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THE HEAT KERNEL TRANSFORM FOR THE HEISENBERG GROUP

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ABSTRACT. The heat kernel transform $\mathcal{H}_t$ is studied for the Heisenberg group in detail. The main result shows that the image of $\mathcal{H}_t$ is a direct sum of two weighted Bergman spaces, in contrast to the classical case of $\mathbb{R}^n$ and compact symmetric spaces, and the weight functions are found to be (surprisingly) not non-negative.

1. INTRODUCTION

Over the last decade one could observe interesting developments on the *heat kernel transform* for various types of homogeneous Riemannian manifolds X. Complete results have been obtained for compact Lie groups (cf. [2, 3]) and, more generally, for compact symmetric spaces (cf. [6]). For non-compact spaces X the situation seems to be more complicated and little research has been undertaken in this direction: There is the well understood Euclidean case (e.g. $X = \mathbb{R}^n$, cf. [1]) and some partial results have been obtained for non-compact Riemannian symmetric spaces (cf. [5]). The objective of this paper is to give a complete and self-contained discussion for the Heisenberg group.

Our concern is with the $(2n+1)$-dimensional Heisenberg group $\mathbb{H}$ and its universal complexification $\mathbb{H}_\mathbb{C}$. For $t > 0$ we write $k_t : \mathbb{H} \to \mathbb{R}^+$ for the heat kernel on $\mathbb{H}$. Contemplating on the spectral resolution of k_t, it is not hard to see that k_t admits an analytic continuation to a holomorphic function $k_t^\sim : \mathbb{H}_\mathbb{C} \to \mathbb{C}$. Consequently, for every $f \in L^2(\mathbb{H})$ the convolution $f * k_t$ continues holomorphically to $H_\mathbb{C}$ and we obtain a map

$$\mathcal{H}_t : L^2(\mathbb{H}) \to \mathcal{O}(\mathbb{H}_\mathbb{C}), \quad f \mapsto (f * k_t)^\sim .$$

We refer to $\mathcal{H}_t$ as the *heat kernel transform* on $\mathbb{H}$ with parameter $t > 0$. The map $\mathcal{H}_t$ is injective, left $\mathbb{H}$-equivariant and becomes continuous if $\mathcal{O}(\mathbb{H}_\mathbb{C})$ is equipped with its natural Fréchet topology of compact convergence. It follows that $\operatorname{im} \mathcal{H}_t$ is a reproducing kernel Hilbert space. Standard abstract arguments readily yield an expression for the kernel function in terms of $k_{2t}^\sim$ (see (3.1.2) below).

In all known cases (e.g. X a compact symmetric space or $X = \mathbb{R}^n$) the image of the heat kernel transform has been a weighted Bergman space $X_\mathbb{C}$ with regard to a postive weight function. It came to our surprise that the Heisenberg group deviates from this pattern. The main result of this paper asserts that

$$\operatorname{im} \mathcal{H}_t = \mathcal{B}_t^+(\mathbb{H}_\mathbb{C}) \oplus \mathcal{B}_t^-(\mathbb{H}_\mathbb{C}) \tag{1.1}$$

is a direct sum of two weighted Bergman spaces on $H_\mathbb{C}$. Most interestingly, the weight functions $W_t^\pm$ for $\mathcal{B}_t^\pm(\mathbb{H}_\mathbb{C})$ have an oscillatory nature and attain positive and *negative* values. This fact forces the use of a certain exhaustion $\mathbb{H}_\mathbb{C} = \bigcup_{R>0} K_R$ to define the inner product a suitable dense subspace $\mathcal{V}_t^\pm(\mathbb{H}_\mathbb{C})$ of $\mathcal{B}_t^\pm(\mathbb{H}_\mathbb{C})$ by

$$\langle f, g\rangle = \lim_{R\to\infty} \int_{K_R} f(z)\overline{g(z)} W_t^\pm(z)\, dz \qquad (f, g \in \mathcal{V}_t^\pm(\mathbb{H}_\mathbb{C})),$$

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quite reminiscent to the familiar notion of principal value.

Let us now describe the contents of this paper in more detail. In Section 2 we introduce our notation and recall some facts on the heat kernel k_t on $\mathbb{H}$ and its analytic continuation to $\mathbb{H}_{\mathbb{C}}$. Subsequently in Section 3 we define the heat kernel transform and give a discussion of its general nature.

For the remainder it is useful to identify $\mathbb{H}$ with $\mathbb{R}^{2n} \times \mathbb{R}$. In Section 4 we introduce for each spectral parameter $\lambda \in \mathbb{R}^\times$ a partial heat kernel transform

$$H_t^\lambda : L^2(\mathbb{R}^{2n}) \to \mathcal{O}(\mathbb{C}^{2n})$$

and show that im H_t^λ is a weighted Bergman space $\mathcal{B}_t^\lambda(\mathbb{C}^{2n})$ associated to an explicitly given positive weight function $W_t^\lambda : \mathbb{C}^{2n} \to \mathbb{R}^{2n}$. With these results we prove in Section 5 that there is a natural left $\mathbb{H}$-equivariant equivalence

$$L^2(\mathbb{H}) \simeq \int_{\mathbb{R}^\times}^{\oplus} \mathcal{B}_t^\lambda(\mathbb{C}^{2n})\, e^{2t\lambda^2}\, d\lambda\,. \tag{1.2}$$

Moreover, within the identification (1.2) the heat kernel transform $\mathcal{H}_t$ becomes the diagonal operator $(H_t^\lambda)_\lambda$.

In Section 6 we combine all previously obtained results to establish our main result (1.1). It turns out that the global weight functions $W_t^\pm$ admit an integral representation in terms of the partial weight functions W_t^λ. Finally, in the appendix we derive explicit expansions of $W_t^\pm$ by Hermite polynomials and explain their oscillatory behavior.

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2. The heat kernel on the Heisenberg group

2.1. **Notation.** Let $\mathfrak{h}$ denote the $(2n+1)$-dimensional Heisenberg algebra with generators, say,

$$X_1, \ldots, X_n, U_1, \ldots, U_n, Z$$

and relations $[X_j, U_j] = Z$. In the sequel we will often identify $\mathfrak{h}$ with $\mathbb{R}^{2n+1} = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$. For that let $(\mathbf{x}, \mathbf{u}, \xi)$ with $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{u} = (u_1, \ldots, u_n)$ denote the canonical coordinates on $\mathbb{R}^{2n+1}$. Then the map

$$\mathbb{R}^{2n+1} \to \mathfrak{h}, \quad (\mathbf{x}, \mathbf{u}, \xi) \mapsto \sum_{j=1}^n x_j X_j + \sum_{j=1}^n u_j U_j + \xi Z$$

is a linear isomorphism providing us with suitable coordinates for $\mathfrak{h}$.

Let $\mathbb{H}$ denote a simply connected Lie group with Lie algebra $\mathfrak{h}$, the *Heisenberg group*. We will identify $\mathbb{H}$ with $\mathfrak{h}$ through the exponential function $\exp = \mathrm{id} : \mathfrak{h} \to \mathbb{H}$. As $\mathbb{H}$ is two step, the Baker-Campbell-Hausdorff formula provides the group law

$$(\mathbf{x}, \mathbf{u}, \xi)(\mathbf{x}', \mathbf{u}', \xi') = (\mathbf{x} + \mathbf{x}', \mathbf{u} + \mathbf{u}', \frac{1}{2}(\mathbf{x} \cdot \mathbf{u}' - \mathbf{u} \cdot \mathbf{x}') + \xi + \xi')\,.$$

Here $\mathbf{x} \cdot \mathbf{u} = \sum_{j=1}^n x_j u_j$, as usual, denotes the standard pairing on $\mathbb{R}^n$. We notice in particular that

$$(\mathbf{x}, \mathbf{u}, \xi)^{-1} = (-\mathbf{x}, -\mathbf{u}, -\xi)\,. \tag{2.1.1}$$

Write dh for a Haar measure on $\mathbb{H}$. We can and will normalize dh in such a way that it coincides with the product of Lebesgue measures, i.e.

$$\int_{\mathbb{H}} f(h)\, dh = \int_{\mathbb{R}^{2n+1}} f(\mathbf{x}, \mathbf{u}, \xi)\, d\mathbf{x}\, d\mathbf{u}\, d\xi$$

for all $f \in C_c(\mathbb{H})$.

Write $\mathbb{H}_{\mathbb{C}}$ for the universal complexification of $\mathbb{H}$. Of course we can identify $\mathbb{H}_{\mathbb{C}}$ with $\mathbb{C}^{2n+1}$ and we will often do so. We will write $(\mathbf{z}, \mathbf{w}, \zeta)$ for the coordinates on $\mathbb{C}^{2n+1}$ where $\mathbf{z} = \mathbf{x} + i\mathbf{y}$, $\mathbf{w} = \mathbf{u} + i\mathbf{v}$ and $\zeta = \xi + i\eta$.

For any simply connected nilpotent Lie group H the polar mapping

$$H \times \mathfrak{h} \to H_{\mathbb{C}}, \quad (h, X) \mapsto h \exp(iX)$$

is a homeomorphism. Furthermore the Haar measure on $H_{\mathbb{C}}$ decomposes as

$$\int_{H_{\mathbb{C}}} f(g)\, dg = \int_H \int_{\mathfrak{h}} f(h \exp(iX))\, dX\, dh \tag{2.1.2}$$

for all $f \in C_c(H_{\mathbb{C}})$.

For the Heisenberg group $\mathbb{H}$, the polar mapping is explicitly given by

$$((\mathbf{x}, \mathbf{u}, \xi), (\mathbf{x}', \mathbf{u}', \xi')) \mapsto (\mathbf{x} + i\mathbf{x}', \mathbf{u} + i\mathbf{u}', \frac{i}{2}(\mathbf{x} \cdot \mathbf{u}' - \mathbf{u} \cdot \mathbf{x}') + \xi + i\xi')$$

where $h = (\mathbf{x}, \mathbf{u}, \xi)$ and $X = (\mathbf{x}', \mathbf{u}', \xi')$. In particular the Haar measure on $\mathbb{H}_{\mathbb{C}}$ can be chosen as the product of Lebesgue measures $d\mathbf{x}\, d\mathbf{y}\, d\mathbf{u}\, d\mathbf{v}\, d\xi\, d\eta$.

For integrable functions f, g on $\mathbb{H}$ we define their convolution by

$$(f * g)(x) = \int_{\mathbb{H}} f(h) g(h^{-1}x)\, dh \qquad (x \in \mathbb{H}).$$

In coordinates this is explicitly given by

$$(f * g)(\mathbf{x}, \mathbf{u}, \xi) = \int_{\mathbb{R}^{2n+1}} f(\mathbf{x}', \mathbf{u}', \xi') g\left((-\mathbf{x}', -\mathbf{u}', -\xi')(\mathbf{x}, \mathbf{u}, \xi)\right)\, d\mathbf{x}'\, d\mathbf{u}'\, d\xi'.$$

2.2. **The heat kernel.** Write $\mathcal{U}(\mathfrak{h})$ for the universal enveloping algebra of $\mathfrak{h}$ and define the Laplace element in $\mathcal{U}(\mathfrak{h})$ by

$$\mathcal{L} = \sum_{j=1}^n X_j^2 + \sum_{j=1}^n U_j^2 + Z^2.$$

For $X \in \mathfrak{h}$ we write $\tilde{X}$ for the left invariant vector field on $\mathbb{H}$, i.e.,

$$(\tilde{X} f)(h) = \frac{d}{dt}\Big|_{t=0} f(h \exp(tX))$$

for f a function on $\mathbb{H}$ which is differentiable at $h \in \mathbb{H}$. Write ρ for the right regular representation of $\mathbb{H}$ on $L^2(\mathbb{H})$, i.e.

$$(\rho(h) f)(x) = f(xh)$$

for $h, x \in \mathbb{H}$ and $f \in L^2(\mathbb{H})$. With $d\rho$ the derived representation we then have $d\rho(X) = \tilde{X}$ for all $X \in \mathfrak{h}$. In particular if

$$\Delta = \sum_{j=1}^n \tilde{X}_j^2 + \sum_{j=1}^n \tilde{U}_j^2 + \tilde{Z}^2$$

denotes the Laplace operator on $\mathbb{H}$, then $d\rho(\mathcal{L}) = \Delta$.

Set $\mathbb{R}^+ = (0, \infty)$. Our concern will be with the heat equation on $\mathbb{H} \times \mathbb{R}^+$

$$\partial_t u(h, t) = \Delta u(h, t)$$

for appropriate functions $u(h, t)$ on $\mathbb{H} \times \mathbb{R}^+$. The fundamental solution is given by the heat kernel $k_t(h)$ which can be computed as follows:

$$k_t(\mathbf{x}, \mathbf{u}, \xi) = c_n \int_{\mathbb{R}} e^{-i\lambda\xi} e^{-t\lambda^2} \left(\frac{\lambda}{\sinh \lambda t}\right)^n e^{-\frac{1}{4}\lambda(\coth t\lambda)(\mathbf{x}\cdot\mathbf{x} + \mathbf{u}\cdot\mathbf{u})}\, d\lambda \tag{2.2.1}$$

with $c_n = (4\pi)^{-n}$ (this follows from a slight modification of [7, Theorem 2.8.1].) It satisfies the usual property of $k_t * k_t = k_{2t}$ (see, for example, [7, (2.87) and Corollary 2.3.4]).

If f is an analytic function on $\mathbb{H}$ which holomorphically extends to $\mathbb{H}_\mathbb{C}$, then we write $f^\sim$ for this holomorphic extension. The explicit formula (2.2.1) now implies that k_t has a holomorphic continuation to $\mathbb{H}_\mathbb{C}$ which is given by

$$k_t^\sim(\mathbf{z}, \mathbf{w}, \zeta) = c_n \int_\mathbb{R} e^{-i\lambda\zeta} e^{-t\lambda^2} \left(\frac{\lambda}{\sinh \lambda t}\right)^n e^{-\frac{1}{4}\lambda(\coth t\lambda)(\mathbf{z}\cdot\mathbf{z}+\mathbf{w}\cdot\mathbf{w})}\, d\lambda \tag{2.2.2}$$

for $(\mathbf{z}, \mathbf{w}, \zeta) \in \mathbb{C}^{2n+1} = \mathbb{H}_\mathbb{C}$. It follows from (2.1.1) and (2.2.2) that

$$k_t^\sim(z) = k_t^\sim(z^{-1}) \qquad (z \in \mathbb{H}_\mathbb{C})\,. \tag{2.2.3}$$

Furthermore, as $k_t \geq 0$ is real, we record

$$\overline{k_t^\sim(z)} = k_t^\sim(\overline{z}) \qquad (z \in \mathbb{H}_\mathbb{C}) \tag{2.2.4}$$

Here, as usual, $z \mapsto \overline{z}$ denotes the complex conjugation of $\mathbb{H}_\mathbb{C}$ with respect to the real form $\mathbb{H}$.

3. The heat kernel transform

3.1. **Definition and basic properties.** Let $C \subseteq \mathbb{H}_\mathbb{C}$ be a compact subset. Then it follows from (2.2.2) that

$$\sup_{z\in C} \int_\mathbb{H} |k_t^\sim(h^{-1}z)|^2\, dh < \infty\,. \tag{3.1.1}$$

Fix $t > 0$. Then (3.1.1) implies that $f * k_t$ has an analytic continuation to $\mathbb{H}_\mathbb{C}$ for all $f \in L^2(\mathbb{H})$. In particular we obtain a linear map

$$\mathcal{H}_t : L^2(\mathbb{H}) \to \mathcal{O}(\mathbb{H}_\mathbb{C}), \quad f \mapsto (f * k_t)^\sim;\ \mathcal{H}_t(f)(z) = \int_{\mathbb{H}_\mathbb{C}} f(h) k_t^\sim(h^{-1}z)\, dh\ .$$

We will call $\mathcal{H}_t$ the *heat kernel transform.*

In the sequel we wish to consider $\mathcal{O}(\mathbb{H}_\mathbb{C})$ as a Fréchet space – the topology being the one of compact convergence. If $h \in H$ and f is a function on $\mathbb{H}$ or $\mathbb{H}_\mathbb{C}$, then we write $\tau(h)f = f(h^{-1}\cdot)$. The following properties of $\mathcal{H}_t$ are immediate:

- $\mathcal{H}_t$ is continuous (because of (3.1.1))
- $\mathcal{H}_t$ is injective (note that $\mathcal{H}_t(f) = e^{t\Delta}f$ and Δ is a negative definite operator).
- $\mathcal{H}_t$ is $\mathbb{H}$-equivariant, i.e. $\mathcal{H}_t \circ \tau(h) = \tau(h) \circ \mathcal{H}_t$ for all $h \in \mathbb{H}$ (this is a general fact for the convolution on a locally compact group).

We will endow $\operatorname{im}\mathcal{H}_t$ with the Hilbert topology induced from $L^2(\mathbb{H})$. As $\mathcal{H}_t$ is continuous we see that $\operatorname{im}\mathcal{H}_t$ is an $\mathbb{H}$-invariant Hilbert space of holomorphic functions on $\mathbb{H}_\mathbb{C}$. As such $\operatorname{im}\mathcal{H}_t$ has continuous point evaluations, i.e. for all $z \in \mathbb{H}_\mathbb{C}$ the map

$$\operatorname{im}\mathcal{H}_t \to \mathbb{C}, \quad f \mapsto f(z)$$

is continuous. Hence $f(z) = \langle f, \mathcal{K}_z^t\rangle$ for a unique element $\mathcal{K}_z^t \in \operatorname{im}\mathcal{H}_t$. We then obtain a positive definite kernel function

$$\mathcal{K}^t : \mathbb{H}_\mathbb{C} \times \mathbb{H}_\mathbb{C} \to \mathbb{C};\ \mathcal{K}^t(z,w) = \langle \mathcal{K}_w^t, \mathcal{K}_z^t\rangle = \mathcal{K}_w^t(z)$$

which is holomorphic in the first and anti-holomorphic in the second variable. Moreover, the $\mathbb{H}$-invariance of $\operatorname{im}\mathcal{H}_t$ translates into $\mathcal{K}^t(hz, hw) = \mathcal{K}^t(z,w)$ for all $h \in \mathbb{H}$ and $z, w \in \mathbb{H}_\mathbb{C}$.

Let us compute $\mathcal{K}^t$. Fix $w \in \mathbb{H}_\mathbb{C}$. Let $g \in \operatorname{im}\mathcal{H}_t$. Then $g = \mathcal{H}_t(f)$ for some $f \in L^2(\mathbb{H})$ and

$$\langle g, \mathcal{K}_w^t\rangle = g(w) = \mathcal{H}_t(f)(w) = (f * k_t)^\sim(w) = \int_\mathbb{H} f(h) k_t^\sim(h^{-1}w)\, dh\,.$$

As this holds for all $g \in \operatorname{im}\mathcal{H}_t$, we thus conclude that

$$\mathcal{H}_t^{-1}(\mathcal{K}_w^t)(h) = \overline{k_t^\sim(h^{-1}w)} = k_t^\sim(\overline{w}^{-1}h) \qquad (h \in \mathbb{H})$$

where for the last equality we used the facts (2.2.3-4). ¿From this we now get for all $w, z \in \mathbb{H}_{\mathbb{C}}$ that

$$\begin{aligned}\mathcal{K}_w^t(z) &= \mathcal{H}_t(k_t^\sim(\overline{w}^{-1}\cdot))(z) = \int_{\mathbb{H}} k_t^\sim(\overline{w}^{-1}h)k_t^\sim(h^{-1}z)\,dh \\ &= \int_{\mathbb{H}} k_t(h)k_t^\sim(h^{-1}\overline{w}^{-1}z)\,dh \\ &= (k_t * k_t)^\sim(\overline{w}^{-1}z) \\ &= k_{2t}^\sim(\overline{w}^{-1}z).\end{aligned}$$

We have thus shown that the kernel function is given by

$$\mathcal{K}^t(z,w) = k_{2t}^\sim(\overline{w}^{-1}z) \qquad (z, w \in \mathbb{H}_{\mathbb{C}})\,. \tag{3.1.2}$$

3.2. **General remarks on integral transforms and Bergman spaces.** The setup for this Section is as follows: We let N be a positive integer and G be a Lie group which acts on $\mathbb{R}^N$ in a measure preserving manner. We assume that the action of G extends to an action on $\mathbb{C}^N$ by measure preserving biholomorphisms. Our next data is a continuous (integral) transform

$$\Phi : L^2(\mathbb{R}^N) \hookrightarrow \mathcal{O}(\mathbb{C}^N)$$

which we assume to be G-equivariant. In this way $\operatorname{im}\Phi$ becomes a G-invariant Hilbert space of holomorphic functions on $\mathbb{C}^N$. We write $\mathcal{K} : \mathbb{C}^N \times \mathbb{C}^N \to \mathbb{C}$ for the corresponding kernel function.

Example 3.1. (a) The heat kernel transform $\mathcal{H}_t : L^2(\mathbb{H}) \to \mathcal{O}(\mathbb{H}_{\mathbb{C}})$ meets the general assumptions from above. In fact, for $N = 2n+1$ we may identify $\mathbb{H}$ with $\mathbb{R}^N$ and $\mathbb{H}_{\mathbb{C}}$ with $\mathbb{C}^N$. Furthermore the group $G = \mathbb{H}$ acts from the left on $\mathbb{H} = \mathbb{R}^N$ and $\mathbb{H}_{\mathbb{C}} = \mathbb{C}^N$ in a measure preserving manner.
(b) The partial heat kernel transforms $H_t^\lambda : L^2(\mathbb{R}^{2n}) \hookrightarrow \mathcal{O}(\mathbb{C}^{2n})$ introduced in Section 4 below satisfy the general assumptions made above. □

For the remainder we will assume that $\operatorname{im}\Phi = \mathcal{B}(\mathbb{C}^N, W)$ is a weighted Bergman space for some measurable weight function $W : \mathbb{C}^N \to \mathbb{R}$, i.e.,

$$\mathcal{B}(\mathbb{C}^N, W) = \{f \in \mathcal{O}(\mathbb{C}^N) : \int_{\mathbb{C}^N} |f(z)|^2\ |W(z)|\,dz < \infty\}$$

Hilbert structure given by

$$\langle f, g\rangle = \int_{\mathbb{C}^n} f(z)\overline{g(z)}W(z)dz \tag{3.2.1}$$

As the action of G on $\mathcal{B}(\mathbb{C}^N, W)$ is unitary, the weight function W should be left G-invariant, i.e.

$$W(g.z) = W(z) \qquad (g \in G, z \in \mathbb{C}^N)\,. \tag{3.2.2}$$

What we cannot expect however is that W is non-negative. It might then be a surprise that (3.2.1) still defines a Hilbert structure. As the following example shows, this is a phenomenon which already appears in one variable.

Example 3.2. We consider the unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$. For a measurable subset $A \subseteq D$ write $\mathbf{1}_A$ for its characteristic function. Define a weight function W on D by

$$W = \mathbf{1}_{\{\frac{1}{2} \leq |z| < 1\}} - \mathbf{1}_{\{|z| < \frac{1}{2}\}}\,.$$

With W we form the weighted Bergman space

$$\mathcal{B}^2(D, W) := \{f \in \mathcal{O}(D) : \int_D |f(z)|^2\ |W(z)|\,dx\,dy < \infty\}$$

and endow it with the sesquilinear bracket

$$\langle f, g\rangle = \int_D f(z)\overline{g(z)}\ W(z)\, dx\, dy\,.$$

We will show that $(\mathcal{B}^2(D,W), \langle\cdot,\cdot\rangle)$ is a Hilbert space. For that we first observe that $\{z^n\}_{n\in\mathbb{N}_0}$ is an orthogonal system in $\mathcal{B}^2(D,W)$. This is because W is rotationally invariant. Next we compute

$$\begin{aligned}\langle z^n, z^n\rangle &= 2\pi\int_{\frac{1}{2}}^1 r^{2n+1}\, dr - 2\pi\int_0^{\frac{1}{2}} r^{2n+1}\, dr\\ &= \frac{\pi}{n+1}\left[1-\left(\frac{1}{2}\right)^{2n+1}\right] > 0\end{aligned}$$

for any $n \in \mathbb{N}_0$. Thus if $f = \sum_n a_n z^n \in \mathcal{B}^2(D,W)$ is an arbitrary element, then

$$\langle f, f\rangle = \sum_n |a_n|^2 \frac{\pi}{n+1}\left[1-\left(\frac{1}{2}\right)^{2n+1}\right] \geq 0 \tag{3.2.1}$$

and $\langle f, f\rangle = 0$ if and only if $f = 0$. This shows that $\langle\cdot,\cdot\rangle$ defines a pre Hilbert structure on $\mathcal{B}^2(D,W)$. Next notice that

$$\int_D |f(z)|^2\ |W(z)|\, dx\, dy = \sum_n |a_n|^2 \frac{\pi}{n+1}\,. \tag{3.2.2}$$

It follows from identiies (3.2.1) and (3.2.2) that $\langle\cdot,\cdot\rangle$ and the Hilbert bracket $(f|g) = \int_D f(z)\overline{g(z)}\ |W(z)|\, dx\, dy$ induce equivalent norms. Hence $(\mathcal{B}^2(D,W), \langle\cdot,\cdot\rangle)$ is a Hilbert space.

Finally we note that W is uniquely characterized by the Hilbert norm on $\mathcal{B}^2(D,W)$, i.e. $\mathcal{B}^2(D,W) = \mathcal{B}^2(D,W')$ if and only if $W = W'$ almost everywhere (use Stone-Weierstraß). □

We conclude this section with some general remarks on how to obtain the weight function W. Define a subspace of $\operatorname{im}\Phi$ by

$$(\operatorname{im}\Phi)_0 = \operatorname{span}\{\mathcal{K}_x : x \in \mathbb{R}^N\}\,.$$

Since a holomorphic function on $\mathbb{C}^N$ which vanishes on $\mathbb{R}^N$ is identically zero, we conclude that $(\operatorname{im}\Phi)_0$ is dense in $\operatorname{im}\Phi$. Hence $\operatorname{im}\Phi = \mathcal{B}(\mathbb{C}^N, W)$ will hold precisely if

$$\mathcal{K}(x,x') = \langle\mathcal{K}_{x'}, \mathcal{K}_x\rangle = \int_{\mathbb{C}^N} \mathcal{K}_{x'}(z)\overline{\mathcal{K}_x(z)}W(z)\, dz \tag{3.2.3}$$

for all $x, x' \in \mathbb{R}^N$. The formula (3.2.3) is actually quite helpful and will be applied in Section 4 below.

4. The λ-twisted heat-kernel transform

For $\lambda \in \mathbb{R}$, $\lambda \neq 0$, we will introduce a λ-twisted heat kernel transform $H_t^\lambda : L^2(\mathbb{R}^{2n}) \to \mathcal{O}(\mathbb{C}^{2n})$. We will show that the image of H_t^λ is a weighted Bergman space $\mathcal{B}_t^\lambda(\mathbb{C}^{2n})$ on $\mathbb{C}^{2n}$. Further we provide an inversion formula for H_t^λ.

The results of this section are the building blocks for our general discussion of the heat kernel transform $\mathcal{H}_t : L^2(\mathbb{H}) \to \mathcal{O}(\mathbb{H}_\mathbb{C})$ in the following sections.

4.1. **Notation.** Let $\lambda \in \mathbb{R}$, $\lambda \neq 0$. For suitable functions F on $\mathbb{H}$ we define a function F^λ on $\mathbb{R}^{2n}$ by

$$F^\lambda(\mathbf{x}, \mathbf{u}) = \int_{\mathbb{R}} e^{i\lambda\xi} F(\mathbf{x}, \mathbf{u}, \xi)\, d\xi\,.$$

For $f, g \in L^1(\mathbb{R}^{2n})$ the λ-*twisted convolution* is defined by

$$(f *_\lambda g)(\mathbf{x}, \mathbf{u}) = \int_{\mathbb{R}^{2n}} f(\mathbf{x}', \mathbf{u}') g(\mathbf{x} - \mathbf{x}', \mathbf{u} - \mathbf{u}') e^{-i\frac{\lambda}{2}(\mathbf{x}'\cdot\mathbf{u} - \mathbf{x}\cdot\mathbf{u}')}\, d\mathbf{x}'\, d\mathbf{u}'.$$

Notice that we have for Schwartz functions $F, G \in S(\mathbb{H}) = S(\mathbb{R}^{2n+1})$ that

$$(F * G)^\lambda = F^\lambda *_\lambda G^\lambda\,. \tag{4.1.1}$$

Let $\Delta_{\mathrm{sub}} = d\rho\left(\sum_{j=1}^n (\tilde{X}_j^2 + \tilde{Y}_j^2)\right)$ denote the sublaplacian on $\mathbb{H}$. The heat kernel for Δ_{sub} is denoted by p_t and its inverse Fourier transform in the central variable is explicitly given by

$$p_t^\lambda(\mathbf{x}, \mathbf{u}) = c_n \left(\frac{\lambda}{\sinh t\lambda}\right)^n e^{-\frac{\lambda}{4}\coth(\lambda t)(|\mathbf{x}|^2 + |\mathbf{u}|^2)} \tag{4.1.2}$$

with $c_n = (4\pi)^{-n}$.

For all $f \in L^2(\mathbb{R}^{2n})$ the twisted convolution $f *_\lambda p_t^\lambda$ has an analytic continuation to $\mathbb{C}^{2n}$. In particular, there is a λ-*twisted heat kernel transform*

$$H_t^\lambda : L^2(\mathbb{R}^{2n}) \to \mathcal{O}(\mathbb{C}^{2n}), \quad f \mapsto (f *_\lambda p_t^\lambda)^\sim\,.$$

In coordinates we have

$$H_t^\lambda(f)(\mathbf{z}, \mathbf{w}) = \int_{\mathbb{R}^{2n}} f(\mathbf{x}', \mathbf{u}') p_t^\lambda(\mathbf{z} - \mathbf{x}', \mathbf{w} - \mathbf{u}') e^{-\frac{i}{2}\lambda(\mathbf{x}'\cdot\mathbf{w} - \mathbf{u}'\cdot\mathbf{z})}\, d\mathbf{x}'\, d\mathbf{u}'\,.$$

We define a unitary representation τ^λ of $\mathbb{R}^{2n}$ on $L^2(\mathbb{R}^{2n})$ by

$$(\tau^\lambda(\mathbf{a}, \mathbf{b}) f)(\mathbf{x}, \mathbf{u}) = e^{-\frac{i\lambda}{2}(\mathbf{a}\cdot\mathbf{u} - \mathbf{b}\cdot\mathbf{x})} f(\mathbf{x} - \mathbf{a}, \mathbf{u} - \mathbf{b})$$

for $(\mathbf{a}, \mathbf{b}) \in \mathbb{R}^{2n}$, $f \in L^2(\mathbb{R}^{2n})$ and $(\mathbf{x}, \mathbf{u}) \in \mathbb{R}^{2n}$. Likewise τ^λ defines an action of $\mathbb{R}^{2n}$ on $\mathcal{O}(\mathbb{C}^{2n})$ via

$$(\tau^\lambda(\mathbf{a}, \mathbf{b}) f)(\mathbf{z}, \mathbf{w}) = e^{-\frac{i\lambda}{2}(\mathbf{a}\cdot\mathbf{w} - \mathbf{b}\cdot\mathbf{z})} f(\mathbf{z} - \mathbf{a}, \mathbf{w} - \mathbf{b})$$

where $(\mathbf{a}, \mathbf{b}) \in \mathbb{R}^{2n}$, $f \in \mathcal{O}(\mathbb{C}^{2n})$ and $(\mathbf{z}, \mathbf{w}) \in \mathbb{C}^{2n}$.

As for functions $F, G \in L^1(\mathbb{H})$ we have $\tau(h)F * G = \tau(h)(F * G)$ for all $h \in \mathbb{H}$, it is immediate from (4.1.1) that H_t^λ becomes $\mathbb{R}^{2n}$-equivariant, i.e

$$H_t^\lambda(\tau^\lambda(\mathbf{a}, \mathbf{b}) f) = \tau^\lambda(\mathbf{a}, \mathbf{b})(H_t^\lambda(f)) \tag{4.1.3}$$

for all $(\mathbf{a}, \mathbf{b}) \in \mathbb{R}^{2n}$ and $f \in L^2(\mathbb{R}^{2n})$.

Remark 4.1. For the proofs in the sequel it is notationally convenient to prove the assertions for the "essential case" $\lambda = 1$ only. Whenever we do so we will use a simplified notation: we write $f \times g$ instead of $f *_1 g$ for the 1-twisted convolution; further we will drop all sub- and superscripts involving $\lambda = 1$, i.e. p_t^1 becomes p_t, H_t^1 becomes H_t etc.

4.2. **Determination of the weight function.** Our objective is to find a non-negative weight function W_t^λ on $\mathbb{C}^{2n}$ such that

$$\int_{\mathbb{C}^{2n}} |H_t^\lambda(f)(\mathbf{z},\mathbf{w})|^2 W_t^\lambda(\mathbf{z},\mathbf{w})\, d\mathbf{z}\, d\mathbf{w} = \int_{\mathbb{R}^{2n}} |f(\mathbf{x},\mathbf{u})|^2\, d\mathbf{x}\, d\mathbf{u} \tag{4.2.1}$$

for all $f \in L^2(\mathbb{R}^{2n})$.

Proposition 4.1. *A weight function W_t^λ which satisfies* (4.2.1) *is given by*

$$W_t^\lambda(\mathbf{x}+i\mathbf{y},\mathbf{u}+i\mathbf{v}) = 4^n e^{\lambda(\mathbf{u}\cdot\mathbf{y}-\mathbf{v}\cdot\mathbf{x})} p_{2t}^\lambda(2\mathbf{y},2\mathbf{v}). \tag{4.2.2}$$

Remark 4.2. The weight function W_t^λ is unique in the sense that is the unique measurable function $W_t^\lambda : \mathbb{C}^{2n} \to \mathbb{R}_{\geq 0}$ which satisfies (4.2.1). This will be shown in Lemma 4.7 below.

Proof. We restrict our attention to the case $\lambda = 1$. As mentioned earlier we will write now p_t and W_t in place of p_t^λ and W_t^λ, respectively, and write $f \times g$ for the 1−twisted convolution of f and g. Via H_t we can transfer the Hilbert space structure of $L^2(\mathbb{R}^{2n})$ to $\operatorname{im} H_t$ and make it into Hilbert space of holomorphic functions. Write $K^t(\mathbf{z},\mathbf{w};\mathbf{z}',\mathbf{w}')$ for the corresponding reproducing kernel. Arguing as in Subsection 3.2, the inner product $\langle\cdot,\cdot\rangle_t$ on the image is uniquely determined by the equality

$$K^t(\mathbf{a},\mathbf{b};\mathbf{a}',\mathbf{b}') = \langle K^t_{(\mathbf{a},\mathbf{b})}, K^t_{(\mathbf{a}',\mathbf{b}')}\rangle_t \tag{4.2.3}$$

for all real pairs $(\mathbf{a},\mathbf{b}),(\mathbf{a}',\mathbf{b}') \in \mathbb{R}^n \times \mathbb{R}^n$.

As the heat kernel transform $f \to H_t(f) = (f \times p_t)^\sim$ commutes with the twisted translation (see equation (4.1.3)), we may assume $(\mathbf{a}',\mathbf{b}') = 0$ in (4.2.3). As $p_t \times p_t = p_{2t}$, arguing as in Subsection 3.1 readily yields

$$K^t_{(\mathbf{a},\mathbf{b})}(\mathbf{z},\mathbf{w}) = p_{2t}(\mathbf{z}-\mathbf{a},\mathbf{w}-\mathbf{b}) e^{-\frac{i}{2}(\mathbf{a}\cdot\mathbf{w}-\mathbf{b}\cdot\mathbf{z})}.$$

In particular, $K^t_{(0,0)} = p_{2t}$ and $K^t(\mathbf{a},\mathbf{b},0,0) = p_{2t}(\mathbf{a},\mathbf{b})$. Thus (4.2.3) translates into

$$p_{2t}(\mathbf{a},\mathbf{b}) = \int_{\mathbb{C}^n}\int_{\mathbb{C}^n} p_{2t}(\mathbf{z}-\mathbf{a},\mathbf{w}-\mathbf{b}) e^{-\frac{i}{2}(\mathbf{a}\cdot\mathbf{w}-\mathbf{b}\cdot\mathbf{z})}\overline{p_{2t}(\mathbf{z},\mathbf{w})} W_t(\mathbf{z},\mathbf{w})\, d\mathbf{z}\, d\mathbf{w}.$$

This is established in Lemma 4.2 below. □

Lemma 4.2. *For $\mathbf{a},\mathbf{b} \in \mathbb{R}^n$ we have*

$$\int_{\mathbb{C}^n}\int_{\mathbb{C}^n} p_{2t}^\lambda(\mathbf{z}+\mathbf{a},\mathbf{w}+\mathbf{b}) e^{\frac{i\lambda}{2}(\mathbf{a}\cdot\mathbf{w}-\mathbf{b}\cdot\mathbf{z})}\overline{p_{2t}^\lambda(\mathbf{z},\mathbf{w})} W_t^\lambda(\mathbf{z},\mathbf{w})\, d\mathbf{z}\, d\mathbf{w} = p_{2t}^\lambda(\mathbf{a},\mathbf{b}).$$

Proof. We will prove the assertion for $\lambda = 1$. Further, by the product nature of the functions involved, we may assume in addition that $n = 1$.

Expanding out and simplifying we have

$$\begin{aligned} p_{2t}(x+a+iy,u+b+iv)\overline{p_{2t}(x+iy,u+iv)} &= (4\pi)^{-2}(\sinh 2t)^{-2} e^{-\frac{1}{2}(\coth 2t)(x^2+u^2)} \\ &\cdot e^{-\frac{1}{4}(\coth 2t)(a^2+b^2)} e^{\frac{1}{2}(\coth 2t)(y^2+v^2)} e^{-\frac{1}{2}(\coth 2t)(a(x+iy)+b(u+iv))}. \end{aligned}$$

We can combine the terms $e^{-\frac{1}{2}(\coth 2t)(x^2+u^2)}$ and

$$e^{(uy-vx)} = e^{(\coth 2t)(uy\tanh(2t)-xv\tanh(2t))}$$

to get

$$\begin{aligned} &p_{2t}(x+a+iy,u+b+iv)\overline{p_{2t}(x+iy,u+iv)} e^{(uy-vx)} \\ &\quad= (4\pi)^{-2}(\sinh 2t)^{-2} e^{-\frac{1}{4}(\coth 2t)(a^2+b^2)} e^{\frac{1}{2}(\coth 2t+\tanh 2t)(y^2+v^2)} \\ &\qquad\cdot e^{-\frac{1}{2}(\coth 2t)\left((x+v\tanh(2t))^2+(u-y\tanh(2t))^2\right)} e^{-\frac{1}{2}(\coth 2t)(a(x+iy)+b(u+iv))}. \end{aligned}$$

Using the identity $\tanh 2t + \coth 2t = 2\coth 4t$ and simplifying further we get

$$\begin{aligned} &p_{2t}(z+a, w+b)e^{\frac{i}{2}(aw-bz)}\overline{p_{2t}(z,w)}W_t(z,w) \\ &\quad = 4^{-2}\pi^{-3}(\sinh 2t)^{-3}e^{-\frac{1}{8}\coth 2t(a^2+b^2)}e^{(\coth 4t-\coth 2t)(y^2+v^2)} \\ &\qquad \cdot e^{-\frac{1}{2}(\coth 2t)\left((x+\frac{a}{2}+\tanh(2t)v)^2+(u+\frac{b}{2}-y\tanh(2t))^2\right)}e^{-\frac{i}{2}(\coth 2t)(ay+bv)}e^{\frac{i}{2}(au-bx)}, \end{aligned}$$

where $z = x+iy$ and $w = u+iv$.

First consider the integral

$$\begin{aligned} &\int_{\mathbb{R}^2} e^{\frac{i}{2}(au-bx)}e^{-\frac{1}{2}(\coth 2t)\left((x+\frac{a}{2}+v\tanh(2t))^2+(u+\frac{b}{2}-y\tanh(2t))^2\right)}\,dx\,du \\ &\quad = e^{\frac{i}{2}(\tanh 2t)(ay+bv)}\int_{\mathbb{R}^2} e^{\frac{i}{2}(au-bx)}e^{-\frac{1}{2}(\coth 2t)(x^2+u^2)}\,dx\,du \\ &\quad = 2\pi(\tanh 2t)e^{\frac{i}{2}(\tanh 2t)(ay+bv)}e^{-\frac{1}{8}(\tanh 2t)(a^2+b^2)}. \end{aligned}$$

Up to an explicit factor the remaining integral is

$$\int_{\mathbb{R}^2} e^{-\frac{i}{2}(\coth 2t-\tanh 2t)(ay+bv)}e^{-(\coth 2t-\coth 4t)(y^2+v^2)}\,dy\,dv.$$

As $\coth 2t - \tanh 2t = 2(\sinh 4t)^{-1}$ and $\coth 2t - \coth 4t = (\sinh 4t)^{-1}$ the above integral reduces to

$$\int_{\mathbb{R}^2} e^{-i(\sinh 4t)^{-1}(ay+bv)}e^{-(\sinh 4t)^{-1}(y^2+v^2)}\,dy\,dv = \pi(\sinh 4t)e^{-\frac{1}{4}(\sinh 4t)^{-1}(a^2+b^2)}.$$

Combining results yields

$$\begin{aligned} &\int_{\mathbb{C}^2} p_{2t}(z+a, w+b)e^{\frac{i}{2}(aw-bz)}\overline{p_{2t}(z,w)}W_t(z,w)\,dz\,dw \\ &= 8^{-1}\pi^{-1}(\sinh 2t)^{-3}(\tanh 2t)(\sinh 4t)e^{-\frac{1}{8}(\coth 2t+\tanh 2t)(a^2+b^2)}e^{-\frac{1}{4}(\sinh 4t)^{-1}(a^2+b^2)}. \end{aligned}$$

Finally using the identities $\coth 2t + \tanh 2t = 2\coth 4t$ and $\coth 4t + (\sinh 4t)^{-1} = \coth 2t$ and simplifying we get

$$\begin{aligned} &\int_{\mathbb{C}^2} p_{2t}(z+a, w+b)e^{\frac{i}{2}(aw-bz)}\overline{p_{2t}(z,w)}W_t(z,w)\,dz\,dw \\ &\qquad\qquad = \frac{1}{4\pi}(\sinh 2t)^{-1}e^{-\frac{1}{4}\coth 2t(a^2+b^2)} = p_{2t}(a,b)\,. \end{aligned}$$

This proves the lemma. □

4.3. **The twisted Bergman space and surjectivity of H_t^λ.** For each $\lambda \in \mathbb{R}$, $\lambda \neq 0$, we define the *λ-twisted Bergman space* by

$$\mathcal{B}_t^\lambda(\mathbb{C}^{2n}) = \{f \in \mathcal{O}(\mathbb{C}^{2n}) : \|f\|_\lambda^2 = \int_{\mathbb{C}^n\times\mathbb{C}^n} |f(\mathbf{z},\mathbf{w})|^2 W_t^\lambda(\mathbf{z},\mathbf{w})\,d\mathbf{z}\,d\mathbf{w} < \infty\}\,.$$

Clearly $\mathcal{B}_t^\lambda(\mathbb{C}^{2n})$ is a Hilbert space of holomorphic functions on $\mathbb{C}^{2n}$. It follows from Proposition 4.1 that $H_t^\lambda : L^2(\mathbb{R}^{2n}) \to \mathcal{B}_t^\lambda(\mathbb{C}^{2n})$ is an isometric embedding.

Our goal for this subsection is to show that H_t^λ is onto. We begin with a description of a useful orthonormal basis for $\operatorname{im} H_t^\lambda$ in terms of the special Hermite functions $\Phi_{\alpha,\beta}^\lambda(\mathbf{x},\mathbf{u})$ (see [7, Section 2.3]). For each $\alpha, \beta \in \mathbb{N}_0^n$, let us consider

$$\widetilde{\Phi}_{\alpha,\beta}^\lambda(\mathbf{z},\mathbf{w}) = (2\pi)^{-n}e^{-(2|\beta|+n)|\lambda|t}\Phi_{\alpha,\beta}^\lambda(\mathbf{z},\mathbf{w})$$

where $\Phi_{\alpha,\beta}^\lambda(\mathbf{z},\mathbf{w})$ is the extension of $\Phi_{\alpha,\beta}^\lambda(\mathbf{x},\mathbf{u})$ to $\mathbb{C}^n \times \mathbb{C}^n$. The functions $\Phi_{\alpha,\beta}^\lambda(\mathbf{x},\mathbf{u})$ satisfy the orthogonal relation

$$(\Phi_{\alpha,\beta}^\lambda *_\lambda \Phi_{\mu,\nu}^\lambda)(\mathbf{x},\mathbf{u}) = \delta_{\beta,\mu}\Phi_{\alpha,\nu}^\lambda(\mathbf{x},\mathbf{u})\,.$$

Lemma 4.3. *The set* $\{\widetilde{\Phi}^{\lambda}_{\alpha,\beta} : \alpha, \beta \in \mathbb{N}_0^n\}$ *is an orthonormal basis for* $\operatorname{im} H_t^{\lambda}$.

Proof. It is enough to prove it for $\lambda = 1$ and we drop the superscript when $\lambda = 1$. As the heat kernel $p_t(\mathbf{x}, \mathbf{u})$ is given by

$$p_t(\mathbf{x}, \mathbf{u}) = (2\pi)^{-n} \sum_{\mu} e^{-(2|\mu|+n)t} \Phi_{\mu,\mu}(\mathbf{x}, \mathbf{u}) \ ,$$

we obtain the relation

$$(\Phi_{\alpha,\beta} \times p_t)(\mathbf{x}, \mathbf{u}) = (2\pi)^{-n} e^{-(2|\beta|+n)t} \Phi_{\alpha,\beta}(\mathbf{x}, \mathbf{u}) .$$

Thus $H_t(\Phi_{\alpha,\beta})(\mathbf{z}, \mathbf{w}) = \widetilde{\Phi}_{\alpha,\beta}(\mathbf{z}, \mathbf{w})$ and, therefore, using Proposition 4.1 we obtain

$$\begin{aligned}
&\int_{\mathbb{C}^{2n}} \widetilde{\Phi}_{\alpha,\beta}(\mathbf{z}, \mathbf{w}) \overline{\widetilde{\Phi}_{\mu,\nu}(\mathbf{z}, \mathbf{w})} W_t(\mathbf{z}, \mathbf{w}) \, d\mathbf{z} \, d\mathbf{w} \\
&\quad = \int_{\mathbb{C}^{2n}} H_t(\Phi_{\alpha,\beta})(\mathbf{z}, \mathbf{w}) \overline{H_t(\Phi_{\mu,\nu})(\mathbf{z}, \mathbf{w})} W_t(\mathbf{z}, \mathbf{w}) \, d\mathbf{z} \, d\mathbf{w} \\
&\quad = \int_{\mathbb{R}^{2n}} \Phi_{\alpha,\beta}(\mathbf{x}, \mathbf{u}) \overline{\Phi_{\mu,\nu}(\mathbf{x}, \mathbf{u})} \, d\mathbf{x} \, d\mathbf{u} .
\end{aligned}$$

Hence $\{\widetilde{\Phi}_{\alpha,\beta} : \alpha, \beta \in \mathbb{N}_0^n\}$ is an orthonormal system in $\operatorname{im} H_t$.

To show that it is an orthonormal basis for $\operatorname{im} H_t$, we only need to show that

$$\int_{\mathbb{C}^{2n}} H_t(f)(\mathbf{z}, \mathbf{w}) \overline{\widetilde{\Phi}_{\alpha,\beta}(\mathbf{z}, \mathbf{w})} W_t(\mathbf{z}, \mathbf{w}) \, d\mathbf{z} \, d\mathbf{w} = 0$$

for all α, β implies $f \equiv 0$. But the above simply means, by Proposition 4.1, that

$$\int_{\mathbb{R}^{2n}} f(\mathbf{x}, \mathbf{u}) \overline{\Phi_{\alpha,\beta}(\mathbf{x}, \mathbf{u})} \, d\mathbf{x} \; d\mathbf{u} = 0$$

for all α, β and we know that $\{\Phi_{\alpha,\beta} : \alpha, \beta \in \mathbb{N}_0^n\}$ is an orthonormal basis for $L^2(\mathbb{R}^{2n})$. Hence $f \equiv 0$ and the proof is complete. □

We will show that $\{\widetilde{\Phi}^{\lambda}_{\alpha,\beta} : \alpha, \beta \in \mathbb{N}_0^n\}$ is also an orthonormal basis for $\mathcal{B}_t^{\lambda}(\mathbb{C}^{2n})$. Clearly this implies that $H_t^{\lambda} : L^2(\mathbb{R}^{2n}) \to \mathcal{B}_t^{\lambda}(\mathbb{C}^{2n})$ is onto.

Note that $\widetilde{\Phi}^{\lambda}_{\alpha,\beta} \in \mathcal{B}_t^{\lambda}(\mathbb{C}^{2n})$ for any $t > 0$ and $\{\widetilde{\Phi}^{\lambda}_{\alpha,\beta} : \alpha, \beta \in \mathbb{N}_0^n\}$ will be an orthonormal basis for any $\mathcal{B}_t^{\lambda}(\mathbb{C}^{2n})$.

As $\mathbf{z} = \mathbf{x} + i\mathbf{y}$ and $\mathbf{w} = \mathbf{u} + i\mathbf{v}$, we note that $\mathbf{u} \cdot \mathbf{y} - \mathbf{v} \cdot \mathbf{x} = \Im(\mathbf{z} \cdot \overline{\mathbf{w}})$ is the symplectic form on $\mathbb{R}^{2n}$. Thus $\Im(\sigma\mathbf{z} \cdot \overline{\sigma\mathbf{w}}) = \Im(\mathbf{z} \cdot \overline{\mathbf{w}})$ for $\sigma \in U(n)$.

We introduce the *twisted Fock space* $\mathcal{F}_t^{\lambda}(\mathbb{C}^{2n})$ by

$$\begin{aligned}
\mathcal{F}_t^{\lambda}(\mathbb{C}^{2n}) = \{G \in \mathcal{O}(\mathbb{C}^{2n}) :& \\
\|G\|^2 = \int_{\mathbb{C}^n \times \mathbb{C}^n} & |G(\mathbf{z}, \mathbf{w})|^2 e^{\lambda \Im(\mathbf{z} \cdot \overline{\mathbf{w}})} e^{-\frac{\lambda}{2}(\coth 2t\lambda)(|\mathbf{z}|^2 + |\mathbf{w}|^2)} \, d\mathbf{z} \, d\mathbf{w} < \infty\} .
\end{aligned}$$

Clearly, the prescription

$$U(n) \times \mathcal{F}_t^{\lambda}(\mathbb{C}^{2n}) \to \mathcal{F}_t^{\lambda}(\mathbb{C}^{2n}), \quad (\sigma, G) \mapsto G^{\sigma}; \ G^{\sigma}(\mathbf{z}, \mathbf{w}) = G(\sigma\mathbf{z}, \sigma\mathbf{w})$$

defines a unitary representation of $U(n)$ on $\mathcal{F}_t(\mathbb{C}^{2n})$.

The Hilbert spaces $\mathcal{B}_t^{\lambda}(\mathbb{C}^{2n})$ and $\mathcal{F}_t^{\lambda}(\mathbb{C}^{2n})$ are related through

$$F(\mathbf{z}, \mathbf{w}) \in \mathcal{B}_t^{\lambda}(\mathbb{C}^{2n}) \quad \text{if and only if} \quad F(\mathbf{z}, \mathbf{w}) e^{\frac{\lambda}{4}(\coth 2t\lambda)(\mathbf{z} \cdot \mathbf{z} + \mathbf{w} \cdot \mathbf{w})} \in \mathcal{F}_t^{\lambda}(\mathbb{C}^{2n}) . \tag{4.3.1}$$

Let $T \simeq (\mathbb{S}^1)^n$ be the diagonal subgroup of $U(n)$. We write the elements of T as $\sigma = (e^{i\varphi_1}, \ldots, e^{i\varphi_n})$. For each $n-$tuple of integers $\mathbf{m} = (m_1, m_2, \ldots, m_n)$ let $\chi_{\mathbf{m}}(\sigma)$ be the character of T defined by $\chi_{\mathbf{m}}(\sigma) = e^{i\sum_{j=1}^n m_j\varphi_j}$. For each $G \in \mathcal{F}_t^\lambda(\mathbb{C}^{2n})$ define

$$G_{\mathbf{m}}(\mathbf{z},\mathbf{w}) = \int_T G(\sigma\mathbf{z},\sigma\mathbf{w})\overline{\chi_{\mathbf{m}}(\sigma)}\, d\sigma\,.$$

As G is holomorphic it is clear that $G_{\mathbf{m}} = 0$ unless $\mathbf{m}$ is a multi-index in $\mathbb{N}_0^n$. By the Fourier expansion

$$G(\sigma\mathbf{z},\sigma\mathbf{w}) = \sum_{\mathbf{m}\in\mathbb{N}_0^n} G_{\mathbf{m}}(\mathbf{z},\mathbf{w})\chi_{\mathbf{m}}(\sigma)$$

and by the Plancherel theorem we have

$$\int_T |G(\sigma\mathbf{z},\sigma\mathbf{w})|^2 d\sigma = \sum_{\mathbf{m}\in\mathbb{N}_0^n} |G_{\mathbf{m}}(\mathbf{z},\mathbf{w})|^2\,. \tag{4.3.2}$$

Note that the functions $G_{\mathbf{m}}$ satisfy the homogeneity condition

$$G_{\mathbf{m}}(\sigma\mathbf{z},\sigma\mathbf{w}) = \chi_{\mathbf{m}}(\sigma)G_{\mathbf{m}}(\mathbf{z},\mathbf{w})\,.$$

For any $G \in \mathcal{F}_t^\lambda(\mathbb{C}^{2n})$ we observe that, as $\Im(\mathbf{z}\cdot\overline{\mathbf{w}}) = \Im(\sigma\mathbf{z}\cdot\sigma\overline{\mathbf{w}})$,

$$\begin{aligned}&\int_{\mathbb{C}^{2n}} G(\mathbf{z},\mathbf{w})e^{\lambda\Im(\mathbf{z}\cdot\overline{\mathbf{w}})}e^{-\frac{\lambda}{2}(\coth 2t\lambda)(|\mathbf{z}|^2+|\mathbf{w}|^2)}d\mathbf{z}\, d\mathbf{w}\\ &\qquad = \int_T\int_{\mathbb{C}^{2n}} G(\sigma\mathbf{z},\sigma\mathbf{w})e^{\lambda\Im(\mathbf{z}\cdot\overline{\mathbf{w}})}e^{-\frac{\lambda}{2}(\coth 2t\lambda)(|\mathbf{z}|^2+|\mathbf{w}|^2)}d\mathbf{z}\, d\mathbf{w}\, d\sigma.\end{aligned}$$

In view of this and the homogeneity condition we arrive at the orthogonality relations

$$\int_{\mathbb{C}^{2n}} G_{\mathbf{m}}(\mathbf{z},\mathbf{w})\overline{G_{\mathbf{m}'}(\mathbf{z},\mathbf{w})}e^{\lambda\Im(\mathbf{z}\cdot\overline{\mathbf{w}})}e^{-\frac{\lambda}{2}(\coth 2t\lambda)(|\mathbf{z}|^2+|\mathbf{w}|^2)}d\mathbf{z}\, d\mathbf{w} = 0\,,$$

whenever $\mathbf{m}$ and $\mathbf{m}'$ are different. We also note that each $G_{\mathbf{m}}$ has an expansion of the form

$$G_{\mathbf{m}}(\mathbf{z},\mathbf{w}) = \sum_{\alpha+\beta=\mathbf{m}} c_{\alpha,\beta}\mathbf{z}^\alpha\mathbf{w}^\beta\,.$$

Hence each $G_{\mathbf{m}}$ is a polynomial.

Lemma 4.4. *The linear span of $P_{\alpha,\beta}(\mathbf{z},\mathbf{w}) = \mathbf{z}^\alpha\mathbf{w}^\beta$, $\alpha,\beta \in \mathbb{N}_0^n$, is dense in $\mathcal{F}_t^\lambda(\mathbb{C}^{2n})$.*

Proof. If $G \in \mathcal{F}_t^\lambda(\mathbb{C}^{2n})$ is orthogonal to all $P_{\alpha,\beta}$ then

$$\int_{\mathbb{C}^{2n}} G(\mathbf{z},\mathbf{w})\overline{G_{\mathbf{m}}(\mathbf{z},\mathbf{w})}e^{\lambda\Im(\mathbf{z}\cdot\overline{\mathbf{w}})}e^{-\frac{\lambda}{2}(\coth 2t\lambda)(|\mathbf{z}|^2+|\mathbf{w}|^2)}d\mathbf{z}\, d\mathbf{w} = 0$$

for any $\mathbf{m} \in \mathbb{N}_0^n$. In view of the homogeneity property of $G_{\mathbf{m}}$ this means that

$$\int_{\mathbb{C}^{2n}} |G_{\mathbf{m}}(\mathbf{z},\mathbf{w})|^2 e^{\lambda\Im(\mathbf{z}\cdot\overline{\mathbf{w}})}e^{-\frac{\lambda}{2}(\coth 2t\lambda)(|\mathbf{z}|^2+|\mathbf{w}|^2)}d\mathbf{z}\, d\mathbf{w} = 0\,.$$

Hence $G_{\mathbf{m}}(\mathbf{z},\mathbf{w}) = 0$ for every $\mathbf{m}$ and so $G = 0$ in view of (4.3.2). □

It follows from Lemma 4.4 and (4.3.1) that every $F \in \mathcal{B}_t^\lambda(\mathbb{C}^{2n})$ has the orthonormal expansion

$$F(\mathbf{z},\mathbf{w}) = \sum_{\mathbf{m}}\sum_{\alpha+\beta=\mathbf{m}} c_{\alpha,\beta}P_{\alpha,\beta}(\mathbf{z},\mathbf{w})e^{-\frac{\lambda}{4}(\coth 2t\lambda)(\mathbf{z}\cdot\mathbf{z}+\mathbf{w}\cdot\mathbf{w})}\,. \tag{4.3.3}$$

The functions

$$\Psi_{\mathbf{m}}(\mathbf{z},\mathbf{w}) = \sum_{\alpha+\beta=\mathbf{m}} c_{\alpha,\beta}P_{\alpha,\beta}(\mathbf{z},\mathbf{w})e^{-\frac{\lambda}{4}(\coth 2t\lambda)(\mathbf{z}\cdot\mathbf{z}+\mathbf{w}\cdot\mathbf{w})}$$

are orthogonal in $\mathcal{B}_t^\lambda(\mathbb{C}^{2n})$ but not orthogonal in any other $\mathcal{B}_s^\lambda(\mathbb{C}^{2n})$ when $s \neq t$. Another crucial property of these functions is proved in the next lemma.

Lemma 4.5. *All the functions* $\Psi^{\mathbf{m}}_{\alpha,\beta}(\mathbf{z},\mathbf{w}) = P_{\alpha,\beta}(\mathbf{z},\mathbf{w})e^{-\frac{\lambda}{4}(\coth 2t\lambda)(\mathbf{z}\cdot\mathbf{z}+\mathbf{w}\cdot\mathbf{w})}$ *belong to the image* $\operatorname{im} H_t^\lambda$ *of the heat kernel transform.*

Proof. We may restrict ourselves to the case of $\lambda = 1$. It will suffice to show that for each pair $\alpha, \beta \in \mathbb{N}_0^n$ there exists a function $f_{\alpha,\beta} \in L^2(\mathbb{R}^{2n})$ such that

$$H_t(f_{\alpha,\beta})(\mathbf{z},\mathbf{w}) = (f_{\alpha,\beta} \times p_t)^{\sim}(\mathbf{z},\mathbf{w}) = \mathbf{z}^\alpha \mathbf{w}^\beta e^{-\frac{1}{4}(\coth 2t)(\mathbf{z}^2+\mathbf{w}^2)}\,.$$

As both sides are holomorphic it is enough to prove this for $\mathbf{z} = \mathbf{x}$ and $\mathbf{w} = \mathbf{u}$ where $\mathbf{x}, \mathbf{u} \in \mathbb{R}^n$. Thus we need to solve the equation

$$(f_{\alpha,\beta} \times p_t)(\mathbf{x},\mathbf{u}) = \mathbf{x}^\alpha \mathbf{u}^\beta p_{2t}(\mathbf{x},\mathbf{u})\,. \tag{4.3.4}$$

In the sequel it will be convenient to identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ via $z = \mathbf{x} + i\mathbf{u}$. Then $\mathbf{x}^\alpha \mathbf{u}^\beta = 2^{-|\alpha|}(2i)^{-|\beta|}(z+\overline{z})^\alpha(z-\overline{z})^\beta$. It is then sufficient to solve the equation

$$(f_{\alpha,\beta} \times p_t)(z) = z^\alpha \overline{z}^\beta p_{2t}(z)$$

where $p_t(z) = p_t(\mathbf{x},\mathbf{u})$. We solve this equation using properties of the Weyl transform.

Recall that the Weyl transform $\mathbb{W}(f)$ of a function $f \in L^1(\mathbb{C}^n)$, is defined to be the bounded operator on $L^2(\mathbb{R}^n)$ given by

$$\mathbb{W}(f)\varphi(\xi) = \int_{\mathbb{C}^n} f(z)\pi(z)\varphi(\xi)\,dz \qquad (\xi \in \mathbb{R}^n)$$

where $\pi(z) = \pi_1(z,0)$ and π_1 is the Schrödinger representation of the Heisenberg group $\mathbb{H}$ with parameter $\lambda = 1$ (see [7, Section 2.2]). Then for $f \in L^1 \bigcap L^2(\mathbb{C}^n)$, $\mathbb{W}(f)$ is a Hilbert-Schmidt operator and $\mathbb{W}$ extends to $L^2(\mathbb{C}^n)$ as an isometry onto the space of Hilbert-Schmidt operators. Moreover $\mathbb{W}(f \times g) = \mathbb{W}(f)\mathbb{W}(g)$ and $\mathbb{W}(p_t) = e^{-tH}$. Here H denotes the Hermite operator

$$H = (-\Delta + |\xi|^2) = \frac{1}{2}\sum_{j=1}^n (A_j A_j^* + A_j^* A_j),$$

in which $A_j = -\frac{\partial}{\partial \xi_j} + \xi_j$ and $A_j^* = \frac{\partial}{\partial \xi_j} + \xi_j$ are the creation and annihilation operators. The eigenfunctions of H are the Hermite functions Φ_α. They satisfy

$$A_j\Phi_\alpha = (2\alpha_j + 2)^{\frac{1}{2}}\Phi_{\alpha+e_j}, \qquad A_j^*\Phi_\alpha = (2\alpha_j)^{\frac{1}{2}}\Phi_{\alpha-e_j}$$

where e_j are the coordinate vectors. Given a bounded linear operator T on $L^2(\mathbb{R}^n)$, define the derivations

$$\delta_j T = [A_j^*, T] = A_j^* T - T A_j^*, \qquad \overline{\delta}_j T = [T, A_j] = T A_j - A_j T\,.$$

Then it can be shown that (see [8])

$$\mathbb{W}(z_j f) = \delta_j \mathbb{W}(f), \quad \text{and} \quad \mathbb{W}(\overline{z}_j f) = \overline{\delta}_j \mathbb{W}(f)\,.$$

By iteration we obtain

$$\mathbb{W}(z^\alpha \overline{z}^\beta f) = \delta^\alpha \overline{\delta}^\beta \mathbb{W}(f)$$

where $\delta^\alpha \overline{\delta}^\beta$ are defined in an obvious way.

Returning to our equation (4.3.4), we take the Weyl transform on both sides and obtain that

$$\mathbb{W}(f_{\alpha,\beta})e^{-tH} = \delta^\alpha \overline{\delta}^\beta e^{-2tH}\,.$$

Testing against the Hermite basis it is easy to see that the densely defined operator

$$T = (\delta^\alpha \overline{\delta}^\beta e^{-2tH})e^{tH}$$

extends to the whole $L^2(\mathbb{R}^n)$ as a Hilbert-Schmidt operator. Hence, $T = \mathbb{W}(f_{\alpha,\beta})$ for some $f_{\alpha,\beta} \in L^2(\mathbb{C}^n)$. This completes the proof of the lemma. □

Theorem 4.6. *Let $t > 0$ and $\lambda \in \mathbb{R}$, $\lambda \neq 0$. Then the λ-twisted heat kernel transform $H_t^\lambda : L^2(\mathbb{R}^{2n}) \to \mathcal{B}_t^\lambda(\mathbb{C}^{2n})$ is an isometric isomorphism. Moreover, $\{\widetilde{\Phi}_{\alpha,\beta}^\lambda : \alpha, \beta \in \mathbb{N}_0^n\}$ is an orthonormal basis for $\mathcal{B}_t^\lambda(\mathbb{C}^{2n})$.*

Proof. As usual we restrict our attention to the case $\lambda = 1$. All what is left to show is that H_t is onto. Suppose that $F \in \mathcal{B}_t(\mathbb{C}^{2n})$ is orthogonal to all $\widetilde{\Phi}_{\alpha,\beta}$. We have to verify that $F \equiv 0$. The function

$$G(\mathbf{z}, \mathbf{w}) = F(\mathbf{z}, \mathbf{w}) e^{\frac{1}{4}(\coth 2t)(\mathbf{z}\cdot\mathbf{z}+\mathbf{w}\cdot\mathbf{w})}$$

is orthogonal in $\mathcal{F}_t$ to all functions of the form

$$f \times p_t(\mathbf{z}, \mathbf{w}) e^{-\frac{1}{4}(\coth 2t)(\mathbf{z}\cdot\mathbf{z}+\mathbf{w}\cdot\mathbf{w})}.$$

In view of Lemma 4.5, G is orthogonal to all $P_{\alpha,\beta}$. Hence by Lemma 4.4 we get $G = 0$ and so $F = 0$ as desired. □

We conclude this subsection with a proof of the uniqueness of the weight function W_t^λ.

Lemma 4.7. *W_t^λ is the unique non-negative measurable weight function for the λ-twisted Bergman space $\mathcal{B}_t^\lambda(\mathbb{C}^{2n})$.*

Proof. In view of (4.3.1), the statement is equivalent to the assertion that

$$\mathcal{W}_t^\lambda(\mathbf{w}, \mathbf{z}) = e^{\lambda \Im(\mathbf{z}\cdot\overline{\mathbf{w}})} e^{-\frac{\lambda}{2}(\coth 2t\lambda)(|\mathbf{z}|^2+|\mathbf{w}|^2)} \tag{4.3.5}$$

is the unique weight function for the twisted Fock space $\mathcal{F}_t^\lambda(\mathbb{C}^{2n})$. This will be verified in the sequel.

We may restrict ourselves to the notationally convenient case $n = 1$, $\lambda = 1$ and drop all sub and superscripts involving λ. Let $\mathcal{U}_t : \mathbb{C}^2 \to \mathbb{R}_{\geq 0}$ be a measurable function such that

$$\int_{\mathbb{C}^2} f(z, w)\overline{g(z, w)}\; \mathcal{W}_t(z, w)\, dz\, dw = \int_{\mathbb{C}^2} f(z, w)\overline{g(z, w)}\; \mathcal{U}_t(z, w)\, dz\, dw \tag{4.3.6}$$

holds for all $f, g \in \mathcal{F}_t(\mathbb{C}^2)$. We have to show that $\mathcal{W}_t = \mathcal{U}_t$ almost everywhere. Recall from Lemma 4.5 that all polynomials $z^m w^n$ lie in $\mathcal{F}_t(\mathbb{C}^2)$. In particular the constant function belongs to $\mathcal{F}_t(\mathbb{C}^2)$ and (4.3.6) implies that $\mathcal{U}_t$ is integrable.

Let us introduce polar coordinates on $\mathbb{C}^2$ by $(z, w) = (re^{i\phi}, se^{i\theta})$. Consider the Fourier expansions of $\mathcal{W}_t$ and $\mathcal{U}_t$ given by

$$\mathcal{W}_t(re^{i\phi}, se^{i\theta}) = \sum_{m,n\in\mathbb{Z}} a_{m,n}(r, s) e^{im\phi} e^{in\theta}$$

and

$$\mathcal{U}_t(re^{i\phi}, se^{i\theta}) = \sum_{m,n\in\mathbb{Z}} b_{m,n}(r, s) e^{im\phi} e^{in\theta}\,.$$

Identity (4.3.6) applied to $f = g = z^k w^l$ yields the estimates

$$\int_0^\infty \int_0^\infty r^{2k+1} s^{2l+1} |a_{m,n}(r, s)|\, dr\, ds \leq \|z^k w^l\|_{\mathcal{F}_t(\mathbb{C}^2)}^2 \tag{4.3.7}$$

$$\int_0^\infty \int_0^\infty r^{2k+1} s^{2l+1} |b_{m,n}(r, s)|\, dr\, ds \leq \|z^k w^l\|_{\mathcal{F}_t(\mathbb{C}^2)}^2 \tag{4.3.8}$$

for all $m, n \in \mathbb{Z}$.

We finish the proof and show $c_{m,n} = a_{m,n} - b_{m,n} = 0$ for all $m, n \in \mathbb{Z}$. In fact for $f = z^{m_1} w^{n_1}$ and $g = z^{m_2} w^{n_2}$ for $m_1, m_2, n_1, n_2 \in \mathbb{N}_0$ we obtain from (4.3.6) that

$$\int_0^\infty \int_0^\infty r^{m_1+m_2+1} s^{n_1+n_2+1} c_{m_2-m_1, n_2-n_1}(r, s)\, dr\, ds = 0\,. \tag{4.3.9}$$

Note that the integral on the left is absolutely convergent by (4.3.7)-(4.3.8). Fix now $m, n \in \mathbb{Z}$. Reformulating (4.3.9) reads

$$\int_0^\infty \int_0^\infty r^{|m|+2k+1} s^{|n|+2l+1} c_{m,n}(r,s)\, dr\, ds = 0 \tag{4.3.10}$$

for all $k, l \in \mathbb{N}_0$. In view of (4.3.7)-(4.3.8), we have the estimate

$$\int_0^\infty \int_0^\infty r^{|m|+2k+1} s^{|n|+2l+1} |c_{m,n}(r,s)|\, dr\, ds \leq 2\|z^{|m|+k} w^{|n|+l}\|^2_{\mathcal{F}_t(\mathbb{C}^2)} + C \tag{4.3.11}$$

with $C = \int_{|z|<1,|w|<1} (\mathcal{W}_t(z,w) + \mathcal{U}_t(z,w))\, dz\, dw > 0$ a constant independent of m, n.

Denote by $\mathcal{R}_+ = \{\zeta \in \mathbb{C} : \Re\zeta > 0\}$ the right halfplane. Let us recall the elementary fact that a bounded holomorphic function $f : \mathcal{R}_+ \to \mathbb{C}$ which vanishes on $\alpha + \beta\mathbb{N}_0$ for some $\alpha \geq 0$, $\beta > 0$ is identically zero (see [4], Lemma A.1 for a proof).

The explicite formula for $\mathcal{W}_t$ in (4.3.5) yields a crude but sufficient estimate for the norm of monomials: there exists constants $c, \gamma > 0$ such that for all $k, l \in \mathbb{N}_0$ one has

$$\|z^k w^l\|^2 \leq c \cdot e^{\gamma(k+l)}. \tag{4.3.12}$$

Now define the function

$$F_{m,n} : \mathcal{R}_+ \times \mathcal{R}_+ \to \mathbb{C},$$
$$(\zeta_1, \zeta_2) \mapsto e^{-3\gamma(\zeta_1+\zeta_2)} \int_0^\infty \int_0^\infty r^{|m|+2\zeta_1+1} s^{|n|+2\zeta_2+1} c_{m,n}(r,s)\, dr\, ds.$$

It is a consequence of (4.3.11) and (4.3.12) that $F_{m,n}$ is bounded and holomorphic on $\mathcal{R}_+ \times \mathcal{R}_+$. As $F_{m,n}|_{\mathbb{N}\times\mathbb{N}} = 0$ by (4.3.10), we conclude that $F_{m,n} = 0$. But then $c_{m,n} = 0$ by the properties of the Mellin transform. □

4.4. **The inversion formula for H_t^λ.** We conclude this section by proving a formula for the inverse map of the λ-twisted heat kernel transform $H_t^\lambda : L^2(\mathbb{R}^{2n}) \to \mathcal{B}_t^\lambda(\mathbb{C}^{2n})$. It is in the nature of the problem that $(H_t^\lambda)^{-1}$ can only be defined nicely on a dense subspace of $\mathcal{B}_t^\lambda(\mathbb{C}^{2n})$. The precise statement is as follows:

Theorem 4.8. *The inverse of* $H_t^\lambda : L^2(\mathbb{R}^{2n}) \to \mathcal{B}_t^\lambda(\mathbb{C}^{2n})$ *is given by*

$$(H_t^\lambda)^{-1}(F) = \lim_{s\to 0+} F_s \qquad (F \in \mathcal{B}_t^\lambda(\mathbb{C}^{2n})),$$

where

$$F_s(\mathbf{a},\mathbf{b}) = \int_{\mathbb{C}^{2n}} F(\mathbf{z}+\mathbf{a}, \mathbf{w}+\mathbf{b}) e^{\frac{i\lambda}{2}(\mathbf{a}\cdot\mathbf{w}-\mathbf{b}\cdot\mathbf{z})} \overline{p_{t+s}^\lambda(\mathbf{z},\mathbf{w})} W_t^\lambda(\mathbf{z},\mathbf{w})\, d\mathbf{z}\, d\mathbf{w}.$$

Proof. As before we only need to handle the case of $\lambda = 1$.

Let $F \in \mathcal{B}_t(\mathbb{C}^{2n})$. Since the space $\mathcal{B}_t(\mathbb{C}^{2n})$ is twisted-translation invariant, it is clear that the function

$$(\tau(-\mathbf{a}, -\mathbf{b})F)(\mathbf{z},\mathbf{w}) = F(\mathbf{z}+\mathbf{a}, \mathbf{w}+\mathbf{b}) e^{\frac{i}{2}(\mathbf{a}\cdot\mathbf{w}-\mathbf{b}\cdot\mathbf{z})}$$

belongs to $\mathcal{B}_t(\mathbb{C}^{2n})$. Hence, by Cauchy-Schwarz ineqaulity, the integral defining F_s converges. According to Theorem 4.6 we have $F = H_t(f) = (f \times p_t)^\sim$ for some $f \in L^2(\mathbb{R}^{2n})$. It is easy to see that $F_s \in L^2(\mathbb{R}^{2n})$

and that F_s converges to f. In fact, we have

$$\begin{aligned} F_s(\mathbf{a},\mathbf{b}) &= \int_{\mathbb{C}^{2n}} (\tau(-\mathbf{a},-\mathbf{b})H_t(f))(\mathbf{z},\mathbf{w})\overline{H_t(p_s)(\mathbf{z},\mathbf{w})}W_t(\mathbf{z},\mathbf{w})\,d\mathbf{z}\,d\mathbf{w} \\ &= \int_{\mathbb{C}^{2n}} H_t(\tau(-\mathbf{a},-\mathbf{b})f)(\mathbf{z},\mathbf{w})\overline{H_t(p_s)(\mathbf{z},\mathbf{w})}W_t(\mathbf{z},\mathbf{w})\,d\mathbf{z}\,d\mathbf{w} \\ &= \int_{\mathbb{R}^{2n}} (\tau(-\mathbf{a},-\mathbf{b})f)(\mathbf{x},\mathbf{u})p_s(\mathbf{x},\mathbf{u})\,d\mathbf{x}\,d\mathbf{u}\,. \end{aligned}$$

As $(p_s)_{s>0}$ is a Dirac sequence, it therefore follows that

$$F_s(\mathbf{a},\mathbf{b}) \to (\tau(-\mathbf{a},-\mathbf{b})f)(0,0) = f(\mathbf{a},\mathbf{b})$$

for $s \to 0^+$. This proves the theorem. □

5. The image of $\mathcal{H}_t$ as a direct integral

The goal of this section is to give a natural $\mathbb{H}$-equivariant identification of the image of the heat kernel transform $\mathcal{H}_t : L^2(\mathbb{H}) \to \mathcal{O}(\mathbb{H}_{\mathbb{C}})$ with a direct integral of twisted Bergman-spaces.

We set $\mathbb{R}^\times = \mathbb{R}\backslash\{0\}$. For each $\lambda \in \mathbb{R}^\times$ we write $\langle\cdot,\cdot\rangle_\lambda$ for the inner product on $\mathcal{B}_t^\lambda(\mathbb{C}^{2n})$. Recall the orthonormal basis $\{\tilde{\Phi}^\lambda_{\alpha,\beta} : \alpha,\beta \in \mathbb{N}_0^n\}$ of $\mathcal{B}_t^\lambda(\mathbb{C}^{2n})$ from Theorem 4.6.

We now introduce a measurable structure on $\coprod_{\lambda\in\mathbb{R}^\times} \mathcal{B}_t^\lambda(\mathbb{C}^{2n})$. By a *section* s of $\coprod_{\lambda\in\mathbb{R}^\times} \mathcal{B}_t^\lambda(\mathbb{C}^{2n})$ we understand an assignment

$$s : \mathbb{R}^\times \to \coprod_{\lambda\in\mathbb{R}^\times} \mathcal{B}_t^\lambda(\mathbb{C}^{2n}), \quad \lambda \mapsto s_\lambda \in \mathcal{B}_t^\lambda(\mathbb{C}^{2n})\,.$$

We declare a section $s = (s_\lambda)$ to be *measurable* if for all $\alpha,\beta \in \mathbb{N}_0^n$ the map

$$\mathbb{R}^\times \to \mathbb{C}, \quad \lambda \mapsto \langle s_\lambda, \tilde{\Phi}^\lambda_{\alpha,\beta}\rangle_\lambda$$

is measurable. With that we can define a direct integral of Hilbert spaces by

$$\begin{aligned} \int_{\mathbb{R}^\times}^{\oplus} \mathcal{B}_t^\lambda(\mathbb{C}^{2n})\, e^{2t\lambda^2} d\lambda = \{s : \mathbb{R}^\times \to \coprod_{\lambda\in\mathbb{R}^\times} \mathcal{B}_t^\lambda(\mathbb{C}^{2n}) : s \text{ measurable}\,, \\ \|s\|^2 = \int_{\mathbb{R}^\times} \|s_\lambda\|_\lambda^2\, e^{2t\lambda^2} d\lambda < \infty\}\,. \end{aligned}$$

Recall the unitary representation τ^λ of $\mathbb{H}$ on $\mathcal{B}_t^\lambda(\mathbb{C}^{2n})$ from Subsection 4.1. We then obtain a unitary representation $\int_{\mathbb{R}^\times} \tau^\lambda\, d\lambda$ on $\int_{\mathbb{R}^\times}^{\oplus} \mathcal{B}_t^\lambda(\mathbb{C}^{2n})\, e^{2t\lambda^2} d\lambda$ by

$$\left(\int_{\mathbb{R}^\times} \tau^\lambda\, d\lambda\right)(h)(s) = (\tau^\lambda(h)s_\lambda)_\lambda$$

for $h \in \mathbb{H}$ and $s = (s_\lambda)$ a square integrable section.

In our next step we will identify $\operatorname{im}\mathcal{H}_t$ with our direct integral from above. For that let $f \in S(\mathbb{H})$ be a Schwartz function. Then $\mathcal{H}_t(f) = (k_t * f)^\sim$ and from $(f * k_t)^\lambda = e^{-t\lambda^2} f^\lambda *_\lambda p_t^\lambda$ it hence follows that

$$(\mathcal{H}_t(f))^\lambda = e^{-t\lambda^2} H_t^\lambda(f^\lambda)\,. \tag{5.1}$$

Theorem 5.1. *Let $t > 0$. The map*

$$\mathcal{J}_t : S(\mathbb{H}) \to \int_{\mathbb{R}^\times}^{\oplus} \mathcal{B}_t^\lambda(\mathbb{C}^{2n})\, e^{2t\lambda^2} d\lambda, \quad f \mapsto \big((\mathcal{H}_t(f))^\lambda\big)_\lambda$$

extends to an $\mathbb{H}$-equivariant unitary equivalence

$$(\tau, L^2(\mathbb{H})) \simeq \left(\int_{\mathbb{R}^\times} \tau^\lambda\, d\lambda\,, \int_{\mathbb{R}^\times}^{\oplus} \mathcal{B}_t^\lambda(\mathbb{C}^{2n})\, e^{2t\lambda^2} d\lambda\right)\,.$$

Proof. Let $f \in S(\mathbb{H})$. Then

$$\|f\|^2 = \int_{\mathbb{R}^{2n+1}} |f(\mathbf{x}, \mathbf{u}, \xi)|^2 \, d\mathbf{x} \, d\mathbf{u} \, d\xi = \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}} |f^\lambda(\mathbf{x}, \mathbf{u})|^2 \, d\mathbf{x} \, d\mathbf{u} \, d\lambda.$$

By Theorem 4.6 we have for each λ that

$$\int_{\mathbb{R}^{2n}} |f^\lambda(\mathbf{x}, \mathbf{u})|^2 \, d\mathbf{x} \, d\mathbf{u} = \|H_t^\lambda(f^\lambda)\|^2 \; .$$

Thus it follows from (5.1) that $\mathcal{J}_t$ extends to an isometric embedding

$$\mathcal{J}_t : L^2(\mathbb{H}) \to \int_{\mathbb{R}^\times}^{\oplus} \mathcal{B}_t^\lambda(\mathbb{C}^{2n}) \, e^{2t\lambda^2} \, d\lambda \, ,$$

denoted by the same symbol. The discussion leading up to (4.1.3) shows that $\mathcal{J}_t$ is $\mathbb{H}$-equivariant.

It remains to show that $\mathcal{J}_t$ is onto. For that observe if

$$f(\mathbf{x}, \mathbf{u}, \xi) = F(\mathbf{x}, \mathbf{u})\varphi(\xi)$$

for Schwartz functions $F \in S(\mathbb{R}^{2n})$, $\varphi \in \mathbb{S}(\mathbb{R})$, then

$$(\mathcal{H}_t(f))^\lambda(\mathbf{x}, \mathbf{u}) = \hat{\varphi}(\lambda) e^{-t\lambda^2} H_t^\lambda(F)(\mathbf{x}, \mathbf{u}) \, .$$

¿From that the surjectivity of $\mathcal{J}_t$ easily follows. □

6. The image of $\mathcal{H}_t$ as a sum of weighted Bergman spaces

In this section we prove the main result of this paper: $\operatorname{im} \mathcal{H}_t = \mathcal{B}_t^+(\mathbb{H}_\mathbb{C}) \oplus \mathcal{B}_t^-(\mathbb{H}_\mathbb{C})$ is a direct sum of two weighted Bergman spaces. Very surprisingly, the corresponding weight functions W_t^+ and W_t^- attain also negative values (see the phenomenon explained in Example 3.1).

We will begin our discussion by showing that $\operatorname{im} \mathcal{H}_t$ is not a weighted Bergman space corresponding to a non-negative weight function. This will lead naturally to the definition of the partial weight functions W_t^+ and W_t^- and to a proof of the main theorem.

6.1. **Non-existence of a non-negative weight function.** The goal of this subsection is to discuss the non-existence of a non-negative weight function W_t on $\mathbb{H}_\mathbb{C}$ such that

$$\|f\|^2 = \int_{\mathbb{H}_\mathbb{C}} |\mathcal{H}_t(f)(z)|^2 W_t(z) \, dz \tag{6.1.1}$$

holds for all $f \in L^2(\mathbb{H})$. In other words, $\operatorname{im} \mathcal{H}_t$ is not a weighted Bergman space corresponding to a non-negative weight function W_t. Subject to the natural assumption that W_t is $\mathbb{H}$-invariant, this will be established in Theorem 6.2 below.

Recall that we identify $\mathbb{H}_\mathbb{C}$ with $\mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}$. If $\mathbf{z} = \mathbf{x} + i\mathbf{y}, \mathbf{w} = \mathbf{u} + i\mathbf{v}, \zeta = \xi + i\eta$ then $(\mathbf{z}, \mathbf{w}, \zeta) = he^{iX}$ with $X = (\mathbf{y}, \mathbf{v}, \eta + \frac{1}{2}(\mathbf{x} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{y}))$ and $h = (\mathbf{x}, \mathbf{u}, \xi)$.

Suppose that (6.1.1) holds. As $\mathcal{H}_t$ is $\mathbb{H}$-equivariant, it is natural to assume that $W_t(he^{iX}) = W_t(e^{iX})$ for all $h \in \mathbb{H}$. In coordinates $(\mathbf{z}, \mathbf{w}, \zeta)$ this means that

$$W_t(\mathbf{x} + i\mathbf{y}, \mathbf{u} + i\mathbf{v}, \xi + i\eta) = W_t\left(i\mathbf{y}, i\mathbf{v}, i\eta + \frac{i}{2}(\mathbf{x} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{y})\right) . \tag{6.1.2}$$

Thus the weight function is uniquely determined by its restriction to $(i\mathbf{y}, i\mathbf{v}, i\eta)$. Furthermore W_t is independent of the ξ variable. Hence (6.1.1) reads as

$$\|f\|^2 = \int_{\mathbb{H}_\mathbb{C}} |\mathcal{H}_t(f)(\mathbf{z}, \mathbf{w}, \zeta)|^2 W_t(\mathbf{z}, \mathbf{w}, i\eta) \, d\mathbf{z} \, d\mathbf{w} \, d\zeta \tag{6.1.3}$$

Proposition 6.1. *Let $W_t(\mathbf{z},\mathbf{w},i\eta)$ be a non-negative measurable function on $\mathbb{H}_\mathbb{C}$. If (6.1.3) holds for all $f \in L^2(\mathbb{H})$, then it is necessary that W_t satisfies*

$$W_t^\lambda(\mathbf{z},\mathbf{w}) = e^{-2t\lambda^2}\int_\mathbb{R} e^{2\lambda\eta}W_t(\mathbf{z},\mathbf{w},i\eta)\,d\eta \tag{6.1.4}$$

for all $\lambda \in \mathbb{R}^\times$ and W_t^λ the function given in (4.2.2).

Proof. Write

$$\mathcal{W}_t^\lambda(\mathbf{z},\mathbf{w}) = e^{-2t\lambda^2}\int_\mathbb{R} e^{2\lambda\eta}W_t(\mathbf{z},\mathbf{w},i\eta)\,d\eta\ .$$

We have to show that $W_t^\lambda = \mathcal{W}_t^\lambda$.

It follows from (2.2.2) that

$$\int_\mathbb{R} k_t^\sim(\mathbf{z},\mathbf{w},\xi+i\eta)e^{i\lambda\xi}\,d\xi = e^{\lambda\eta}e^{-t\lambda^2}p_t^\lambda(\mathbf{z},\mathbf{w})\ .$$

An easy calculation shows that

$$\begin{aligned}\int_\mathbb{R} \mathcal{H}_t(f)(\mathbf{z},\mathbf{w},\xi+i\eta)e^{i\lambda\xi}\,d\xi &= e^{\lambda\eta}e^{-t\lambda^2}(f^\lambda *_\lambda p_t^\lambda)(\mathbf{z},\mathbf{w})\\ &= e^{\lambda\eta}e^{-t\lambda^2}H_t^\lambda(f^\lambda)(\mathbf{z},\mathbf{w})\ .\end{aligned} \tag{6.1.5}$$

Therefore, upon applying Plancherel theorem in the ξ-variable, the equation (6.1.3) becomes

$$\|f\|^2 = \int_\mathbb{R}\int_{\mathbb{C}^{2n}}\int_\mathbb{R} |H_t^\lambda(f^\lambda)(\mathbf{z},\mathbf{w})|^2 e^{-2t\lambda^2}e^{2\lambda\eta}W_t(\mathbf{z},\mathbf{w},i\eta)\,d\eta\,d\mathbf{x}\,d\mathbf{u}\,d\mathbf{y}\,d\mathbf{v}\,d\lambda\ .$$

Here we applied Fubini's theorem which is justified as W_t is by assumption non-negative. Employing the definition of $\mathcal{W}_t$ we therefore get

$$\int_\mathbb{R}\int_{\mathbb{R}^{2n}} |f^\lambda(\mathbf{x},\mathbf{u})|^2\,d\mathbf{x}\,d\mathbf{u}\,d\lambda = \int_\mathbb{R}\int_{\mathbb{C}^{2n}} |H_t^\lambda(f^\lambda)(\mathbf{z},\mathbf{w})|^2\mathcal{W}_t^\lambda(\mathbf{z},\mathbf{w})\,d\mathbf{x}\,d\mathbf{u}\,d\mathbf{y}\,d\mathbf{v}\,d\lambda\,.$$

Let now φ be a Schwartz class function on $\mathbb{R}$ with unit L^2-norm and define f by $f(\mathbf{x},\mathbf{u},\xi) = \widehat{\varphi}(\xi)F(\mathbf{x},\mathbf{u})$ with $F \in L^2(\mathbb{R}^{2n})$. Then $f^\lambda(\mathbf{x},\mathbf{u}) = \varphi(\lambda)F(\mathbf{x},\mathbf{u})$ and $H_t^\lambda(f^\lambda) = \varphi(\lambda)H_t^\lambda(F)$. For such f the above displayed equation becomes

$$\int_{\mathbb{R}^{2n}} |F(\mathbf{x},\mathbf{u})|^2\,d\mathbf{x}\,d\mathbf{u} = \int_\mathbb{R}\int_{\mathbb{C}^{2n}} |\varphi(\lambda)|^2|H_t^\lambda(F)(\mathbf{z},\mathbf{w})|^2\mathcal{W}_t^\lambda(\mathbf{z},\mathbf{w})\,d\mathbf{x}\,d\mathbf{u}\,d\mathbf{y}\,d\mathbf{v}\,d\lambda\,. \tag{6.1.6}$$

¿From (6.1.6) it is easy to see that for every $\lambda \neq 0$ and all $F \in L^2(\mathbb{R}^{2n})$

$$\int_{\mathbb{R}^{2n}} |F(\mathbf{x},\mathbf{u})|^2\,d\mathbf{x}\,d\mathbf{u} = \int_{\mathbb{C}^{2n}} |H_t^\lambda(F)(\mathbf{z},\mathbf{w})|^2\mathcal{W}_t^\lambda(\mathbf{z},\mathbf{w})\,d\mathbf{x}\,d\mathbf{u}\,d\mathbf{y}\,d\mathbf{v}\,.$$

By Lemma 4.7, the weight function $\mathcal{W}_t^\lambda$ is given by (4.2.2). □

Theorem 6.2. *There is no non-negative left $\mathbb{H}$-invariant weight function W_t for which (6.1.3) holds for all $f \in L^2(\mathbb{H})$, i.e. $\operatorname{im}\mathcal{H}_t$ is not a weighted Bergman spaces corresponding to a left $\mathbb{H}$-invariant non-negative weight function.*

Proof. By (6.1.2), W_t is uniquely determined by its restriction to $(i\mathbf{y},i\mathbf{v},i\eta)$. By (4.2.2) and (6.1.4),

$$\int_\mathbb{R} e^{2\lambda\eta}W_t(i\mathbf{y},i\mathbf{v},i\eta)\,d\eta = e^{2t\lambda^2}p_{2t}^\lambda(2\mathbf{y},2\mathbf{v}) \qquad (\lambda \in \mathbb{R}^\times)\,.$$

If W_t were non-negative, then for fixed $\mathbf{y},\mathbf{v}$ and λ the function $\eta \mapsto e^{2\lambda\eta}W_t(i\mathbf{y},i\mathbf{v},i\eta)$ would belong to $L^1(\mathbb{R})$. Consequently, we would have

$$\int_\mathbb{R} e^{2(\lambda+is)\eta}W_t(i\mathbf{y},i\mathbf{v},i\eta)\,d\eta = e^{2t(\lambda+is)^2}p_{2t}^{\lambda+is}(2\mathbf{y},2\mathbf{v})\,. \tag{6.1.7}$$

The left hand side of (6.1.7) would be holomorphic in $\lambda + is$ since for every $n \in \mathbb{N}_0$ there exists an $\varepsilon > 0$ such that $|\eta|^n e^{2\lambda\eta} W_t(i\mathbf{y}, i\mathbf{v}, i\eta) \le e^{2\lambda\eta+\varepsilon|\eta|} W_t(i\mathbf{y}, i\mathbf{v}, i\eta)$. However, the right side of (6.1.7) is holomorphic only for $\lambda \neq 0$. If $\lambda = 0$, it becomes

$$p_{2t}^{is}(2\mathbf{y}, 2\mathbf{v}) = c_n \left(\frac{s}{\sin(2st)}\right)^n e^{-s(\cot 2st)(\mathbf{y}^2+\mathbf{v}^2)},$$

which has an essential singularity at the points $s \in \mathbb{Z}^\times(\pi/t)$. Therefore there is no non-negative W_t that will satisfy (6.1.4) or (6.1.3). □

6.2. **The partial weight functions W_t^+ and W_t^-.** Recall the twisted weight function W_t^λ from (4.2.2).

Let $\lambda > 0$ and define a function W_t^+ on $\mathbb{H}_\mathbb{C}$ by

$$W_t^+(\mathbf{z}, \mathbf{w}, \zeta) = \int_\mathbb{R} e^{2t(\lambda+\frac{i}{2}s)^2} e^{-2\eta(\lambda+\frac{i}{2}s)} W_t^{\lambda+\frac{i}{2}s}(\mathbf{z}, \mathbf{w})\, ds. \tag{6.2.1}$$

It is easy to see that W_t^+ is well-defined. Notice that W_t^+ does not depend on ξ. In Proposition 6.3 below we will show that W_t^+ is independent of the choice of $\lambda > 0$.

Proposition 6.3. *The function W_t^+ satisfies the following properties:*
(i) *W_t^+ is independent of the choice of $\lambda > 0$. In particular,*

$$W_t^+(\mathbf{z}, \mathbf{w}, \zeta) = \lim_{\lambda\to 0^+} \int_\mathbb{R} e^{2t(\lambda+\frac{i}{2}s)^2} e^{-2\eta(\lambda+\frac{i}{2}s)} W_t^{\lambda+\frac{i}{2}s}(\mathbf{z}, \mathbf{w})\, ds.$$

(ii) *Let $a > 0$ and $Q \subseteq \mathbb{C}^{2n}$ be a compact set. Then there exists a constant $C = C(Q, a) > 0$ such that for all $\varepsilon \in [a^{-1}, a]$ and $\xi \in \mathbb{R}$*

$$\sup_{(\mathbf{z},\mathbf{w})\in Q} \int_\mathbb{R} \left|e^{2\varepsilon\eta} W_t^+(\mathbf{z}, \mathbf{w}, \xi + i\eta)\right| d\eta \le C\,.$$

(iii) *W_t^+ satisfies (6.1.4) with $\lambda > 0$, i.e.*

$$W_t^\lambda(\mathbf{z}, \mathbf{w}) = e^{-2t\lambda^2} \int_\mathbb{R} e^{2\eta\lambda} W_t^+(\mathbf{z}, \mathbf{w}, i\eta)\, d\eta \tag{6.2.2}$$

for $\lambda > 0$.
(iv) *W_t^+ is real valued and left $\mathbb{H}$-invariant.*

Proof. (i) Let $\lambda > 0$. We have to show that

$$W_t^+(\mathbf{z}, \mathbf{w}, \zeta) = \int_\mathbb{R} e^{2t(\lambda+\frac{i}{2}s)^2} e^{-2\eta(\lambda+\frac{i}{2}s)} W_t^{\lambda+\frac{i}{2}s}(\mathbf{z}, \mathbf{w})\, ds$$

is independent of the choice of $\lambda > 0$. This will be a consequence of Cauchy's theorem. Indeed, let us denote the right hand side by $I(\lambda)$. For $R > 0$ and $\lambda_2 > \lambda_1 > 0$, let Γ_R be the contour consisting four lines, $\Gamma_R(\lambda_1) := \{\lambda_1 + is/2 : -2R < s < 2R\}$, $\gamma_{-R} = \{\lambda - iR : \lambda_1 \le \lambda \le \lambda_2\}$, $\Gamma_R(\lambda_2) = \{\lambda_2 + is/2 : -2R < s < 2R\}$ and $\gamma_R = \{\lambda + iR : \lambda_1 \le \lambda \le \lambda_2\}$, going counterclockwise. As $R \to \infty$, the integral on $\Gamma_R(\lambda)$ becomes $I(\lambda)$. Cauchy's theorem shows that

$$\int_{\Gamma_R} e^{-2\eta z} e^{2tz^2} W_t^z(\mathbf{z}, \mathbf{w})\, dz = 0.$$

It is easy to see that $|\sinh(\lambda + iR)t| \ge \sinh(\lambda t)$ and $|\cosh(\lambda + iR)t| \le \cosh(\lambda t)$. Thus,

$$|p_{2t}^{\lambda\pm iR}(2\mathbf{y}, 2\mathbf{v})| \le \left(\frac{\lambda + R}{\sinh \lambda t}\right)^n e^{(\lambda+R)\coth(\lambda t)(|\mathbf{y}|^2+|\mathbf{v}|^2)}.$$

Together with $|e^{2t(\lambda\pm iR)^2}| = e^{2t\lambda^2} e^{-2tR^2}$, this shows that the integrals on γ_{-R} and on γ_R go to zero as $R \to +\infty$. Thus, taking $R \to \infty$ shows that $I(\lambda_1) = I(\lambda_2)$. This completes the proof of (i).

(ii) It follows from (i) that W_t^+ satisfies the bound

$$|W_t^+(\mathbf{z},\mathbf{w},\xi+i\eta)| \le e^{-2\eta\lambda}e^{2t\lambda^2}\int_{\mathbb{R}} e^{-\frac{1}{2}ts^2}\left|W_t^{\lambda+\frac{i}{2}s}(\mathbf{z},\mathbf{w})\right| \, ds \ .$$

for any $\lambda > 0$. Notice that the integral on the right is independent of η. Thus if we let $\lambda > \epsilon$ if $\eta > 0$ and $\lambda < \epsilon$ if $\eta < 0$, we see that $\eta \mapsto e^{2\varepsilon\eta}W_t^+(\mathbf{z},\mathbf{w},\xi+i\eta)$ is integrable. This implies (ii).

(iii) This is immediate from the definition (6.2.1) and Fourier inversion (which is justified by (ii)). In fact, we have

$$W_t^+(\mathbf{z},\mathbf{w},\zeta) = e^{-2\eta\lambda}\int_{\mathbb{R}} e^{-i\eta s}e^{2t(\lambda+\frac{i}{2}s)^2}W_t^{\lambda+\frac{i}{2}s}(\mathbf{z},\mathbf{w})\, ds$$

and so

$$\int_{\mathbb{R}} e^{2\lambda\eta}W_t^+(\mathbf{z},\mathbf{w},\xi+i\eta)e^{i\eta s}\, d\eta = e^{2t(\lambda+\frac{i}{2}s)^2}W_t^{\lambda+\frac{i}{2}s}(\mathbf{z},\mathbf{w}) \ .$$

Setting $s = 0$ gives the the stated result.

(iv) We first show that W_t^+ is real valued. In fact, taking the conjugate of the integral (6.2.1) and then changing variable $s \to -s$ shows that the weight function W_t^+ is real. Finally, the fact that W_t^λ is twisted-translation invariant forces that W_t^+ is left $\mathbb{H}$-invariant. □

The function W_t^+ has a natural counterpart W_t^-. For $\lambda < 0$ we define W_t^- by

$$W_t^-(\mathbf{z},\mathbf{w},\zeta) = \int_{\mathbb{R}} e^{2t(\lambda+\frac{i}{2}s)^2}e^{-2\eta(\lambda+\frac{i}{2}s)}W_t^{\lambda+\frac{i}{2}s}(\mathbf{z},\mathbf{w})\, ds. \tag{6.2.3}$$

It is more or less obvious that W_t^- satisfies the same properties as W_t^+ listed in Proposition (6.3), i.e. W_t^- is independent of the choice of $\lambda < 0$ etc. In fact, a simple change of variable in the integral and the fact that $p_t^\lambda(2\mathbf{y},2\mathbf{v})$ is even in λ leads to the relation

$$W_t^+(\mathbf{z},\mathbf{w},i\eta) = W_t^-(\mathbf{z},\mathbf{w},-i\eta).$$

We refer to W_t^+ and W_t^- as the *partial weight functions*. Their importance will become clear in the next subsection.

Remark 6.1. We will show in the appendix that both W_t^+ and W_t^- attain positive and negative values. In addition we shall discuss their oscillatory behaviour. A more heuristic explanation of these phenomena might be the following: Both $W_t^+(i\mathbf{y},i\mathbf{v},i\eta)$ and $W_t^-(i\mathbf{y},i\mathbf{v},i\eta)$ satisfy the differential equation

$$2\frac{\partial}{\partial t}U = \left(\Delta + (1-|\mathbf{y}|^2-|\mathbf{v}|^2)\frac{\partial^2}{\partial\eta^2}\right)U. \tag{6.2.4}$$

Indeed, this follows from a straightforward computation starting from

$$\frac{\partial}{\partial t}p_t^\lambda(\mathbf{y},\mathbf{v}) = \left(\Delta - \frac{\lambda^2}{4}(|\mathbf{y}|^2+|\mathbf{v}|^2)\right)p_t^\lambda(\mathbf{y},\mathbf{v})$$

for all $\lambda \neq 0$ (see [7]). We note that the differential equation (6.2.4) is parabolic only for $|\mathbf{y}|^2+|\mathbf{v}|^2 < 1$. If $|\mathbf{y}|^2+|\mathbf{v}|^2 > 1$, then the right hand side of (6.2.4) resembles a wave equation which in turn might explain the oscillatory behaviour of W_t^+ and W_t^- on the large scale.

6.3. **The image of the heat kernel transform.** The objective of this section is to prove our main theorem: $\operatorname{im}\mathcal{H}_t = \mathcal{B}_t^+(\mathbb{H}_\mathbb{C}) \oplus \mathcal{B}_t^-(\mathbb{H}_\mathbb{C})$ is a sum of two weighted Bergman spaces.

To exhibit the Bergman structure of the spaces $\mathcal{B}_t^+(\mathbb{H}_\mathbb{C})$ and $\mathcal{B}_t^-(\mathbb{H}_\mathbb{C})$ needs some preparation.

First we define subspaces of $L^2(\mathbb{H})$ by

$$L^2_+(\mathbb{H}) = \{f \in L^2(\mathbb{H}) : f^\lambda = 0, \quad \lambda \le 0\}$$

and

$$L^2_-(\mathbb{H}) = \{f \in L^2(\mathbb{H}) : f^\lambda = 0, \quad \lambda \ge 0\}.$$

Notice that both subspaces are $\mathbb{H}$-invariant and

$$L^2(\mathbb{H}) = L^2_+(\mathbb{H}) \oplus L^2_-(\mathbb{H}).$$

Next we recall some facts on the heat kernel transform on the real line. The heat kernel on $\mathbb{R}$ is given by

$$q_t(x) = (4\pi t)^{-\frac{1}{2}} e^{-\frac{x^2}{4t}} \qquad (x \in \mathbb{R}).$$

Define a weighted Bergman space on $\mathbb{C}$ by

$$\mathcal{B}_t(\mathbb{C}) = \{g \in \mathcal{O}(\mathbb{C}) : \|g\|^2 = \int_{\mathbb{C}} |g(x+iy)|^2 e^{-\frac{y^2}{2t}}\, dx\, dy < \infty\}$$

and recall that the mapping

$$h_t : L^2(\mathbb{R}) \to \mathcal{B}_t(\mathbb{C}), \quad g \mapsto (f * q_t)^{\sim}$$

is (up to scale) an $\mathbb{R}$-equivariant isometric isomorphism.

Set $\mathbb{R}^+ = (0,\infty)$ and $\mathbb{R}^- = (-\infty, 0)$. With $L^2_+(\mathbb{R}) = \{f \in L^2(\mathbb{R}) : \operatorname{supp}\hat{f} \subseteq \mathbb{R}^+\}$ and $L^2_-(\mathbb{R}) = \{f \in L^2(\mathbb{R}) : \operatorname{supp}\hat{f} \subseteq \mathbb{R}^-\}$ we have $L^2(\mathbb{R}) = L^2_+(\mathbb{R}) \oplus L^2_-(\mathbb{R})$. Finally, let us write $\mathcal{B}_t^{\pm}(\mathbb{C}) = h_t(L^2_{\pm}(\mathbb{R}))$. Clearly we have $\mathcal{B}_t(\mathbb{C}) = \mathcal{B}_t^+(\mathbb{C}) \oplus \mathcal{B}_t^-(\mathbb{C})$.

Let $R > 0$. Denote by B_R the open ball centered at 0 with radius R in $\mathbb{C}^n$. Further define $K_R = B_R \times B_R \times \mathbb{C} \subseteq \mathbb{H}_\mathbb{C}$ and note that $\bigcup_{R>0} K_R = \mathbb{H}_\mathbb{C}$.

We define $\mathcal{V}_t^+(\mathbb{H}_\mathbb{C})$ as the vector space consisting of all holomorphic functions F on $\mathbb{H}_\mathbb{C}$ such that

- $F|_{K_R} \in L^2(K_R, |W_t^+| dz)$ for all $R > 0$,
- $\lim_{R\to\infty} \int_{K_R} |F(z)|^2 W_t^+(z)\, dz < \infty$,
- $F(\mathbf{z}, \mathbf{w}, \cdot) \in \mathcal{B}_t^+(\mathbb{C})$ for all $\mathbf{z}, \mathbf{w} \in \mathbb{C}^n$.

We endow $\mathcal{V}_t^+(\mathbb{H}_\mathbb{C})$ with a sesquilinear bracket

$$\langle F, G\rangle_+ = \lim_{R\to\infty} \int_{K_R} F(z)\overline{G(z)} W_t^+(z)\, dz\,, \tag{6.3.1}$$

for $F, G \in \mathcal{V}_t^+(\mathbb{H}_\mathbb{C})$. Similarly one defines $\mathcal{V}_t^-(\mathbb{H}_\mathbb{C})$ and $\langle\cdot,\cdot\rangle_-$.

Remark 6.2. One might ask if one cannot define $\mathcal{V}_t^{\pm}(\mathbb{H}_\mathbb{C})$ in a simpler manner: avoid the exhaustion $\bigcup_{R>0} K_R = \mathbb{H}_\mathbb{C}$ and just require $|F|^2 W_t^{\pm}$ to be absolutely integrable on $\mathbb{H}_\mathbb{C}$. However, this will not work, and the reason for this is the bad oscillatory behaviour of $W_t^{\pm}$ (see the appendix).

A priori it is not clear that $\langle F, F\rangle_{\pm} \geq 0$. This will be shown next.

Lemma 6.4. *The bracket $\langle\cdot,\cdot\rangle_{\pm}$ induces on $\mathcal{V}_t^{\pm}(\mathbb{H}_\mathbb{C})$ a pre Hilbert space structure.*

Proof. It is sufficient to treat the case "+" only. All what is left to show is that $\langle F, F\rangle_+ \geq 0$ and $\langle F, F\rangle_+ = 0$ if and only if $F = 0$.

Fix $F \in \mathcal{V}_t^+(\mathbb{H}_\mathbb{C})$. Then $F(\mathbf{z}, \mathbf{w}, \cdot) \in \mathcal{B}_t^+(\mathbb{C})$ implies the existence of a function $g(\mathbf{z}, \mathbf{w}, \cdot) \in L^2_+(\mathbb{R})$ such that

$$F(\mathbf{z}, \mathbf{w}, \zeta) = h_t(g(\mathbf{z}, \mathbf{w}, \cdot))(\zeta) = \int_{\mathbb{R}} g(\mathbf{z}, \mathbf{w}, s) q_t(\zeta - s)\, ds\,.$$

Therefore, up to an irrelevant constant only depending on t, the following equality holds:

$$\int_{\mathbb{R}} F(\mathbf{z}, \mathbf{w}, \xi + i\eta) e^{i\lambda\xi}\, d\xi = e^{\lambda\eta} e^{-t\lambda^2} g^\lambda(\mathbf{z}, \mathbf{w})\ .$$

Consequently, as W_t^+ is independent of ξ,

$$\int_{K_R} |F(z)|^2 W_t^+(z)\, dz = \int_{B_R^2} \int_0^\infty \int_{\mathbb{R}} |g^\lambda(\mathbf{z}, \mathbf{w})|^2 e^{2\lambda\eta} e^{-2t\lambda^2} W_t^+(\mathbf{z}, \mathbf{w}, i\eta)\, d\eta\, d\lambda\, d\mathbf{z}\, d\mathbf{w}\ .$$

In view of (6.2.2) we thus get

$$\int_{K_R} |F(z)|^2 W_t^+(z)\,dz = \int_{B_R}\int_{B_R}\int_0^\infty |g^\lambda(\mathbf{z},\mathbf{w})|^2 W_t^\lambda(\mathbf{z},\mathbf{w})\,d\lambda\,d\mathbf{z}\,d\mathbf{w}\ .$$

But $W_t^\lambda \geq 0$ and so

$$\langle F,F\rangle_+ = \lim_{R\to\infty}\int_{B_R}\int_{B_R}\int_0^\infty |g^\lambda(\mathbf{z},\mathbf{w})|^2 W_t^\lambda(\mathbf{z},\mathbf{w})\,d\lambda\,d\mathbf{z}\,d\mathbf{w} \geq 0$$

and $\langle F,F\rangle_+ = 0$ if and only if $g^\lambda = 0$ for all λ, i.e. $F=0$. This completes the proof of the lemma. □

Let us write $\mathcal{H}_t^\pm$ for the heat kernel transform when restricted to $L^2_\pm(\mathbb{H})$. Define Hilbert spaces of holomorphic functions by $\mathcal{B}_t^\pm(\mathbb{H}_\mathbb{C}) = \operatorname{im}\mathcal{H}_t^\pm$ and note that

$$\operatorname{im}\mathcal{H}_t = \mathcal{B}_t^+(\mathbb{H}_\mathbb{C}) \oplus \mathcal{B}_t^-(\mathbb{H}_\mathbb{C})\,.$$

Let us remark that this decomposition can be also achieved using the Hilbert transform in the last variable.

Theorem 6.5. *Let $t>0$. Then $\mathcal{B}_t^\pm(\mathbb{H}_\mathbb{C})$ is the Hilbert completion of $(\mathcal{V}_t^\pm(\mathbb{H}_\mathbb{C}), \langle\cdot,\cdot\rangle_\pm)$ with $\langle\cdot,\cdot\rangle_\pm$ given by (6.3.1).*

Proof. We restrict ourselves to the "+"-case. Define a dense subspace of $L^2_+(\mathbb{H})^0$ of $L^2_+(\mathbb{H})$ by

$$L^2_+(\mathbb{H})^0 = \{f\in L^2_+(\mathbb{H}) : \lambda\mapsto f^\lambda \ \text{ compactly supported in } (0,\infty)\}$$

We claim that $\mathcal{H}_t(L^2_+(\mathbb{H})^0) \subset \mathcal{V}_t^+(\mathbb{H})$. Let $f\in L^2_+(\mathbb{H})^0$ and set $F=\mathcal{H}_t^+(f)$. Choose $a>0$ such that $f^\lambda = 0$ for λ outside of (a^{-1},a). Proceeding as in Lemma 6.4 and using the estimate Proposition 6.3 (ii) we see that $\mathcal{H}_t^+(f)$ satisfies the first condition in the definition of $\mathcal{V}_t^+(\mathbb{H}_\mathbb{C})$. Furthermore (6.1.5) implies that

$$\int_{K_R} |F(z)|^2 W_t^+(z)\,dz = \int_{B_R}\int_{B_R}\int_0^\infty |H_t^\lambda(f^\lambda)(\mathbf{z},\mathbf{w})|^2 W_t^\lambda(\mathbf{z},\mathbf{w})\,d\lambda\,d\mathbf{z}\ d\mathbf{w}\,.$$

As $W_t^\lambda\geq 0$, it hence follows that $\int_{K_R}|F(z)|^2W_t^+(z)\,dz$ is increasing in R. Similar reasoning as in (6.1.6) now shows that

$$\lim_{R\to\infty}\int_{K_R}|F(z)|^2 W_t^+(z)\,dz = \|f\|^2 < \infty\ .$$

Furthermore, for fixed $(\mathbf{z},\mathbf{w})$ we have $F(\mathbf{z},\mathbf{w},\cdot)\in\mathcal{B}_t(\mathbb{C})$ as a quick inspection of (6.1.5) shows. This proves our claim.

As a byproduct of our reasoning above we have shown that $\mathcal{H}_t^+ : L_2^+(\mathbb{H})^0 \to \mathcal{V}_t^+(\mathbb{H})$ is an isometric map. It remains to verify that each function $F\in\mathcal{V}_t^+(\mathbb{H}_\mathbb{C})$ can be written as $\mathcal{H}_t^+(f)$ for some $f\in L^2_+(\mathbb{H})$. Let $g^\lambda(\mathbf{z},\mathbf{w})$ be the function associated to F as in the proof of Lemma 6.4. Then for almost all λ there exists an $f^\lambda\in L^2(\mathbb{R}^{2n})$ such that $g^\lambda = H_t^\lambda(f^\lambda)$. It is easy to check that the prescription

$$f(\mathbf{x},\mathbf{u},\xi) = \int_\mathbb{R} e^{-i\lambda\xi} f^\lambda(\mathbf{x},\mathbf{u})\,d\lambda$$

defines a function in $L^2_+(\mathbb{H})$ such that $\mathcal{H}_t^+(f) = F$. This completes the proof of the theorem. □

7. Appendix: The oscillatory behaviour of the partial weight functions

This appendix is devoted to a closer study of the partial weight functions $W_t^{\pm}$. In particular we will detect "good" and "bad" directions for $W_t^{\pm}$, meaning rays in $H_{\mathbb{C}}$ on which $W_t^{\pm}$ stays positive resp. starts to oscillate. It is no loss of generality to treat the case of W_t^+ only.

We start with an expicit formula for the function W_t^+. Recall that the kernel p_t^λ admits an expansion of the type [7, p. 85]

$$p_t^\lambda(\mathbf{y},\mathbf{v}) = (2\pi)^{-n}\lambda^n \sum_{k=0}^{\infty} e^{-(2k+n)|\lambda|t} L_k^{n-1}(\frac{|\lambda|}{2}(|\mathbf{y}|^2+|\mathbf{v}|^2))e^{-\frac{|\lambda|}{4}(|\mathbf{y}|^2+|\mathbf{v}|^2)},$$

where L_k^{n-1} is the Laguerre polynomial of degree k with parameter $n-1$, which can be extended analytically to $\lambda+is$ for $\lambda\neq 0$. Let $H_k(x)$ denote the Hermite polynomial, which can be defined by Rodrigue's formula $H_k(x) = (-1)^k e^{x^2}\frac{d^k}{dx^k}e^{-x^2}$.

Proposition 7.1. *For $n=1$ and $\beta := (y^2+v^2)$,*

$$W_{t/2}^+(iy,iv,i\eta) = c\sqrt{\frac{\pi}{t}}\sum_{k=0}^{\infty} e^{-\frac{1}{4}\mu_k^2}$$

$$\times\left[\mu_k\sum_{j=0}^{k}\frac{1}{j!}\left(\frac{\beta}{\sqrt{t}}\right)^j H_j(-\mu_k\sqrt{t})\binom{k}{j} + \frac{\beta}{t}\sum_{j=0}^{k-1}\frac{1}{j!}\left(\frac{\beta}{\sqrt{t}}\right)^j H_j(-\mu_k\sqrt{t})\binom{k}{j+1}\right]$$

where $\mu_k = (2k+1+(2\eta+\beta)/t)/2$.

Proof. The integral formula of W_t^+ shows that, for a fixed $\lambda>0$,

$$\begin{aligned}
W_{t/2}^+(iy,iv,i\eta) &= c\int_{\mathbb{R}} e^{t(\lambda+is)^2}e^{-2\eta(\lambda+is)}p_t^{\lambda+is}(2y,2v)\,ds\\
&= c\,e^{-(t+2\eta)\lambda+t\lambda^2-\lambda\beta}\sum_{k=0}^{\infty}e^{-2kt\lambda}\int_{\mathbb{R}}(\lambda+is)e^{-ts^2}L_k^{n-1}(2\lambda\beta+2is\beta)\\
&\qquad\qquad\times e^{-ist(2\lambda-2k-\beta-2\eta/t)}e^{-is\beta}\,ds\\
&= c\,e^{-(t+2\eta)\lambda+t\lambda^2-\lambda\beta}\sum_{k=0}^{\infty}e^{-2kt\lambda}(\lambda+\partial_\alpha)L_k^{n-1}(2\beta(\lambda+\partial_\alpha))\int_{\mathbb{R}}e^{i\alpha s}e^{-ts^2}\,ds\\
&= c\sqrt{\frac{\pi}{t}}e^{-(t+2\eta)\lambda+t\lambda^2-\lambda\beta}\sum_{k=0}^{\infty}e^{-2kt\lambda}(\lambda+\partial_\alpha)L_k^{n-1}(2\beta(\lambda+\partial_\alpha))e^{-\frac{1}{4t}\alpha^2}
\end{aligned}$$

where $\alpha = t(2\lambda-2k-1-(2\eta+\beta)/t)$ and $\partial_\alpha = \partial/\partial_\alpha$.

Using the Rodrigue's formula of the Hermite polynomials and the explicit formula of L_k^γ, we conclude that

$$\begin{aligned}
L_k^{n-1}(2\beta(\lambda+\partial_\alpha))e^{-\frac{1}{4t}\alpha^2} &= \sum_{l=0}^{k}\frac{(-k)_l}{l!l!}(2\beta)^l(\lambda+\partial_\alpha)^l e^{-\frac{1}{4t}\alpha^2}\\
&= \sum_{l=0}^{k}\frac{(-k)_l}{l!l!}(2\beta)^l\sum_{j=0}^{l}\binom{l}{j}\frac{1}{(2\sqrt{t})^j}(-1)^j e^{-\frac{1}{4t}\alpha^2}H_j\left(\frac{\alpha}{2\sqrt{t}}\right)\\
&= e^{-\frac{1}{4t}\alpha^2}\sum_{j=0}^{k}\frac{1}{j!}\frac{\beta}{(\sqrt{t})^j}H_j\left(\frac{\alpha}{2\sqrt{t}}\right)L_{k-j}^j(2\beta\lambda),
\end{aligned}$$

upon changing summations, simplifying and using the explicit formula of L_k^j. Let α be fixed. It turns out that the generating function of the above quantity is given by

$$e^{-\frac{1}{4t}\alpha^2}\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}\frac{1}{j!}\frac{\beta}{(\sqrt{t})^j}H_j\left(\frac{\alpha}{2\sqrt{t}}\right)L_{k-j}^j(2\beta\lambda)\right)s^k=\exp\left[-\frac{2\beta s\mu}{1-s}-\frac{\beta s}{(1-s)\sqrt{t}}\right]$$

where $\alpha = 2t(\lambda-\mu)$. Since the generating function is independent of λ, this shows that the inner sum is in fact independent of λ. We can, in particular, set $\lambda = 0$ in the inner sum and set $\mu = (2k+1+(2\eta+\beta)/t)/2$. Recall that $L_{k-j}^j(0) = \binom{k}{j}$. The change of variable from α to μ also leads to $\partial_\alpha = -\frac{1}{2t}\partial_\mu$. A simple computation then leads to

$$\begin{aligned}(\lambda+\partial_\alpha)L_k^{n-1}(2\beta(\lambda+\partial_\alpha))e^{-\frac{1}{4t}\alpha^2} &= e^{-t(\lambda-\mu)^2}\\ &\times\left[\mu\sum_{j=0}^{k}\frac{1}{j!}\left(\frac{\beta}{\sqrt{t}}\right)^j H_j(-\sqrt{t}\,\mu)\binom{k}{j}+\frac{\beta}{t}\sum_{j=0}^{k-1}\frac{1}{j!}\left(\frac{\beta}{\sqrt{t}}\right)^j H_j(-\sqrt{t}\,\mu)\binom{k}{j+1}\right]\end{aligned}$$

from which the stated formula follows readily. □

We note that the formula proved above shows explicitly that W_t^+ is independent of λ without using the contour integral and Cauchy's theorem.

Proposition 7.2. *The function W_t^+ is positive in a neighborhood of $(0,0,0)$. Furthermore, $W_t^+(0,0,i\eta)$ is non-negative for all η.*

Proof. Setting $\beta = 0$ in the explicit formula of W_t^+ gives

$$W_{t/2}^+(0,0,i\eta) = c\sqrt{\frac{\pi}{t}}\sum_{k=0}^{\infty}e^{-\frac{1}{4}(2k+1+2\eta/t)^2}\left(2k+1+\frac{2\eta}{t}\right),$$

which is clearly positive if $\eta \geq 0$. Furthermore, if $\eta/t = -m$ for $m \in \mathbb{N}$ then the sum can be written as

$$\sum_{k=0}^{\infty}e^{-\frac{t}{4}(2k+1-2m)^2}(2k+1-2m)=\sum_{k=0}^{\infty}e^{-\frac{t}{4}(2k+1)^2}(2k+1)-\sum_{k=1}^{m}e^{-\frac{t}{4}(2k-1)^2}(2k-1)$$

which is strictly positive. Similarly, the sum is strictly positive if $\eta/t = -m - 1/2$. Hence, we are left with the case of $2\eta/t = -2m - 1 + r$, where $0 < r < 1$. In this case, the sum becomes

$$S_m := 2\sum_{k=0}^{\infty}e^{-(k-m+r/2)^2t}(k-m+r/2) = 2\sum_{k=0}^{\infty}e^{-(k+s)^2t}(k+s)-\sum_{k=1}^{m}e^{-(k-s)^2t}(k-s)$$

where $0 < s = r/2 < 1/2$. Set $g_k(s) = e^{-(k+s)^2t}(k+s) - e^{-(k+1-s)^2t}(k+1-s)$. It is easy to see that $g_k'(s) > 0$ for $0 < s < 1$. Hence g_k is increasing. It follows that

$$\sum_{k=0}^{\infty}e^{-\frac{t}{4}(2k+1)^2}(2k+1)-\sum_{k=1}^{\infty}e^{-\frac{t}{4}(2k-1)^2}(2k-1)=\sum_{k=0}^{\infty}g_k(s)\geq\sum_{k=0}^{\infty}g_k(0)=0,$$

from which the stated result follows. □

However, the weight function $W_t^+(iy,iv,i\eta)$ is not non-negative for all (y,v,η). In fact, if $2\eta = -(y^2+v^2)$, then $2\eta+\beta = 0$ and

$$\begin{aligned}W_{t/2}^+(iy,iv,i\eta) = c\sqrt{\frac{\pi}{t}}\sum_{k=0}^{\infty}e^{-(k+\frac{1}{2})^2}&\left[(k+\frac{1}{2})\sum_{j=0}^{k}\frac{1}{j!}\left(\frac{\beta}{\sqrt{t}}\right)^j H_j(-\sqrt{t}(k+\frac{1}{2}))\binom{k}{j}\right.\\ &\left.+\frac{\beta}{t}\sum_{j=0}^{k-1}\frac{1}{j!}\left(\frac{\beta}{\sqrt{t}}\right)^j H_j(-\sqrt{t}(k+\frac{1}{2}))\binom{k}{j+1}\right].\end{aligned}$$

For each fixed t, this is a function of β and it appears to be oscillatory. The graph for $t = 1$ is shown below.

The function oscillates in growing intervals and increasing amplitudes. To demonstrate the oscillatory nature of the function, what we have shown above is the function $W(iy, iv, -i\beta/2)/\log(2+\beta^2)$ without the factor $c\sqrt{\pi}$. It is a function of β, where $\beta = y^2 + v^2$.

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