Workshop on Group Theory: Classification of Reductive Algebraic Groups
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Introduction to Lie Algebras

Amber Habib
Mathematical Sciences Foundation, Delhi

Lecture 1 - Basic Definitions and Examples

For our basic example we consider the vector space $L(V)$ of linear operators on a vector space V (over a field $\mathbb{F}$). Besides the vector space operations of addition and scaling, this has another natural operation: composition of linear operators. This operation is not commutative: in general, $f \circ g \neq g \circ f$. We can try to capture the amount of non-commutativity by defining

$$[f, g] = f \circ g - g \circ f.$$

Now we have a new operation, $[\cdot\,, \cdot] : L(V) \times L(V) \to L(V)$. First, we easily see it is bilinear:

$$[\alpha f + \beta g, \alpha' f' + \beta' g'] = \alpha\alpha'[f, f'] + \alpha\beta'[f, g'] + \beta\alpha'[g, f'] + \beta\beta'[g, g'],$$

where $\alpha, \beta, \alpha', \beta' \in \mathbb{F}$. Next, it is not commutative. In fact, we have

$$[f, g] = -[g, f] \quad \text{and} \quad [f, f] = 0.$$

Finally, let us consider associativity:

$$\begin{aligned}
[f, [g, h]] - [[f, g], h] &= f[g, h] - [g, h]f - [f, g]h + h[f, g] \\
&= fgh - fhg - ghf + hgf - fgh + gfh + hfg - hgf \\
&= g[f, h] - [f, h]g \ = \ [g, [f, h]].
\end{aligned}$$

So the bracket is not associative either. However, the last calculation can be rewritten in a form which is quite useful:

$$[f, [g, h]] + [g, [h, f]] + [h, [f, g]] = 0.$$

Note the cyclic pattern.

The properties listed above lead to the following abstract notion:

Definition 1 A *Lie algebra* $\mathfrak{g}$ is a vector space (over a field $\mathbb{F}$) with a bilinear operation $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ called the *bracket* or *commutator*, and denoted $(X, Y) \mapsto [X, Y]$, such that:

1. $[X, X] = 0 \qquad \forall X, Y \in \mathfrak{g}$.
2. $[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0 \qquad \forall X, Y, Z \in \mathfrak{g}$.

The first property of the bracket is called *anti-commutativity* while the second is the *Jacobi identity*.

Exercise 2 *Show that in a Lie algebra,* $[X, Y] = -[Y, X]$.

Lie algebras can be studied for their own sake, but *our* interest in them arises out of their applications to the study of certain groups. Roughly, to each such group we will assign a Lie algebra which will contain local information about this group. Its job will be to convert problems about group structure to problems in linear algebra.

Exercise 3 *Let A be an associative algebra over $\mathbb{F}$. Define $[a, b] = ab - ba$ for $a, b \in A$. Show that this bracket makes A a Lie algebra.*

Example 4 Consider $L(V)$ with the bracket $[f, g] = f \circ g - g \circ f$. We have seen that it becomes a Lie algebra, and we shall call this Lie algebra the *general Lie algebra* and denote it by $\mathfrak{gl}(V)$. $\square$

Definition 5 Let $\mathfrak{g}$ be a Lie algebra. We have the following definitions.

1. The Lie algebra $\mathfrak{g}$ is *abelian* if the bracket is trivial: $[X, Y] \equiv 0$.
2. A subset $\mathfrak{h} \subset \mathfrak{g}$ is a *Lie subalgebra* of $\mathfrak{g}$ if it is a vector subspace and is closed under the bracket operation.
3. A subset $\mathfrak{h} \subset \mathfrak{g}$ is an *ideal* of $\mathfrak{g}$ if it is a vector subspace and $H \in \mathfrak{h}, X \in \mathfrak{g}$ implies $[H, X] \in \mathfrak{h}$.
4. If $\mathfrak{h}$ is another Lie algebra, then $\varphi : \mathfrak{g} \to \mathfrak{h}$ is a *Lie algebra homomorphism* if it is linear and preserves the bracket:

$$\varphi[X, Y] = [\varphi X, \varphi Y] \quad \forall X, Y \in \mathfrak{g}.$$

5. A Lie algebra $\mathfrak{h}$ is *isomorphic* to $\mathfrak{g}$ if there is a bijective Lie algebra homomorphism $\varphi : \mathfrak{g} \to \mathfrak{h}$. Then φ is called an *isomorphism.* (Note that φ^{-1} is then an isomorphism from $\mathfrak{h}$ to $\mathfrak{g}$.)

6. Let V be a vector space. A Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{gl}(V)$ is called a *representation* of $\mathfrak{g}$ in V.

Exercise 6 *Classify the one and two dimensional Lie algebras up to isomorphism.*

Exercise 7 *Let* $\varphi : \mathfrak{g} \to \mathfrak{h}$ *be a Lie algebra homomorphism. Show that* $\operatorname{im}\varphi$ *is a Lie subalgebra of* $\mathfrak{h}$ *and* $\ker\varphi$ *is an ideal in* $\mathfrak{g}$*.*

Example 8 If the vector space V has a basis of size n, it becomes identified with $\mathbb{F}^n$ and $L(V)$ with $M(n, \mathbb{F})$ - the $n \times n$ matrices with entries in $\mathbb{F}$. Under this identifcation, composition becomes matrix multiplication and so the bracket is now defined by

$$[A, B] = AB - BA.$$

$M(n, \mathbb{F})$ with this bracket is denoted by $\mathfrak{gl}(n, \mathbb{F})$. Clearly $\mathfrak{gl}(V)$ and $\mathfrak{gl}(n, \mathbb{F})$ are isomorphic. □

Example 9 With the Lie algebra $\mathfrak{gl}(n, \mathbb{F})$ in hand, we obtain others by considering various familiar subspaces:

1. $\mathfrak{sl}(n, \mathbb{F}) = \{X \in \mathfrak{gl}(n, \mathbb{F}) : \mathrm{Trace}(X) = 0.\}$. (*Special Linear Algebra*)
2. $\mathrm{skew}(n, \mathbb{F}) = \{X \in \mathfrak{gl}(n, \mathbb{F}) : X + X^t = 0.\}$.
3. $\mathfrak{t}(n, \mathbb{F}) = \{X \in \mathfrak{gl}(n, \mathbb{F}) : X \text{ is upper triangular}\}$.
4. $\mathfrak{n}(n, \mathbb{F}) = \{X \in \mathfrak{gl}(n, \mathbb{F}) : X \text{ is strictly upper triangular}\}$.
5. $\mathfrak{d}(n, \mathbb{F}) = \{X \in \mathfrak{gl}(n, \mathbb{F}) : X \text{ is diagonal}\}$.

□

Exercise 10 *Which of the above Lie algebras depend on the choice of basis, and to what extent?*

Since $\mathfrak{sl}(n, \mathbb{F})$ is independent of the choice of basis, we can denote it by $\mathfrak{sl}(V)$.

Definition 11 A Lie algebra is called *linear* if it is a Lie subalgebra of $\mathfrak{gl}(n, \mathbb{F})$.

Example 12 We shall describe a machine for generating many linear Lie algebras. Let $V = \mathbb{F}^n$ and $J \in M(n, \mathbb{F})$. Then define

$$\mathfrak{g}_J := \{X \in \mathfrak{gl}(n, \mathbb{F}) : JX + X^t J = 0\}.$$

It is easily verified that $\mathfrak{g}_J$ is a vector subspace and also closed under bracket, hence it is a Lie subalgebra of $\mathfrak{gl}(n, \mathbb{F})$. For example, $J = I$ gives $\mathfrak{g}_I = \mathfrak{o}(n, \mathbb{F})$.

□

Exercise 13 *Show that if J and K are orthogonally similar, then $\mathfrak{g}_J$ and $\mathfrak{g}_K$ are isomorphic.*

Example 14 Let us consider various choices of J. (Note: The explicit descriptions of the Lie algebras below involve the assumption that $\mathrm{char}(\mathbb{F}) \neq 2$.)

1. Let $n = 2p$ and consider

   $$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

   where I is the $p \times p$ identity matrix. Then

   $$\mathfrak{g}_J = \left\{ \begin{pmatrix} X & Y \\ Z & -X^t \end{pmatrix} : X, Y, Z \in M(p, \mathbb{F}),\ Y = Y^t,\ Z = Z^t \right\}$$

   is called the *symplectic algebra* and denoted by $\mathfrak{sp}(n, \mathbb{F})$.

2. Let $n = 2p$ and consider

   $$K = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},$$

   where I is the $p \times p$ identity matrix. Then

   $$\mathfrak{g}_K = \left\{ \begin{pmatrix} X & Y \\ Z & -X^t \end{pmatrix} : X, Y, Z \in M(p, \mathbb{F}),\ Y + Y^t = Z + Z^t = 0 \right\}$$

   is called the *orthogonal algebra* and denoted by $\mathfrak{o}(n, \mathbb{F}) = \mathfrak{o}(2p, \mathbb{F})$.

3. Let $n = 2p + 1$ and consider

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{pmatrix},$$

where I is the $p \times p$ identity matrix. Then

$$\mathfrak{g}_L = \left\{ \begin{pmatrix} 0 & -b^t & -c^t \\ c & X & Y \\ b & Z & -X^t \end{pmatrix} : \begin{array}{c} b, c \in \mathbb{F}^p,\ X, Y, Z \in M(p, \mathbb{F}), \\ Y + Y^t = Z + Z^t = 0 \end{array} \right\}$$

is also called the *orthogonal algebra* and denoted $\mathfrak{o}(n, \mathbb{F}) = \mathfrak{o}(2p+1, \mathbb{F})$.

$\square$

Exercise 15 *Show that $\mathfrak{o}(n, \mathbb{F})$ is isomorphic to skew$(n, \mathbb{F})$, provided that $\mathbb{F}$ is algebraically closed.*

Exercise 16 *Consider $\mathfrak{g} = \mathbb{R}^3$ with the vector cross-product*

$$[X, Y] := X \times Y.$$

Verify $\mathfrak{g}$ is a Lie algebra. Show it is isomorphic to $\mathfrak{o}(3, \mathbb{R})$.

Exercise 17 *Let $E_{ij} \in M(n, \mathbb{F})$ be defined as having all entries equal 0, except that the (i, j) one equals 1. Show that*

$$[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{li} E_{kj}.$$

Exercise 18 *Show that* $\dim(\mathfrak{sp}(n, \mathbb{F})) = \frac{1}{2}n(n+1)$, $\dim(\mathfrak{o}(n, \mathbb{F})) = \frac{1}{2}n(n-1)$.

Exercise 19 *Prove the isomorphisms $\mathfrak{sl}(2, \mathbb{F}) \cong \mathfrak{o}(3, \mathbb{F}) \cong \mathfrak{sp}(2, \mathbb{F})$, assuming* $\mathrm{char}(\mathbb{F}) \neq 2$.