Workshop on Group Theory: Classification of Reductive Algebraic Groups
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Introduction to Lie Algebras

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Lecture 4: Theorems of Engel and Lie

Exercise 1 *A Lie algebra* $\mathfrak{g}$ *is solvable if and only if it has a sequence of Lie subalgebras*

$$\mathfrak{g} = \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \cdots \supset \mathfrak{g}_k = 0,$$

such that each $\mathfrak{g}_{i+1}$ *is an ideal in* $\mathfrak{g}_i$ *and* $\mathfrak{g}_i/\mathfrak{g}_{i+1}$ *is abelian.*

Exercise 2 *Let* $\mathfrak{h}$ *be an ideal of* $\mathfrak{g}$*. Then* $\mathfrak{g}$ *is solvable if and only if* $\mathfrak{h}$ *and* $\mathfrak{g}/\mathfrak{h}$ *are solvable.*

Exercise 3 *If* $\mathfrak{h}_1, \mathfrak{h}_2$ *are solvable ideals of* $\mathfrak{g}$*, so is* $\mathfrak{h}_1 + \mathfrak{h}_2$*.*

These results don't hold for nilpotent Lie algebras:

Example 4 Let $\mathfrak{g}$ be the non-abelian two dimensional Lie algebra. It has a basis $\{X, Y\}$ such that $[X, Y] = X$. Now, let $\mathfrak{h} = \mathbb{F}X$ and $\mathfrak{j} = \mathbb{F}Y$. Then $\mathfrak{h}$ is an ideal and $\mathfrak{h}, \mathfrak{g}/\mathfrak{h}$ are nilpotent because they are one dimensional. However $\mathfrak{g}$ is not nilpotent: $\mathfrak{g}^i = \mathbb{F}X \ \forall i > 0$.

Similarly, $\mathfrak{h}$ and $\mathfrak{j}$ are nilpotent but $\mathfrak{g} = \mathfrak{h} + \mathfrak{j}$ is not. □

However, there is the following result.

Exercise 5 *If* $\mathfrak{g}/Z(\mathfrak{g})$ *is nilpotent, so is* $\mathfrak{g}$*. (Hint:* $(\mathfrak{g}/Z(\mathfrak{g}))^i = \mathfrak{g}^i/Z(\mathfrak{g})$*.)*

Exercise 6 *A Lie algebra* $\mathfrak{g}$ *is solvable or nilpotent iff* $\mathrm{ad}(\mathfrak{g})$ *is so.*

Exercise 7 *Every Lie algebra* $\mathfrak{g}$ *has a unique maximal solvable ideal.*

Definition 8 The unique maximal solvable ideal of the Lie algebra $\mathfrak{g}$ is called its *radical* and is denoted $\mathrm{Rad}(\mathfrak{g})$.

Definition 9 If $\mathrm{Rad}(\mathfrak{g}) = 0$, $\mathfrak{g}$ is called *semisimple.*

A semisimple Lie algebra must have zero center, hence its adjoint representation is injective.

Simple Lie algebras are semisimple. So is the 0 algebra.

Exercise 10 *For any Lie algebra $\mathfrak{g}$, $\mathfrak{g}/\mathrm{Rad}(\mathfrak{g})$ is semisimple.*

Let us turn to nilpotent Lie algebras. The condition $\mathfrak{g}^n = 0$ means that for any $X_1, \ldots, X_{n+1} \in \mathfrak{g}$ we have

$$[X_1, [X_2, \ldots [X_n, X_{n+1}] \ldots]] = 0, \quad \text{or} \quad \mathrm{ad}(X_1)\mathrm{ad}(X_2) \cdots \mathrm{ad}(X_n) = 0$$

In particular: $\mathrm{ad}(X)^n = 0 \ \forall X \in \mathfrak{g}$.

Definition 11 If $\mathrm{ad}(X)$ is nilpotent, we say X is *ad-nilpotent.*

We have just observed that if $\mathfrak{g}$ is nilpotent, then each element of $\mathfrak{g}$ is ad-nilpotent. Amazingly, the converse is also true.

Theorem 12 *Let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be a linear Lie algebra, $V \neq 0$. If each $X \in \mathfrak{g}$ is nilpotent, then $\exists v \in V$ such that $v \neq 0$ and $Xv = 0 \ \forall X \in \mathfrak{g}$.*

Proof. We proceed by induction on $\dim(\mathfrak{g})$. When $\dim(\mathfrak{g}) = 0$, the result is easy. Now consider an arbitrary $\mathfrak{g}$ satisfying the hypotheses of the theorem.

We first show $\mathfrak{g}$ has an ideal $\mathfrak{h}$ of codimension one. Choose $\mathfrak{h}$ to be any maximal proper subalgebra of $\mathfrak{g}$. For any $H \in \mathfrak{h}$, $\mathrm{ad}(H)$ preserves $\mathfrak{h}$, hence we can define

$$\overline{\mathrm{ad}} : \mathfrak{h} \to \mathfrak{gl}(\mathfrak{g}/\mathfrak{h}), \qquad \overline{\mathrm{ad}}(H)(X + \mathfrak{h}) = \mathrm{ad}(H)X + \mathfrak{h}.$$

Since each $\mathrm{ad}(H)$ is nilpotent, so is each $\overline{\mathrm{ad}}(H)$. Applying the induction hypothesis to $\overline{\mathrm{ad}}(\mathfrak{h})$, we find a non-zero $Y + \mathfrak{h}$ such that $\overline{\mathrm{ad}}(H)(Y + \mathfrak{h}) = \mathfrak{h}$ for every $H \in \mathfrak{h}$. Then $Y \notin \mathfrak{h}$ and $\mathrm{ad}(H)Y \in \mathfrak{h}$ for each $H \in \mathfrak{h}$.

This shows $\mathfrak{h} + \mathbb{F}Y$ is a subalgebra, properly containing $\mathfrak{h}$. Hence we must have $\mathfrak{g} = \mathfrak{h} + \mathbb{F}Y$. So $\mathfrak{h}$ has codimension one, and is also an ideal.

Now apply the induction hypothesis to $\mathfrak{h}$: $\exists v \in V$, $v \neq 0$, such that $Hv = 0$ for every $H \in \mathfrak{h}$. Let

$$W = \{v \in V : Hv = 0 \ \forall H \in \mathfrak{h}\}.$$

Also, fix $Y \notin \mathfrak{h}$: then $\mathfrak{g} = \mathfrak{h} + \mathbb{F}Y$. We have to find a non-zero $v \in W$ such that $Yv = 0$. For any $w \in W$ and $H \in \mathfrak{h}$,

$$HYw = [H, Y]w + YHw = 0,$$

and so $Yw \in W$. Therefore Y acts on W. Since the action must be nilpotent, there is a non-zero $v \in W$ such that $Yv = 0$. □

Corollary 13 *Let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be a linear Lie algebra. If each $X \in \mathfrak{g}$ is nilpotent, then there is a basis of V such that each member of $\mathfrak{g}$ is represented by a strictly upper triangular matrix.*

Corollary 14 (Engel's Theorem) *Let $\mathfrak{g}$ be a Lie algebra such that each element is ad-nilpotent. Then $\mathfrak{g}$ is nilpotent.*

In fact, the entire family of results 12-14 is generally grouped under the name of Engel's Theorem.

Exercise 15 *Let $\mathfrak{g}$ be a nilpotent Lie algebra and $\mathfrak{h}$ a non-zero ideal in $\mathfrak{g}$. Then $\mathfrak{h} \cap Z(\mathfrak{g}) \neq 0$.*

Now we shall start exploring the structure of Lie algebras via eigenvectors and eigenvalues, hence:

We assume that the underlying field $\mathbb{F}$ is algebraically closed.

We have seen that prime characteristic creates various exceptions to general patterns, and so **we also assume that** $\mathrm{char}(\mathbb{F}) = 0$.

Example 16 Let $\mathfrak{g}$ be the non-abelian two dimensional Lie algebra, with basis $\{X, Y\}$ such that $[X, Y] = X$. Then it is solvable: $\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}] = \mathbb{F}X$, $\mathfrak{g}^{(2)} = [\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}] = 0$. Consider its adjoint representation:

$$\begin{aligned} \mathrm{ad}(X) &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ \mathrm{ad}(Y) &= \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

Hence for a general element $T = aX + bY \in \mathfrak{g}$,

$$\mathrm{ad}(aX + bY) = \begin{pmatrix} -b & a \\ 0 & 0 \end{pmatrix}.$$

Clearly, there is no non-zero vector Z such that $\mathrm{ad}(T)Z = 0$ for every $T \in \mathfrak{g}$. However X is atleast a common eigenvector for each $\mathrm{ad}(T)$:

$$\mathrm{ad}(aX + bY)X = -bX.$$

□

Theorem 17 *Let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be a solvable Lie algebra. If $V \neq 0$ then it contains a common eigenvector for all the elements of $\mathfrak{g}$.*

Proof. We proceed by induction on $\dim(\mathfrak{g})$, broadly following the scheme used for Engel's Theorem. If $\dim(\mathfrak{g}) = 0$, the claim is trivial.

For a general $\mathfrak{g}$ we first locate an ideal of codimension one. Since $\mathfrak{g}$ is solvable, $[\mathfrak{g}, \mathfrak{g}] \neq \mathfrak{g}$. Hence we can choose some vector subspace $\mathfrak{k} \subset \mathfrak{g}$ such that it has codimension one and contains $[\mathfrak{g}, \mathfrak{g}]$. But then,

$$[\mathfrak{k}, \mathfrak{g}] \subset [\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{k},$$

hence $\mathfrak{k}$ is an ideal.

By the induction hypothesis, there is a common eigenvector $v \in V$ for each element of $\mathfrak{k}$:

$$Kv = \lambda(K)v, \quad \forall K \in \mathfrak{k}.$$

It is easy to see that $\lambda : \mathfrak{k} \to \mathbb{F}$ is linear. Now define:

$$W = \{w \in V : Kw = \lambda(K)w, \quad \forall K \in \mathfrak{k}\}.$$

Since $v \in W$, W is a non-zero subspace of V. We shall show $\mathfrak{g}$ preserves W. Let $X \in \mathfrak{g}$ and $w \in W$. Then for any $K \in \mathfrak{k}$,

$$KXw = [K, X]w + XKw = \lambda([K, X])w + \lambda(K)Xw.$$

To show $Xw \in W$ we have to prove that $\lambda([K, X]) = 0$. Let $w \in W$ be non-zero and consider the sequence $\{w, Xw, X^2w \dots\}$. Let n be largest such that

$$W_n = \{w, Xw, \dots, X^n w\}$$

is linearly independent. We have:

$$\begin{aligned} KX^i w &= KXX^{i-1}w \\ &= [K, X]X^{i-1}w + XKX^{i-1}w \\ &= \lambda([K, X])X^{i-1}w + \lambda(K)X^i w. \end{aligned}$$

Hence the action of K on W_n is given by an upper triangular matrix whose diagonal entries are all $\lambda(K)$. Applying this to the action of $[K,X]$, we find that its trace is $\mathrm{Tr}\,[K,X]|_{W_n} = (n+1)\lambda([K,X])$. By our choice of n, we also have $XW_n \subset W_n$, and so $[K,X]|_{W_n} = [K|_{W_n}, X|_{W_n}]$. But then $\mathrm{Tr}\,[K,X]|_{W_n} = 0$, and so $\lambda([K,X]) = 0$ (Since $\mathrm{char}(\mathbb{F}) = 0$).

So we have shown that $\mathfrak{g}$ preserves W. Let $\mathfrak{g} = \mathfrak{k} + \mathbb{F}Z$. Since $\mathbb{F}$ is algebraically closed, Z has an eigenvector $v \in W$. Then v is clearly a common eigenvector for $\mathfrak{g}$. □

Corollary 18 (Lie's Theorem) *Let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be a solvable Lie algebra. Then V has a basis such that the matrix of each $X \in \mathfrak{g}$ is upper triangular.*

Proof. Proceed by induction on $n = \dim(V)$. The $n = 0$ case is trivial. Assume $n > 0$. We have a common eigenvector v for the $\mathfrak{g}$ action on V. Let $V_1 = V/\mathbb{F}v$. Define $\pi : \mathfrak{g} \to \mathfrak{gl}(V_1)$ by

$$\pi(X)(w + \mathbb{F}v) = Xw + \mathbb{F}v.$$

Since $\pi(\mathfrak{g})$ is solvable, by the induction hypothesis, there is a basis $\{v_1 + \mathbb{F}v, \ldots, v_n + \mathbb{F}v\}$ of V_1 such that each $\pi(\mathfrak{g})$ is upper triangular. Then

$$\{v, v_1, \ldots, v_n\}$$

is a basis of V which makes $\mathfrak{g}$ upper triangular. □

Exercise 19 *Let $\mathfrak{g}$ be solvable and $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$ a representation of $\mathfrak{g}$. Then V has a basis in which each $\pi(X)$ is upper triangular.*

Exercise 20 *Let $\mathfrak{g}$ be solvable. Then there are ideals $\mathfrak{h}_i$ of $\mathfrak{g}$ such that*

$$0 = \mathfrak{h}_0 \subset \mathfrak{h}_1 \subset \mathfrak{h}_2 \subset \cdots \subset \mathfrak{h}_n = \mathfrak{g}, \quad \textit{and} \quad \dim(\mathfrak{h}_i) = i.$$

Exercise 21 *Let $\mathfrak{g}$ be solvable. Then $\mathrm{ad}(X)$ is nilpotent for each $X \in [\mathfrak{g},\mathfrak{g}]$.*

Exercise 22 *A Lie algebra $\mathfrak{g}$ is solvable if and only if $[\mathfrak{g},\mathfrak{g}]$ is nilpotent.*

Exercise 23 *The sum of two nilpotent ideals is nilpotent. Hence each $\mathfrak{g}$ has a maximal nilpotent ideal.*

Exercise 24 *Let $X, Y \in L(V)$ commute. Let their Jordan decompositions be $X_s + X_n$ and $Y_s + Y_n$ respectively. Then the Jordan decomposition of $X + Y$ is $(X_s + Y_s) + (X_n + Y_n)$.*

Exercise 25 *Show the previous result can fail if* X, Y *do not commute.*

Exercise 26 *Let* $\mathfrak{g} \subset \mathfrak{gl}(V)$ *be solvable. Show that* $\mathrm{Tr}\,(XY) = 0$ *for all* $X \in [\mathfrak{g}, \mathfrak{g}]$ *and* $Y \in \mathfrak{g}$.

Exercise 27 *Let* $\mathrm{char}(\mathbb{F}) = 2$ *and let* $\mathfrak{g} \subset \mathfrak{gl}(2, \mathbb{F})$ *be the span of*

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Show $\mathfrak{g}$ *is a solvable Lie algebra but its elements have no common eigenvector in* $\mathbb{F}^2$.

Exercise 28 *Let* $\mathfrak{g}$ *be as in the previous exercise. Consider the vector space direct sum* $\mathfrak{h} = \mathfrak{g} \oplus \mathbb{F}^2$ *and define a bracket on it by*

$$[X \oplus x, Y \oplus y] = [X, Y] \oplus (Xy - Yx).$$

Show that $\mathfrak{h}$ *is a solvable Lie algebra but* $[\mathfrak{h}, \mathfrak{h}]$ *is not nilpotent.*

Exercise 29 *If* $\mathfrak{g}$ *is a real Lie algebra, its* complexification *is the complex vector space* $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$, *with bracket defined by* $[X \otimes w, Y \otimes z] = [X, Y] \otimes wz$. *Verify* $\mathfrak{g}_{\mathbb{C}}$ *is a Lie algebra over* $\mathbb{C}$.

Exercise 30 *Let* $\mathfrak{g}$ *be a real Lie algebra. Show it is solvable if and only if* $\mathfrak{g}_{\mathbb{C}}$ *is.*

Exercise 31 *If* $\mathfrak{g}$ *is a solvable real Lie algebra, then* $[\mathfrak{g}, \mathfrak{g}]$ *is nilpotent.*