Representation Theory of Lie Algebras

Classification of Reductive Algebraic Groups
Workshop on Group Theory
18 Dec 2006 - 5 Jan 2007, Bangalore

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Contents

These notes are intended to be an introduction to the Representation theory of Lie algebras. The subject is very rich and there are good books available e.g. [**Hu2**], [**FH**], [**GW**] to which we refer most of the time. The interested reader is advised to refer to these books. In what follows I assume familiarity with Lie algebras specially Cartan decomposition of a semisimple Lie algebra.

CHAPTER 1

Representations and Weyl's Theorem

Here we give definition of Lie algebra and its representation. In this connection we also mention the Weyl's theorem concerning finite dimensional representations.

Definition 1. A **Lie algebra** L is a vector space over a field F together with a binary operation

$$[,]: L \times L \to L,$$

called the Lie bracket, which satisfies the following properties:

1. **Bilinearity:** For $a, b \in F$ and $x, y, z \in L$

$$[ax + by, z] = a[x, z] + b[y, z], [z, ax + by] = a[z, x] + b[z, y],$$

2. For all $x \in L$ we have $[x, x] = 0$,
3. **Jacobi identity:** $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in L$.

Note that the first and second properties together imply

$$[x, y] = -[y, x]$$

for all $x, y \in L$ (anti-symmetry). Conversely, the antisymmetry property implies property 2 above as long as F is not of characteristic 2. Also note that the multiplication represented by the Lie bracket is not in general associative, that is, [[x,y],z] need not equal [x,[y,z]].

Let L be a Semisimple Lie algebra over F (an algebraically closed field of characteristic 0). A vector space V (of infinite dimension unless stated otherwise) over F is called a **representation** (or an L**-module**) if we have a Lie algebra homomorphism $L \to \mathfrak{gl}(V)$. An L-module V is called **irreducible** if it has no proper submodule, i.e., no submodule other than 0 and L itself. A representation V is called **completely reducible** if V is a direct sum of irreducible L-submodules or equivalently each L-submodule has a direct complement.

The ad **Representation :** For a Lie algebra L the map $ad\colon L \to gl(L)$ defined by $ad(x)(y) = [x, y]$ is a representation of L. In the case L is semisimple we have a maximal toral subalgebra H contained in L. Since H consists of commuting semisimple elements it gives rise to a decomposition of L, called **Cartan**

decomposition, i.e.,

$$L = H \bigoplus \bigoplus_{\alpha \in \Phi} L_\alpha$$

where $\Phi \subset H^*$ is a root system and $L_\alpha = \{x \in L \mid h.x = \alpha(h)x \; \forall h \in H\}$ a root space corresponding to the root α. We will explicitly write down the roots in the case of classical Lie algebras later.

Representations of $\mathfrak{sl}(2)$: In the next chapter we will write down representations of $\mathfrak{sl}(2)$.

Contragradient Representation : Let V be an L-module. Then V^* is the dual or contragradient representation given by $(x.f)(v) = -f(x.v)$ for $x \in L, f \in V^*, v \in V$.

Tensor Product of Representations : Let V and W be L-modules. The Lie algebra L acts on $V \otimes W$ by $A.(v \otimes w) = Av \otimes w + v \otimes Aw$.

The finite dimensional representations can be broken in smaller representations for a semisimple Lie algebra.

Theorem 1.0.1 (Weyl's Theorem)**.** *Let V be a nonzero finite dimensional representation of a semisimple Lie algebra L. Then V is completely reducible.*

It need not be true for infinite dimensional representation. For example the representation $Z(\lambda)$ of $\mathfrak{sl}(2)$ considered in the section 2.2 when $\lambda + 1$ is a nonnegative integer.

CHAPTER 2

Representations of $\mathfrak{sl}(2)$

We begin with the simplest example of Lie algebra. It turns out that this example gives the general idea of the subject. The Lie Algebra

$$\mathfrak{sl}(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(F) \mid a + d = 0 \right\}$$

is a special linear algebra of dimension 3. The bracket operation is induced from matrix multiplication and is given by $[A, B] = AB - BA$ for $A, B \in \mathfrak{sl}(2)$. We choose a basis of $\mathfrak{sl}(2)$ as follows:

$$\left\{ x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, h \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$$

which satisfies $[h, x] = 2x, [h, y] = -2y, [x, y] = h$. In this chapter we describe all finite dimensional irreducible representations of this Lie algebra and give some infinite dimensional ones.

2.1. Classification of $\mathfrak{sl}(2)$-modules

Let V be an $\mathfrak{sl}(2)$-module, i.e., we have a Lie algebra homomorphism

$$\phi \colon \mathfrak{sl}(2) \to \mathfrak{gl}(V).$$

As h is semisimple we get a decomposition of V as follows:

$$V = \bigoplus_{\mu \in F} V_\mu$$

where $V_\mu = \{v \in V \mid h.v = \mu v\}$. Whenever $V_\mu \neq 0$ we call μ a **weight** and V_μ a **weight space**. How does x and y act on weight spaces? Let $v \in V_\mu$. Then

$$xv \in V_{\mu+2}, yv \in V_{\mu-2}.$$

Since V is finite dimensional there exists V_λ such that $V_{\lambda+2} = 0$. For such λ, any nonzero vector in V_λ will be called a **maximal vector** of weight λ.

Let V be an irreducible $\mathfrak{sl}(2)$-module. Let $v_0 \in V_\lambda$ be a maximal vector; set $v_{-1} = 0, v_i = \frac{1}{i!} y^i v_0$. Then for $i \geq 0$,

$$hv_i = (\lambda - 2i)v_i, yv_i = (i + 1)v_{i+1}, xv_i = (\lambda - i + 1)v_{i-1}.$$

Theorem 2.1.1. *With notation as above, we get the description of irreducible $\mathfrak{sl}(2)$-module as follows:*

(a) *relative to h, V is the direct sum of weight spaces $V_\mu, \mu = m, m-2, \dots, -(m-2), -m$, where $m+1 = \dim(V)$ and $\dim(V_\mu) = 1$ for each μ.*
(b) *V has (up to nonzero scalar multiples) a unique maximal vector, whose weight (called the highest weight of V) is m.*
(c) *The action of $\mathfrak{sl}(2)$ on V is given explicitly by the above formulas, if the basis is chosen in the prescribed fashion. In particular, there exists at most one irreducible $\mathfrak{sl}(2)$-module of each possible dimension $m+1, m \geq 0$.*

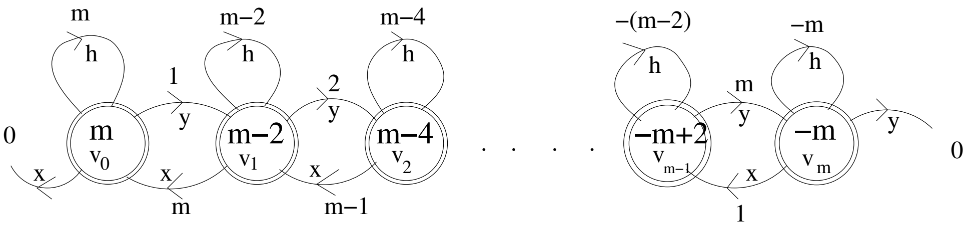
The theorem can be summarised in the matrix form as

$$x = \begin{pmatrix} 0 & m & 0 & \dots & 0 \\ 0 & 0 & m-1 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, y = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & & \dots & 0 \\ 0 & 2 & 0 & & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & m & 0 \end{pmatrix}$$

and

$$h \begin{pmatrix} m & & & & \\ & m-2 & & & \\ & & \ddots & & \\ & & & -m+2 & \\ & & & & -m \end{pmatrix}.$$

This information can be also put in picture.



In the view of Weyl's theorem we can describe all finite dimensional representations by taking direct sum of the irreducible ones described above.

Theorem 2.1.2. *Let V be any (finite dimensional) $\mathfrak{sl}(2)$-module. Then the eigenvalues of h on V are integers, and each occurs along with its negative (an equal number of times). Moreover, in any decomposition of V into direct sum of irreducible submodules, the number of summands is precisely $\dim(V_0) + \dim(V_1)$.*

We describe these representations in another way. Take the standard representation of $\mathfrak{sl}(2)$ on V, a 2-dimensional vector space with basis X, Y. Then $\mathfrak{sl}(2)$

acts on $V \otimes V$ by $A.(v \otimes w) = Av \otimes w + v \otimes Aw$. Moreover it acts on $Sym^2(V)$ which has a basis $\{X^2, XY, Y^2\}$. A simple calculation shows that this is the 3-dimensional irreducible representation obtained above. In general $Sym^n(V)$ with basis $\{X^n, X^{n-1}Y, \ldots, XY^{n-1}, Y^n\}$ is the $n+1$ dimensional irreducible representation of $\mathfrak{sl}(2)$. Notice that, in this case one representation (the standard one) generates all other finite dimensional irreducible representations.

2.2. More Representations of $\mathfrak{sl}(2)$

We give a method to construct some representations (a priori infinite dimensional) of $\mathfrak{sl}(2)$ and it turns out that this gives all finite dimension ones. In Chapter 4 we will see that this method is part of the general theory for any semisimple Lie algebra.

In what follows for every $\lambda \in F$ we associate an irreducible representation $V(\lambda)$ (possibly of infinite dimension) and classify among them the finite dimensional ones depending on λ. Let $Z(\lambda)$ be a vector space with countable basis $\{v_0, v_1, v_2, \ldots\}$. Define the action of $\mathfrak{sl}(2)$ by formulas: $hv_i = (\lambda - 2i)v_i, yv_i = (i+1)v_{i+1}, xv_i = (\lambda - i + 1)v_{i-1}$. Then,

(a) The space $Z(\lambda)$ is an $\mathfrak{sl}(2)$-module and every proper submodule contains at least one maximal vector.
(b) $Z(\lambda)$ is an irreducible representation if and only if $\lambda + 1$ is not a nonnegative integer.
(c) For r a nonnegative integer define a map $\phi\colon Z(\mu) \to Z(\lambda)$ by $v_i \mapsto v_{i+r}$ where $\mu = \lambda - 2r$. Then ϕ is an injective $\mathfrak{sl}(2)$-module homomorphism. In case $\lambda + 1 = r$ the $Im(\phi)$ and $V(\lambda) = Z(\lambda)/Im(\phi)$ are irreducible whereas $Z(\lambda)$ is not.

To prove part (b) if $\lambda + 1 = r$ is a nonnegative integer then $xv_r = 0$ and the subspace generated by $\{v_r, v_{r+1}, \ldots\}$ is an $\mathfrak{sl}(2)$-submodule of $Z(\lambda)$ and hence it's not irreducible. Conversely suppose $\lambda + 1$ is not a nonnegative integer. Let $v \in Z(\lambda)$ be a nonzero vector. Let $v = a_s v_s + \cdots + a_r v_r$ with $a_s, a_r \neq 0$. Let W be the subspace of $Z(\lambda)$ generated by v under the action of $\mathfrak{sl}(2)$, i.e., $\mathfrak{sl}(2)$-submodule generated by v. By applying h repeatedly on v we can assume $v_s \in W$. And applying y repeatedly we see $v_i \in W$ for all $i \geq s$. Look at $xv_s = (\lambda - s + 1)v_{s-1}$. Since $\lambda + 1$ is not a nonnegative integer $xv_s \in W$ implies $v_{s-1} \in W$. Applying this argument repeatedly we get $v_0 \in W$ hence $W = V$.

Proposition 2.2.1. *$V(\lambda)$ is finite dimensional if and only if λ is a nonnegative integer.*

Observe that we have associated an irreducible module $V(\lambda)$ to every element $\lambda \in H^*$ where H is an one dimensional space generated by h and moreover we categorise finite dimensional ones.

2.3. Weyl Group Action on the Weights

Let L be a Linear Lie algebra, i.e., $L \subset gl(V)$. For $x \in L$ suppose $ad(x)$ is nilpotent, say $(ad(x))^k = 0$. We can define

$$exp(ad(x)) = 1 + ad(x) + \cdots + (ad(x))^{k-1}/(k-1)!$$

and $exp(ad(x))$ is an automorphism of L. In fact,

$$exp(ad(x))(y) = (exp(x))y(exp(x))^{-1}.$$

To prove this note that $ad(x) = \lambda_x + \rho_{-x}$. These automorphisms are called **inner automorphism** and form a normal subgroup of Aut(L).

In case of $L = \mathfrak{sl}(2)$ with basis $\{x, y, h\}$ we define

$$\sigma = exp(ad(x))exp(ad(-y))exp(ad(x))$$

an element of Aut(L). This automorphism is same as (in view of above) conjugation by $s = exp(x)exp(-y)exp(x)$ on L. One can calculate and see that $s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\sigma(x) = -y, \sigma(y) = -x$ and $\sigma(h) = -h$ (i.e., $shs^{-1} = -h$).

Now let us consider V an irreducible representation of highest weight m of $L = \mathfrak{sl}(2)$. Suppose $m \geq 1$ so that the map $\phi\colon L \to \mathfrak{gl}(V)$ is injective. Then $exp(\phi(x))$ and $exp(\phi(y))$ are automorphisms of V. Let $\tau = exp(\phi(x))exp(\phi(-y))exp(\phi(x))$. Since the map ϕ is injective the action of τ on $\phi(h)$ is same as the action s on h in previous paragraph, i.e., $\tau\phi(h)\tau^{-1} = -\phi(h)$. Let V_{m-2i} be a weight space. Then $\tau(V_{m-2i}) \subset V_{-(m-2i)}$.

Proposition 2.3.1. *There exist an automorphism τ of V which maps positive weights to negative weights and vice-versa.*

CHAPTER 3

Root Space Decomposition

In this section we take another example of representations namely ad representation of a Lie algebra. We fix a semisimple Lie algebra L with a maximal toral subalgebra (or Cartan subalgebra) H. We have $ad\colon L \to \mathfrak{gl}(L)$. Since H consists of commuting semisimple elements of L we obtain a following decomposition (called **Cartan decomposition**) of L:

$$L = H \bigoplus \bigoplus_{\alpha \in \Phi} L_\alpha$$

where Φ is a root system and L_α a root space corresponding to the root α. Here we give description of this representation for the classical lie algebras.

3.1. Special Linear Algebra $A_l, l \geq 1$

Let V be a vector space of dimension $l+1$. The Lie Algebra $\mathfrak{sl}(V)$ or $\mathfrak{sl}(l+1)$ is

$$\mathfrak{sl}(l+1) = \{A \in M_{l+1}(F) \mid tr(A) = 0\}$$

called a special linear algebra.

A Basis and Dimension:

$$\mathfrak{B} = \{e_{i,j}(i \neq j, 1 \leq i, j \leq l+1), h_i = e_{i,i} - e_{i+1,i+1}\}(1 \leq i \leq l)$$

is a basis and it has dimension $l^2 + 2l$.

Bracket Operation: $[x, y] = xy - yx$ for $x, y \in \mathfrak{sl}(l+1)$.

A Maximal Toral Subalgebra:

$$H = \left\{ \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \ldots & 0 & a_{l+1} \end{pmatrix} \mid a_1 + \cdots a_{l+1} = 0 \right\}.$$

Notation: For $h \in H$ write $h = \sum_{i=1}^{l+1} a_i e_{i,i}$ and $\epsilon_i \in H^*$ is given by $\epsilon_i(h) = a_i$.

Cartan Decomposition:

$$sl(l+1) = H \bigoplus \bigoplus_{i>j} (F.e_{i,j} \bigoplus F.e_{j,i})$$

Root Vectors and Roots:

root vectors	roots
$e_{i,j}$ $(i \neq j, 1 \leq i, j \leq l+1)$	$\epsilon_i - \epsilon_j$

check the calculation: $ad(h)(e_{i,j}) = [h, e_{i,j}] = (a_i - a_j)e_{i,j} = (\epsilon_i - \epsilon_j)(h)e_{i,j}$ using $e_{i,j}e_{k,l} = \delta_{jk}e_{i,l}$.

A base and the Weyl group: The set of roots $\{\epsilon_i - \epsilon_{i+1}\ (1 \leq i \leq l)\}$ is a base and the Weyl group is the symmetric group S_{l+1}.

3.2. Orthogonal Algebra (odd dimension) $B_l, l \geq 2$

Let V be a vector space of dimension $2l+1$ with a symmetric bilinear form. We denote a vector in V by $x = (x_0, x_1, \ldots, x_i, \ldots, x_l, x_{l+1}, \ldots, x_{l+i}, \ldots, x_{2l})$. Let

$$s = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 0 & I_l \\ 0 & I_l & 0 \end{array}\right)$$

be the matrix of the bilinear form. The Lie Algebra $\mathfrak{o}(V)$ or $\mathfrak{o}(2l+1)$ is

$$\mathfrak{o}(2l+1) = \{x \in M_{2l+1}(F) \mid {}^txs = -sx\}$$

called orthogonal algebra. In fact $x \in \mathfrak{o}(2l+1)$ looks like

$$x = \left(\begin{array}{c|cc} 0 & b_1 & b_2 \\ \hline -{}^tb_2 & m & n = -{}^tn \\ -{}^tb_1 & p = -{}^tp & -{}^tm \end{array}\right).$$

Dimension: $2l^2 + l$.

Bracket Operation: $[x, y] = xy - yx$ for $x, y \in \mathfrak{o}(2l+1)$.

A Maximal Toral Subalgebra:

$$H = \left\{ \left(\begin{array}{c|cccccccc} 0 & & & & & & & & \\ \hline & a_1 & & & & & & & \\ & & a_2 & & & & & & \\ & & & \ddots & & & & & \\ & & & & a_l & & & & \\ & & & & & -a_1 & & & \\ & & & & & & -a_2 & & \\ & & & & & & & \ddots & \\ & & & & & & & & -a_l \end{array}\right) \right\}.$$

Notation: For $h \in H$ write $h = \sum_{i=1}^{l} a_i e_{i,i} - \sum_{i=1}^{l} a_i e_{l+i,l+i}$ and $\epsilon_i \in H^*$ is given by $\epsilon_i(h) = a_i$.

Root Vectors and Roots:

root vectors	roots
$X_{i,0} = e_{i,0} - e_{0,l+i}$ $(1 \leq i \leq l)$	ϵ_i
$X_{l+i,0} = e_{l+i,0} - e_{0,i}$ $(1 \leq i \leq l)$	$-\epsilon_i$
$X_{i,j} = e_{i,j} - e_{l+j,l+i}$ $(i \neq j, 1 \leq i, j \leq l)$	$\epsilon_i - \epsilon_j$
$Y_{i,j} = e_{i,l+j} - e_{j,l+i}$ $(1 \leq i < j \leq l)$	$\epsilon_i + \epsilon_j$
$Z_{i,j} = e_{l+i,j} - e_{l+j,i}$ $(1 \leq i < j \leq l)$	$-(\epsilon_i + \epsilon_j)$

check the calculations: $ad(h)$ acts as in the following diagram:

$$\left(\begin{array}{c|c|c} & -\epsilon_i & \epsilon_i \\ \hline \epsilon_i & \epsilon_i - \epsilon_j & \epsilon_i + \epsilon_j \\ \hline -\epsilon_i & -(\epsilon_i + \epsilon_j) & -(\epsilon_i - \epsilon_j) \end{array}\right)$$

A base and the Weyl group: The set of roots $\{\epsilon_i - \epsilon_{i+1}\ (1 \leq i \leq l-1), \epsilon_l\}$ is a base and the Weyl group is $(\mathbb{Z}/2\mathbb{Z})^l \rtimes S_l$.

3.3. Symplectic Algebra $C_l, l \geq 3$

Let V be a vector space of dimension $2l$ with a skew-symmetric bilinear form. We denote a vector in V by $x = (x_1, \ldots, x_i, \ldots, x_l, x_{l+1}, \ldots, x_{l+i}, \ldots, x_{2l})$. Let

$$s = \begin{pmatrix} 0 & I_l \\ -I_l & 0 \end{pmatrix}$$

be the matrix of the form. The Lie Algebra $\mathfrak{sp}(V)$ or $\mathfrak{sp}(2l)$ is

$$\mathfrak{sp}(2l) = \{x \in M_{2l}(F) \mid {}^t xs = -sx\}$$

called symplectic algebra. In fact $x \in \mathfrak{sp}(2l)$ looks like

$$x = \begin{pmatrix} m & n = {}^t n \\ p = {}^t p & -{}^t m \end{pmatrix}.$$

Dimension: $2l^2 + l$.
Bracket Operation: $[x, y] = xy - yx$ for $x, y \in \mathfrak{sp}(2l)$.

A Maximal Toral Subalgebra:

$$H = \left\{ \begin{pmatrix} a_1 & & & & & & & \\ & a_2 & & & & & & \\ & & \ddots & & & & & \\ & & & a_l & & & & \\ & & & & -a_1 & & & \\ & & & & & -a_2 & & \\ & & & & & & \ddots & \\ & & & & & & & -a_l \end{pmatrix} \right\}.$$

Notation: For $h \in H$ write $h = \sum_{i=1}^{l} a_i e_{i,i} - \sum_{i=1}^{l} a_i e_{l+i,l+i}$ and $\epsilon_i \in H^*$ is given by $\epsilon_i(h) = a_i$.

Root Vectors and Roots:

root vector	root
$X_{i,j} = e_{i,j} - e_{l+j,l+i}$ $(i \neq j, 1 \leq i,j \leq l)$	$\epsilon_i - \epsilon_j$
$Y_{i,j} = e_{i,l+j} - e_{j,l+i}$ $(1 \leq i < j \leq l)$	$\epsilon_i + \epsilon_j$
$Z_{i,j} = e_{l+i,j} - e_{l+j,i}$ $(1 \leq i < j \leq l)$	$-(\epsilon_i + \epsilon_j)$
$Y_{i,i} = e_{i,l+i}$ $(1 \leq i \leq l)$	$2\epsilon_i$
$Z_{i,i} = e_{l+i,i}$ $(1 \leq i \leq l)$	$-2\epsilon_i$

A base and the Weyl group: The set of roots $\{\epsilon_i - \epsilon_{i+1}\ (1 \leq i \leq l-1), 2\epsilon_l\}$ is a base and the Weyl group is $(\mathbb{Z}/2\mathbb{Z})^l \rtimes S_l$.

3.4. Orthogonal Algebra (even dimension) $D_l, l \geq 4$

Let V be a vector space of dimension $2l$ with a symmetric bilinear form. We denote a vector in V by $x = (x_1, \ldots, x_i, \ldots, x_l, x_{l+1}, \ldots, x_{l+i}, \ldots, x_{2l})$. Let

$$s = \begin{pmatrix} 0 & I_l \\ I_l & 0 \end{pmatrix}$$

be the matrix of the bilinear form. The Lie Algebra $\mathfrak{o}(V)$ or $\mathfrak{o}(2l)$ is

$$\mathfrak{o}(2l) = \{x \in M_{2l}(F) \mid {}^t x s = -sx\}$$

called orthogonal algebra. In fact $x \in \mathfrak{o}(2l)$ looks like

$$x = \begin{pmatrix} m & n = -{}^t n \\ p = -{}^t p & -{}^t m \end{pmatrix}.$$

Dimension: $2l^2 - l$.

Bracket Operation: $[x, y] = xy - yx$ for $x, y \in \mathfrak{o}(2l)$.

A Maximal Toral Subalgebra:

$$H = \left\{ \begin{pmatrix} a_1 & & & & & & & \\ & a_2 & & & & & & \\ & & \ddots & & & & & \\ & & & a_l & & & & \\ & & & & -a_1 & & & \\ & & & & & -a_2 & & \\ & & & & & & \ddots & \\ & & & & & & & -a_l \end{pmatrix} \right\}.$$

Notation: For $h \in H$ write $h = \sum_{i=1}^{l} a_i e_{i,i} - \sum_{i=1}^{l} a_i e_{l+i,l+i}$ and $\epsilon_i \in H^*$ is given by $\epsilon_i(h) = a_i$.

Root Vectors and Roots:

root vector	root
$X_{i,j} = e_{i,j} - e_{l+j,l+i}$ $(i \neq j, 1 \leq i, j \leq l)$	$\epsilon_i - \epsilon_j$
$Y_{i,j} = e_{i,l+j} - e_{j,l+i}$ $(1 \leq i < j \leq l)$	$\epsilon_i + \epsilon_j$
$Z_{i,j} = e_{l+i,j} - e_{l+j,i}$ $(1 \leq i < j \leq l)$	$-(\epsilon_i + \epsilon_j)$

A base and the Weyl group: The set of roots $\{\epsilon_i - \epsilon_{i+1}\ (1 \leq i \leq l-1), \epsilon_{l-1} + \epsilon_l\}$ is a base and the Weyl group is $(\mathbb{Z}/2\mathbb{Z})^{l-1} \rtimes S_l$.

CHAPTER 4

Representation Theory

Keeping in the mind the two examples i.e., the representation theory of $\mathfrak{sl}(2)$ and the ad representation of a semisimple Lie algebra we develop the representation theory in general. For a semisimple Lie algebra we ask following questions:

(a) Classify all representations (finite or infinite dimensional) of L. In case of finite dimensional, in view of Weyl's theorem, it is enough to classify the irreducible ones.
(b) For a given representation what are the weights appearing in the representation?

4.1. Universal Enveloping Algebra

Let L be a Lie algebra over field F. Any associative algebra A can be made into Lie algebra by operation $[x, y] = xy - yx$ for $x, y \in A$. Roughly speaking, to a Lie algebra L we will associate an associative algebra $U(L)$ which contains L and the Lie algebra operation on L becomes usual bracket operation in $U(L)$.

Definition 2. An associative algebra $U(L)$ with a map $i\colon L \to U(L)$ which is a Lie algebra homomorphism (i.e. $i([x, y]) = i(x)i(y) - i(y)i(x)$) is called **universal enveloping algebra** if it satisfies following universal property: for any associative algebra A if we have a Lie algebra map $\phi\colon L \to A$ then there exists an algebra homomorphism $\tilde{\phi}\colon U(L) \to A$ such that $\phi = \tilde{\phi}i$.

Now we prove the existence and uniqueness of this algebra. Let $T(L)$ be the tensor algebra of L. Consider the ideal J generated by elements $[x, y] - (x \otimes y - y \otimes x)$ in $T(L)$ for $x, y \in L$. Define

$$U(L) = \frac{T(L)}{J}$$

and the map $i\colon L \to U(L)$ by sending elements of x to in the 1st component of the tensor algebra. Then $U(L)$ is the required universal enveloping algebra. Note that if L is Abelian then $U(L)$ is the symmetric algebra.

Theorem 4.1.1 (Poincare-Birkhoff-Witt). *Let L be of countable dimension with $\{x_1, x_2, \ldots\}$, a basis. Then $\{1, x_{i_1}x_{i_2}\cdots x_{i_m} \mid m \in \mathbb{Z}^+, i_1 \le i_2 \le \ldots \le i_m\}$ is a basis of $U(L)$.*

The map i in the definition of $U(L)$ is injective and hence L can be identified with its image. Also if H is a subalgebra with basis $\{h_1, h_2, \ldots\}$ and extend this basis to that of L, say, $\{h_1, h_2, \ldots, x_1, x_2, \ldots\}$ then the map $U(H) \to U(L)$ is injective and in fact, $U(L)$ is free $U(H)$-module with basis $\{1, x_{i_1}x_{i_2}\cdots x_{i_m}\}$.

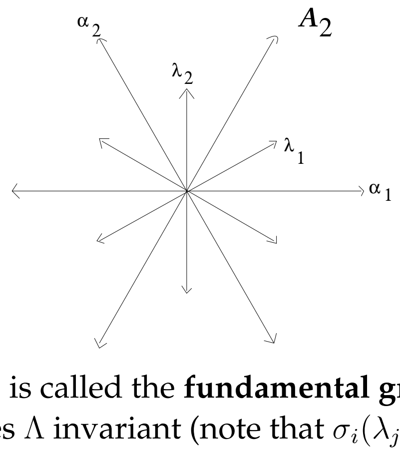
4.2. Abstract Theory of Weights

Let Φ be a root system in an Euclidean space E with Weyl group W. Let

$$\Lambda = \{\lambda \in E \mid \langle \lambda, \alpha \rangle \in \mathbb{Z},\ \forall \alpha \in \Phi\}$$

where $\langle \lambda, \alpha \rangle = 2\frac{(\lambda,\alpha)}{\alpha,\alpha}$. Then Λ is a lattice (abelian subgroup containing a basis of E) called **weight lattice** and elements are called weights. Note that Λ contains Φ. Let Λ_r be the lattice generated by Φ, called **root lattice**. Fix a base $\Delta \subset \Phi$. This is equivalent to defining an order ($>$) on E and in turn on Λ. An element $\lambda \in \Lambda$ is called **dominant** if $\langle \lambda, \alpha \rangle \ge 0\ \forall \alpha \in \Delta$ and **strongly dominant** if $\langle \lambda, \alpha \rangle > 0\ \forall \alpha \in \Delta$. We denote by Λ^+ the set of dominant weights.

Let $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ then the vectors $2\frac{\alpha_i}{(\alpha_i,\alpha_i)}$ also form a basis. Let $\lambda_1, \ldots, \lambda_l$ be the dual basis, i.e., $2\frac{(\lambda_i,\alpha_j)}{(\alpha_j,\alpha_j)} = \delta_{ij}$. Note that λ_i are dominant weights called **fundamental dominant weights**. Every element $\lambda \in E$ can be written as $\lambda = \sum m_i\lambda_i$ where $m_i = \langle \lambda, \alpha_i \rangle$. Then $\Lambda = \mathbb{Z}\lambda_1 \oplus \cdots \oplus \mathbb{Z}\lambda_l$ and $\Lambda^+ = \mathbb{Z}_{\ge 0}\lambda_1 \oplus \cdots \oplus \mathbb{Z}_{\ge 0}\lambda_l$.



The finite group Λ/Λ_r is called the **fundamental group** of Φ.

The Weyl group leaves Λ invariant (note that $\sigma_i(\lambda_j) = \lambda_j - \delta_{ij}\alpha_i$).

Proposition 4.2.1. *Each weight is conjugate under W to one and only one dominant weight. If λ is dominant, then $\sigma(\lambda) < \lambda$ for all $\sigma \in W$. Moreover for $\lambda \in \Lambda^+$ the number of dominant weights $\mu < \lambda$ is finite.*

4.3. Weights and Maximal Vectors

Notation: L semisimple Lie algebra (over algebraically closed field F of characteristic 0) of rank l, H a maximal toral subalgebra of L, Φ the root system, Φ^+ a positive root system and $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ corresponding base, W the Weyl group.

Let V be a finite dimensional L-module. We can decompose V with respect to H and have $V = \oplus_{\lambda \in H^*} V_\lambda$ where $V_\lambda = \{v \in V \mid hv = \lambda(h)v \, \forall h \in H\}$. But if V is of infinite dimension V_λ still makes sense (though there need not be a direct sum decomposition of V) and is called **weight space** if it is non-trivial and λ a **weight**.

Proposition 4.3.1. *Let V be an L-module (need not be of finite dimension). Then*

(a) *L_α maps V_λ into $V_{\lambda+\alpha}$ ($\lambda \in H^*, \alpha \in \Phi$).*
(b) *The sum $V' = \sum_{\lambda \in H^*} V_\lambda$ is direct, and V' is an L-submodule of V.*
(c) *If V is finite dimensional then $V = V'$.*

PROOF. Let $x \in L_\alpha, v \in V_\lambda$ and $h \in H$. Then $h.x.v = x.h.v + [h, x].v = \lambda(h)x.v + \alpha(h)x.v = (\lambda(h) + \alpha(h))x.v$, i.e., $L_\alpha V_\lambda \subset V_{\lambda+\alpha}$.

From Part (a) it is clear that V' is an L-submodule. Let $v \in V_\lambda \cap V_\mu$ for $\lambda \neq \mu \in H^*$. There exist $h \in H$ such that $\lambda(h) \neq \mu(h)$. Then $h.v = \lambda(h)v = \mu(h)v$ implies $v = 0$.

If dimension of V is finite it is direct sum of weight spaces. □

Exercise: Do exercise 1 and 2 from **[Hu2]** section 20.

4.4. Construction of L-modules

A **maximal vector of weight** λ in an L-module V is a vector $0 \neq v^+ \in V_\lambda$ such that $L_\alpha.v^+ = 0 \, \forall \alpha \in \Delta$ (equivalently $\forall \alpha \in \Phi^+$). In case V is finite dimensional the Boral subalgebra $B(\Delta) = H \oplus \oplus_{\alpha \in \Phi^+} L_\alpha$ has a common eigenvector (thanks to Lie's theorem) which is a maximal vector.

Since V is an L-module it is also an $U(L)$ module. Fix v^+ a maximal vector of weight λ and denote $V = U(L).v^+$. These are called **standard cyclic modules** of **highest weight** λ. We will first study structure of such modules. Recall the notation $x_\alpha \in L_\alpha, y_\alpha \in L_{-\alpha}$ and $h_\alpha = [x_\alpha, y_\alpha]$ for $\alpha \in \Phi^+$.

Theorem 4.4.1. *With notation as above, let $\Phi^+ = \{\beta_1, \ldots, \beta_m\}$, Then,*

(a) *V is spanned by the vectors $y_{\beta_1}^{i_1} \cdots y_{\beta_m}^{i_m}.v^+$ for $i_j \in \mathbb{Z}$; in particular V is direct sum of its weight spaces with weights of the form $\mu = \lambda - \sum_{j=1}^{l} k_j \alpha_j$.*
(b) *For each $\mu \in H^*$, V_μ is finite dimensional and $\dim(V_\lambda) = 1$. Moreover, each submodule of V is direct sum of its weight spaces.*

(c) *V is an indecomposable L-module with a unique maximal proper submodule and the corresponding unique irreducible quotient module. Every homomorphic image of V is also standard cyclic of weight λ.*

PROOF. See **[Hu2]** section 20.2. □

In addition suppose V was irreducible. Then v^+ is the unique maximal vector in V upto scalar multiples.

Now we explicitly construct irreducible standard cyclic modules of highest weight λ which may be of infinite dimension. It is unique up to isomorphism.

Theorem 4.4.2. *Let V, W be standard cyclic modules of highest weight λ. If both are irreducible then they are isomorphic.*

Next we construct a cyclic L-module of highest weight λ for any $\lambda \in H^*$.

Construction 1: Let $D_\lambda = F.v^+$ be a one dimensional vector space. We define an action of $B = B(\Delta) = H \oplus_{\alpha \in \Phi^+} L_\alpha$ on D_λ by $h.v^+ = \lambda(h)v^+$ and $x_\alpha.v^+ = 0$. Hence D_λ is an $U(B)$-module. Consider

$$Z(\lambda) = U(L) \otimes_{U(B)} D_\lambda$$

an $U(L)$ module. Then $Z(\lambda)$ is standard cyclic of weight λ. Since $U(L)$ is free $U(B)$ module the element $1 \otimes v^+$ is non zero and generates $Z(\lambda)$.

Construction 2: Consider the left ideal $I(\lambda)$ in $U(L)$ generated by $\{x_\alpha, \alpha \in \Phi^+\}$ and $\{h_\alpha - \lambda(h_\alpha), \alpha \in \Phi\}$. By the construction of D_λ we get a map $U(L)/I(\lambda) \to Z(\lambda)$ which maps $1 \mapsto 1 \otimes v^+$. Using PBW basis we can show that this map is isomorphism.

The modules $Z(\lambda)$ are called **Verma modules** after D.-N. Verma. Let $Y(\lambda)$ be the maximal proper submodule of $Z(\lambda)$. We consider $V(\lambda) = Z(\lambda)/Y(\lambda)$.

Theorem 4.4.3. *For any $\lambda \in H^*$, $V(\lambda)$ is an irreducible standard cyclic modules of weight λ.*

4.5. All finite dimensional Representations

Theorem 4.5.1. *Every finite dimensional irreducible L-module V is isomorphic to $V(\lambda)$ for some $\lambda \in H^*$. Moreover $\lambda(h_i)$ is a nonnegative integer for all $1 \leq i \leq l$ and any weight μ takes integer values on h_i.*

An element $\lambda \in H^*$ such that $\lambda(h_i) \in \mathbb{Z}$ is called **integral** and if all $\lambda(h_i)$ are nonnegative integers then it is called **dominant integral**. Then the set Λ of integral linear functions on H is a lattice containing root lattice. The set of dominant integral linear functions is denoted as Λ^+.

Theorem 4.5.2. *If $\lambda \in \Lambda^+$ then the irreducible L-module $V(\lambda)$ is finite dimensional and its set of weights $\Pi(\lambda)$ is permuted by the Weyl group W, with $\dim(V_\mu) = \dim(V_{\sigma\mu})$ for $\sigma \in W$.*

Hence the map $\lambda \mapsto V(\lambda)$ induces a one-one correspondence between Λ^+ and the isomorphism classes of finite dimensional irreducible L-modules. To prove the theorem we will first prove several lemmas.

Lemma 4.5.3. *The following identities hold in $U(L)$, for $k \geq 0, 1 \leq i, j \leq l$:*

a. $[x_j, y_i^{k+1}] = 0$ *when* $i \neq j$;
b. $[h_j, y_i^{k+1}] = -(k+1)\alpha_i(h_j)y_i^{k+1}$;
c. $[x_i, y_i^{k+1}] = -(k+1)y_i^k(k.1 - h_j)$.

PROOF. First one follows from the fact that $\alpha_i - \alpha_j$ is not a root for $i \neq j$. Others follow by induction. □

We fix some notation for the proof. We denote the representation $V(\lambda)$ by V and write $\phi\colon L \to \mathfrak{gl}(V)$. We fix a maximal vector v^+ of V of weight λ and set $m_i = \lambda(h_i), 1 \leq i \leq l$. Note that m_i are nonnegative integers by hypothesis. We also denote the set of weights appearing in V by $\Pi(\lambda)$.

Lemma 4.5.4. *The representation V is sum of finite dimensional S_i-modules for all i where S_i is a copy of $\mathfrak{sl}(2)$ generated by x_i, y_i and h_i in L.*

PROOF. Let V' be sum of all S_i-submodules of V for all i. First of all V' is nonzero. Check following:

(1) $y_i^{m_i+1}.v^+ = 0$.
(2) The subspace spanned by $\{v^+, y_i.v^+, \ldots, y_i^{m_i}.v^+\}$ is a finite dimensional S_i-module.
(3) For any non-zero finite dimensional submodule W of V the space spanned by $x_\alpha W$ for $\alpha \in \Phi$ is finite dimensional and S_i-stable.

Since V is irreducible we get $V = V'$. □

Lemma 4.5.5. *The elements $\phi(x_i)$ and $\phi(y_i)$ are locally nilpotent endomorphisms on V.*

PROOF. Let $v \in V$. Then v is contained in finite dimensional submodule from previous Lemma. □

Lemma 4.5.6. *If μ is any weight of V then there exist an automorphism s_i of V such that $s_i(V_\mu) = V_{\sigma_i\mu}$ where σ_i is the reflection relative to α_i (which generate the Weyl group W).*

PROOF. We consider the automorphism $s_i = exp\phi(x_i)exp\phi(-y_i)exp\phi(x_i)$. Then $s_i(V_\mu) = V_{\sigma_i\mu}$ where σ_i is reflection with respect to α_i. □

Lemma 4.5.7. *The set* $\Pi(\lambda)$ *is stable under* W *and* $\dim(V_\mu) = \dim(V_{\sigma\mu})$. *Moreover* $\Pi(\lambda)$ *is finite.*

PROOF. From previous lemma it is clear that the set $\Pi(\lambda)$ is stable under W. And finiteness follows from Proposition 4.2.1. □

PROOF OF THEOREM 4.5.2. □

CHAPTER 5

Some Representation of Classical Lie Algebras

Here we give a lots of example. Mainly for classical Lie algebras we describe the fundamental representations. To get any irreducible representation we take their tensor products.

Proposition 5.0.8. *Let $V = V(\lambda)$ and $W = V(\mu)$ with $\lambda, \mu \in \Lambda^+$. Then the weights of $V \otimes W$ is $\Pi(V \otimes W) = \{\nu + \nu' \mid \nu \in \Pi(\lambda), \nu' \in \Pi(\mu)\}$ and*

$$\dim(V \otimes W)_{\nu+\nu'} = \sum_{\pi+\pi'=\nu+\nu'} \dim V_\pi . \dim W_{\pi'}.$$

In particular, $\lambda + \mu$ occurs with multiplicity one, so $V(\mu + \lambda)$ occurs exactly once as a direct summand of $V \otimes W$.

Let $\lambda_1, \ldots, \lambda_l$ be the fundamental dominant weights. Let $\lambda = a_1\lambda_1 + \cdots + a_l\lambda_l \in \Lambda^+$, i.e., $a_i \geq 0$ integers. Then $V(\lambda)$ is a direct summand in $V(\lambda_1)^{\otimes a_1} \otimes \cdots \otimes V(\lambda_l)^{\otimes a_l}$. Hence knowing fundamental representation will yield any finite dimensional representation.

5.1. Fundamental Representations of $\mathfrak{sl}(n)$

Consider the vector space F^n with natural action of $\mathfrak{g} = \mathfrak{sl}(n)$. We fix the notation as $\mathfrak{h} = \{\mathrm{diag}(a_1, \ldots, a_n) \mid a_1 + \cdots + a_n = 0\}$ and $\mathfrak{n}$ for strictly upper triangular matrices and $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$. We consider the vector space $V = \wedge^r \mathbb{C}^n$. For a fixed basis $\{e_1, \ldots, e_n\}$ of F^n we have $\{e_{i_1} \wedge \cdots \wedge e_{i_r} \mid 1 \leq i_1 < \ldots < i_r \leq n\}$ a basis of V. We define the action of $x \in sl(n)$ on V by

$$x.e_{i_1} \wedge \cdots \wedge e_{i_r} = xe_{i_1} \wedge \cdots \wedge e_{i_r} + \cdots + e_{i_1} \wedge \cdots \wedge xe_{i_r}.$$

We note that $h \in \mathfrak{h}$ acts as follows:

$$h.e_{i_1} \wedge \cdots \wedge e_{i_r} = (a_{i_1} + \cdots + a_{i_r})e_{i_1} \wedge \cdots \wedge e_{i_r} = (\epsilon_{i_1} + \cdots + \epsilon_{i_r})(h)e_{i_1} \wedge \cdots \wedge e_{i_r}.$$

Hence the possible weights of the representation are $\{\epsilon_{i_1} + \cdots + \epsilon_{i_r} \mid 1 \leq i_1 < \ldots < i_r \leq n\}$. And the highest weight (with respect to fixed base as done in previous section) is $\epsilon_1 + \cdots + \epsilon_r$.

For a base $\Delta = \{\epsilon_1 - \epsilon_2, \ldots, \epsilon_{n-1} - \epsilon_n\}$ of the root system we get the fundamental dominant weights as $\{\lambda_1 = \epsilon_1, \lambda_2 = \epsilon_1 + \epsilon_2, \ldots, \lambda_{n-1} = \epsilon_1 + \cdots + \epsilon_{n-1}\}$.

Theorem 5.1.1. *The representation $V = \wedge^r \mathbb{C}^n$ for $r = 1, \dots, n-1$ is irreducible representation of $\mathfrak{g} = \mathfrak{sl}(n)$ corresponding to fundamental weights $\epsilon_1 + \cdots + \epsilon_r$.*

Example ($\mathfrak{sl}(3)$) : For $V_1 = \wedge^1 \mathbb{C}^3$ a basis is given by $\{e_1, e_2, e_3\}$ and the weights are $\{\epsilon_1, \epsilon_2, \epsilon_3\}$ where ϵ_1 is the highest weight. Notice that $\epsilon_2 = \epsilon_1 - (\epsilon_1 - \epsilon_2)$ and $\epsilon_3 = \epsilon_1 - (\epsilon_1 - \epsilon_2) - (\epsilon_2 - \epsilon_3)$.

For $V_2 = \wedge^2 \mathbb{C}^3$ a basis is given by $\{e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3\}$ and the weights are $\{\epsilon_1 + \epsilon_2, \epsilon_1 + \epsilon_3, \epsilon_2 + \epsilon_3\}$ where $\epsilon_1 + \epsilon_2$ is the highest weight. We can again write that $\epsilon_1 + \epsilon_3 = (\epsilon_1 + \epsilon_2) - (\epsilon_2 - \epsilon_3)$ and $\epsilon_2 + \epsilon_3 = (\epsilon_1 + \epsilon_2) - (\epsilon_1 - \epsilon_2) - (\epsilon_2 - \epsilon_3)$.

5.2. Fundamental Representations of $\mathfrak{so}(n)$

Let $L = \mathfrak{so}(n)$ and σ_1 be the defining representation of L on F^n. Let us denote the representation of L on the rth exterior power $\wedge^r F^n$ by σ_r.

Theorem 5.2.1. *Let $n = 2l + 1 \geq 3$ be odd. For $1 \leq r \leq l$ the representation $(\sigma_r, \wedge^r F^n)$ is an irreducible representation of $\mathfrak{so}(n)$ with highest weight $\epsilon_1 + \cdots + \epsilon_r$.*

Theorem 5.2.2. *Let $n = 2l \geq 4$ be even.*

(1) *For $1 \leq r \leq l-1$ the representation $(\sigma_r, \wedge^r F^n)$ is an irreducible representation of $\mathfrak{so}(n)$ with highest weight $\epsilon_1 + \cdots + \epsilon_r$.*
(2) *For $r = l$ the space $\wedge^l F^n$ is direct sum of two irreducible representations of highest weights $\epsilon_1 + \cdots + \epsilon_l$ (corresponding to weight vector $e_1 \wedge \cdots \wedge e_{l-1} \wedge e_l$) and $\epsilon_1 + \cdots + \epsilon_{l-1} \wedge e_l$ (corresponding to weight vector $e_1 \wedge \cdots \wedge e_{l-1} \wedge e_{-l}$).*

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