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On unitaries in Banach spaces

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Abstract. This is a write-up of the 'First Professor K. R. Unni Memorial Lecture' delivered at the Cochin University of Science and Technology during November 2004.

In the first part of the lecture we give complete and detailed proofs of certain geometric aspects of unitaries of C^*-algebras as can be found in standard sources like [9], [7]. In the second half of the lecture we consider for a Banach space E various notions of a unitary that have been recently introduced. We consider conditions under which various versions of the Russo-Dye theorem can be proved. We show that convex transitivity of E is equivalent to the validity of a Russo-Dye type theorem for algebraic unitaries, in the weak operator topology of $\mathcal{L}(E)$.

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1. Certain aspects of the geometry of C^*-algebras

Let X be a Banach space and let $\mathcal{L}(X)$ denote the space of bounded linear operators. This is a well-known and typical example of a Banach algebra. For example any Banach algebra B with a unit e is algebraicaly isometric (via for example the left multiplication operator) to a subalgebra of $\mathcal{L}(X)$ with e mapped to the identity operator I. In these lectures we only consider complex Banach spaces.

Let X_1 denote the closed unit ball of X. Recall that a $x \in X_1$ is an extreme point if $x = \lambda y + (1-\lambda)z$ where $y, z \in X_1$ and $\lambda \in [0,1]$ implies, $x = y = z$.

Proposition 1.1. *Let B be a Banach algebra with unit e. e is an extreme point of the unit ball of B.*

Proof. Let us first observe that I is an extreme point of $\mathcal{L}(X)_1$. Suppose $I = \lambda R + (1-\lambda)S$ for $R, S \in \mathcal{L}(X)_1$ and $\lambda \in [0,1]$. By taking adjoints we have, $I = \lambda R^* + (1-\lambda)S^*$. Now for any extreme point $x^* \in X_1^*$, $x^* = \lambda R^*(x^*) + (1-\lambda)S^*(x^*)$. Therefore $I = R^* = S^*$ on the extreme points of X_1^*. Hence we conclude from the Krein-Milman theorem that $I = R^* = S^*$ so that $I = R = S$. From our remarks above we get that the unit in a Banach algebra is an extreme point of the unit ball. □

We will see soon that e has geometric properties which are much stronger than being an extreme point.

Let us now consider $\mathcal{L}(H)$ for a Hilbert space H. Here we have an additional notion of an involution given by the Hilbert space adjoint of an operator. For any $T \in \mathcal{L}(H)$ and $x \in H$ since $\|T(x)\|^2 =< T(x), T(x) >=< x, T^*T(x) >$ we have $\|T\|^2 \leq \|T^*T\|$. Thus the involution is related to the norm by $\|T\| = \|T^*\|$, $\|T^*T\| = \|T\|^2$.

T is said to be an isometry if $\|T(x)\| = \|x\|$ for all x and is said to be a co-isometry if T^* is an isometry.

Remark 1.2. *The extreme points of $\mathcal{L}(H)_1$ are precisely isometries and co-isometries.*

A Banach algebra is said to be a C^*-algebra if there is an involution $*$ such that $\|x^*\| = \|x\|$ and $\|xx^*\| = \|x\|^2$ for all x. A well-known result of Gelfand and Naimark asserts that any C^* algebra is $*$-isometric to a $*$-closed subalgebra of a $\mathcal{L}(H)$. Our next result gives an algebraic description of extreme points of the unit ball of a unital C^*-algebra.

Theorem 1.3. *Let A be a C^*-algebra with unit e. $b \in A_1$ is an extreme point if and only if $(e - b^*b)A(e - bb^*) = \{0\}$.*

Proof. We will only prove that the condition implies that b is an extreme point. Note that $0 = \|b(e - b^*b)b^*(e - bb^*)\| = \|(b^* - b^*bb^*)|^2$. Thus $b^* = b^*bb^*$. Similarly $b = bb^*b$. Let $p = b^*b$ and $q = bb^*$. Then p, q are projections. Thus p is the identity of the Banach algebra pAp and hence by our proposition is an extreme point of the unit ball.

Now suppose $b = \lambda a + (1-\lambda)c$ for some $a, c \in A_1$ and $\lambda \in [0,1]$. Now as $p = b^*bp = \lambda b^*ap + (1-\lambda)b^*cp$ we get $p = b^*ap = b^*cp$. Also $(b-ap)^*(b-ap) = p - b^*ap - pa^*b + pa^*ap = -p(e-a^*a)p$ and hence is the 0 vector. Thus $b = ap$ and similarly $b = qa$. Finally $a = a - ((e-p)a^*(e-q))^* = ap + qa - qap = b$ since $qap = bp = b$. □

We recall that $S = \{x^* \in A_1^* : x^*(e) = 1\}$ is called the State space. For example in the case of a commutative C^* algebra this corresponds to the set of probability measures on a compact set K.

For a $\phi \in A^*$ define, ϕ^* by $\phi^*(x) = \overline{\phi(x^*)}$. Then $\phi^* \in A^*$ and ϕ is said to be self adjoint if $\phi = \phi^*$. The equation $\phi = \frac{\phi+\phi^*}{2} + \frac{\phi-\phi^*}{2}$ decomposes ϕ into self-adjoint and skew-adjoint parts. One can show that any $\phi \in A^*$ can be decomposed as $\alpha_1\phi_1 + \alpha_2\phi_2 + \alpha_3\phi_3 + \alpha_4\phi_4$ where $\phi_j \in S$ and α_j's are scalars. In other words the vector space spanned by S, $spanS = A^*$.

$u \in A$ is said to be a unitary if $uu^* = u^*u = e$. It is easy to see that $a \to ua$ is a surjective isometry of A mapping e to u. Thus u has all the geometric properties that the identity has. In particular if $S_u = \{x^* \in A_1^* : x^*(u) = 1\}$ then $spanS_u = A^*$.

We are now ready to prove the main theorem of this section which provides a description of unitaries involving only Banach space theoretic conditions.

Theorem 1.4. *Let A be a unital C^*-algebra. Suppose u is a unit vector such that $spanS_u = A^*$. Then u is a unitary.*

Proof. We first note that u under the canonical embedding is in fact an extreme point of A_1^{**}. If $u = \frac{\Lambda_1+\Lambda_2}{2}$ for some $\Lambda_j \in A_1^{**}$ then for any $x^* \in S_u$, $1 = \frac{\Lambda_1(x^*)+\Lambda_2(x^*)}{2}$. Thus $\Lambda_1 = \Lambda_2$ on S_u and hence u is an extreme point.

Thus as in the proof of the Proposition if $q = uu^*$ then $(e-q)u = 0 = u^*(e-q)$. Now for any complex number λ and $x^* \in S_u$, $|1 + \lambda x^*(e-q)|^2 = |x^*(u + \lambda(e-q))|^2 \leq \|u + \lambda(e-q)\|^2 = \|u^*u + |\lambda|^2(e-q)\| \leq 1 + |\lambda|^2$.

In particular letting $\lambda = \overline{x^*(e-q)}$ we get that $x^*(e-q) = 0$. Since this is true for all $x^* \in S_u$ we get that $uu^* = e$. Similarly one can show that $u^*u = e$. Thus u is a unitary. □

Let A be a Banach algebra with unit e. Let $x \in A$ be an invertible element such that $\|x\| = 1 = \|x^{-1}\|$. Note that $a \to ax^{-1}$ is an onto isometry of A mapping x to e. Thus x has the same geometric properties as e. For the Banach algebra $\mathcal{L}(X)$ these elements are precisely isometries.

The next corollary describes unitaries of a C^*-algebra without involving the involution.

Corollary 1.5. *Let A be a C^*-algebra with unit e. If $x \in A$ is an invertible element such that $\|x\| = 1 = \|x^{-1}\|$ then x is a unitary. In particular any invertible extreme point is a unitary.*

Proof. As remarked above x has the same geometric properties as e. Since being a unitary is a geometric property, x is a unitary.

When x is invertible so is x^*. Now as in the proof of Theorem 1.3, $x = xx^*x$ thus by multiplying both sides by x^{-1} we get $x^{-1} = x^*$. Hence $\|x^{-1}\| = 1$ so that x is a unitary. □

2. Unitaries in general spaces

We are now ready to define two types of unitareis, see [2].

Definition 2.1. *Let A be a Banach algebra with identity e. An invertible unit vector $x \in A$ is said to be a algebraic unitary if $\|x^{-1}\| = 1$.*

Definition 2.2. *Let X be a Banach space $u \in S_X$ is said to be a unitary if the state space $S_u = \{x^* \in X_1^* : x^*(u) = 1\}$ spans X^*.*

Remark 2.3. *From the results proved so far we see that the two notions of unitary coincide in the case of a C^*-algebra. However they do not coincide for a general Banach algebra as illustrated by the following example.*

Example 2.4. *Let K be a compact Hausdorff space and let $A \subset C(K)$ be a closed subalgebra containing 1. If $f \in A$ is an algebraic unitary then since $\|f^{-1}\| = 1$ we get that $f^{-1} = \overline{f}$. Let Δ be the closed unit disc. Now let A be the closed sub algebra of $C(\Delta)$, consisting of functions that are analytic in the open unit disc. Consider A via the restriction map as an algebra on the unit circle Γ. Since $f(z) = z$ is a unitary in $C(\Gamma)$, we see from Proposition 2.7 below that that it is a unitary in A. As this is not invertible in A, it is not an algebraic unitary.*

Let X be a Banach space and let $u \in S_X$. As in the case of C^*-algebras, put $S_u = \{x^* \in X_1^* : x^*(u) = 1\}$. Define a semi-norm p on X by $p(x) = sup\{|x^*(x)| : x \in S_u\}$. The following theorem characterizes unitary in terms of this semi-norm.

Theorem 2.5. *u is a unitary if and only if p is a complete norm.*

Proof. Suppose u is a unitary. If $p(x) = 0$ then since $x^*(x) = 0$ for all $x^* \in S_u$ and hence for all $x^* \in X^*$, we get that $x = 0$. Thus p is a norm. Clearly $p(x) \leq \|x\|$.

Since S_u is a weak* compact set, consider $\Phi : X \to C(S_u)$ defined by $\Phi(x)(x^*) = x^*(x)$ for $x \in X$ and $x^* \in S_u$. Since $\Phi(u) = 1$ clearly Φ^* maps probability measures onto S_u. Therefore by our assumption Φ^* is onto. Hence by the closed range theorem, range of Φ is closed. Thus p is a complete norm.

Conversely suppose that p is a complete norm. In particular this norm is equivalent to the given norm. If Φ is defined as above then range of Φ is closed and as Φ is one-one we get again by the closed range theorem that Φ^* is onto. Now since 1 is a unitary in $C(S_u)$ we conclude that u is a unitary. □

Remark 2.6. *Let A be a Banach algebra with unit 1 then it is known that $p(x) \geq \frac{1}{e}\|x\|$. Thus the identity element of a unital Banach algebra is a unitary. By our earlier remarks it follows that any algebraic unitary is a unitary.*

Next set of propositions give properties of unitaries. The following proposition follows from the Hahn-Banach theorem.

Proposition 2.7. *Let $M \subset X$ be a closed subspace. If $x \in M \subset X$ is a unitary in X then it is a unitary in M.*

Remark 2.8. *Since in the above example $f(z) = z$ is not a unitary in $C(\Delta)$ we see that the converse of the above proposition is not true. However in some special cases unitaries do get lifted from a subspace to the whole space as illustrated below.*

For a Banach algebra A its bidual A^{**} is again a Banach algebra under the Arens product (either one) and algebraic unitaries of A continue to be algebraic unitaries of the bidual. In the case of a C^*-algebra the Arens product is unique and if $uu^* = e = u^*u$ in A the same relation holds in A^{**}.

Therefore it is natural to ask for a Banach space X if $x \in X$ is a unitary, is it also a unitary in X^{**}?

Before proving the affirmative result we recall the notion of a weak* unitary in a dual space from [2].

Definition 2.9. *$x^* \in X_1^*$ is said to be a weak* unitary if $S = \{x \in X_1 : x^*(x) = 1\}$ spans X.*

It follows from Theorem 4 in [1] that in a von Neumann algebra (i.e, C^*-algebras which are duals) any unitary is a weak* unitary. Thus all the notions defined so far coincide here. In [8] the authors have constructed a class of dual spaces where unitaries need not be weak* unitaries.

Proposition 2.10. *In a dual space X^* any weak*-unitary is a unitary.*

Proof. Let $x_0^* \in S_{X^*}$ be a weak* unitary. Let S be as above. As before let p be the semi-norm on X^* associated with $S_{x_0^*}$ as in the proof of Theorem 2.5 . We define a new semi-norm on X^* by $p'(x^*) = sup\{|x^*(x)| : x \in S\}$. Using the hypothesis as in the proof of Theorem 2.5 it is easy to see that p' is an equivalent norm on X^*. But as $S \subset S_{x_0^*}$ we have $p'(x^*) \leq p(x^*)$. Thus p is an equivalent norm. Therefore by Theorem 2.5 we get that x_0^* is a unitary.

□

Corollary 2.11. *Let $x \in S_X$ be a unitary. Then x is a unitary in X^{**}.*

Proof. We note that x is by definition a weak* unitary in X^{**}. Thus the conclusion follows from the above proposition. □

3. Russo-Dye type theorems

It is easy to produce unitaries in a unital C^*-algebra A . For any hermitian vector x (i,e., $x^* = x$), the exponential vector e^{ix} is a unitary. The well-known Russo-Dye theorem asserts that the unit ball of A is the norm closed convex hull of unitaries. In fact it can be shown that A_1 is the norm closed convex hull of $\{e^{ix} : x \in A,\ x\ hermitian\}$ [7], [9]. It is now a natural question to consider validity of Russo-Dye type results w. r. t various forms of unitaries we have defined here. This is now an active area of research and some of the versions can be found in [2], [4] and [5]. We shall now present two versions related to algebraic unitaries from [6].

The following form of the Russo-Dye theorem for algebraic unitaries follows easily from the Stone-Weierstrass theorem as the hypothesis implies that real-valued functions in A separate points of X.

Theorem 3.1. *Let $A \subset C(K)$ be a closed subalgebra separating points and containing 1. If A_1 is the norm closed convex hull of algebraic unitaries, then $A = C(K)$.*

For a Banach space E now consider the Banach algebra $\mathcal{L}(E)$. Here the algebraic unitaries are precisely the group of isometries. Thus a version of the Russo-Dye theorem is to ask for what Banach spaces E, is $\mathcal{L}(E)_1$ the norm closed convex hull of the group of isometries ? This investigation was done in [4]. The main result here says that if there is a biholomorphic automorphism of the open unit ball of E that is not an isometry then the validity of the Russo-Dye theorem of the above form implies that E is a Hilbert space.

For a Banach space E we next examine the weaker notion that $\mathcal{L}(E)_1$ is the closed convex hull in the weak operator topology (W. O. T) of the group of isometries. In order to relate to the structure of E we first prove a general result in $\mathcal{L}(X,Y)$ for Banach spaces X, Y. Let σ denote the weak operator topology on $\mathcal{L}(X,Y)$. For $x \in S_X$ and $f \in S_{Y^*}$, let $x \otimes f$ denote the functional $(x \otimes f)(T) = f(T(x))$. We recall that the weak operator topology is determined by these functionals.

Theorem 3.2. *Let $\mathcal{U} \subseteq S_{(\mathcal{L}(X,Y))}$ be a Γ invariant set. The following are equivalent.*

(*a*) $\mathcal{L}(X,Y)_1 = \overline{co}^{\sigma}(\mathcal{U})$

(*b*) *For every $\Lambda \in S_{X^{**}}$ we have,*

$$Y_1^{**} = \overline{co}^*(T^{**}\Lambda :\ T \in \mathcal{U})$$

(*c*) *For every $f \in S_{Y^*}$ we have,*

$$X_1^* = \overline{co}^*(T^*f :\ T \in \mathcal{U})$$

(*d*) *For every $x \in S_X$ we have,*

$$Y_1 = \overline{co}(Tx :\ T \in \mathcal{U})$$

Proof. $(a) \Rightarrow (c)$ Let $g \in X_1^*, x \in X$ and $\varepsilon > 0$. Take $f \in S_{Y^*}$ and an $y \in Y_1$ such that $|1 - f(y)| < \varepsilon$. By (a), there is a net $T_\alpha \in \overline{co}(\mathcal{U})$ such that $T_\alpha \to y \otimes g$ in W. O. T. Thus

$$|g(x) - f(T_\alpha x)| \leq |g(x)||1 - f(y)| + |f(y)g(x) - f(T_\alpha x)| \leq \varepsilon|g(x)| + |f((y \otimes g)(x)) - f(T_\alpha x)|$$

Hence the result.

$(d) \Rightarrow (a)$ Suppose there exists $T \in \mathcal{L}(X,Y)_1$ such that $T \notin \overline{co}^\sigma(\mathcal{U})$. Then by the separation theorem applied to the weak operator topology, we have $x \in S_X$ and $f \in S(Y^*)$ such that $re(f)(Tx) > \sup_{T \in \mathcal{U}} re(f(Tx))$. But by (d), $\{Tx : T \in \mathcal{U}\}$ is norming set for Y^*, and this contradicts the choice of T.

The statement (c) implies (b) follows from a weak*-separation theorem. As $T^{**} = T$ on X, and since weak* convergence for elements of X in X^{**} is same as weak convergence, (b) implies (d). □

Remark 3.3. (a) In the case $X = Y$, taking $\mathcal{U}$ to be the set of onto isometries of X we observe that condition (d) is the definition of X being convex transitive (see [8]) . Thus we get an equivalent formulation of convex transitivity of a Banach space in terms of its algebra of operators. Examples of convex transitive Banach space include Hilbert spaces, the space $C(K)$ where K is a Cantor set or the unit circle.

(b) It is known that (see [10]) for $X = \ell^1$, $\mathcal{L}(X)_1$ is the closed convex hull of its extreme points in W. O. T. It was shown in [2] that these are precisely the geometric unitaries of $\mathcal{L}(X)_1$. It is easy to see that ℓ^1 is not a convex transitive Banach space. From Theorem 3.2 it follows that $\mathcal{L}(X)_1$ is not even closed convex hull of onto isometries in W. O. T.

Thus we have an example of a Banach algebra in which Russo-Dye theorem holds for geometric unitaries w. r. t to a certain topology but fails to hold for algebraic unitaries. Also as noted in [10], $\mathcal{L}(\ell^1)_1$ is not the norm closed convex hull of its extreme points thus Russo-Dye theorem fails to hold w. r. t norm topology for either class of unitaries.

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