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February, 25th 2005
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NON-ORIENTABLE SEIFERT SURFACES, STABILISATION AND KNOT INVARIANTS

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Abstract. We show that two non-orientable Seifert surfaces of a knot $K \subset S^3$ are isotopic after taking band-connected sums with Möbius bands and finitely many stabilisations. The proof is based on a non-orientable Thom-Pontjagin construction. We use this to construct invariants of knots and give some applications.

1. Introduction

Any knot $K \subset S^3$ bounds an orientable Seifert surface Σ. The Seifert surface is not unique. However after finitely many *stabilisations*, any two Seifert surfaces are isotopic [2]. Here a stabilisation of Σ is a 1-surgery about an arc γ in S^3 with interior disjoint from Σ and both endpoints in the interior of Σ. This result allows one to construct various invariants of knots.

We consider here the question of uniqueness up to stabilisation for non-orientable Seifert surfaces. Let $K \subset S^3$ be a knot and let Σ be a non-orientable Seifert surface for Σ. Let $N = S^3 - int(\mathcal{N}(K))$ be the knot exterior. We first make some observations.

The longitude $\lambda = \Sigma \cap \partial N$ is a curve isotopic to K which bounds a non-orientable surface $\Sigma \cap N$ in the complement of K. This implies that λ is trivial as an element of $H_1(N, \mathbb{Z}/2\mathbb{Z})$ and hence the linking number of λ and K is even. Unlike the orientable case we cannot conclude that the linking number is zero. We shall call this linking number the *framing* $\mathcal{F}(\Sigma)$ associated to the Seifert surface.

In the case of the standard embedding $\mathcal{M}$ of the Möbius band in S^3 with one twist, the boundary is an unknot U and the associated framing is ± 2, with the sign depending on whether the twist was to the left or the right. Further, if K is an unknot with a non-orientable Seifert surface Σ, we can regard K as $K\#U$ with the Seifert surface $\Sigma\#\mathcal{M}$ obtained by gluing Σ and $\mathcal{M}$ along an arc. It is easy to see that $\mathcal{F}(\Sigma\# M) = \mathcal{F}(\Sigma) \pm 2$. It follows that given an even integer $2k$, by taking band-connected sums with Möbius bands, we can obtain from Σ a Seifert surface with framing $2k$.

Note that stabilisation does not alter the framing of the Seifert surface, and hence if two Seifert surfaces are isotopic after stabilisation, they have the same framing. We show that so long as Seifert surfaces have the same framing, after stabilisation they are isotopic.

Theorem 1.1. *Let $K \subset S^3$ be a knot and let Σ_1 and Σ_2 be non-orientable Seifert surfaces for K with $\mathcal{F}(\Sigma_1) = \mathcal{F}(\Sigma_2)$. Then after finitely many stabilisations, Σ_1 and Σ_2 are isotopic.*

As band-connected sums with Möbius bands can change the framing to any even integer, we obtain the following corollary.

Corollary 1.2. *Let $K \subset S^3$ be a knot and let Σ_1 and Σ_2 be orientable or non-orientable Seifert surfaces for K. Then after taking band-connected sum with unknotted Möbius bands and finitely many stabilisations, Σ_1 and Σ_2 are isotopic.*

Date: April 1, 2005.

1991 *Mathematics Subject Classification.* Primary ; Secondary .

We now outline the proof of Theorem 1.1. We consider the 4-manifold $S^3 \times [0,1]$ and regard Σ_1 and Σ_2 as surfaces in $S^3 \times \{0\}$ and $S^3 \times \{1\}$ respectively. Let F be the surface $\Sigma_1 \cup (K \times [0,1]) \cup \Sigma_2$. We construct a 3-manifold M whose boundary is F using a non-orientable version of the Thom-Pontrjagin construction. Standard Morse theory arguments (very similar to the orientable case) can now be used to complete the proof.

We shall use Theorem 1.1 to construct invariants of knots. To do this we construct a symmetric bilinear linking pairing on the homology of a non-orientable Seifert surface. After appropriate normalisations (in terms of $\mathcal{F}(\Sigma)$ and $\chi(\Sigma)$) these give invariants of knots. We shall see that these co-incide with analogous classical invariants defined using (orientable) Seifert surfaces. In the final section we give some applications.

2. The Thom-Pontrjagin model

Let K and Σ_i, $i = 1,2$ be as in the hypothesis of Theorem 1.1. Let $N = S^3 - int(\mathcal{N}(K))$ be the knot exterior and let $W = N \times [0,1] \subset S^3 \times [0,1]$. Note that as $\mathcal{F}(\Sigma_1) = \mathcal{F}(\Sigma_2)$, we can assume that $\partial N \cap \Sigma_1 = \partial N \cap \Sigma_2$. Let $K' = \partial N \cap \Sigma_1 = \partial N \cap \Sigma_2$. We can identify K with K' and Σ_i with $N \cap \Sigma_i$. Hence $F = \Sigma_1 \cup (K \times [0,1]) \cup \Sigma_2$ is a surface embedded in ∂W. We shall construct a 3-manifold $M \subset W$ with boundary F using a variant of the Thom-Pontrjagin construction.

Consider the triple of spaces $(W, \partial W, F)$. We cannot apply the classical Thom-Pontrjagin construction to this triple as the normal bundle of F in ∂W is not trivial. However we can make a non-orientable Thom-Pontrjagin construction. Namely, let $\mathcal{K}$ be the Klein bottle, $C \subset \mathcal{K}$ be a longitude and let $V = D^2 \tilde{\times} S^1$ be the solid Klein bottle. Consider the triple $(V, \mathcal{K}, C)$. We shall construct a map $\varphi\colon (W, \partial W, F) \to (V, \mathcal{K}, C)$ with $\varphi^{-1}(C) = F$. This is the Thom-Pontrjagin map.

We shall show that we can construct a map $\psi : W \to \mathcal{K}$ with $\varphi|_{\partial W} = \psi|_{\partial W}$. Then $M = \psi^{-1}(C)$ gives the required 3-manifold.

3. Construction of φ

We shall now construct $\varphi\colon (W, \partial W, F) \to (V, \mathcal{K}, C)$ in several steps. We first construct φ on F, then on a neighbourhood of F, then on ∂W and finally on W. By abuse of notation we shall sometimes denote various restrictions of φ as φ.

Observe that as $H_1(N) = \mathbb{Z}$ and $W = N \times [0,1]$, we have an identification $H_1(W) \cong \mathbb{Z} \cong H_1(C) \cong H_1(V)$. We fix such an isomorphism. Assuming K is given an orientation, the isomorphism $H_1(N) = \mathbb{Z}$ is given by taking the linking number with K. In the case of $V = D^2 \tilde{\times} S^1$, we use the natural identification $H_1(V) = H_1(S^1)$ and the standard isomorphism $H_1(S^1) \cong \mathbb{Z}$.

The map on F. As $C = S^1$ is a $K(\mathbb{Z}, 1)$, to define $\varphi\colon F \to C$ it suffices to define the homomorphism $\varphi_* : \pi_1(F) \to \pi_1(C) \cong H_1(C)$. We define this to be the composition $\pi_1(F) \to H_1(F) \to H_1(W) \to H_1(C)$ with the first homomorphism being the Hurewicz map, the second being induced by inclusion and the third the isomorphism fixed in the previous paragraph. We define φ to be the map that induces this orientation.

Lemma 3.1. *The map $\varphi\colon F \to C$ maps orientation reversing curves on F to orientation reversing curves on the Klein bottle.*

Proof. As $\pi_1(F)$ is generated by curves supported in Σ_1 and those supported in Σ_2, it suffices to consider such curves. Let γ be a curve in $\Sigma_1 \subset N \times \{0\}$ that is orientation reversing and identify $N \times \{0\}$ with N. Then after a perturbation γ intersects Σ_1 in a single point, and hence is non-zero as an element of $H_1(N, \mathbb{Z}/2\mathbb{Z}) = H_1(W, \mathbb{Z}/2\mathbb{Z})$. It follows that γ maps to an odd element in $H_1(W)$, hence $H_1(C)$.

But odd elements in $H_1(C)$ are orientation reversing curves on $\mathcal{K}$. The argument for curves in Σ_2 is similar. $\square$

The map on $\mathcal{N}(F)$. We next extend φ to a neighbourhood of F. We first introduce some notation. Let $\tilde{F}$ be the orientable 2-fold cover of F and let $\tilde{C}$ be the connected 2-fold cover of C. Let τ_F and τ_C denote the deck transformations corresponding to these covers. Define involutions α_F and α_C on $\tilde{F}\times[-1,1]$ (respectively $\tilde{C}\times[-1,1]$) by $\alpha_F(x,y)=(\tau_F(x),-y)$ (respectively $\alpha_C(x,y)=(\tau_C(x),-y)$). Let $F\tilde{\times}[-1,1]$ (respectively $C\tilde{\times}[-1,1]$) be the quotient of $\tilde{F}\times[-1,1]$ (respectively $\tilde{C}\times[-1,1]$) by α_F (respectively α_C).

Observe that a regular neighbourhood of F is homeomorphic to $F\tilde{\times}[-1,1]$. We fix such a neighbourhood $\mathcal{N}(F)$. On the other hand the boundary of $C\tilde{\times}[-1,1]$ can be naturally identified with $\tilde{C}$ and the quotient of $C\tilde{\times}[-1,1]$ by the action of τ_C on its boundary gives the Klein bottle $\mathcal{K}$. Let C' denote the image in $\mathcal{K}$ of the boundary of $C\tilde{\times}[-1,1]$.

As φ maps orientation reversing curves to orientation reversing curves, φ extends to a bundle map $\varphi\colon\ F\tilde{\times}[-1,1]\to C\tilde{\times}[-1,1]$. This in turn gives a map $\mathcal{N}(F)=\varphi\colon\ F\tilde{\times}[-1,1]\to\mathcal{K}$ defined on a regular neighbourhood of F. Note that $\varphi^{-1}(C)=F$

The map on ∂W. We next extend φ to the rest of ∂W. Observe that by construction $\varphi(\partial\mathcal{N}(F))\subset C'$. We shall extend φ so that the complement of $\mathcal{N}(F)$ also maps into C'. As C' is a $K(\mathbb{Z},1)$, to construct this it suffices to extend $\varphi_*:\pi_1(\partial\mathcal{N}(F))\to\pi_1(C')$ to a homomorphism $\varphi_*:\pi_1(\partial W)\to\pi_1(C')$. By construction the map $\varphi_*:\pi_1(\partial\mathcal{N}(F))\to\pi_1(C')$ is the composition of homomorphisms $\pi_1(\partial\mathcal{N}(F))\to H_1(\partial\mathcal{N}(F))\to H_1(W)\to H_1(C')\cong\pi_1(C')$ where each of the maps is a Hurewicz homomorphism or the map induced by inclusion. This extends to the map defined by the composition $\pi_1(\partial W)\to H_1(\partial W)\to H_1(W)\to H_1(C')\cong\pi_1(C')$. Hence we can extend φ to ∂W. As the complement of $\mathcal{N}(F)$ is mapped into C', we still have the property $\varphi^{-1}(C)=F$.

The map on W. .

We extend φ cell-by-cell. As $\pi_k(V)=0$ for $k\geq 2$ and the only non-vanishing relative homotopy group $\pi_k(V,\mathcal{K})$ is $\pi_2(V,\mathcal{K})=ker(\pi_1(K)\to\pi_1(V))$, it suffices to construct a homomorphism $\pi_1(W)\to\pi_1(V)$ so that the diagram

$$\begin{array}{ccc}\pi_1(\partial W) & \longrightarrow & \pi_1(\mathcal{K})\\ \downarrow & & \downarrow\\ \pi_1(W) & \longrightarrow & \pi_1(V)\end{array}$$

commutes. We define the homomorphism $\pi_1(W)\to\pi_1(V)$ as the the composition $\pi_1(W)\to H_1(W)\to\pi_1(V)$ using the identification made earlier. By construction the above diagram commutes.

We can ensure that $\varphi(int(W))\subset int(V)$. Thus we still have the property that $\varphi^{-1}(C)=F$.

4. Construction of the map ψ

We need to construct $\psi:W\to\mathcal{K}$ with $\psi|_{\partial W}=\varphi|_{\partial W}$. Equivalently, we let $\psi|_{\partial W}=\varphi|_{\partial W}$ and extend this to $\psi:W\to\mathcal{K}$. As $\mathcal{K}$ is a $K(\pi,1)$, it suffices to extend the homomorphism on fundamental groups $\psi_*:\pi_1(\partial W)\to\pi_1(\mathcal{K})$ to a homomorphism $\psi_*:\pi_1(W)\to\pi_1(\mathcal{K})$.

Such a homomorphism exists if and only if the image under ψ_* of the kernel of the map $\pi_1(\partial W)\to\pi_1(W)$ induced by inclusion is trivial. We shall show this by obtaining a better understanding of the map ψ_*.

We identify both $N \times \{0\}$ and $N \times \{1\}$ with N. Pick a base point x_0 in ∂N and fix $y_0 = (x_0, 0)$ as the base point in W. Let $\nu : [0,1] \to \partial W$ be the path $\nu(t) = (x_0, t)$. We identify a loop $\gamma \in \pi_1(N)$ with loops $\gamma_1 = (\gamma, 0)$ and γ_2 in $\pi_1(N \times \{0\}, y_0)$ and $\pi_1(N \times \{0\}, y_0)$ respectively, with γ_2 the composition of paths $\nu * (\gamma, 1), \bar{\nu}$. Then $\gamma_1 \gamma_2^{-1}$ maps to the trivial element in $\pi_1(W)$. Further, by the Van Kampen theorem, the kernel of the homomorphism $\pi_1(\partial W) \to \pi_1(W)$ induced by inclusion is generated by elements of this form.

Thus, we need to show that for γ_1 and γ_2 as above, $\psi_*(\gamma_1) = \psi_*(\gamma_2)$. In terms of the identification of N with $N \times \{0\}$ and $N \times \{1\}$, this is equivalent to showing that $\psi_*(\gamma)$ is independent of the Seifert surface $\Sigma = \Sigma_i$ of the knot K. We shall show this by computing $\psi_*(\gamma)$ in terms of the linking number between γ and K.

Recall that the Klein bottle has the fundamental group $\langle \lambda, \mu; \lambda\mu\lambda^{-1} = \mu^{-1} \rangle$. Under the inclusion map to V, the element μ maps to the identity and λ to the generator of $\pi_1(V) \cong \mathbb{Z}$. We pick a base point x_0 on C'. We can identify λ with the standard generator of $\pi_1(C')$.

Lemma 4.1. $\psi_*(\gamma) = (\mu\lambda)^{lk(K,\gamma)}$

Proof. Let m be a meridian of K. Then m is the union of two arcs $\alpha \subset \mathcal{N}(F) = F \tilde{\times} [-1,1]$ and β, with $\alpha(t) = (y_0, t)$, $t \in [-1,1]$ for a point $y_0 \in F$ and the interior of β disjoint from $\mathcal{N}(F)$. Without loss of generality we can assume that $\varphi(y_0) = x_0$.

By construction α maps to the meridian μ of $\mathcal{K}$ and β maps to a loop in C'. Further by construction of the map $\varphi : \partial(W) - int(\mathcal{N}(F)) \to C'$, it follows that, as $lk(m,K) = 1$, β maps to λ. Hence m maps to $\mu\lambda$ and the lemma holds for m.

Next. let γ be disjoint from Σ. Then by construction γ maps to a loop in C'. Further, by construction of the map $\varphi : \partial(W) - int(\mathcal{N}(F)) \to C'$, $\psi_*(\gamma) = (\lambda)^{lk(K,\gamma)}$. Observe that $lk(K,\gamma)$ is even as Σ is dual to the generator of $H_1(N, \mathbb{Z}/2\mathbb{Z})$. Now, in $\pi_1(\mathcal{K})$, $(\lambda\mu) = \lambda^2$, hence for $k = 2m$ even, $(\lambda\mu)^k = \lambda^k$. In particular, if γ is disjoint from Σ, $\psi_*(\gamma) = (\mu\lambda)^{lk(K,\gamma)}$

As m and the curves γ disjoint from Σ generate $\pi_1(N)$, the lemma follows. □

5. Proof of Theorem 1.1

We let $M = \psi^{-1}(C)$. This is a 3-manifold in $W \subset S^3 \times [0,1]$ with boundary F. The rest of the proof is as in the orientable case (see, for instance, [1]).

By perturbing M if necessary, we can assume that the projection $S^3 \times [0,1] \to [0,1]$ restricts to a Morse function on M. We can further assume that there are no critical points of index 0 and index 3, and that all critical points of index 1 are below the critical points of index 2. Let $t \in (0,1)$ separate critical points of index 1 from critical points of index 2.

Let $\Sigma_0 = M \cup S^3 \times \{t\}$. By standard Morse theory, Σ is obtained from each of Σ_1 and Σ_2 by finitely many stabilisations. □

6. The linking pairing

We define a linking pairing on the homology of a non-orientable Seifert surface Σ for a K in S^3. This will be used to construct invariants of the knot K.

Consider a regular neighbourhood of Σ. Its boundary is the orientable 2-fold cover $\tilde{\Sigma}$ of Σ. Let $\tau\colon \tilde{\Sigma} \to \tilde{\Sigma}$ be the non-trivial deck transformation of $\tilde{\Sigma}$ and let $p : \tilde{\Sigma} \to \Sigma$ be the covering map. Observe that τ is an involution.

Consider the subgroup A of $H_1(\Sigma)$ consisting of orientation preserving curves. Then $A = p_*(H_1(\tilde{\Sigma}))$. We shall define a bilinear pairing on A and extend to $H_1(\Sigma)$ using linearity. We need the following proposition.

Proposition 6.1. *We have $A = H_1(\tilde{\Sigma})/\{y - \tau(y) : y \in H_1(\tilde{\Sigma})\}$. In particular, if $x \in ker(p_*)$ then $x + \tau(x) = 0$.*

Proof. The group $\pi_(\tilde{\Sigma})$ is an index 2 subgroup of $\pi_1(S)$, with any orientation reversing curve γ being a representative of the non-trivial coset. The element γ acts by conjugation on $\pi_1(\tilde{\Sigma})$, with the induced action on the abelianisation $H_1(\tilde{\Sigma})$ being by τ. Hence the image A of $\pi_1(\tilde{\Sigma})$ in the abelianisation $H_1(\Sigma)$ of $\pi_1(\Sigma)$ is the quotient $H_1(\tilde{\Sigma})/\{y - \tau(y) : y \in H_1(\tilde{\Sigma})\}$.

As τ is an involution, the second statement follows by a simple calculation. □

Let α, β be elements of A and let $x \in H_1(\tilde{\Sigma})$ be an element with $p(x) = \alpha$. Note that $x \subset \tilde{\Sigma}$ which is disjoint from β. Let $lk(,)$ denote the linking pairing in S^3.

Definition 6.2. We define $l(\alpha, \beta) = (lk(x, \beta) + lk(\tau(x), \beta))/2$

Remark 6.3. The above definition makes sense for any $\beta \in H_1(\Sigma)$ (not necessarily in A) and is consistent with extending by linearity.

Proposition 6.4. *The pairing $l(,)$ is well-defined.*

Proof. As the definition is linear in x. it suffices to show that if $x \in ker(p_*)$, then $lk(x, \beta) + lk(\tau(x), \beta) = 0$. By Proposition 6.1, $x + \tau(x) = 0$ in $H_1(\tilde{\Sigma})$. As $\tilde{\Sigma}$ is disjoint from β, $x + \tau(x) = 0$ in $H_1(S^3 - \beta)$. It follows that $lk(x, \beta) + lk(\tau(x), \beta) = 0$. □

Proposition 6.5. *If α and β are disjoint curves in Σ, then $l(\alpha, \beta) = lk(\alpha, \beta)$.*

Proof. If α and β are in A, then $lk(x, \beta) = lk(\alpha, \beta)$ and $lk(\tau(x), \beta) = lk(\alpha, \beta)$ as x can be obtained from α by a small perturbation. The result follows. In general we use this argument for 2α and 2β and linearity. □

Proposition 6.6. $l(\alpha, \beta) = l(\beta, \alpha)$

Proof. By linearity it suffices to check this for a basis. But we can find a basis of disjoint simple closed curves as Σ can be expressed as the band connected sum of Móbius bands. For these curves, symmetry follows by the above proposition and the symmetry of linking numbers. □

Lemma 6.7. *Let α be an orientation reversing simple closed curve on Σ. Then $l(\alpha, \alpha) \in 1/2 + \mathbb{Z}$.*

Proof. This is equivalent to showing that $l(2\alpha, \alpha)$ is odd. We can represent the homology class $[2\alpha]$ by a simple closed curve γ disjoint from α. This bounds a Möbius band $\mathcal{M}$ in Σ which is a non-orientable Seifert surface. After perturbation α intersects $\mathcal{M}$ in a single point. Thus $l(\alpha, 2\alpha) = lk(\alpha, \gamma)$ is odd. □

Proposition 6.8. *The pairing $l(,)$ is non-degenerate.*

Proof. As Σ is non-orientable, it can be decomposed as a band connected sum of Möbius bands. Further, if α_i, $1 \leq i \leq n$ are the central circles of these Möbius bands, then these form a basis of $H_1(\Sigma)$.

Now consider an element $\alpha = \Sigma n_i \alpha_i \in H_1(\Sigma)$. We need to show that $l(\alpha, \cdot)$ is not identically zero. By linearity it suffices to consider the case when α is primitive, in particular some n_i (say n_1) is odd. Then $l(\alpha, \alpha_1) = \Sigma_i n_i l(\alpha_i, \alpha_1)$. The first term of this sum is in $1/2 + \mathbb{Z}$ while all other terms are non-zero. It follows that $l(\alpha, \alpha_1) \neq 0$. □

Observe that in the case of an orientable Seifert surface, our definition still makes sense and the pairing $l(,)$ is the symmetrised Seifert pairing $l(\alpha,\beta) = (\theta(\alpha,\beta)+\theta(\beta,\alpha))/2$. Further all the results of this section are standard facts in that case.

7. The signature and the determinant

We can use Theorem 1.1 together with the above linking pairing to construct invariants of knots. Let K be a knot and Σ a non-orientable Seifert surface. Let l be the corresponding bilinear pairing and let $sign(l)$ be its signature. Then we define the following invariants.

Definition 7.1. The signature of K is

$$\sigma(K) = sign(l) - \mathcal{F}(\Sigma)/2$$

Definition 7.2. The determinant of K is

$$D(K) = \boldsymbol{i}^{\mathcal{F}(\Sigma)/2+\chi(\Sigma)} det(2l)$$

Proposition 7.3. *The definitions of σ and D are independent of Σ*

Proof. A straightforward computation shows that these quantities are invariant under taking the band connected sum with Möbius bands. The proof of invariance under Stabilisation is the same as in the orientable case. □

Theorem 7.4. *The invariants σ and D are the same as the analogous invariants defined using orientable Seifert surfaces.*

Proof. As remarked in the previous section, the pairing l can be defined for an orientable Seifert surface Σ and co-incides with the symmetrised Seifert pairing. Hence in this case our definition coincides with the classical definition.

By taking a band-connected sum with a Möbius band, Σ is replaced by a non-orientable Seifert surface Σ'. As in the above Proposition, $\sigma(\Sigma) = \sigma(\Sigma')$ and $D(\Sigma) = D(\Sigma')$. The result follows. □

8. Applications

We now give some applications. The first result is that there are knots which bound a Möbius band with arbitrarily high Seifert genus. This result can be obtained in various other ways but our methods give a simple and elegant proof.

Theorem 8.1. *The $(2,n)$ torus knot K_n bounds a Möbius band but does not bound a Seifert surface of genus less than $|n-1|/2$*

Proof. K_n is the boundary of an unknotted Möbius band $\mathcal{M}$ with K twists. It is easy to see that $\mathcal{F}(\mathcal{M}) = 2n$, hence $\sigma(K_n) = 1-n$. It follows that an orientable Seifert surface Σ for K_n has signature $1-n$ and hence $H_1(\Sigma)$ has rank at least $|n-1|$. Thus the genus of Σ is at least $|n-1|/2$. □

For our second application, recall that a knot K has non-orientable Seifert surfaces with every even framing $2n$. We consider a minimal genus non-orientable Seifert surface Σ_n with framing $2n$. Let $r(n) = rank(H_1(\Sigma_n))$.

Theorem 8.2. *There are constants $k, m \in \mathbb{Z}$ such that $|n| + k \le r(n) \le |n| + m$.*

Proof. We consider the case when $n \geq 0$ (the other case is similar). First, let $m = r(0)$. Consider the non-orientable Seifert surface F_n with framing n obtained from Σ_0 by taking a band-connected sum with Möbius bands. It follows that $r(n) \leq rank(H_1(F_n)) = n + m$.

Conversely let l_n denote the bilinear form corresponding to Σ_n and let $l = \sigma(K)$. Using properties of the signature of K, we have

$$r(n) = rank(\Sigma_n) \geq sign(l_n) = \sigma(K) + \mathcal{F}(\Sigma_n)/2 = l + n$$

as claimed. □

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