \mathcal{J}_\beta^\kappa = \mathcal{J}_{\alpha+\beta}^\kappa, \qquad \alpha > 0, \quad \beta > 0.$$

5. Weighted Riesz Transforms

5.1. Definition of Riesz transforms. For $P \in \mathcal{H}_n^d(h_\kappa^2)$, we consider the transform

$$T^\kappa f(x) = \lim_{\varepsilon \to 0} \int_{\|y\| \ge \varepsilon} \tau_y f(x) \frac{P(y)}{\|y\|^{\gamma_\kappa + d + n}} h_\kappa^2(y) dy$$

defined for $f \in S$. The equation (4.3) and the Plancherel theorem shows that $T^\kappa f$ is well defined. We want to show that T^κ is a multiplier operator under the Dunkl transform. A linear operator T is a multiplier operator if $\widehat{Tf}(x) = m(x)\widehat{f}(x)$ in the sense that

$$Tf(x) = \int_{\mathbb{R}^d} m(y) \widehat{f}(y) E(x, iy) h_\kappa^2(y) dy$$

for all smooth f with compact support.

Theorem 5.1. *Let $P \in \mathcal{H}_n^d(h_\kappa^2)$, $n \ge 1$. Then the multiplier corresponding to the transform $T^\kappa f$ is given by $d_{n,\kappa} P(x)/\|x\|^n$, $d_{n,\kappa} = i^{-n} 2^{-\gamma_\kappa - d/2} \Gamma(\frac{n}{2})/\Gamma(\gamma_\kappa + \frac{n+d}{2})$.*

Proof. Since $P_n \in \mathcal{H}_n^d(h_\kappa^2)$, its integral with resepct to h_κ^2 on S^{d-1} is zero. Hence,

$$\int_{\|x\| \le 1} \frac{P_n(x)}{\|x\|^{2\gamma_\kappa + d + n - \alpha}} h_\kappa^2(x) dx = \int_0^1 r^{\alpha - 1} dr \int_{S^{d-1}} P_n(x') h_\kappa^2(x') d\omega(x') = 0.$$

Let ϕ be a function that satisfies the condition in the Lemma 4.1 and the additional assumption that $\widehat{\phi}$ is differentiable near origin. Then we can write

$$\begin{aligned}\int_{\mathbb{R}^d} \frac{P_n(x)}{\|x\|^{2\gamma_\kappa + d + n - \alpha}} \widehat{\phi}(x) h_\kappa^2(x) dx &= \int_{\|x\| \ge 1} \frac{P_n(x)}{\|x\|^{2\gamma_\kappa + d + n - \alpha}} \widehat{\phi}(x) h_\kappa^2(x) dx \\ &\quad + \int_{\|x\| \le 1} \frac{P_n(x)}{\|x\|^{2\gamma_\kappa + d + n - \alpha}} \left[\widehat{\phi}(x) - \widehat{\phi}(0)\right] h_\kappa^2(x) dx.\end{aligned}$$

Since $[\widehat{\phi}(x) - \widehat{\phi}(0)]/\|x\|$ is locally integrable, we can take limit $\alpha \to 0$ to get

$$\begin{aligned}&\int_{\|x\| \le 1} \frac{P_n(x)}{\|x\|^{2\gamma_\kappa + d + n}} \left[\widehat{\phi}(x) - \widehat{\phi}(0)\right] h_\kappa^2(x) dx \\ &\qquad = \lim_{\varepsilon \to 0} \int_{\varepsilon \le \|x\| \le 1} \frac{P_n(x)}{\|x\|^{2\gamma_\kappa + d + n}} \widehat{\phi}(x) h_\kappa^2(x) dx.\end{aligned}$$

Consequently, we conclude that

$$\lim_{\alpha \to 0+} \int_{\mathbb{R}^d} \frac{P_n(x)}{\|x\|^{2\gamma_\kappa + d + n - \alpha}} \widehat{\phi}(x) h_\kappa^2(x) dx = \lim_{\varepsilon \to 0} \int_{\|x\| \ge \varepsilon} \frac{P_n(x)}{\|x\|^{2\gamma_\kappa + d + n}} \widehat{\phi}(x) h_\kappa^2(x) dx. \tag{5.1}$$

Let f be a C^∞ function with compact support. For a fixed x, set $\widehat{\phi}(y) = \tau_y f(x) = \tau_{-x} f(-y)$. Then

$$\begin{aligned}(\widehat{\phi})\widehat{\ }(y) &= \int_{\mathbb{R}^d} \tau_{-x} f(-z) E(z, -iy) h_\kappa^2(z) dz = \int_{\mathbb{R}^d} \tau_{-x} f(z) E(z, iy) h_\kappa^2(z) dz \\ &= \widehat{\tau_{-x} f}(-y) = E(-y, ix) \widehat{f}(-y).\end{aligned}$$

Hence, it follows that $\phi(y) = (\widehat{\phi})\widehat{\ }(-y) = E(y, ix)\widehat{f}(y)$, so that by (4.2) and (5.1)

$$\lim_{\varepsilon \to 0} \int_{\|x\| \ge \varepsilon} \frac{P_n(y)}{\|y\|^{2\gamma_\kappa + d + n}} \tau_y f(x) h_\kappa^2(y) dy = d_{n,\kappa} \int_{\mathbb{R}^d} \frac{P_n(y)}{\|y\|^n} E(x, iy) \widehat{f}(y) h_\kappa^2(y) dy.$$

By the definition of the multiplier m, we conclude that $m(y) = d_{n,\kappa} P_n(y)/\|y\|^n$. □

The proof of this theorem follows the argument for the ordinary Fourier transform as given in [9, p. 73 – 74].

The special case that $P(x) = x_j$ defines the weighted Riesz transform.

Definition 5.2. *For $f \in S$ the Riesz transform $R_j^\kappa f$ is defined by*

$$R_j^\kappa f(x) = \lim_{\varepsilon \to 0} c_j \int_{\|y\| \geq \varepsilon} \tau_y f(x) \frac{y_j}{\|y\|^{2\gamma_\kappa + d + 1}} h_\kappa^2(y) dy,$$

where $1 \leq j \leq d$ and $c_j = 2^{\gamma_\kappa + d/2} \Gamma(\gamma_\kappa + (d+1)/2)/\sqrt{\pi}$.

Theorem 5.3. *The Riesz transform is a multiplier operator with*

$$\widehat{R_j^\kappa f}(x) = -i \frac{x_j}{\|x\|} \widehat{f}(x), \qquad 1 \leq j \leq d.$$

Proof. Since $x_j \in \mathcal{H}_1^d(h_\kappa^2)$, this is just the previous theorem with $P(x) = x_j$. □

5.2. **The boundedness of weighted Riesz transform.** The Riesz transforms are important singular integral operators. One would like to prove the boundedness of the weighted Riesz transforms, just as in the case of the ordinary Riesz transforms. This, however, turns out to be a rather difficult task. The effort is hindered by the lack of information on $\tau_y f$. Furthermore, currently no theory of singular integrals with reflection invariant weight functions is available.

For some special parameters, however, the weighted Riesz transforms on the radial functions can be related to the classical Riesz transforms. This is given in the next proposition.

Proposition 5.4. *If $f(x) = f_0(\|x\|)$ is a radial function in $L^p(\mathbb{R}^d, h_\kappa^2)$ and $2\gamma_\kappa \in \mathbb{N}_0$, then for $1 < p < \infty$,*

$$\|R_j^\kappa f\|_{\kappa,p} \leq c\|f\|_{\kappa,p}.$$

Proof. It is enough to prove the inequality for radial Schwartz class functions. Since f is radial, the explicit formula of $\tau_y f$ in (2.5) and the Funk-Hecke formula ([13]) shows that

$$\begin{aligned} R_j^\kappa f(x) &= c_h \int_{\mathbb{R}^d} \tau_y f(x) \frac{y_j}{\|y\|^{d+2\gamma_\kappa+1}} h_\kappa^2(y) dy \\ &= c_h \int_0^\infty \int_{S^{d-1}} V_\kappa f_0 \left(\sqrt{\|x\|^2 + s^2 - 2\|x\| s \langle x, \cdot \rangle} \right) (y') y_j' h_\kappa^2(y') d\omega(y') \frac{ds}{s} \\ &= c\, x_j' \int_0^\infty \int_{-1}^1 f_0 \left(\sqrt{\|x\|^2 + s^2 - 2\|x\| st} \right) t(1-t^2)^{\gamma_\kappa + (d-3)/2} dt \frac{ds}{s}. \end{aligned}$$

Therefore, since $2\gamma_\kappa \in \mathbb{N}_0$, we conclude by (3) that $R_j^\kappa f(x) = c x_j' \widetilde{R}_j f_0(\|x\|)$, where $\widetilde{R}_j$ corresponds to the ordinary Riesz transform $R_j f$ defined on $\mathbb{R}^m$ with $m = d + 2\gamma_\kappa$. Consequently, by Proposition (3.1), we have

$$\begin{aligned} \int_{\mathbb{R}^d} |R_j^\kappa f(x)|^p h_\kappa^2(x) dx &= c \int_0^\infty r^{2\gamma_\kappa + d - 1} |\widetilde{R}_j f_0(r)|^p dr \\ &= c \int_0^\infty r^{m-1} |\widetilde{R}_j f_0(r)|^p dr \\ &\leq c \int_{\mathbb{R}^m} |f_0(\|x\|)|^p dx = c\|f\|_{\kappa,p}^p \end{aligned}$$

which completes the proof. □

In the rest of this section, we consider only the case $d = 1$ and $G = \mathbb{Z}_2$, for which the weight function is simply $h_\kappa(x) = |x|^\kappa$ and the Riesz transform is the integral operator on the real line

$$R^\kappa f(x) = c_\kappa \int_{-\infty}^\infty \tau_y f(x) \frac{y}{\|y\|^{2\kappa+2}} h_\kappa^2(y) dy, \qquad c = \Gamma(\kappa+1)/\sqrt{\pi},$$

where the integral holds in the principal value sense. There is a multiplier theorem in this setting of Dunkl transform on the real line ([8]). However, it does not apply to the Riesz transform. We have the following result.

Theorem 5.5. *Let $G = \mathbb{Z}_2$. If $f \in L^p(\mathbb{R}, h_\kappa^2)$ and $2\gamma_\kappa \in \mathbb{N}_0$, then for $1 < p < \infty$,*

$$\|R^\kappa f\|_{\kappa,p} \le c\|f\|_{\kappa,p}.$$

Proof. In this case f is radial means that f is even, so that the stated result holds for f being even. Every function f on $\mathbb{R}$ can be split as $f = f_e + f_o$ where $f_e(r) = (f(r) + f(-r))/2$ is even and $f_o(r) = (f(r) - f(-r))/2$ is odd. Evidently, we have $\|f_e\|_{\kappa,p} \le \|f\|_{\kappa,p}$ and $\|f_o\|_{\kappa,p} \le \|f\|_{\kappa,p}$, we only need to prove the stated inequality for f being an odd function.

Let f be an odd function and define $g(r) = f(r)/r$ for $r \ne 0$. Then g is even. The explicit formula of $\tau_r f$ in (2.6) shows that

$$\tau_r f(s) = (s-r)\tau_r g(s) = s\tau_r g(s) - r\tau_r g(s)$$

so that the Riesz operator can be written as a sum of two terms,

$$R^\kappa f(s) = c_\kappa s \int_{\mathbb{R}} \tau_r g(s) \frac{r}{|r|^2} dr - c_\kappa \int_{\mathbb{R}} \tau_r g(s) dr := c_\kappa R_1^\kappa f(s) - c_\kappa R_2^\kappa f(s).$$

For the first term we start with the following observation. Let $\kappa = (m-1)/2$, where $m \in \mathbb{N}$. Define $F(x) = g(\|x\|)$ for $x \in \mathbb{R}^m$. Let Ω be a harmonic polynomial of first degree on $\mathbb{R}^m$ and let $K(x) = \Omega(x)/\|x\|^{m+1}$. We consider the convolution $F * K$ in $L^1(\mathbb{R}^m)$. Using the spherical-polar coordinates and the ordinary Funk-Hecke formula, we get

$$\begin{aligned}(F*K)(x) &= \int_0^\infty r^{m-1} \int_{S^{m-1}} g\left(\sqrt{\|x\|^2 + r^2 - 2\|x\|\, r\langle x', y'\rangle}\right) \Omega(y') d\omega(y') \frac{dr}{r^m} \\ &= c\,\Omega(x') \int_0^\infty \int_{-1}^1 g\left(\sqrt{\|x\|^2 + s^2 - 2\|x\| rt}\right) t(1-t^2)^{\frac{m-3}{2}} dt \frac{dr}{r} \\ &= c\,\Omega(x') \int_{-\infty}^\infty \int_{-1}^1 g\left(\sqrt{\|x\|^2 + s^2 - 2\|x\| st}\right) t(1-t^2)^{\frac{m-3}{2}} dt \,\mathrm{sign}(r) \frac{dr}{|r|}.\end{aligned}$$

Since $\mathrm{sign}(r)/|r| = r/r^2$ is odd in r, changing variables $t \to -t$ and $r \to -r$ shows that we can replace $t(1-t^2)^{m-3}/2$ by $(1+t)(1-t^2)^{\kappa-1}$ in the last expression, so that the inner integral becomes $\tau_r g(\|x\|)$, from which we get

$$(F*K)(x) = c\,\Omega(x') \int_{-\infty}^\infty \tau_r g(\|x\|) \frac{r}{|r|^2} dr.$$

Hence, $\|x\|(F*K)(x) = c\,\Omega(x') R_1^\kappa f(\|x\|)$. Using the spherical-polar coordinates and integrating over $\mathbb{R}^m$ we get

$$\begin{aligned}\int_{\mathbb{R}^m} |(F*K)(x)|^p \|x\|^p dx &= c \int_0^\infty |R_1^\kappa f(s)|^p s^{m-1} ds \int_{S^{d-1}} \Omega(x') d\omega(x') \\ &= c \int_0^\infty |R_1^\kappa f(s)|^p s^{2\kappa} ds.\end{aligned}$$

Since $R_1^\kappa f$ is an even function, this gives

$$\begin{aligned}\int_{-\infty}^\infty |R_1^\kappa f(s)|^p |s|^{2\kappa} ds &\le c \int_{\mathbb{R}^m} |(F*K)(x)|^p \|x\|^p dx \\ &\le c \int_{\mathbb{R}^m} |F(x)|^p \|x\|^p dx\end{aligned}$$

for $p > m/(m-1) = (2\kappa+1)/(2\kappa)$ since $\|x\|^p \in A_p(\mathbb{R}^m)$ for $p > m/(m-1)$. By definition, $F(x) = g(\|x\|)$ and $g(s) = f(s)/s$, so that

$$\int_{\mathbb{R}^m} |F(x)|^p \|x\|^p dx = \int_{\mathbb{R}^m} |f(\|x\|)|^p dx = \int_0^\infty |f(r)|^p r^{m-1} dr = \frac{1}{2}\int_{-\infty}^\infty |f(r)|^p |r|^{2\kappa} dr.$$

This takes care of the first term $R_1^\kappa f$ for $p > (2\kappa+1)/(2\kappa)$.

For the second term we consider the operator $F \mapsto F * K_1$, where F is as above and $K_1(x) = \|x\|^{1-m}$, which agrees with the notation in (3.1). Similar to $F * K$ we get

$$\begin{aligned} F * K_1(x) &= c\int_0^\infty \int_{-1}^1 g\left(\sqrt{\|x\|^2 + s^2 - 2\|x\|rt}\right)(1-t^2)^{\frac{m-3}{2}} dt dr \\ &= c\int_{-\infty}^\infty \tau_r g(\|x\|) dr = c R_2^\kappa(\|x\|). \end{aligned}$$

Therefore, since R_2^κ is even,

$$\begin{aligned} \int_{-\infty}^\infty |R_2^\kappa(s)|^p |s|^{2\kappa} ds &= 2\int_0^\infty |R_2^\kappa(s)|^p s^{m-1} ds \\ &= c\int_{\mathbb{R}^m} |F * K_1(x)|^p dx \le c\int_{\mathbb{R}^m} |F(x)|^p \|x\|^p dx \end{aligned}$$

for $p > m/(m-1)$ using Proposition 3.2. Again, the last integral is the same as $\|f\|_{\kappa,p}^p$, which takes care of the second term for $p > (2\kappa+1)/(2\kappa)$. Together, we have proved that $\|R^\kappa\|_{\kappa,p} \le c\|f\|_{\kappa,p}$ or $p > (2\kappa+1)/(2\kappa)$. For $1 < p \le (2\kappa+1)/(2\kappa)$ we use the standard duality argument. □

References

[1] C. F. Dunkl, Differential-difference operators associated to reflection groups, *Trans. Amer. Math. Soc.* **311** (1989), 167-183.

[2] C. F. Dunkl, Hankel transforms associated to finite reflection groups, in *Hypergeometric functions on domains of positivity, Jack polynomials, and applications* (Tampa, FL, 1991), 123-138, Contemporary Mathematics **138**, American Math. Society, Providence, RI. 1992.

[3] C. F. Dunkl and Yuan Xu, *Orthogonal polynomials of several variables*, Cambridge Univ. Press, 2001.

[4] M. F. E. de Jeu, The Dunkl transform, *Invent. Math.* **113** (1993), 147-162.

[5] M. Rösler, Bessel-type signed hypergroups on $\mathbb{R}$, in: H. Heyer, A. Mukherjea (eds.), Probability measures on groups and related structures XI, Proc. Oberwolfach 1994, World Scientific, Singapore, 1995, 292-304.

[6] M. Rösler, Positivity of Dunkl's intertwining operator, *Duke Math. J.*, **98** (1999), 445–463.

[7] M. Rösler, A positive radial product formula for the Dunkl kernel, *Trans. Amer. Math. Soc.*, **355** (2003), 2413-2438.

[8] F. Soltani, L^p-Fourier multipliers for the Dunkl operator on the real line, *J. Funct. Analysis*, **209** (2004), 16-35.

[9] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Univ. Press, Princeton, NJ, 1971.

[10] E. M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Univ. Press, Princeton, NJ, 1993.

[11] S. Thangavelu and Yuan Xu, Convolution operator and maximal functions for Dunkl transform, to appear. ArXiv math.CA/0403049

[12] K. Trimèche, Paley-Wiener theorems for the Dunkl transform and Dunkl translation operators, *Integral Transforms and Special Functions*, **13** (2002), 17-38.

[13] Yuan Xu, Funk-Hecke formula for orthogonal polynomials on spheres and on balls, *Bull. London Math. Soc.* **32** (2000), 447–457.

Stat-Math Division, Indian Statistical Institute, 8th Mile, Mysore Road, Bangalore-560 059, India.
E-mail address: veluma@isibang.ac.in

Department of Mathematics, University of Oregon, Eugene, Oregon 97403-1222.
E-mail address: yuan@math.uoregon.edu