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Bounded Cohomology and Bounded Generation of Wreath Products

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Abstract

We establish a criterion for triviality of the singular part of second bounded cohomology of (standard restricted) wreath products of groups. We also investigate when wreath products of groups have bounded generation.

1. Introduction

Wreath products of groups (for definitions and notation see §2) naturally arise in the study of Sylow subgroups of appropriate symmetric groups. They also often provide examples of certain groups with rather unexpected properties. The goal of this paper is to investigate bounded cohomology and bounded generation of (standard restricted) wreath products of groups.

We recall that bounded cohomology $H_b^*(G)$ of a group G (we will be considering only cohomology with coefficients in the additive group of reals $\mathbb{R}$ with trivial action, so in our notations for cohomology the coefficient module will be omitted) is defined using the complex

$$\cdots \longleftarrow C_b^{n+1}(G) \xleftarrow{\delta_b^n} C_b^n(G) \longleftarrow \cdots \longleftarrow C_b^2(G) \xleftarrow{\delta_b^1} C_b^1(G) \xleftarrow{\delta_b^0=0} \mathbb{R} \xleftarrow{\delta_b^{-1}=0} 0$$

of bounded cochains $f\colon G\times\cdots\times G \to \mathbb{R}$, and $\delta_b^n = \delta^n|_{C_b^n(G)}$ is the bounded differential operator. Since $H_b^0(G) = \mathbb{R}$ and $H_b^1(G) = 0$ for any group G, investigation of bounded cohomology starts in dimension 2. One observes that $H_b^2(G)$ contains a subspace $H_{b,2}^2(G)$ (called the singular part of the second bounded cohomology group), which has a simple algebraic description in terms of quasicharacters and pseudocharacters, and the quotient space $H_b^2(G)/H_{b,2}^2(G)$ is canonically isomorphic to the bounded part of the ordinary cohomology group $H^2(G)$. For background on bounded cohomology of groups see [4], for bounded cohomology of topological spaces see [5]. Special interest in $H_{b,2}^2$ is motivated in part by its connections with other structural properties of groups such as commutator length [2] and bounded generation [4]. In particular, it is important to know when $H_{b,2}^2$ vanishes.

We recall that a function $F\colon G \to \mathbb{R}$ is called a *quasicharacter* if there exists a constant $C_F \geqslant 0$ such that

$$|F(xy) - F(x) - F(y)| \leqslant C_F \quad \text{for all } x, y \in G.$$

A function $f\colon G \to \mathbb{R}$ is called a *pseudocharacter* if f is a quasicharacter and in addition $f(g^n) = nf(g)$ for all $g \in G$ and $n \in \mathbb{Z}$. The notions of a quasicharacter and a pseudocharacter originally arose from the questions of stability of solutions of functional equations [6, 7, 8] and continuous representations of groups [9]. We will use the following notation:

(i) $X(G)$ is the space of additive characters $G \to \mathbb{R}$;

(ii) $QX(G)$ is the space of quasicharacters;

(iii) $PX(G)$ is the space of pseudocharacters;

(iv) $B(G)$ is the space of bounded functions.

Then

$$H^2_{b,2}(G) \cong QX(G)/(X(G) \oplus B(G)) \cong PX(G)/X(G) \tag{1}$$

as vector spaces (cf. [4, Prop. 3.2 and Thm. 3.5]). We establish the following criterion for triviality of $H^2_{b,2}$ of wreath products.

Theorem 1.1 *$H^2_{b,2}(A \wr B) = 0$ if and only if the following conditions hold:*

(i) *$H^2_{b,2}(B) = 0$;*

(ii) *$H^2_{b,2}(A) = 0$ or B is infinite.*

An abstract group G is said to have *bounded generation* if there exist (not necessarily distinct) elements $g_1, \ldots, g_k \in G$ such that $G = \langle g_1 \rangle \cdots \langle g_k \rangle$, where $\langle g_i \rangle$ is the cyclic subgroup generated by g_i. Even though defined as simple combinatorial notion, bounded generation turns out to imply a number of remarkable structural properties:

(i) the pro-p completion of a boundedly generated group is a p-adic analytic group for every prime p [3, 10];

(ii) if G in addition satisfies condition (FAb), then it has only finitely many inequivalent completely reducible representations in every dimension over any field (see [11, 14, 15] for representations in characteristic zero, and [1] for arbitrary characteristic);

(iii) if G is an S-arithmetic subgroup of an absolutely simple simply connected algebraic group over a number field, then G has the congruence subgroup property [12, 13].

We establish the following results on bounded generation of wreath products.

Theorem 1.2 *If B is finite, then $A \wr B$ has bounded generation if and only if A does. If both A and B are infinite, then $A \wr B$ does not have bounded generation.*

Theorem 1.3 *Let A be a finite group which is not perfect and let B be a free abelian group of finite rank. Then $A \wr B$ does not have bounded generation.*

Remarks. 1. It would be interesting to see if $A \wr B$ does not have bounded generation whenever A is finite and B is infinite.

2. Since for the wreath product $A \wr B$ to have bounded generation it is necessary that B be boundedly generated, in Theorem 1.3 it suffices to consider the case when B has finite rank.

3. According to [16, Thm. 10.21], a wreath product $A \wr B$ with A finite and B infinite is linear only when A is a finite abelian p-group for some prime p and B is a virtually free abelian group. Thus, Theorem 1.3 includes also nonlinear groups like $A \wr \mathbb{Z}^n$ where $A = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z}$ for primes $p \neq q$.

4. The question of bounded generation of complete wreath products is trivial: if B is finite, then the complete wreath product is the same as the restricted wreath product, and if B is infinite, then the complete wreath product $A \operatorname{Wr} B$ is not finitely generated, hence cannot be boundedly generated.

2. Preliminaries

Wreath products are defined in the context of permutation groups when a group A acts on a set X and a group B acts on a set Y. We will consider the so-called *standard* wreath products, in which case the groups A and B act on themselves via right regular representations. Moreover, we will work with *restricted* standard wreath products, which are defined as follows.

Let A^* be the direct sum of copies of A indexed by elements of B (for the *complete* wreath product one takes the direct product). We will write this as $A^* = \sum_{b \in B} A_b$, where each group A_b is a copy of A. Elements of A^* can be thought of as functions from B to A with finite support. An element $f \in A^*$ such that

$$f(b) = \begin{cases} a & \text{if } b = b_0 \in B, \\ e_A & \text{otherwise} \end{cases}$$

will be denoted by $\sigma_a(b_0)$. We will say that the element $b_0 \in B$ is an *argument* of $\sigma_a(b_0)$. In this notation, every element of A^* can be uniquely written in the form

$$\sigma_{a_1}(b_1) \cdots \sigma_{a_s}(b_s),$$

where $b_1, \ldots, b_s$ are *distinct* elements of B, and $a_1, \ldots, a_s$ are any elements of A. Such a presentation will be called a *canonical form* or a *canonical word*. We will also say that s is the *length* of the canonical word. We define an action of B on A^* by

$$f^c(b) = f(bc^{-1}), \quad c \in B, \ b \in B. \tag{2}$$

The (standard restricted) wreath product of A and B, denoted by $A \wr B$, is the semidirect product of A^* and B with the action of B on A^* given by (2). If we denote the elements of a copy of B in $A \wr B$ by τ_c, $c \in B$, then (2) becomes

$$\tau_c \sigma_a(b) = \sigma_a(bc)\tau_c,$$

whence every element of $A \wr B$ can be uniquely written in the canonical form

$$\sigma_{a_1}(b_1) \cdots \sigma_{a_s}(b_s)\tau_b,$$

where $\sigma_{a_1}(b_1) \cdots \sigma_{a_s}(b_s)$ is a canonical word in A^*.

Next, we list some well-known facts about pseudocharacters that will be used in the sequel.

Lemma 2.1 *Any pseudocharacter is constant on conjugacy classes; a bounded pseudocharacter is trivial.*

Proof. Let $f \in PX(G)$ and suppose that $f(yxy^{-1}) - f(x) = \alpha \neq 0$ for some $x, y \in G$. Then the difference $f(yx^ny^{-1}) - f(x^n) = n\alpha$ is unbounded when $n \to \infty$. On the other hand,

$$|f(yx^ny^{-1}) - f(x^n)| = |f(yx^ny^{-1}) - f(y) - f(x^n) - f(y^{-1})| \leqslant 2C_f,$$

a contradiction. The second assertion is obvious. ☐

Lemma 2.2 *If G is any group, $f \in PX(G)$, and $x_1, \ldots, x_n$ are pairwise commuting elements of G, then*

$$f(x_1 \cdots x_n) = f(x_1) + \cdots + f(x_n).$$

Proof. Let $\alpha = f(x_1x_2) - f(x_1) - f(x_2)$. Then for any positive integer n,

$$|n\alpha| = |nf(x_1x_2) - nf(x_1) - nf(x_2)| = |f(x_1^nx_2^n) - f(x_1^n) - f(x_2^n)| \leqslant C_f$$

which implies that $\alpha = 0$. The general case follows by induction on n. ☐

Remark. Of course, the result follows from the general fact that bounded cohomology of amenable groups vanishes, but we chose to give a short elementary proof.

Finally, we will need the following standard fact about bounded generation. We provide the proof for reader's convenience.

Lemma 2.3 *If H is a subgroup of finite index of a group G, then G has bounded generation if and only if H does.*

Proof. If H has bounded generation, then $H = \langle h_1 \rangle \cdots \langle h_n \rangle$ for some $h_1, \ldots, h_n \in H$. Let $\{g_1, \ldots, g_m\}$ be a system of left coset representatives of H in G. Then $G = \langle g_1 \rangle \cdots \langle g_m \rangle \langle h_1 \rangle \cdots \langle h_n \rangle$.

For the converse suppose that $G = \langle g_1 \rangle \cdots \langle g_n \rangle$. Since every subgroup of finite index contains a normal subgroup of finite index, it suffices to prove our claim assuming in addition that H is normal in G.

An arbitrary element $h \in H$ can be written in the form $h = g_1^{r_1} \cdots g_n^{r_n}$ for some $r_1, \ldots, r_n \in \mathbb{Z}$. Say, $[G : H] = m$ and for $i = 1, \ldots, n$ write $r_i = e_i + ma_i$ with $0 \leqslant e_i < m$. Since H is normal, we have $h_i := g_i^m \in H$. In this notation,

$$\begin{aligned} h &= g_1^{r_1} \cdots g_n^{r_n} = g_1^{e_1} h_1^{a_1} g_2^{e_2} h_2^{a_2} \cdots g_n^{e_n} h_n^{a_n} \\ &= g_1^{e_1} \cdots g_n^{e_n} \left[\prod_{i=1}^{n-1} [(g_{i+1}^{e_{i+1}} \cdots g_n^{e_n})^{-1} h_i (g_{i+1}^{e_{i+1}} \cdots g_n^{e_n})]^{a_i} \right] h_n^{a_n}. \end{aligned} \tag{3}$$

We now introduce the following finite set

$$\Lambda = \{g_1^{e_1} \cdots g_n^{e_n} \mid 0 \leqslant e_i < m\}.$$

Then it follows from (3) that

$$H = \left(\prod_{y \in \Lambda \cap H} \langle y \rangle \right) \cdot \prod_{i=1}^{n} \left(\prod_{x \in \Lambda} \langle x^{-1} h_i x \rangle \right),$$

so H has bounded generation. $\square$

3. Proof of Theorem 1.1

Suppose that $H^2_{b,2}(A \wr B) = 0$. We begin by showing that (i) must hold.

Lemma 3.1 $\dim H^2_{b,2}(A \wr B) \geqslant \dim H^2_{b,2}(B)$.

Proof. Let $f_1, \ldots, f_n$ be pseudocharacters of B linearly independent modulo characters. For each $i = 1, \ldots, n$ define a function F_i on $A \wr B$ by

$$F_i(h\tau_b) = f_i(b),$$

where $h \in A^*$ and $b \in B$. Then

$$F_i((h\tau_b)^n) = F_i(hh^{\tau_b} \cdots h^{\tau_{b^{n-1}}} \tau_{b^n}) = f_i(b^n) = nf_i(b) = nF_i(h\tau_b)$$

and

$$|F_i(h_1\tau_{b_1} h_2 \tau_{b_2}) - F_i(h_1\tau_{b_1}) - F_i(h_2\tau_{b_2})| = |f_i(b_1b_2) - f_i(b_1) - f_i(b_2)| \leqslant C_{f_i}.$$

Therefore, $F_1, \ldots, F_n$ are pseudocharacters on $A \wr B$ which are linearly independent modulo characters of $A \wr B$, whence our claim. $\square$

Condition (ii) must hold in view of the following.

Lemma 3.2 *If B is finite, then* $\dim H^2_{b,2}(A \wr B) \geqslant \dim H^2_{b,2}(A)$.

Proof. Given pseudocharacters $f_1, \ldots, f_n$ on A linearly independent modulo characters, define functions F_i, $1 \leqslant i \leqslant n$, on $A \wr B$ as follows:

$$F_i(\sigma_{a_1}(b_1) \cdots \sigma_{a_s}(b_s)\tau_b) = f_i(a_1) + \cdots + f_i(a_s).$$

If $|B| = m$, then

$$\begin{aligned} &|F_i([\sigma_{a_1}(b_1) \cdots \sigma_{a_s}(b_s)\tau_u][\sigma_{c_1}(d_1) \cdots \sigma_{c_t}(d_t)\tau_v]) - F_i(\sigma_{a_1}(b_1) \cdots \sigma_{a_s}(b_s)\tau_u) \\ &\qquad - F_i(\sigma_{c_1}(d_1) \cdots \sigma_{c_t}(d_t)\tau_v)| \\ &= |F_i(\sigma_{a_1}(b_1) \cdots \sigma_{a_s}(b_s)\sigma_{c_1}(d_1u) \cdots \sigma_{c_t}(d_tu)\tau_{uv}) - F_i(\sigma_{a_1}(b_1) \cdots \sigma_{a_s}(b_s)\tau_u) \\ &\qquad - F_i(\sigma_{c_1}(d_1) \cdots \sigma_{c_t}(d_t)\tau_v)| \\ &= \left| \sum_{b_k = d_l u} (f_i(a_kc_l) - f_i(a_k) - f_i(c_l)) \right| \leqslant mC_{f_i}, \end{aligned}$$

which shows that F_i is an unbounded quasicharacter of $A \wr B$ with constant mC_{f_i}. Hence, there is a bounded function G_i on $A \wr B$ such that $F_i + G_i \in PX(A \wr B)$. Note that the restriction $(F_i + G_i)|_{A_e}$ is a pseudocharacter of $A_e \cong A$ which differs from f_i by a bounded function, hence coincides with f_i. It follows that the pseudocharacters $F_1 + G_1, \ldots, F_n + G_n$ are linearly independent modulo characters of $A \wr B$. □

Now it remains to show that (i) and (ii) imply that $H^2_{b,2}(A \wr B) = 0$. We begin with a couple of observations.

Lemma 3.3 *If $F \in PX(A \wr B)$, then $F(\sigma_a(b_1)) = F(\sigma_a(b_2))$ for all b_1, $b_2 \in B$.*

Proof. Every pseudocharacter is constant on conjugacy classes (Lemma 2.1) and $\sigma_a(b_2) = \tau_{b_1^{-1}b_2}\sigma_a(b_1)\tau^{-1}_{b_1^{-1}b_2}$. □

Lemma 3.4 *If $H^2_{b,2}(A) = 0$, then $H^2_{b,2}(A^*) = 0$.*

Proof. If $F \in PX(A^*)$ and $\sigma_{a_1}(b_1) \cdots \sigma_{a_s}(b_s)$ is a "canonical word" in A^*, then by Lemma 2.2,

$$F(\sigma_{a_1}(b_1) \cdots \sigma_{a_s}(b_s)) = F(\sigma_{a_1}(b_1)) + \cdots + F(\sigma_{a_s}(b_s)).$$

Since $H^2_{b,2}(A) = 0$, the restriction of F to every A_b is a character of A_b, whence F is a character of A^*. □

We now continue with the proof of the theorem.

Lemma 3.5 *If $H^2_{b,2}(A) = 0$ and $H^2_{b,2}(B) = 0$, then $H^2_{b,2}(A \wr B) = 0$.*

Proof. If $F \in PX(A \wr B)$, then by Lemma 3.4, the restriction $F|_{A^*}$ is a character of A^*; also $F|_B$ is a character of B. To prove that $F \in X(A \wr B)$, it thus remains to show that

$$F(h\tau_b) = F(h) + F(\tau_b) \quad \text{for all } h \in A^*,\ b \in B.$$

Let $\alpha = F(h\tau_b) - F(h) - F(\tau_b)$. Then for any positive integer n, we have

$$\begin{aligned} |n\alpha| &= |nF(h\tau_b) - nF(h) - nF(\tau_b)| \\ &= |F(hh^{\tau_b} \cdots h^{\tau_{b^{n-1}}}\tau_{b^n}) - nF(h) - F(\tau_{b^n})| \\ &\leqslant |F(hh^{\tau_b} \cdots h^{\tau_{b^{n-1}}}\tau_{b^n}) - F(hh^{\tau_b} \cdots h^{\tau_{b^{n-1}}}) - F(\tau_{b^n})| \\ &\quad + |F(hh^{\tau_b} \cdots h^{\tau_{b^{n-1}}}) - nF(h)| \\ &\leqslant C_F + |F(h) + F(h^{\tau_b}) + \cdots + F(h^{\tau_{b^{n-1}}}) - nF(h)| \\ &= C_F \end{aligned}$$

which shows that $\alpha = 0$. □

Remark. One can also derive the result of Lemma 3.5 from the observation that if N and K are subgroups of G with N normal such that $G = NK$ and $H^2_{b,2}(N) = H^2_{b,2}(K) = 0$, then $H^2_{b,2}(G) = 0$. The proof is identical to that of the lemma.

Lemma 3.6 *If $H^2_{b,2}(B) = 0$ and B is infinite, then $H^2_{b,2}(A \wr B) = 0$.*

Proof. Let $F \in PX(A \wr B)$. Proof of Lemma 3.5 shows that in order to establish that F is a character of $A \wr B$, it suffices to show that $F \in X(A^*)$. Lemma 3.3 implies that there exists a pseudocharacter f of A such that $F|_{A_b} = f$ for all $b \in B$, i.e.,

$$F(\sigma_a(b)) = f(a) \quad \text{for all } b \in B.$$

If $b_1, \dots, b_s$ are distinct element of B, then Lemma 2.2 implies that

$$F(\sigma_{a_1}(b_1) \cdots \sigma_{a_s}(b_s)) = f(a_1) + \cdots + f(a_s)$$

for any elements $a_1, \dots, a_s \in A$. Therefore, to prove that F is a character of A^*, it suffices to show that f is a character of A. Suppose that f is not a character of A. Then there exist a_1, $a_2 \in A$ such that

$$\alpha = f(a_1a_2) - f(a_1) - f(a_2) \neq 0.$$

Choose an infinite sequence $\{b_i\}$ of distinct elements of B. Then

$$\begin{aligned}
&|F([\sigma_{a_1}(b_1) \cdots \sigma_{a_1}(b_n)][\sigma_{a_2}(b_1) \cdots \sigma_{a_2}(b_n)]) \\
&\qquad - F([\sigma_{a_1}(b_1) \cdots \sigma_{a_1}(b_n)]) - F([\sigma_{a_2}(b_1) \cdots \sigma_{a_2}(b_n)])| \\
&= |F(\sigma_{a_1a_2}(b_1) \cdots \sigma_{a_1a_2}(b_n)) - F(\sigma_{a_1}(b_1) \cdots \sigma_{a_1}(b_n)) - F(\sigma_{a_2}(b_1) \cdots \sigma_{a_2}(b_n))| \\
&= |nf(a_1a_2) - nf(a_1) - nf(a_2)| = |n\alpha| \to \infty \quad \text{as} \quad n \to \infty
\end{aligned}$$

On the other hand, it is bounded by C_F, a contradiction. □

The proof of Theorem 1.1 is now complete.

4. Proof of Theorem 1.2

First assume that B is finite. Then A^* has finite index in $A \wr B$ and in view of Lemma 2.3 it suffices to show that A^* has bounded generation if and only if A does. But A^* is isomorphic to a direct sum of finitely many copies of A, so the first assertion of Theorem 1.2 follows.

Now assume that both A and B are infinite and suppose that there are some elements $g_1, \dots, g_k \in A \wr B$ (not necessarily distinct) such that

$$A \wr B = \langle g_1 \rangle \cdots \langle g_k \rangle. \tag{4}$$

Write

$$g_i = \sigma_{a_{i,1}}(b_{i,1}) \cdots \sigma_{a_{i,r_i}}(b_{i,r_i}) \tau_{c_i} \tag{5}$$

in the canonical form, $i = 1, \dots, k$. If $u_1, \dots, u_k$ are integers, then

$$g_1^{u_1} g_2^{u_2} \cdots g_k^{u_k} = R_1 R_2 \cdots R_k \tau_{c_1^{u_1} \cdots c_k^{u_k}},$$

where

$$\begin{aligned}
R_1 &= [\sigma_{a_{1,1}}(b_{1,1})\cdots\sigma_{a_{1,r_1}}(b_{1,r_1})] \\
&\quad [\sigma_{a_{1,1}}(b_{1,1}c_1)\cdots\sigma_{a_{1,r_1}}(b_{1,r_1}c_1)]\cdots \\
&\quad [\sigma_{a_{1,1}}(b_{1,1}c_1^{u_1-1})\cdots\sigma_{a_{1,r_1}}(b_{1,r_1}c_1^{u_1-1})]; \\
R_2 &= [\sigma_{a_{2,1}}(b_{2,1}c_1^{u_1})\cdots\sigma_{a_{2,r_2}}(b_{2,r_2}c_1^{u_1})] \\
&\quad [\sigma_{a_{2,1}}(b_{2,1}c_1^{u_1}c_2)\cdots\sigma_{a_{2,r_2}}(b_{2,r_2}c_1^{u_1}c_2)]\cdots \\
&\quad [\sigma_{a_{2,1}}(b_{2,1}c_1^{u_1}c_2^{u_2-1})\cdots\sigma_{a_{2,r_2}}(b_{2,r_2}c_1^{u_1}c_2^{u_2-1})]; \\
&\vdots \\
R_k &= [\sigma_{a_{k,1}}(b_{k,1}c_1^{u_1}\cdots c_{k-1}^{u_{k-1}})\cdots\sigma_{a_{k,r_k}}(b_{k,r_k}c_1^{u_1}\cdots c_{k-1}^{u_{k-1}})] \\
&\quad [\sigma_{a_{k,1}}(b_{k,1}c_1^{u_1}\cdots c_{k-1}^{u_{k-1}}c_k)\cdots\sigma_{a_{k,r_k}}(b_{k,r_k}c_1^{u_1}\cdots c_{k-1}^{u_{k-1}}c_k)]\cdots \\
&\quad [\sigma_{a_{k,1}}(b_{k,1}c_1^{u_1}\cdots c_{k-1}^{u_{k-1}}c_k^{u_k-1})\cdots\sigma_{a_{k,r_k}}(b_{k,r_k}c_1^{u_1}\cdots c_{k-1}^{u_{k-1}}c_k^{u_k-1})].
\end{aligned}$$

Note that these presentations are not necessarily "canonical words" in the wreath product. For example, if some $c_i = 1$, then the canonical form of the corresponding R_i is

$$R_i = \sigma_{a_{i,1}^{u_i}}(b_{i,1}c_1^{u_1}\cdots c_{i-1}^{u_{i-1}})\cdots\sigma_{a_{i,r_i}^{u_i}}(b_{i,r_i}c_1^{u_1}\cdots c_{i-1}^{u_{i-1}}),$$

in particular, the length of each such R_i is r_i. Similarly, if some c_i has finite order, say m_i, then the length of the corresponding R_i is at most $m_i r_i$. All of this holds independently of the choice of the powers u_i. It follows that there is a constant N such that for any choice of u_i's the sum of the lengths of R_i's which correspond to c_i of finite order is $\leqslant N$.

Equations (4) and (5) imply that $B = \langle c_1\rangle\cdots\langle c_k\rangle$, and since B is infinite, at least one of the c_i's must have infinite order. Let $g = \sigma_{\alpha_1}(\beta_1)\cdots\sigma_{\alpha_s}(\beta_s)$, where $s > N$, be a canonical word in $A \wr B$. Then there is some β_t, $1 \leqslant t \leqslant s$, which appears as an argument only in the R_i's which correspond to c_i's of infinite order. Consider such an R_i where β_t occurs. Now, R_i has u_i "blocks", and in each of the blocks an argument can occur at most once. Moreover, as c_i has infinite order, if β_t is the argument $b_{i,l}c_1^{u_1}\cdots c_{i-1}^{u_{i-1}}c_i^m$ in the $(m+1)$st block and is also the argument $b_{i,l'}c_1^{u_1}\cdots c_{i-1}^{u_{i-1}}c_i^{m'}$ in the $(m'+1)$st block with $m \neq m'$, then $l \neq l'$.

Therefore, if we gather together all the terms in R_i which have β_t as their argument, the corresponding product is $\sigma_a(\beta_t)$, where a is a subproduct of $\prod_{j\leqslant r_i} a_{i,j}$. Gathering together all the terms in all R_i's which correspond to c_i's of infinite order with β_t as the argument, we see that α_t must be equal to a subproduct of the product

$$\prod_{i\leqslant k}\prod_{j\leqslant r_i} a_{i,j}. \qquad (6)$$

Since A is infinite, it is possible to choose α_t to be different from any of the finitely many subproducts of the product (6). Hence g cannot be represented in the form $g_1^{u_1}\cdots g_k^{u_k}$, and consequently $A \wr B$ does not have bounded generation. This proves the second assertion of Theorem 1.2.

5. Proof of Theorem 1.3

The proof is based on the observation that if C is a quotient group of A, then $C \wr B$ is a quotient group of $A \wr B$. Therefore, in view of the fact that a finite non-perfect group admits a cyclic group of prime order as a quotient, to prove the theorem it suffices to show that $A \wr B$ does not have bounded generation if B is free abelian of finite rank and A is a cyclic group of prime order. This is established in the next lemma. The proof uses the explicit construction of faithful representations of such wreath products.

Lemma 5.1 *$\mathbb{Z}/p\mathbb{Z} \wr \mathbb{Z}^n$ does not have bounded generation.*

Proof. Let $y_1, \dots, y_n$ be independent indeterminates. Then the group $G = \mathbb{Z}/p\mathbb{Z} \wr \mathbb{Z}^n$ admits a faithful representation

$$\rho \colon G \to GL_2(\mathbb{F}_p(y_1, \dots, y_n))$$

which is described explicitly as follows:

$$\rho(\tau_{(m_1,\dots,m_n)}) = \begin{pmatrix} y_1^{m_1} \cdots y_n^{m_n} & 0 \\ 0 & 1 \end{pmatrix}, \quad m_i \in \mathbb{Z};$$

$$\rho(\sigma_a((m_1, \dots, m_n))) = \begin{pmatrix} 1 & a y_1^{m_1} \cdots y_n^{m_n} \\ 0 & 1 \end{pmatrix}, \quad m_i \in \mathbb{Z}, a \in \mathbb{F}_p.$$

The image of every element of G in $GL_2(\mathbb{F}_p(y_1, \dots, y_n))$ can be uniquely expressed as $X(f)H(y_1^{m_1} \cdots y_n^{m_n})$, where $m_1, \dots, m_n \in \mathbb{Z}$, $f \in \mathbb{F}_p[y_1, y_1^{-1}, \dots, y_n, y_n^{-1}]$,

$$X(f) = \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix},$$

and

$$H(y_1^{m_1} \cdots y_n^{m_n}) = \begin{pmatrix} y_1^{m_1} \cdots y_n^{m_n} & 0 \\ 0 & 1 \end{pmatrix}.$$

Suppose that $G = \langle g_1 \rangle \cdots \langle g_k \rangle$ and write

$$\rho(g_j) = X(f_j)\, H\left(y_1^{m_{1j}} \cdots y_n^{m_{nj}}\right), \quad f_j \in \mathbb{F}_p[y_1, y_1^{-1}, \dots, y_n, y_n^{-1}].$$

Then

$$\rho(g_1^{u_1} \cdots g_k^{u_k}) = X(U(y_1, \dots, y_n))\, H\Big(\prod_{j=1}^{n} y_j^{\sum_{i=1}^{k} u_i m_{ji}}\Big).$$

Here

$$U = \sum_{s=1}^{k} \left(U_s(y_1, \dots, y_n) f_s\right),$$

where

$$U_s(y_1, \dots, y_n) = \begin{cases} \left(\prod_{j=1}^{n} y_j^{\sum_{t<s} u_t m_{jt}}\right) u_s & \text{if } m_{1s} = \cdots = m_{ns} = 0, \\ \prod_{j=1}^{n} y_j^{\sum_{t<s} u_t m_{jt}} \cdot \frac{\prod_{j=1}^{n} y_j^{u_s m_{js}} - 1}{\prod_{j=1}^{n} y_j^{m_{js}} - 1} & \text{otherwise.} \end{cases}$$

To make the proof transparent, we first discuss the case when $n=1$. Let us write y instead of y_1 and $f_i = y^{n_i}P_i(y)$ for $1 \leqslant i \leqslant k$, where $P_i \in \mathbb{F}_p[y]$ and $n_i \in \mathbb{Z}$. With this simpler notation, we have

$$\rho(g_1^{u_1}\cdots g_k^{u_k}) = X\left(\sum_{i=1}^{k} y^{n_i+\sum_{j<i}u_jm_j}P_i(y)U_i(y)\right)H\left(y^{\sum_{i=1}^{k}u_im_i}\right),$$

where

$$U_i(y) = \begin{cases} \frac{y^{u_im_i}-1}{y^{m_i}-1} & \text{if } m_i \neq 0, \\ u_i & \text{if } m_i = 0. \end{cases}$$

Note that at least one of m_i's must be nonzero. Moreover, if $m_i < 0$, then $y^{m_i}-1 = y^{m_i}(1-y^{|m_i|})$ and

$$U_i(y) = y^{-m_i}\frac{y^{u_im_i}-1}{1-y^{|m_i|}}.$$

Let

$$\prod_{m_i\neq 0}(y^{|m_i|}-1) = \sum_{j=0}^{N-1} a_jy^j$$

where $N-1 = \sum_i |m_i|$ and $a_{N-1} = 1$. Consider the polynomial

$$Q(y) = \left(1+y^N+y^{2N}+\cdots+y^{(d-1)N}\right)\sum_{j=0}^{N-1}a_jy^j \in \mathbb{F}_p[y],$$

where $d > 2(N-1)(\sum_{i=1}^{k}\deg P_i)$. Observe that the number of monomials of distinct degrees in $Q(y)$ is $\geqslant d$. If

$$\rho(g_1^{u_1}\cdots g_k^{u_k}) = X\left(1+y^N+y^{2N}+\cdots+y^{(d-1)N}\right)$$

then

$$\sum_{i=1}^{k} y^{n_i+\sum_{j<i}u_jm_j}P_i(y)U_i(y) = 1+y^N+y^{2N}+\cdots+y^{(d-1)N}.$$

Multiply both sides by $\prod_{m_i\neq 0}(y^{|m_i|}-1)$ to get

$$\sum_{i=1}^{k} y^{n_i+\sum_{j<i}u_jm_j}P_i(y)V_i(y) = Q(y), \tag{7}$$

where

$$V_i(y) = \begin{cases} (y^{u_im_i}-1)\prod_{m_j\neq 0, j\neq i}(y^{|m_j|}-1) & \text{if } m_i > 0, \\ u_i\prod_{m_j\neq 0}(y^{|m_j|}-1) & \text{if } m_i = 0, \\ -y^{-m_i}\,(y^{u_im_i}-1)\prod_{m_j\neq 0, j\neq i}(y^{|m_j|}-1) & \text{if } m_i < 0. \end{cases}$$

In any case, $V_i(y)$ contains at most $2(N-1)$ monomials. It follows that the left hand side of (7) is a sum of at most $2(N-1)(\sum_{i=1}^{k}\deg P_i)$ monomials, while the right hand side of (7) contains a larger number of monomials of distinct degrees. This contradiction shows that the group $(\mathbb{Z}/p\mathbb{Z})\wr\mathbb{Z}$ cannot have bounded generation.

For general n, using the same technique of clearing denominators, we see that U can involve only a bounded number of monomials in $y_1,\ldots,y_n$, hence a contradiction. ∎

Remark. One can give a constructive proof similar to that of Lemma 5.1 of the fact that $A \wr B$ does not have bounded generation if A is a finite abelian p-group and B is a free abelian group of finite rank using explicit faithful representations of such wreath products. If A is a finite abelian p-group, then

$$A = \left(\oplus_{I_1}\mathbb{Z}/p^{d_1}\mathbb{Z}\right) \bigoplus \left(\oplus_{I_2}\mathbb{Z}/p^{d_2}\mathbb{Z}\right) \bigoplus \cdots \bigoplus \left(\oplus_{I_k}\mathbb{Z}/p^{d_k}\mathbb{Z}\right)$$

for some $1 \leqslant d_1 < d_2 < \cdots < d_k$. Let $\{x_i(r) \mid i \in I_r,\ 1 \leqslant r \leqslant k\}$ and $y_1, \dots, y_n$ be independent indeterminates. For each $1 \leqslant r \leqslant k$, the direct sum $\oplus_{I_r}\mathbb{Z}/p^{d_r}\mathbb{Z}$ can be represented as a group of $(p^{d_r-1}+1) \times (p^{d_r-1}+1)$ matrices using the indeterminates $x_i(r)$ for $i \in I_r$. The image of $\oplus_{I_r}\mathbb{Z}/p^{d_r}\mathbb{Z}$ is generated by the matrices

$$Y(x_i(r)) = \begin{pmatrix} 1 & x_i(r) & 0 & \cdots & 0 \\ 0 & 1 & x_i(r) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & x_i(r) \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}, \quad i \in I_r.$$

Let $N = \sum_{r=1}^{k}(p^{d_r-1}+1)$. Then a faithful representation ρ of $G = A \wr B$, where $B = \mathbb{Z}^n$, into $GL_N(\mathbb{F}_p(x_i(r), y_j))$ can be obtained as follows. The image $\rho(G)$ consists of block-diagonal matrices with blocks of sizes $p^{d_1-1}+1, p^{d_2-1}+1, \dots, p^{d_k-1}+1$. The image of $\oplus_{i\in I_r}\mathbb{Z}/p^{d_r}\mathbb{Z}$ is generated by matrices whose only nontrivial blocks $Y(x_i(r))$, $i \in I_r$, are in position r. The image of $\mathbb{Z}^n$ is generated by matrices of the form

$$\begin{pmatrix} D_1(y_j) & 0 & \cdots & 0 \\ 0 & D_2(y_j) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & D_k(y_j) \end{pmatrix}, \quad 1 \leqslant j \leqslant n,$$

where $D_r(y_j) = \operatorname{diag}\,(y_j^{p^{d_r-1}}, y_j^{p^{d_r-1}-1}, \dots, y_j, 1) \in GL_{p^{d_r-1}+1}(\mathbb{F}_p(y_1, \dots, y_n))$. The proof is conducted by counting monomials in position $(1,2)$.

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