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On growth, recurrence and the Choquet-Deny Theorem for p-adic Lie groups

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Abstract

We first study the growth properties of p-adic Lie groups and its connection with p-adic Lie groups of type R and prove that a non-type R p-adic Lie group has compact neighbourhoods of identity having exponential growth. This is applied to prove the growth dichotomy for a large class of p-adic Lie groups which includes p-adic algebraic groups. We next study p-adic Lie groups that admit recurrent random walks and prove the natural growth conjecture connecting growth and the existence of recurrent random walks, precisely we show that a p-adic Lie group admits a recurrent random walk if and only if it has polynomial growth of degree at most two. We prove this conjecture for some other classes of groups also. We also prove the Choquet-Deny Theorem for compactly generated p-adic Lie groups of polynomial growth and also show that polynomial growth is necessary and sufficient for the validity of the Choquet-Deny for all spread-out probabilities on Zariski-connected p-adic algebraic groups. Counter example is also given to show that certain assumptions made in the main results can not be relaxed.

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1 Introduction and Preliminaries

Let G be a locally compact group with a left-invariant Haar measure m. For a subset E in G and $n \geq 1$, let $E^n = \{x_1x_2\cdots x_n \mid x_i \in E \text{ for all } i\}$ and $E^{-1} = \{x^{-1} \mid x \in E\}$. Let $\mathcal{P}(G)$ be the space of all regular Borel probability measures on G. We first fix the following notations. Let $x \in G$ and $\mu, \lambda \in \mathcal{P}(G)$. Then

(i) δ_x denotes measure concentrated at the point x,

(ii) $\mu * \lambda$ as well as $\mu\lambda$ denotes the convolution of μ and λ,

(iii) $x\mu$ and μx denote $\delta_x * \mu$ and $\mu * \delta_x$ respectively and

(iv) for any $n \geq 1$, μ^n denotes the n-th convolution power of μ.

For $\mu \in \mathcal{P}(G)$, $\check{\mu} \in \mathcal{P}(G)$ is defined to be $\check{\mu}(E) = \mu(E^{-1})$ for any Borel subset E of G. We say that a measure $\mu \in \mathcal{P}(G)$ is *symmetric* if $\check{\mu} = \mu$.

1.1 Growth and various classes of groups

It is a well-acknowledged fact that growth properties of groups play a vital role in analysis and probability (see [11], [15], [32] and [34]). We first introduce different notions of growth.

Definition 1.1 A compact neighborhood V of e in a locally compact group G is said to generate G if $G = \cup_{n=1}^{\infty} V^n$. A locally compact group is called *compactly generated* if it is generated by a compact neighborhood of e.

Definition 1.2 We say that a compactly generated group with a left-invariant Haar measure m has *polynomial growth* if there exists an integer $k \geq 1$ such that for each compact neighborhood V of e, there exists a constant $c > 0$ so that $m(V^n) \leq cn^k$ for all $n \geq 1$. We say that a locally compact group G has polynomial growth if every compactly generated open subgroup of G has polynomial growth.

Definition 1.3 A compact neighborhood V of e in a locally compact group G with a left-invariant Haar measure m is said to have *exponential growth* if there exist constants $c > 0$ and $a > 1$ such that $m(V^n) > ca^n$ for all $n \geq 1$ and we say that G has *exponential growth* if every compact neighborhood of e has exponential growth.

The structure of compactly generated groups of polynomial growth is well-studied in [10], [11], [15]-[21] and the references therein: see also Chapter 6 of [23]. Growth of general class of groups, especially non-compactly generated groups is less developed.

In this article we consider p-adic Lie groups. Let $\mathbb{Q}_p$ be the field of p-adic numbers and $|\cdot|$ be denote the p-adic absolute value. Let G be a p-adic Lie group with Lie algebra $\mathcal{G}$: see [29] for generalities on p-adic Lie groups. Let Ad be the adjoint representation of a p-adic Lie group G in its Lie algebra $\mathcal{G}$. It is known that there are many p-adic Lie groups which are not compactly generated, in fact, the additive group $\mathbb{Q}_p$ of p-adic numbers is not compactly generated.

It can be easily seen that the additive group $\mathbb{Q}_p$ has growth zero as any compact subset is contained in a compact group. Consider the multiplicative group $\mathbb{Q}_p^*$ of non-zero elements in $\mathbb{Q}_p$. Then the map $x \mapsto (|x|, \frac{x}{|x|})$ defines an isomorphism of $\mathbb{Q}_p^*$ onto $\mathbb{Z} \times \{x \in \mathbb{Q}_p \mid |x| = 1\}$. Since $\{x \in \mathbb{Q}_p \mid |x| = 1\}$ is a compact group, $\mathbb{Q}_p^*$ has growth of degree one.

Any countable group can be easily seen to be a p-adic Lie group. This motivates the following definition.

Definition 1.4 A p-adic Lie group G is called *Ad-regular* if the kernel of the adjoint representation, Ker (Ad) is the center of G.

It is a fact that Zariski-connected p-adic algebraic groups such as $GL(n, \mathbb{Q}_p)$ are Ad-regular p-adic Lie groups. Recall that a locally compact group G is called p-adic algebraic group if G is the group of $\mathbb{Q}_p$-points of an algebraic group defined over $\mathbb{Q}_p$: see [4] and [6] for details on algebraic groups. The following class of groups plays a crucial role in understanding the growth of p-adic Lie groups.

Definition 1.5 We say that a p-adic Lie group is of *type R* if the eigenvalues of Ad (g) are of absolute value one for any $g \in G$.

Type R p-adic Lie groups are studied in [24] and this class was shown to coincide with many other classes (see also [7] and [14]). In [24], it is shown that compactly generated p-adic algebraic group has polynomial growth if and only if it is of type R. Here we study the growth of general p-adic Lie groups and prove that a type R p-adic Lie group with Ad (G) closed has polynomial growth if and only if the kernel of adjoint representation has polynomial growth and a p-adic Lie group of polynomial growth is type R. As a consequence we get that a p-adic algebraic group G has either polynomial growth or has a compact neighborhood of identity in G having exponential growth according to G is of type R or not. This growth dichotomy is known to be valid for connected groups (see [18]) but there are discrete groups of intermediate growth (see [9]). So, it is interesting to see growth dichotomy for p-adic algebraic groups.

We now describe a class of groups which will be needed in the sequel.

Definition 1.6 A locally compact group G is called *pro-discrete* if there exists a basis at e consisting of compact open normal subgroups and G is called *sigma-pro-discrete* if every compactly generated closed subgroup of G is pro-discrete.

We now produce p-adic Lie groups that are sigma-pro-discrete which will be needed in the sequel.

Proposition 1.1 *Let G be a p-adic Lie group of type R. Then G is sigma-pro-discrete.*

Proof Suppose G is a p-adic Lie group of type R. Let H be a compactly generated closed subgroup of G. By Corollary to Theorem 1, Section 9, Chapter V of [29], H is also a p-adic Lie group. Now by Theorem 1 of [24], G is non-contracting, hence H is non-contracting. Again by Theorem 1 of [24], H is of type R. Then by Theorem 1 of [24] and by Theorem 5.2 of [14], H is pro-discrete. Thus, G is sigma-pro-discrete.

1.2 Recurrence of random walks

We next consider the recurrence of random walks defined on p-adic Lie groups. A left random walk on a locally compact group G starting at a point $g \in G$ is defined to be

$$Y_0 = g \text{ and } X_n = Y_n Y_{n-1} \cdots Y_2 Y_1 Y_0, \quad n \geq 0$$

where $(Y_n)_{n \geq 1}$ is a sequence of independent and identically distributed random variables with common law μ and the sequence $(X_n)_{n \geq 0}$ is called the (left) random walk on G defined by μ starting at $g \in G$. When $g = e$, we say that (X_n) is a random walk on G defined by μ. We first recall the following notions.

Definition 1.7 Let μ be a probability measure on G. We say that μ is *adapted* if the closed subgroup generated by the support of μ is G.

Definition 1.8 A random walk (X_n) defined by an adapted probability measure $\mu \in \mathcal{P}(G)$ is said to be *recurrent* if

$$\sum_{n \geq 0} P[X_n \in U] = \infty$$

for every neighborhood U of e in G and a locally compact group G is called *recurrent* if there exists an adapted probability measure $\mu \in \mathcal{P}(G)$ such that random walk defined by μ is recurrent.

For basic results in recurrent random walks on groups and its significance see [13], [27] and [31]. In [8], R. M. Dudley proved that an abelian group is recurrent if and only if it is of rank at most two. Guivarc'h and Keane formulated a conjecture for general locally compact groups which resembles Dudley's result and their precise conjecture is that a locally compact group is recurrent if and only if it has polynomial growth of degree at most two, that is, for every compact neighborhood K of identity in G there exists a constant a such that $m(K^n) \leq an^2$ for all $n \geq 1$ (see [12]). An affirmative answer to this conjecture for connected Lie groups is given in [13]: see also [2] and [3].

In [19], Kesten questioned whether a countable group admitting a recurrent random walk can have non-exponential growth and Varapoulous answered this question (see [32]) which proves Guivarch-Keane conjecture for finitely generated groups.

In Section 3 the afore-discussed Guivarc'h-Keane's growth conjecture is solved for groups whose compactly generated open subgroups admit compact open normal subgroups. We apply this to prove the growth conjecture for p-adic Lie groups. We also show that growth conjecture is valid for groups of polynomial growth whose connected component of identity is compact.

1.3 The Choquet-Deny Theorem

We next consider the Choquet-Deny Theorem.

Definition 1.9 For $\mu \in \mathcal{P}(G)$, a bounded continuous function f on G is called μ-harmonic if

$$\mu * f(x) = \int f(g^{-1}x)d\mu(g) = f(x)$$

for all $x \in G$.

We are interested in studying measures that do not admit non-constant bounded continuous harmonic functions. This type of result is known as the Choquet-Deny Theorem and we say that the Choquet-Deny Theorem is valid for a measure $\mu \in \mathcal{P}(G)$ if μ does not admit non-constant continuous bounded harmonic functions and we say that the Choquet-Deny Theorem is valid for a group G if the Choquet-Deny Theorem is valid for any adapted probability measure on G: see [17] and the references therein for results on the Choquet-Deny Theorem for various classes of locally compact groups. In Section 4 we prove the Choquet-Deny Theorem for compactly generated p-adic Lie groups of polynomial growth and also show that polynomial growth is necessary and sufficient for the validity of the Choquet-Deny Theorem for all spread-out probability measures on Zariski-connected p-adic algebraic groups: a measure μ on a locally compact group G with a left-invariant Haar measure m is called *spread-out* if there exists $n \geq 1$ such that μ^n is not singular with respect to m.

2 Type R and growth

In this section we explore the relation between growth properties of p-adic Lie groups and the eigenvalues of the adjoint representation of the group, that is we study the relation between growth and type R.

Theorem 2.1 *Let G be a p-adic Lie group.*

(1) *Suppose G is of type R and Ad (G) is closed. Then G has polynomial growth if and only if* $\ker(\mathrm{Ad})$ *has polynomial growth. In particular, if G is of type R such that Ad (G) is closed and G is Ad-regular, then G has polynomial growth.*

(2) *Suppose G is not of type R. Then there exists a compact neighborhood of e having exponential growth.*

(3) *Suppose G is not of type R and G is compactly generated. Then any generating compact neighborhood of e has exponential growth.*

(4) *In particular, any p-adic Lie group of polynomial growth is of type R.*

Remark 2.1 Since totally disconnected groups have compact open subgroups, no totally disconnected group can have exponential growth and so we consider the exponential growth of particular compact neighborhoods.

Proof Let G be a p-adic Lie group and Z be the kernel of the adjoint representation of G. Suppose Ad (G) is closed and G is of type R. Let V be a compact symmetric neighborhood of e. Let G_0 be the group generated by V and Z. Then G_0 is an open subgroup of G and hence the adjoint representation of G_0 is the restriction of the adjoint representation of G to G_0. This implies that G_0 is also a p-adic Lie group of type R. Since Ad (G) is closed and Ad (G_0) is an open subgroup of Ad (G), Ad (G_0) is also closed. Since $Z \subset G_0$, Ad $(G_0) \simeq G_0/Z$ which is compactly generated. It follows from Theorem 1 of [24] and Theorem 1 of [22], that Ad (G_0) is compact. Now the result follows from Theorem 1.4 of [11] (see also Theorem 3.4 of [15]). This proves (1).

Suppose G is not of type R. By Theorem 1 of [24], there exist x and g_0 in G such that $x^{-n}g_0x^n \to e$ as $n \to \infty$ and $g_0 \neq e$. Let α denote the inner automorphism defined by x. Let $C(\alpha) = \{g \in G \mid \alpha^n(g) \to e \text{ as } n \to \infty\}$, $C(\alpha^{-1}) = \{g \in G \mid \alpha^{-n}(g) \to e \text{ as } n \to \infty\}$ and $M(\alpha) = \{g \in G \mid (\alpha^n(g))_{n\in\mathbb{Z}} \text{ is relatively compact }\}$. Then by Theorem 3.5 of [33], we have

(i) $C(\alpha)$, $C(\alpha^{-1})$ and $M(\alpha)$ are closed subgroups,

(ii) $M(\alpha)$ normalizes $C(\alpha)$ and $C(\alpha^{-1})$ and

(iii) the product map $C(\alpha) \times M(\alpha) \times C(\alpha^{-1}) \to G$ is a homeomorphism onto an open subset of G.

Now by (i) and Proposition 2.1 of [33], the restriction of α to $C(\alpha)$ and the restriction of α^{-1} to $C(\alpha^{-1})$ are compactly contracting. By 3.1 and 3.2 of [30] there exist a compact open subgroup K_1 of $C(\alpha^{-1})$ and a compact open subgroup K_2 of $C(\alpha)$ such that

$\alpha^{-n}(K_1) \downarrow e$ and $\alpha^n(K_2) \downarrow e$ as $n \to \infty$. Let K_3 be a compact open subgroup of M normalized by x. By (iii) we get that $K_1K_2K_3$ is a compact neighborhood of e in G. Take $K = K_1K_2K_3 \cup \{x, x^{-1}\}$. Then for any $n \geq 1$,

$$\begin{array}{rcl} K^{2n+2} & \supset & (x^n)(K_1K_2)(x^{-n})(K_2K_3) \\ & = & (x^nK_1x^{-n})(x^nK_2x^{-n})K_2K_3 \\ & = & \alpha^n(K_1)\alpha^n(K_2)K_2K_3 \\ & = & \alpha^n(K_1)K_2K_3 \end{array}$$

because $K_2 \supset \alpha^n(K_2)$.

Now by 3.1 of [30] $\alpha^{n-1}(K_1)$ is normal in $\alpha^n(K_1)$ for all $n \geq 1$. Let $a = |\alpha(K_1)/K_1|$, the number of elements in the quotient group $\alpha(K_1)/K_1$. Then $a = |\alpha^n(K_1)/\alpha^{n-1}(K_1)|$ for any $n \geq 1$. Since $g_0 \neq e \in C(\alpha^{-1})$, $C(\alpha^{-1})$ is a non-trivial unipotent group (see Theorem 3.5 (ii) of [33]) and hence K_1 is non-trivial. Since $\alpha^{-n}(K_1) \downarrow e$ as $n \to \infty$, $a > 1$.

Suppose for some elements g and h in $C(\alpha^{-1})$ and for some $i \geq 0$, $g\alpha^i(K_1)K_2K_3 \cap h\alpha^i(K_1)K_2K_3 \neq \emptyset$, then $\alpha^i(K_1)h^{-1}g\alpha^i(K_1) \cap K_2K_3K_2 \neq \emptyset$. Since $MC(\alpha) \cap C(\alpha^{-1}) = (e)$ by (iii), we get that $e \in \alpha^i(K_1)h^{-1}g\alpha^i(K_1)$ and hence $g\alpha^i(K_1) = h\alpha^i(K_1)$. This implies that for $n \geq 1$, $m(\alpha^n(K_1)K_2K_3) = am(\alpha^{n-1}(K_1)K_2K_3)$ and hence $m(K^{2n+2}) \geq a^nm(K_1K_2K_3) = a^nm(K)$. Now by taking $c = m(K) > 0$, we get that $m(K^{2n+2}) > ca^n$. This shows that K has exponential growth. Thus proving (2).

Suppose G is not of type R and V is a compact neighborhood of e generating G. Let K be as in (2). Then since $K \subset G = \cup_{n\geq 1}V^n$, $K \subset V^i$ for some $i \geq 1$. This implies that for any $n \geq 1$, $K^n \subset V^{ni}$. Thus, V also has exponential growth. This proves (3).

Let G be a p-adic Lie group of polynomial growth. Suppose G is not of type R. Then by (2) there exists a compact neighborhood V of e in G such that $m(V^n) \geq c_1a^n$ for all $n \geq 1$ where $c_1 > 0$ and $a > 1$. Since G has polynomial growth, there exist $c_2 > 0$ and an integer $k \geq 0$ such that $m(V^n) \leq c_2n^k$ for all $n \geq 1$. This implies that the sequence $(\frac{a^n}{n^k})$ is bounded. This is a contradiction to $a > 1$. Thus, any p-adic Lie group of polynomial growth is of type R.

Remark 2.2 Theorem 2.1 says that any p-adic Lie group of polynomial growth should be of type R. This may also be proved in the following way. For any $x \in G$, let K be a compact symmetric neighborhood of e containing x, then the group generated by K, say H, is an open subgroup of G and hence H also has polynomial growth. Now by Theorem 2 of [20], H has a compact open normal subgroup. In particular, there exists a compact open subgroup in G normalized by x. This shows that G is of type R (see Theorem 1 of [24]). (In general, we may prove in an exactly similar way that totally disconnected groups of polynomial growth are uniscalar in the sense of [14]).

Remark 2.3 The following example shows that the afore-mentioned result, that is (4) of Theorem 2.1 is in sharp contrast to the real Lie group situation. Let T be the two-dimensional torus and α be an automorphism of T given by the 2×2 integer matrix

$$\begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}.$$

Then there exists an eigenvalue of α of absolute value strictly less than one. This implies that for a proper connected subgroup D of T, $\alpha^n(x) \to e$ as $n \to \infty$ for all $x \in D$. This shows that the semi-direct product of $\mathbb{Z}$ and T, say G, where the $\mathbb{Z}$-action on T is given by α is not of type R but G is a real Lie group of polynomial growth because G has a compact normal subgroup T such that $G/T \simeq \mathbb{Z}$. It may also be pointed out that here - in contrast to the p-adic situation - the contractible parts $C(\alpha)$ and $C(\alpha^{-1})$ are not closed.

We now prove a dichotomy result on growth of p-adic Lie groups.

Corollary 2.1 *Let G be a p-adic Lie group.*

(i) If G is Ad-regular and Ad (G) is closed, then G has polynomial growth or there are compact neighborhoods of e having exponential growth according to G is of type R or not.

(ii) If G is a p-adic algebraic group, then the conclusion of (i) holds. In case G is a p-adic algebraic group of polynomial growth, G is a compact extension of its nilradical and the degree of growth is the dimension of the maximal split torus in its nilradical.

Proof Suppose G is Ad-regular and Ad (G) is closed. Then (i) follows from Theorem 2.1.

We now prove (ii). Suppose G is a Zariski-connected algebraic group. Then G is Ad-regular and Ad (G) is closed. So, first part of (ii) follows from (i). Suppose G has polynomial growth. Then by Theorem 2.1, G is of type R, hence by Theorem 2 of [24], G is a compact split extension of its nilradical. This implies that the degree of growth of G is equal to the degree of growth of its nilradical. So we may assume that G is a nilpotent group. Then G has a subgroup of the form T_sU where T_s is the maximal split torus in G and U is the unipotent radical of G such that G/T_sU is compact (see 10.6, Chapter III of [4]). So, the degree of growth of G is same as the degree of growth of T_sU. Now any compact neighborhood V of e in T_sU is contained in a subgroup of the form T_sK for some compact subgroup K of U. So, the growth of V is equal to the growth of $V \cap T_s$. It is easy to see that $T_s \simeq M \times \mathbb{Z}^k$ where M is a compact group and k is the dimension of T_s (in fact, $M = \{x \in \mathbb{Q}_p \mid |x| = 1\}^k$). Thus, the degree of growth is equal to the dimension of the maximal split torus.

Suppose G is any p-adic algebraic group. Let G_0 be the Zariski-connected component of e in G. Then G/G_0 is finite. So, G has polynomial growth if and only if G_0 has polynomial growth (see Theorem 1.4 of [10] or Theorem 3.4 of [15]) and G is of type R if and only if G_0 is of type R (see Theorem 1 of [24]). Now, the result follows from the previous case.

3 Recurrent groups

In this section we study recurrent groups for certain class of totally disconnected groups which includes p-adic Lie groups and we prove Guivarc'h-Keane conjecture for totally disconnected groups of polynomial growth and p-adic Lie groups. We first prove results on finiteness of second moment.

Definition 3.1 Let G be a compactly generated group. A probability measure μ on G is said to have *finite second moment* if for a generating compact neighborhood V of e,

$$M_2(\mu) = \sum_{n\geq 1} n^2 \mu(V^n \setminus V^{n-1}) < \infty.$$

Suppose G is a compactly generated group and K is a compact symmetric neighborhood of e generating G. Then define

$$\phi_K(x) = \inf\{n \mid x \in K^n\}$$

for any $x \in G$. Then ϕ_K is a non-negative Borel function on G such that for $x, y \in G$, $\phi_K(xy) \leq \phi_K(x) + \phi_K(y)$, that is, ϕ_K is subadditive on G. It is easy to see that a measure $\mu \in \mathcal{P}(G)$ has finite second moment if and only if $\int \phi_K(x)^2 d\mu(x) < \infty$ for some compact symmetric generating neighborhood K of e.

We first make the following observations.

Lemma 3.1 *If μ is a probability measure on a compactly generated group G with compact support, then μ has finite second moment.*

Proof Let K be a compact symmetric neighborhood of identity generating G. Then ϕ_K is bounded on compact subsets of G. Thus, $\int \phi_K(x)^2 d\mu(x) < \infty$ for any μ with compact support.

It may also be easily seen that for a compactly generated group G admitting compact open normal subgroups, a measure $\mu \in \mathcal{P}(G)$ has finite second moment if and only if for any compact open normal subgroup K of G, the projection of μ onto G/K has finite second moment.

It follows easily from the result on finitely generated groups that Guivarc'h-Keane conjecture is valid for compactly generated groups admitting compact open normal subgroups. It is a well-know fact that totally disconnected locally compact groups have a basis of compact open subgroups at identity. We now prove Guivarc'h-Keane conjecture in case a compact open normal subgroup exists for every compactly generated open subgroup (our proof makes use of some of the ideas of [8]). The following non-degeneracy condition is useful for recurrence of random walks.

Definition 3.2 A measure $\mu \in \mathcal{P}(G)$ is called *irreducible* in G if $G = \overline{\cup_{n\geq 0} S^n}$, that is, the closed subsemigroup generated by S is G where S is the support of μ.

It may be easily seen that irreducible measures are adapted and any symmetric adapted measure is irreducbile.

Proposition 3.1 *Let G be a locally compact σ-compact group in which every compactly generated open subgroup of G admits compact open normal subgroups. Then G is a recurrent group if and only if G has polynomial growth of degree at most two.*

Proof Since G is a locally compact σ-compact group, there exists a compact normal subgroup K of G such that G/K is second countable. Now, any compactly generated open subgroup of G/K lifts to a compactly generated open subgroup of G and hence any compactly generated subgroup of G/K also admits compact open normal subgroups. It may be easily seen that G has polynomial growth of degree at most two if and only if G/K has polynomial growth of degree at most two and G is recurrent if and only if G/K is recurrent. Thus, we may assume that G is second countable.

Suppose G has polynomial growth of degree at most two. Let V be a compact symmetric neighborhood of e in G. Let G_1 be the subgroup generated by V. Then G_1 is a compactly generated open subgroup of G, hence G_1 is of polynomial growth of degree at most two. Now, G_1 has a compact open normal subgroup K. This implies that G_1/K is a finitely generated group of polynomial growth of degree at most two. Let λ be a symmetric irreducible probability measure on G_1/K supported on a finite generating set. Then by Lemma 3.1, λ has finite second moment. Since λ is symmetric and irreducible, by Proposition 3.24 of [34], λ is recurrent.

Let μ_1 be the K-biinvariant measure on G_1 such that the projection μ_1 onto G_1/K is λ. This implies that μ_1 is also irreducible, symmetric and recurrent. If G is compactly generated, then we have a recurrent random walk on G by taking V to be a generating compact neighborhood of e in G. So, we may assume that G is not compactly generated.

Since G/G_1 is countable, there exists a sequence (a_n) of points in G such that with $a_1 = e$, if G_m is the subgroup generated by $a_1, a_2, \cdots, a_m$ and G_1, then $a_{m+1} \notin G_m$ and $G = \cup_{m\geq 1} G_m$.

For a given sequence $(q_m)_{m>1}$ with $0 < q_m < 1$ and $\Pi_{m=2}^{\infty}(1-q_m)$ converges, we define inductively measures μ_m in $\mathcal{P}(G_m)$ by

$$\mu_m = (1-q_m)\mu_{m-1} + \frac{1}{2}q_m[\delta_{a_m} + \delta_{a_m^{-1}}]$$

for $m > 1$. Let S_m be the support of μ_m for any $m \geq 1$. Then it is easy to see that

$$S_m = S_{m-1} \cup \{a_m, a_m^{-1}\}$$

for all $m > 1$. This implies since μ_1 is irreducible in G_1 and has compact support, that μ_m is irreducible in G_m and has compact support for all $m > 1$. Since μ_1 symmetric, it can be easily seen that μ_m is also symmetric for all $m > 1$.

We now claim that each μ_m is recurrent, in fact we claim that each μ_m has finite second moment in G_m. By our choice of μ_1 and μ_m for $m > 1$, μ_1 is recurrent and μ_m is an irreducible symmetric measure supported on a generating compact symmetric neighborhood of e in G_m. This implies by Lemma 3.1, that μ_m has finite second moment in G_m for all $m > 1$. For $m > 1$, G_m has compact open normal subgroups, let K_m be a compact open normal subgroup in G_m and μ'_m denote the projection of μ_m onto G_m/K_m. This implies that the irreducible symmetric measure μ'_m has finite second moment. Thus, by Proposition 3.24 of [34], μ'_m is recurrent and hence μ_m is recurrent for all $m > 1$.

We now define a measure μ on G by defining it on compact subsets of G. Let E be any compact subset of G. Since G is the increasing union of open subgroups (G_m), $E \subset G_m$ for some $m > 0$. Define

$$\mu(E) = \Pi_{i>m}(1-q_i)\mu_m(E).$$

Suppose E is contained in G_m and G_n for $m > n \geq 1$. Then $\mu_m(E) = (1-q_m)\mu_{m-1}(E) = (1-q_m)(1-q_{m-1})\mu_{m-2}(E) = \cdots = \Pi_{n<i\leq m}(1-q_i)\mu_n(E)$. This implies that $\Pi_{i>m}(1-q_i)\mu_m(E) = \Pi_{i>n}(1-q_i)\mu_n(E)$. Thus, $\mu(E)$ is well-defined. Let $0 < \epsilon < 1$ be given. Since $\Pi(1-q_i)$ converges, there exists a m such that

$$\Pi_{i\geq m+1}(1-q_i) > \sqrt{(1-\epsilon)}.$$

Since μ_m is a probability measure on the metric space G_m, μ_m is tight and hence there exists a compact subset B of G_m such that

$$\mu_m(B) > \sqrt{(1-\epsilon)}.$$

Then

$$\mu(B) = \Pi_{i \geq m+1}(1 - q_i)\mu_m(B) > 1 - \epsilon.$$

This implies that μ is a probability measure on G. Since the support of μ_m is contained in the support of μ for all $m \geq 1$, μ is adapted on G. We now choose the sequence (q_m) so that μ becomes recurrent. We first fix a sequence (r_n) such that $0 < r_n < 1$ and $\Pi_{i \geq 1}(1 - r_i)$ converges. Recall that K is a compact open subgroup contained in $G_1 \subset G_m$ for all $m \geq 1$. Since μ_1 generates a recurrent random walk on G_1, we can choose a number N_1 and a sequence of numbers $(A_{1,j})$ such that

$$0 < A_{1,j} < 1 \quad \text{and} \quad \Pi_{j \geq 1}(1 - A_{1,j})^{N_1} \sum_{k=1}^{N_1} \mu_1^k(K) > 1$$

and set $q_2 = \min\{r_1, A_{1,1}\}$. Now, μ_2 is fixed and μ_2 induces a recurrent random walk on G_2, so we can choose a number N_2 and a sequence of numbers $(A_{2,j})$ such that

$$0 < A_{2,j} < 1 \quad \text{and} \quad \Pi_{j \geq 1}(1 - A_{2,j})^{N_2} \sum_{k=N_1+1}^{N_2} \mu_2^k(K) > 1$$

and set $q_3 = \min\{r_2, A_{1,2}, A_{2,2}\}$. We now define q_m inductively. Suppose q_m is chosen as in the above process we now choose q_{m+1}. Since μ_m generates a recurrent random walk on G_m, we can choose a number N_m and a sequence of numbers $(A_{m,j})$ such that

$$0 < A_{m,j} < 1 \quad \text{and} \quad \Pi_{j \geq 1}(1 - A_{m,j})^{N_m} \sum_{k=N_{m-1}+1}^{N_m} \mu_m^k(K) > 1$$

and set $q_{m+1} = \min\{r_{m-1}, A_{1,m}, A_{2,m}, \cdots, A_{m,m}\}$. Since $0 < q_m \leq r_{m-1} < 1$ for all $m > 1$, $\Pi_{i \geq m}(1 - r_i) \leq \Pi_{i > m}(1 - q_i)$ for any $m \geq 1$. Since $\Pi_{i \geq 1}(1 - r_i)$ converges, $\Pi_{i \geq 2}(1 - q_i)$ converges.

We now claim that μ is recurrent. We first claim for any $n \geq 1$, by induction on k, that for all $k \geq 1$, $\mu^k(B) \geq \Pi_{i>n}(1 - q_i)^k \mu_n^k(B)$ for any Borel subset B contained in G_n. Let $n \geq 1$ be fixed. For $k = 1$, it follows from the definition of μ that $\mu(B) = \Pi_{i>n}(1 - q_i)\mu_n(B)$ for any Borel subset B contained in G_n. Suppose for any Borel subset B contained in G_n, $\mu^k(B) \geq \Pi_{i>n}(1 - q_i)^k \mu_n^k(B)$ for some $k \geq 1$. Then for any Borel subset B contained in G_n,

$$\begin{array}{rcl}
\mu^{k+1}(B) & = & \int \mu^k(gB) d\mu(g) \\
& = & \int_{G_n} \mu^k(gB) d\mu(g) + \int_{G \setminus G_n} \mu^k(gB) d\mu(g) \\
& \geq & \int \Pi_{i>n}(1 - q_i)^k \mu_n^k(gB) \Pi_{i>n}(1 - q_i) d\mu_n(g) \\
& = & \Pi_{i>n}(1 - q_i)^{k+1} \mu_n^{k+1}(B)
\end{array}$$

and hence for any $n \geq 1$ and for any Borel subset B contained in G_n

$$\mu^k(B) \geq \Pi_{i>n}(1-q_i)^k \mu_n^k(B)$$

for all $k \geq 1$. Recall that $K \subset G_n$ for all $n \geq 1$, we now get by taking $N_0 = 0$ that

$$\begin{array}{rcl}\sum_{k\geq 1} \mu^k(K) & = & \sum_{n\geq 1} \sum_{k=N_{n-1}+1}^{N_n} \mu^k(K) \\ & \geq & \sum_{n\geq 1} \sum_{k=N_{n-1}+1}^{N_n} \Pi_{i>n}(1-q_i)^k \mu_n^k(K) \\ & \geq & \sum_{n\geq 1} \sum_{k=N_{n-1}+1}^{N_n} \Pi_{i\geq 1}(1-A_{n,i})^{N_n} \mu_n^k(K) = \infty\end{array}$$

since each of the term in the first sum is strictly greater than 1. This shows that μ is recurrent.

Conversely, suppose G is a recurrent group. Let V be a compact symmetric neighborhood of e in G. Let H be the subgroup generated by V. Since H is open in G, H is also recurrent. By assumption, H has a compact open normal subgroup K. Since H is recurrent, H/K is also recurrent. This implies that by Theorem 3.24 of [34], H/K has polynomial growth of degree at most two. Let $V' = VK$. Since $V \subset V'$ and the restriction of the Haar measure of G to H is a Haar measure on H, there exists a constant $c > 0$ such that $m(V^n) \leq m(V'^n) \leq cn^2$ for all $n \geq 1$. This proves that G has polynomial growth of degree at most two.

Corollary 3.1 *Suppose a locally compact σ-compact group G has polynomial growth and the connected component of identity in G is compact. Then G is a recurrent group if and only if the degree of growth is at most two.*

Proof Suppose G has polynomial growth and the connected component of identity in G is compact. We first prove that every compactly generated open subgroup of G admits compact open normal subgroups. Let H be a compactly generated open subgroup of G. Let H_0 be the connected component of identity in H. Then H_0 is also compact, hence H/H_0 has polynomial growth. This implies by Theorem 1 of [20] that H/H_0 has a compact open normal subgroup. Since H_0 is compact, H itself has a compact open normal subgroup. Now the result follows from Proposition 3.1.

We now prove the Guivarc'h-Keane conjecture for p-adic Lie groups.

Lemma 3.2 *Let G be a recurrent p-adic Lie group. For any continuous automorphism α of G, let $C(\alpha) = \{g \in G \mid \alpha^n(g) \to e \text{ as } n \to \infty\}$. Suppose α is an inner automorphism of G and the elements of $C(\alpha)$ and $C(\alpha^{-1})$ commute with each other. Then $C(\alpha) = (e)$.*

Proof Let $M(\alpha) = \{h \in G_1 \mid (\alpha^n(h))_{n\in\mathbb{Z}} \text{ is relatively compact }\}$. Then $M(\alpha) \cap C(\alpha)$ and $M(\alpha) \cap C(\alpha^{-1})$ are trivial and $C(\alpha)M(\alpha)C(\alpha^{-1})$ is an open subset of G (see [33]).

By assumption, elements of $C(\alpha)$ and $C(\alpha^{-1})$ commute with each other and hence $N(C(\alpha^{-1}))$ contains $C(\alpha)$ as well as $M(\alpha)$. Thus, $C(\alpha)M(\alpha)C(\alpha^{-1})$ is an open subgroup of G and $C(\alpha^{-1})$ is a normal subgroup of $C(\alpha)M(\alpha)C(\alpha^{-1})$. Since G is recurrent, the open subgroup $C(\alpha)M(\alpha)C(\alpha^{-1})$ is also recurrent and hence $C(\alpha)M(\alpha) \simeq C(\alpha)M(\alpha)C(\alpha^{-1})/C(\alpha^{-1})$ is also recurrent. This implies by [5] that $C(\alpha)M(\alpha)$ is unimodular. Let m be a biinvariant Haar measure on $C(\alpha)M(\alpha)$. Then by 3.1 and 3.2 of [30], there exists a compact open subgroup K_1 in $C(\alpha)$ such that $\alpha^n(K_1) \downarrow e$ and $\alpha(K_1)$ is normal in K_1. Let K_2 be a compact open subgroup in $M(\alpha)$ such that $\alpha(K_2) = K_2$. Since α is an inner automorphism, $m(K_1K_2) = m(\alpha(K_1K_2)) = m(\alpha(K_1)K_2) = am(K_1K_2)$ because for g and h in K_1, $g\alpha(K_1)K_2 \cap h\alpha(K_1)K_2 \neq \emptyset$ implies $g\alpha(K_1) = h\alpha(K_1)$ where $a^{-1} = |K_1/\alpha(K_1)|$, the number of elements in the quotient group $K_1/\alpha(K_1)$. Since $0 < m(K_1K_2) < \infty$, we have $a = 1$ which implies $K_1 = (e)$ and hence $C(\alpha) = (e)$.

Lemma 3.3 *Let G be a p-adic Lie group. Suppose Ad (G) is a p-adic Lie group of type R. Then G is also of type R.*

Proof Suppose Ad (G) is of type R. Let α be an inner automorphism of G and $C(\alpha) = \{g \in G \mid \alpha^n(g) \to e \ \text{ as } \ n \to \infty\}$ be the contractible part of α. Then by assumption, Ad $(C(\alpha))$ is trivial for any inner automorphism α of G. Since $C(\alpha)$ and $C(\alpha^{-1})$ are unipotent algebraic groups, elements of $C(\alpha)$ and $C(\alpha^{-1})$ commute with each other for any inner automorphism of G. By Lemma 3.2, we get that $C(\alpha)$ is trivial for any inner automorphism of G. Thus, G is of type R.

Proposition 3.2 *Let G be a p-adic Lie group. Then G is recurrent implies G is of type R.*

Proof We first assume that G is a linear group, that is, G is a closed subgroup of $GL(V)$ for some finite-dimensional vector space V. Suppose G is recurrent. Then by 5.23 of [27], G is an amenable subgroup of $GL(V)$. By Corollary 2 of [25], there exists a closed subgroup H of $GL(V)$ and a closed solvable normal subgroup S of H such that H/S is a compact group and $G \subset H$. For any $x \in G$, let $C(x) = \{h \in G \mid x^nhx^{-n} \to e \ \text{ as } \ n \to \infty\}$. Let $S_1 = [S,S]$ and $S_k = [S_{k-1}, S_{k-1}]$ for all $k > 1$. For each $k \geq 1$, let $\phi_k: G \to H/S_k$ be the restriction of the canonical projection and let G_k be the closure of $\phi_k(G)$. Then each G_k is recurrent. For any $x \in G$, let $C_k(x) = \{g \in G_k \mid \phi_k(x)^n g \phi_k(x)^{-n} \to e \ \text{ as } \ n \to \infty\}$ for $k \geq 1$. We now claim by induction that $C_k(x)$ is trivial for all $k \geq 1$ and all $x \in G$. Since H/S is compact and $C_k(x)$ is an unipotent p-adic algebraic group, we get that $C_k(x) \subset S/S_k$ for all $k \geq 1$ and all $x \in G$ (see Proposition 1.2 of [26]). Since H/S is compact and S/S_1 is abelian, we get that $C_k(x) \subset S_1/S_k$ for all $k \geq 1$ and all $x \in G$. In particular, we get that $C_1(x)$ is trivial for all $x \in G$. Suppose for some $i \geq 1$, $C_i(x)$ is trivial for all $x \in G$. Let $\psi\colon H/S_{i+1} \to H/S_i$ be the canonical projection onto H/S_i. It can be easily seen that $\psi(C_{i+1}(x)) \subset C_i(x)$ for all $x \in G$. Thus, $C_{i+1}(x) \subset S_i/S_{i+1}$ for all $x \in G$. Since S_i/S_{i+1} is abelian, elements of $C_{i+1}(x)$ and $C_{i+1}(x^{-1})$ commute with each other for any $x \in G$. Since G_{i+1} is recurrent, by Lemma 3.2, $C_{i+1}(x)$ is trivial for all $x \in G$. It follows by induction that $C_k(x)$ is trivial for all $k \geq 1$ and all $x \in G$. Since S is solvable, S_j is trivial for some $j \geq 1$ and hence $C(x) = C_j(x)$ is trivial for all $x \in G$. Thus, G is of type R.

Let G be any recurrent p-adic Lie group. Then Ad (G) is also recurrent. Then the closure of Ad (G) is also a recurrent group. Thus, by the previous case, closure of Ad (G) is of type R and hence Ad (G) is also of type R. Now by Lemma 3.3, G is of type R.

Theorem 3.1 *Let G be a p-adic Lie group. Then G is a recurrent group if and only if G is of polynomial growth of degree at most two.*

Proof Suppose G is a p-adic Lie group having polynomial growth of degree at most two. Then by Theorem 2.1 G is of type R. Thus, by Propositions 1.1 and 3.1, we get that G is a recurrent group.

Suppose G is a recurrent group. Then by Proposition 3.2, G is of type R. Thus, by Propositions 1.1 and 3.1, we get that G is of polynomial growth of degree at most two.

4 The Choquet-Deny Theorem

In this section we study the Choquet-Deny Theorem for p-adic Lie groups of polynomial growth. We first recall that a locally compact group G is called *FC^--nilpotent*, if there exists a series $(e) = G_n \subset G_{n-1} \subset \cdots \subset G_1 \subset G_0 = G$ of closed normal subgroups of G such that $\{x^{-1}gxG_{i+1} \mid x \in G\}$ is relatively compact in G_i/G_{i+1} for all $g \in G_i$ and for all i.

It may be easily seen from the results in [17] that the Choquet-Deny Theorem is valid for spread-out adapted probabilities on FC$^-$-nilpotent groups. Here we prove that the Choquet-Deny Theorem is valid for any adapted probability on compactly generated p-adic Lie group of polynomial growth (or equivalently FC$^-$-nilpotent groups by Theorem 1 of [21]).

Theorem 4.1 *Let G be a p-adic Lie group and μ be an adapted probability measure on G. Then if G is a compactly generated group of polynomial growth, then any continuous bounded μ-harmonic function on G is constant.*

Proof Suppose G is a compactly generated group of polynomial growth. Then by Theorem 2.1, G is of type R. By Theorem 1.1, G is pro-discrete, hence G has a basis

(K_i) of compact open normal subgroups of G. For $i \geq 1$, let m_i be the normalized Haar measure on K_i and for any continuous bounded function f on G, let $f_i(xK_i) = \int f(xk)dm_i(k)$ for any $x \in G$. Then for $i \geq 1$, f_i is a continuous bounded function on G/K_i. Let μ_i be the projection of μ onto G/K_i for $i \geq 1$. Then μ_i is an adapted probability measure on G/K_i for $i \geq 1$. If f is μ-harmonic, it may be easily seen that f_i is μ_i-harmonic on G/K_i for $i \geq 1$. For $i \geq 1$, G/K_i is a finitely generated group of polynomial growth and hence f_i is is constant on G/K_i (see [17]). Thus, any bounded continuous μ-harmonic function is constant on the cosets of K_i for $i \geq 1$.

Let f be a continuous bounded μ-harmonic function on G. Suppose f is not constant on G. Then we may assume that for some $x \in G$, $f(x)-f(e) > 0$. Since $g \mapsto f(xg)-f(g)$ is a continuous function on G, there exists a $j \geq 1$ such that $f(xk) > f(k)$ for all $k \in K_j$. But we have $\int f(xk)dm_j(k) = \int f(k)dm_j(k)$. This is a contradiction. Thus, f is constant on G.

We now produce a class of groups on which the Choquet-Deny Theorem fails for certain measures which may be compared with Corollary 3.15 of [17].

Lemma 4.1 *Suppose S is a Zariski-connected algebraic group such that $S = TU$ where T is a split torus and U be a unipotent normal subgroup. Then S/U_1 is of type R implies S is of type R where $U_1 = [U, U]$.*

Proof Let $U_1 = [U, U]$. Suppose S/U_1 is of type R. Let $U_0 = U$ and $U_k = [U, U_{k-1}]$ for $k \geq 1$. Let $m \geq 1$ be such that $U_m = (e)$ but $U_{m-1} \neq (e)$. Then each U_k is a normal subgroup of S. The proof is based on induction on m. Suppose $m = 1$. Then there is nothing to prove. Suppose $m > 1$. Let $U' = U/U_{m-1}$. Then by induction, $S' = TU'$ is of type R. Therefore, $aua^{-1}u^{-1} \in U_{m-1}$ for all $a \in T$ and $u \in U$. Now since U_{m-1} is contained in the center of U, for any $a \in T$, $u \mapsto aua^{-1}u^{-1}$ is a homomorphism of U into U_{m-1}. Since U_{m-1} is abelian and $U_{m-1} \subset U_1$, we have $aua^{-1}u^{-1} = e$ for all $a \in T$ and for all $u \in U_{m-1}$. Also the T action on U/U_{m-1} is also trivial. Since T is a split torus, T centralizes U. This implies that S is also of type R.

Lemma 4.2 *Let G be a Zariski-connected p-adic algebraic group. Suppose G is not of type R. Then there exists an open subsemigroup G_s of G such that*

(i) $G = G_s \cup G_s^{-1}$ and

(ii) any probability measure μ on G with support contained in G_s admits non-constant continuous bounded μ-harmonic functions.

Proof Let G be a Zariski-connected algebraic group. Suppose G is not amenable. Then every adapted probability measure on G admits non-constant harmonic functions (see

Corollary 43 of [1] or Proposition 1.9 of [28]): note that in [28] probability measures that do not admit non-constant continuous bounded harmonic functions are called ergodic. So, we may assume that G is amenable. Then reductive Levi subgroup of G is a compact extension of a torus in G. This implies that there exists a unipotent normal subgroup U of G, a split torus T in G and a compact algebraic subgroup K of G such that $G = KTU$ and K centralizes T (see 11.23, Chapter IV of [4]). Let $S = TU$. Then G/S is compact.

Suppose G is not of type R. Then S is not of type R. Let $U_1 = [U, U]$. Then by Lemma 4.1, S/U_1 is also not of type R. We now consider the action of G on U/U_1 induced by the conjugacy action, that is, $g(vU_1) = gvg^{-1}U_1$ for all $v \in U$. Since S/U_1 is not of type R there exists a $g_0 \in T$ such that $C(g_0) = \{v \in U/U_1 \mid g_0^n(v) \to e \ \text{ as } \ n \to \infty\}$ is non-trivial. Since T is in the center of KT, we get that $C(g_0)$ is invariant under the conjugacy action of KT. Since KT is a Zariski-connected reductive algebraic group, there exists a subspace V_1 of U/U_1 which is invariant under KT such that $U/U_1 = C(g_0) \oplus V_1$ (see Theorem 4(a), Chapter IV of [6]). Let U_2 be an unipotent normal subgroup of U (of G) such that $U_2/U_1 = V_1$. Then $C(g_0) = U/U_2$. Since S is not of type R, $C(g_0)$ is a non-trivial vector space on which G acts by automorphisms.

Since T is a split torus, $C(g_0)$ has a basis $\{e_1, \cdots, e_m\}$ consisting of eigenvectors of elements in T. Now for any $1 \leq i \leq m$, let $\chi_i: T \to \mathbb{Q}_p \setminus (0)$ be such that $g(e_i) = \chi_i(g)e_i$ for all $g \in T$. Let i be such that χ_i is non-trivial and let $V = \{v \in C(g_0) \mid g(v) = \chi_i(g)v \ \text{ for all } \ g \in T\}$. Since K centralizes T, V is invariant under K, hence V is G-invariant. As in the previous paragraph V has a G-invariant complement. Thus, $V = U/U_3$ for some unipotent normal subgroup U_3 of G (and hence of U).

Let $v_0 = g_0U_3 \in S/U_3$. Then for any $g \in G$, there exists a $g_1 \in KT$ and $g_2 \in U$ such that $g = g_2g_1$. Then

$$g(v_0)v_0^{-1} = gg_0g^{-1}g_0^{-1}U_3 = g_2g_1g_0g_1^{-1}g_2^{-1}g_0^{-1}U_3 = g_2g_0g_2^{-1}g_0^{-1}U_3 \in U/U_3 = V$$

for all $g \in G$. Now, define the following G-action on V, for $g \in G$, $g \cdot v$ is defined to be

$$g \cdot v = g(vv_0)v_0^{-1}$$

for all $v \in V$. It may be easily seen that $(g, v) \to g \cdot v$ is a well-defined action of G on U by homeomorphisms, in fact by rational morphisms.

Now for $v \in V$, $g_0^n \cdot v = g_0^n(v) \to e$ as $n \to \infty$. Let $x_0 \in U \setminus U_3$ and let $w_0 = x_0g_0x_0^{-1}g_0^{-1}U_3 \in V$. Then w_0 is a non-trivial element in V. Also for $g \in KT$ and $v \in U$,

$$g \cdot vU_3 = gvg^{-1}gg_0g^{-1}g_0^{-1}U_3 = gvg^{-1}U_3$$

because KT centralizes T. Thus, the KT-action on V is linear. Let $|| \cdot ||$ be a p-adic norm on V. Since K is compact and the action of K on V is linear we may assume that

$|| \cdot ||$ is invariant under K. Let $j \in \mathbb{Z}$ be such that $w_0 \notin M = \{v \in V \mid ||v|| \leq p^j\}$. Then M is a compact open K-invariant subgroup of V. Now for any $g \in T$ and any $v \in U$, $g \cdot vU_3 = g(v)U_3 = \chi_i(g)vU_3$. This implies that for $g \in T$, either g or g^{-1} fixes M according to $|\chi_i(g)| \leq 1$ or $|\chi_i(g)| \geq 1$. Let $G_s = \{g \in G \mid g \cdot M \subset M\}$. Then G_s is an open subsemigroup in G containing K and U. Also for $g \in T$ either $g \in G_s$ or $g^{-1} \in G_s$. Thus, $G = G_s \cup G_s^{-1}$.

Let μ be any probability measure on G with support contained in G_s. Let

$$\rho_n = \frac{1}{n} \sum_{i=1}^{n} \mu^i * \delta_e$$

be a measure in $\mathcal{P}(V)$ for any $n \geq 1$. Then $\rho_n(M) = 1$ for all $n \geq 1$. This implies since M is compact that (ρ_n) is relatively compact in $\mathcal{P}(V)$. Let $\rho \in \mathcal{P}(V)$ be a limit point of (ρ_n). Then $\mu * \rho = \rho$.

Let $\phi: V \to [0,1]$ be such that $\phi(g) = 1$ for all $g \in M$ and $\phi(g) = 0$ for all $g \notin M$. Then ϕ is a continuous function with compact support. Now, let

$$\psi(g) = \int \phi(g \cdot v) d\rho(v) = \int_M \phi(g \cdot v) d\rho(v)$$

for any $g \in G$. Then ψ is a continuous bounded μ-harmonic function on G. For any $g \in G_s$,

$$\psi(g) = \int_M \phi(g \cdot v) d\rho(v) = 1$$

and for any $v \in V$,

$$x_0 \cdot v = x_0(v)(x_0 g_0 x_0^{-1} g_0^{-1} U_3) = v w_0$$

because $x_0 \in U$. This implies that for $v \in M$, $\phi(x_0 \cdot v) = \phi(v w_0) = 0$ and hence $\psi(x_0) = 0$. Thus, ψ is a non-constant continuous bounded μ-harmonic function on G.

Theorem 4.2 *Let G be a Zariski-connected p-adic algebraic group. Then the following are equivalent:*

(1) G has polynomial growth;

(2) for any adapted spread-out probability measure μ on G, continuous bounded μ-harmonic functions are constant.

Proof Let G be a Zariski-connected p-adic algebraic group. Suppose G has polynomial growth. Then by Corollary 2.1 (ii), G is a compact extension of a nilpotent group. This implies that G is FC^--nilpotent. Then it may be easily seen from the results in [17] that the Choquet-Deny Theorem is valid for any spread-out probability on G. Thus, proving (1) implies (2).

Suppose G does not have polynomial growth, by Corollary 2.1 of (i), G is not of type R. By Lemma 4.2, there exists an open subsemigroup G_s of G satisfying conditions (i) and (ii) of Lemma 4.2. Let $\mu \in \mathcal{P}(G)$ be such that μ is absolutely continuous and the support of μ is all of G_s. Then by (i) μ is spread-out and adapted and by (ii) μ admits non-constant continuous bounded μ-harmonic functions. This proves (2) implies (1).

5 Remarks on exponential boundedness

We would like to make few remarks regarding p-adic analogue of results in [16].

Definition 5.1 A locally compact group G with a left-invariant Haar measure m is called *exponentially bounded* if $\lim m(V^n)^{\frac{1}{n}} = 1$ for any compact neighborhood V of e in G.

By Theorem I.1 of [11], for any compact neighborhood V of e in G, $\lim m(V^n)^{\frac{1}{n}}$ exists and $\lim m(V^n)^{\frac{1}{n}} \geq 1$ and hence a locally compact group G is either exponentially bounded or G has compact neighborhoods of e having exponential growth. It can be easily seen that any group of polynomial growth is exponentially bounded but the converse need not be true (see [9]).

It is shown in [16] that every open subsemigroup of an exponentially bounded locally compact group is amenable and the converse is true for connected groups (see Theorem 3.2 of [16]). Here, we prove the p-adic analogue of this result and the equivalence of translate property and exponential boundedness within the class of amenable groups.

Definition 5.2 A locally compact group G is said to have *translate property* if for every Borel subset $A \subset G$ with $m(A) \neq 0$ and every $f \in L^1(G)$ with compact support the condition that $\int_A f(xg)dm(x) > 0$ for all $g \in G$ implies $\int f(x)dm(x) > 0$.

Theorem 5.1 *Let G be a p-adic algebraic group. Then we have the following:*

(i) G is exponentially bounded if and only if every open subsemigroup of G is amenable;

(ii) an amenable G has translate property if and only if it is exponentially bounded.

Proof Let G be a p-adic algebraic group. Suppose G is exponentially bounded. Then by Theorem 1.2 of [16], every open subsemigroup of G is amenable. Conversely, suppose G is not exponentially bounded. Then there exists a neighborhood of e in G having exponential growth. This implies by Corollary 2.1 that G is not of type R. Let G_0 be the Zariski-connected component of e in G. Then G/G_0 is finite, hence G_0 is also not of type R. By Lemma 4.2, there exists a open subsemigroup G_s of G_0 such that

$G_0 = G_s \cup G_s^{-1}$ and any measure $\mu \in \mathcal{P}(G_0)$ with $\mu(G_s) = 1$ admits a non-constant continuous bounded μ-harmonic function. This shows that G_s generates G_0 and G_s does not support any probability measure in $\mathcal{P}(G_0)$ for which the Choquet-Deny Theorem is valid. It may be easily seen that G_s is σ-compact. Thus, by Theorem 2.4 of [16], G_s is not amenable. Since G_0 is open in G, G_s is an open non-amenable subsemigroup of G. This proves (i).

It can easily be seen that (ii) follows from Theorem 4.1 of [16] and (i).

6 Counter example

We now give examples to show that the conditions in Theorem 2.1 (1) and in Theorem 4.2 for p-adic Lie groups cannot be removed. It is easy to see that countable groups are not Ad-regular p-adic Lie groups but they are type R. It is known that a finitely generated group has polynomial growth if and only if it has a nilpotent subgroup of finite index (see Gromov's Theorem [10]). So the condition Ad-regular can not be removed.

Now let F be a free (non-abelian) group on two generators and U be a non-commutative unipotent algebraic group such that

(i) F-acts on U faithfully by automorphisms,

(ii) F-action on U has bounded orbits and

(iii) the F-action on the center of U is trivial.

Let G be the semi-direct product of F and U. Then G is a p-adic Lie group of type R. Suppose for $g \in G$, Ad (g) is trivial. Let $g = au$ for $a \in F$ and $u \in U$. Then Ad $(a) =$ Ad (u^{-1}). Since F is contained in a compact group, it may be easily shown that Ad (u) is trivial and hence Ad (a) is also trivial. Since U is an unipotent group, u is in the center of U. Since the K-action on U is faithful, $a = e$. Thus, the center of U contains the kernel of the adjoint representation of G. Since F-acts trivially on the center of U, the center of U is contained in the center of G. Thus, the kernel of the adjoint representation of G is the center of G. This shows that G is Ad-regular. It may be easily seen that Ad (G) is not closed. It is well known that F is not amenable and hence F does not have polynomial growth. Since $G/U \simeq F$, G also does not have polynomial growth. Also, since F has adapted probability measures that admit continuous bounded harmonic functions, the condition that G has polynomial growth can not be removed for the validity of the Choquet-Deny Theorem for p-adic Lie groups.

Let us consider the following p-adic Heisenberg group. Let U be the unipotent group of $(n+2)\times(n+2)$-unipotent matrices given by

$$\begin{pmatrix} 1 & x_1 & \cdots & x_n & x \\ 0 & 1 & \cdots & 0 & y_1 \\ . & . & \cdots & . & . \\ . & . & \cdots & . & . \\ 0 & 0 & \cdots & 1 & y_n \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

where x_i, y_i and x are all in $\mathbb{Q}_p$ for all $i \geq 1$. Then the group of automorphisms of U is given by

$$\{\begin{pmatrix} A & 0 \\ c & \det(A) \end{pmatrix} \mid A \in GL(2n, \mathbb{Q}_p) \text{ and } c \in \mathbb{Q}_p^{2n}\}$$

where c denotes a map from $\mathbb{Q}_p^{2n} \to \mathbb{Q}_p$. Let F be a free (non-abelian) group on two generators contained in a compact subgroup of $SL(2n, \mathbb{Q}_p)$. F may be choosen as the group generated by the matrices

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 1 \\ 0 & 1 & \cdots & 0 & 0 \\ . & . & \cdots & . & . \\ . & . & \cdots & . & . \\ . & . & \cdots & . & . \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ . & . & \cdots & . & . \\ . & . & \cdots & . & . \\ . & . & \cdots & . & . \\ 0 & 0 & \cdots & 1 & 0 \\ 1 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

We now treat F as a subgroup of automorphisms of U by identifying $\alpha \in F$ with the automorphism

$$\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}$$

of U. Then F acts on U faithfully by automorphisms and the action of F on the center of U is trivial. Thus, we have provided U and F satisfying the requirements (i), (ii) and (iii).

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