What is the Tits index and how to work with it

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December 10th,11th of 2012
Workshop on Groups and Geometries
Indian Statistical Institute
Bangalore, India

Layout

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The group itself can be built from the abstract root datum via what is called a Steinberg presentation.
To study reductive groups over an arbitrary field K, the root datum over K_{sep} (the separable closure of K) gets jacked up to an 'indexed' root datum called the Tits index - here, the action of the Galois group Γ of the separable closure of K on the root datum is incorporated.

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The Tits index of a K-group is represented by a diagram from which one can determine the corresponding objects like K-root system, the K-anisotropic part, the K-Weyl group, the anisotropic kernel etc.

The theme of these two talks is to demonstrate the oft-repeated slogan that a picture is worth a thousand words - we will show that each Tits index (and each Dynkin diagram itself) is worth several hundred words at least!

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$$s_\alpha(x) = x - < x, \alpha^\vee > \alpha$$

$$s_{\alpha^\vee}(y) = y - < \alpha, y > \alpha^\vee$$

which *stabilize* $R, R^\vee$ respectively.
As a consequence of the above, s_α has order 2 and sends α to $-\alpha$.

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$X = X^*(T), X^\vee = X_*(T), < ., . >: X \times X^\vee \to \mathbf{Z}; (\chi, \lambda) \mapsto n$ where $\chi \circ \lambda : t \mapsto t^n$.

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Also, $R, R^\vee$ are finite sets in bijection by a map $\alpha \mapsto \alpha^\vee$ which satisfies :
(i) $< \alpha, \alpha^\vee >= 2$, and
(ii) $s_\alpha(R) \subset R$, where $s_\alpha \in Aut(X)$ is : $x \mapsto x- < x, \alpha^\vee > \alpha$.

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The subgroup of X generated by R is of finite index if and only if G is semisimple.

Examples : SL_2, PGL_2.

$G = SL_2$:

Here $T = \{diag(t, t^{-1}) : t \in K^*\}$ and $X^*(T) = \mathbf{Z}\chi$ with $\chi : diag(t, t^{-1}) \mapsto t$.

Also, $X_*(T) = \mathbf{Z}\lambda$ where $\lambda : t \mapsto diag(t, t^{-1})$.

The set of roots is $R = \{\alpha, -\alpha\}$ where $\alpha = 2\chi$.

The set of coroots is $R^\vee = \{\alpha^\vee, -\alpha^\vee\}$ where $\alpha^\vee = \lambda$.

Clearly $< \alpha, \alpha^\vee >= 2$ as the composite $\alpha \circ \alpha^\vee$ takes any t to t^2.

The automorphism $s_\alpha = s_{-\alpha}$ interchanges α and $-\alpha$ and the Weyl group is $\{1, s_\alpha\}$.

Identifying $X^*(T)$ and $X_*(T)$ with $\mathbf{Z}$ by means of $\chi \mapsto 1$ and $\lambda \mapsto 1$ respectively, the roots are $R = \{2, -2\}$ and the coroots are $R^\vee = \{1, -1\}$.

The pairing $\mathbf{Z} \times \mathbf{Z} \to \mathbf{Z}$ is the standard one $(x, y) \mapsto xy$ and the bijection between roots and coroots is $2 \mapsto 1, -2 \mapsto -1$.

$G = PGL_2$:

Recall that this group is the quotient group of GL_2 by its center (more precisely, $G(R) = GL_2(R)/R^*$ for all rings R). Here T is the image of the diagonal torus of GL_2 modulo the nonzero scalar matrices over K. One has $X^*(T) = \mathbf{Z}\alpha$, where $\alpha : diag(t_1, t_2) \mapsto \frac{t_1}{t_2}$. The roots are $R = \{\alpha, -\alpha\}$. Identifying $X^*(T)$ with $\mathbf{Z}$ by sending α to 1, the roots are $\{1, -1\}$. It is clear that $X_*(T) = \mathbf{Z}\lambda$ and the set of coroots is $R^\vee = \{2\lambda, -2\lambda\}$, where $\lambda : t \mapsto diag(t, 1)$. Note that the diagonal matrix above is to be interpreted modulo the scalar matrices. Identifying λ with 1 for an isomorphism of X_* with $\mathbf{Z}$, the coroots become $\{2, -2\}$. The pairing is the standard one $(x, y) \mapsto xy$, as in the SL_2 case and, the bijection $\alpha \mapsto \alpha^\vee$ from R to $R^\vee$ is the obvious one $1 \mapsto 2, -1 \mapsto -2$.

Example: GL_n **for** $n \geq 2$.

The diagonal torus of $G = GL_n$ over K is T. Now $X^*(T) = \bigoplus_{i=1}^n \mathbf{Z}\chi_i$, where $\chi_i : diag(x_1, \cdots, x_n) \mapsto x_i$. The adjoint action on $\mathcal{G} = Lie(G) = M_n$ is simply conjugation. The set R of roots is $\alpha_{i,j} = \chi_i - \chi_j; i \neq j$ and $\mathcal{G}_{\alpha_{i,j}} =< E_{i,j} >$, generated by the elementary matrices. If we identify $X^*(T)$ with $\mathbf{Z}^n$ by means of the basis $\{\chi_i\}$, the subset of roots is identified with $\{e_i - e_j : 1 \leq i \neq j \leq n\}$, where $\{e_i\}$ is the canonical basis of $\mathbf{Z}^n$. The group $X_*(T)$ of cocharacters is the free abelian group with basis
$\lambda_i : t \mapsto diag(1, \cdots, 1, t, 1, \cdots, 1)$ where t is at the i-th place in the diagonal matrix. Let us write $\alpha_{i,j}^\vee = \lambda_i - \lambda_j$ for all $i \neq j$. It should be noted that we are writing $X^*(T)$ and $X_*(T)$ additively; this means, for example that
$\alpha_{i,j}^\vee(t) = diag(1, \cdots, 1, t, 1, \cdots, 1, t^{-1}, 1, \cdots, 1)$ where t is at the i-th place and t^{-1} is at the j-th place. Note
$\alpha_{i,j} \circ \alpha_{i,j}^\vee : t \mapsto t^2$ for each $i \neq j$.

For $\chi \in X^(T), \lambda \in X_*$, one writes $(\chi \circ \lambda)(t) = t^{<\chi,\lambda>}$. Thus, $<\alpha_{i,j}, \alpha_{i,j}^\vee >= 2$. The above map $< ., . >$ is actually defined on $X^*(T) \times X_*$ and maps to $\mathbf{Z}$ as $(\chi, \lambda) \mapsto n$ where $\chi \circ \lambda : t \mapsto t^n$ for all $t \in K^*$.

This notation $< ., . >$ is deliberately chosen so as to point out that the group of characters and cocharacters are in duality by means of a pairing

$$< ., . >: (X^*(T) \otimes \mathbf{R}) \times (X_*(T) \otimes \mathbf{R}) \to \mathbf{R}.$$

For each root $\alpha_{i,j}$, we have an automorphism

$$s_{\alpha_{i,j}} : X^*(T) \to X^*(T) \ ;$$

$$x \mapsto x - < x, \alpha_{i,j}^\vee > \alpha_{i,j}.$$

The important point to note is that these automorphisms map Φ into itself; indeed, if i, j, k are distinct, then

$$s_{\alpha_{i,j}}(\alpha_{i,k}) = \alpha_{j,k}.$$

Also $s_{\alpha_{i,j}}(\alpha_{i,j}) = -\alpha_{i,j}$ and $s_{\alpha_{i,j}}$ fixes the other $\alpha_{k,l}$.

Let $X_{i,j}$ denote the permutation matrix where the i-th and the j-th rows of the identity matrix have been interchanged. Clearly, the conjugation action of $X_{i,j}$ on GL_n leaves T invariant and acts on a diagonal matrix by interchanging the i-th and the j-th entries. Observe that if g normalizes T, then the left coset of g modulo T acts on T and there is an induced action on $X^*(T)$ given as $g.\chi : x \mapsto \chi(g^{-1}xg)$. In other words, the induced action of $X_{i,j}$ on $X^*(T)$ is the map $s_{\alpha_{i,j}}$. Therefore, the Weyl group - the group of automorphisms of $X^*(T)$ which is generated by the $s_{\alpha_{i,j}}$'s - is S_n.

Chevalley's isogeny theorem

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There is a bijection $b : R \longrightarrow R'$; $\alpha \mapsto \alpha'$ and, for each $\alpha \in R$, there is a power $q(\alpha)$ of the characteristic exponent p of K such that

$$\theta(b(\alpha)) = q(\alpha)\alpha \ , \ \theta^{\vee}(\alpha^{\vee}) = q(\alpha)(b(\alpha))^{\vee}.$$

If there is an isogeny (that is, a surjective homomorphism with finite kernel) f from an algebraic group G to G', and T, T' are respective maximal tori, then f induces an injective homomorphism from $X(T')$ to $X(T)$; it turns out that the corresponding root data will be isogenous.

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Chevalley's isogeny theorem.

Let G, G' be connected reductive algebraic groups and T, T' be their maximal tori respectively. Let $\Psi = \Psi(G, T)$ and $\Psi' = \Psi(G', T')$ denote their root data. Let $\theta : X^*(T') \to X^*(T)$ give an isogeny of the two root data. Then, there exists an isogeny f of G onto G', mapping T onto T' and inducing θ. This isogeny is uniquely determined upto composition with the inner automorphism of G effected by an(y) element of T.

K-forms

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That is, there are two homomorphisms θ, θ_0 of $\mathrm{Gal}(L/K)$ into Aut $(G_0(L))$.
The map $\beta(x) = \theta_0(x)\theta(x)^{-1}$ from $\mathrm{Gal}(L/K)$ to Aut $G_0(L)$ satisfies the one-cocycle identity

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Conversely, given an L-group G_0 and an one cocycle β from Gal(L/K) to Aut $G_0(L)$, there is a K-group G defined by it,

Twisting

If G is an algebraic group over K and $c \in Z^1(K, Aut(G))$, (that is, if it is a one-cocycle from $\mathrm{Gal}(K_{sep}/K)$ to Aut(G)), then we have the twisted group G_c to be that K-form of G such that $G_c(K_{sep}) = G(K_{sep})$ and Γ acts on $G_c(K_{sep})$ as

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In fact, given an one-cocycle c, it induces a new action of $\mathrm{Gal}(K_{sep}/K)$ on $K[G] \otimes_K K_{sep}$ and the corresponding fixed points determine the co-ordinate ring of G_c over K. Any K-form of G is of the form G_c for some c as above.

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One can twist $Aut(G)$ itself also and obtain the K-group $Aut(G)_c$ by twisting $Aut(G)$ by an one-cocycle $c \in Z^1(K.Aut(G))$, where the group acts on itself by inner conjugation. The action of Γ is then

$$\gamma * \sigma = c(\gamma)(\gamma\sigma)c(\gamma)^{-1}.$$

Tits Index

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There is a maximal torus T defined over K containing a maximal K-split torus S.
Any two maximal K-split tori are conjugate by $G(K)$.

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The K-split torus S gives a root system ${}_K R$ (called the K-root system of (G, S)) and the K-Weyl group; the choice of a minimal parabolic subgroup defined over K containing T gives a basis for the K-root system. Indeed, the nontrivial restrictions of the (absolute) roots of G with respect to T, to the subgroup S, give the K-roots.

The minimal parabolic subgroups defined over K are conjugate. Using these, one has Bruhat and Levi decompositions akin to those using Borels over K_{sep}.
The K-split torus S gives a root system ${}_KR$ (called the K-root system of (G, S)) and the K-Weyl group; the choice of a minimal parabolic subgroup defined over K containing T gives a basis for the K-root system. Indeed, the nontrivial restrictions of the (absolute) roots of G with respect to T, to the subgroup S, give the K-roots.
If the absolute root system R of G with respect to T is irreducible, then so is the K-root system ${}_KR$.

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The absolute Galois group Γ acts on D; this $*$-action can be recognized as follows.
For $\gamma \in \Gamma$, the set $\gamma(D)$ is a simple system of roots for another ordering of the character group of T; so, there exists w in the Weyl group $W(G, T)$ such that $w(\gamma(D)) = D$; then, $\gamma * D := w \circ \gamma$.

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The triple (D, D_0, τ) *is called the index of* G *with respect to* K, S, T.

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The triple (D, D_0, τ) is called the index of G with respect to K, S, T.
The Dynkin diagram of $Z_G(S)$ with respect to T is the full subgraph $\mathcal{D}_0$ of the Dynkin diagram $\mathcal{D}$ on the set of vertices D_0.

One starts with the Dynkin diagram $\mathcal{D}$, and considers the orbits of the simple roots under the action of $\Gamma :=$Gal(K_{sep}/K). Orbits of the simple roots in $D - D_0$ are distinguished by circling them together. Each such circled orbit gives one simple K-root. Also, the roots in an orbit are shown close by putting one below the other.

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The classification/reduction theorem asserts that (the semi-simple K-group) G is determined up to K-isomorphism by its Tits index. In particular, the K-root system of G with respect to S and the absolute root system of its anisotropic kernel (the derived group of $Z_G(S)$ which is K-anisotropic) can be determined from its Tits index.

Some admissiblity conditions

Before we describe the list of Tits indices, we point out some admissibility criteria which will enable us to deduce that certain diagrams *cannot* arise as Tits indices of a group over *any* field.

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Some admissiblity conditions

Before we describe the list of Tits indices, we point out some admissibility criteria which will enable us to deduce that certain diagrams *cannot* arise as Tits indices of a group over *any* field. The *"Opposition involution"* is an involution on the set of simple roots :

$$\alpha \mapsto -w_0(\alpha)$$

It acts as diagram automorphisms on the Dynkin diagrams. It can bs seen that this action is non-trivial only for types A_n, D_{2n+1}, E_6.
HERE IS A FIGURE:

Two necessary conditions for valid index

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From the index of a group, if we remove a distinguished orbit $\mathcal{O}$ along with the other roots connected to it, this gives a valid index - the index of the centralizer of the torus which is the connected component of the identity of the intersection of kernels of all simple K-roots not in $\mathcal{O}$.

Admissibility criterion for anisotropic kernel

There are necessary and sufficient conditions to decide whether the anisotropic kernel is admissible; this depends on the so-called Brauer invariant which can be used to ensure that the list of Tits indices in Tits's paper is complete.

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Later, we will list and describe in some detail all the admissible indices of exceptional groups of K-rank 2.

Admissibility criterion for anisotropic kernel

There are necessary and sufficient conditions to decide whether the anisotropic kernel is admissible; this depends on the so-called Brauer invariant which can be used to ensure that the list of Tits indices in Tits's paper is complete.
Later, we will list and describe in some detail all the admissible indices of exceptional groups of K-rank 2.
HERE ARE the extended Dynkin diagrams of all groups. This will be useful in many ways. Let us see some immediate uses.

Look at any extended Dynkin diagram with n vertices, say. Writing the highest root $-\mu = \sum_{i=1}^{n} c_i \alpha_i$, here is a quick way to compute the coefficients c_i's (which will turn out to be useful in our discussion of Tits indices):

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Apply in turn α_j^* one by one for $j \leq n$; recall that:

α_i and α_j are not connected if $\alpha_j^*(\alpha_i) = 0 = \alpha_i^*(\alpha_j)$;
α_i and α_j are connected by a single edge and of equal length if $\alpha_j^*(\alpha_i) = -1 = \alpha_i^*(\alpha_j)$;
α_i and α_j are connected by a double edge with α_j longer if $\alpha_j^*(\alpha_i) = -1, \alpha_i^*(\alpha_j) = -2$;
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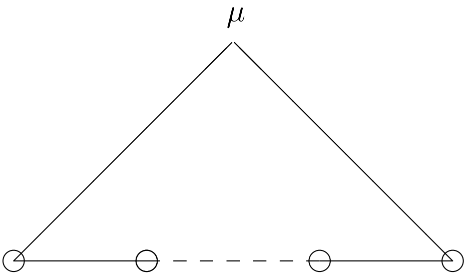
Further, the dimension of the corresponding absolutely simple group G is obtained as $\dim(G) = n(\sum_{i=1}^n c_i + 2)$.

Type A_n



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Finally, applying α_n^*, we get
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$c_n = nc_1 - (n-1)$ gives $c_i = 1$ for all i.
Thus, $-\mu = \sum_{i=1}^{n} \alpha_i$ and
dimension $A_n = n(n+2) = (n+1)^2 - 1$.

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Inductively, until the application of α_i for $i \leq n - 2$ we get

$$c_{i+1} = (i + 1)(c_1 - 2) + 2$$

Application of α^*_{n-1} and α^*_n yield

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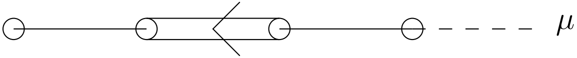
Thus, $-\mu = 2\sum_{i=1}^{n-1} \alpha_i + \alpha_n$ and
dimension $C_n = n(\sum c_i + 2) = n(2n+1).$

Type F_4



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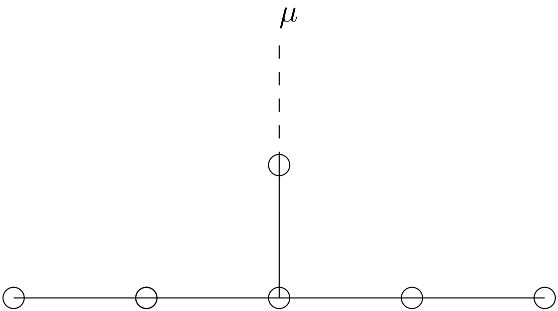
Hence $-\mu = 2\alpha_1 + 4\alpha_2 + 3\alpha_3 + 2\alpha_4$ and
dimension $F_4 = 4(2 + 4 + 3 + 2 + 2) = 52$.

Type E_6



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These give $-\mu = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6$ and dimension
$E_6 = 6(\sum c_i + 2) = 6(1 + 2 + 3 + 2 + 1 + 2 + 2) = 78.$

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In his paper, Tits points out the list of admissible indices over local and global fields.

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HERE ARE the indices of absolute type A.

Reading K-root system, K-Weyl group

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Consider the matrix whose entries are the inner products (α, β) of pairs of (absolute) simple roots and let $a_{\alpha,\beta}$ be the entries of the inverse matrix.

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Define

$$b_{\gamma,\delta} := \sum_{\alpha \in O_\gamma} \sum_{\beta \in O_\delta} a_{\alpha,\beta}$$

Then, the inverse of the matrix with entries $b_{\gamma,\delta}$ gives the matrix whose entries are the inner products (γ_K, δ_K).

The above procedure determines the K-root system excepting the fact that we need to show a way of finding for each simple K-root, the largest n (equal to 1 or 2) such that $n\gamma_K$ is a K-root.

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HERE ARE several examples.

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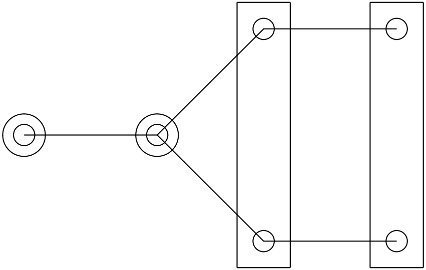
Then $(\beta_i, \beta_j) = a_{id,jd} = di(r+1-j)/(r+1)$ since
$d(r+1) = n+1$.
This is just the matrix analogous to the matrix (a_{ij}) and hence the K-root system is of type A_r.

Type ${}^2E_{6,4}^2$



The matrix of scalar products can be taken to be the Cartan matrix:

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}$$

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The inverse matrix (a_{ij}) is $\frac{1}{3}$ of

$$\begin{pmatrix} 4 & 5 & 6 & 4 & 2 & 3 \\ 5 & 10 & 12 & 8 & 4 & 6 \\ 6 & 12 & 18 & 12 & 6 & 9 \\ 4 & 8 & 12 & 10 & 5 & 6 \\ 2 & 4 & 6 & 5 & 4 & 3 \\ 3 & -6 & 9 & 6 & 3 & 6 \end{pmatrix}$$

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The simple K-roots $\beta_1, \beta_2, \beta_3, \beta_4$ correspond to the orbits $\{1,5\}, \{2,4\}, \{3\}, \{6\}$ respectively.

Using the algorithm, we compute the matrix of scalar products of the β's to be:

$$\begin{pmatrix} 4 & 6 & 4 & 2 \\ 6 & 12 & 8 & 4 \\ 4 & 8 & 6 & 3 \\ 2 & 4 & 3 & 2 \end{pmatrix}$$

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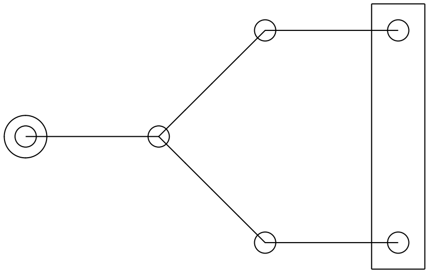
This is the matrix of scalar products of the root system of type F_4 because the Cartan matrix $c_{ij} = 2\frac{(\beta_j,\beta_i)}{(\beta_j,\beta_j)}$ of F_4 is

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

since $(\beta_3, \beta_3) = (\beta_4, \beta_4) = 2(\beta_1, \beta_1) = 2(\beta_2, \beta_2)$ in F_4.

Type ${}^2E_{6,2}^{14}$



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The K-root system is thus either of type C_2 or of type BC_2; we claim that it is the latter.

The diagram with vertices $D_0 \cup O_{\beta_1}$ is A_5 and the sum of the coefficients of α_1, α_5 in the expression for the highest root of A_5 is 2. So, $2\beta_1$ is a root.

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The diagram for $D_0 \cup O_{\beta_4}$ is D_4 and the coefficient of α_6 for the highest root of D_4 is 1; so $2\beta_4$ is not a root.

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The diagram for $D_0 \cup O_{\beta_4}$ is D_4 and the coefficient of α_6 for the highest root of D_4 is 1; so $2\beta_4$ is not a root.
Hence the K-root system is BC_2 with β_1 multipliable.
To determine the K-Weyl group ${}_KW$, one need only determine the order t of $s_{\beta_1}s_{\beta_4}$ in ${}_KW$. For, then

$${}_KW = < s_1, s_2 | s_1^2, s_2^2, (s_1 s_2)^t >$$

Presenting the K-Weyl group

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This can be determined from the inner product matrix as:

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But instead of finding the scalar products etc., there is a shorter method to find this order:

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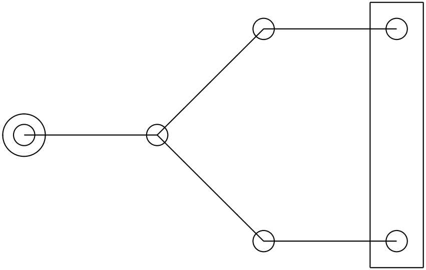
HERE IS AN EXAMPLE.

Example of ${}^2E_{6,2}^{14}$



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Order of $s_{\beta_1} s_{\beta_4}$ is $\frac{2(f_{\gamma,\delta} - f_0)}{f_\gamma + f_\delta - 2f_0} = 2(72 - 12)/(30 + 24 - 24) = 4.$

Proof of the above formula in general:
Write Φ for the K-root system formed by linear combinations of the K-roots γ, δ and $W(\Phi)$ the corresponding Weyl group.
We wish to find the order $2m_{\gamma,\delta}$ of $W(\Phi)$.
We will talk of proportionality in ${}_K D$ as equality.
For each K-root x, let $f(x)$ denote the number of absolute roots which restrict on S to the K-root x.
The restriction to S of an absolute root η is trivial (respectively, proportional to γ, proportional to γ, linear combination of γ, δ) if η is a linear combination of the simple roots in D_0 (respectively, $D_0 \cup O_\gamma, D_0 \cup O_\delta, D_0 \cup O_\gamma \cup O_\delta$).
In other words,

$$f(\gamma) = \frac{1}{2}(f_\gamma - f_0), f(\delta) = \frac{1}{2}(f_\delta - f_0),$$

$$\sum_{x \in \Phi} f(x) = f_{\gamma,\delta} - f_0$$

Now $f(w(x)) = f(x)$ for all $w \in_K W$ and,

$$W(\Phi)\gamma \cup W(\Phi)\delta$$

produces each $x \in \Phi$ exactly twice, we have

$$2 \sum_{x \in \Phi} f(x) = |W(\Phi)|(f(\gamma) + f(\delta))$$

which clearly implies the expression for $m_{\gamma,\delta}$.

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Also, recall that in the index of a group, if a distinguished orbit $\mathcal{O}$ along with the other roots connected to it is removed, this gives the index of a group (the centralizer of the torus which is the connected component of the identity of the intersection of kernels of all simple K-roots not in $\mathcal{O}$).

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Using this, let us look at a possible inner type A_n index.

Suppose the distinguished vertices are at $a_1 < \cdots < a_r$; we show that $a_i = \frac{i(n+1)}{r+1}$.

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Proof.

The $*$-action is trivial.

If $a_1 < a_2 < \cdots < a_r$ are the distinguished vertices, we will prove by induction that $a_i = \frac{i(n+1)}{r+1}$.

Indeed, this is ok for $r = 1$ as $a_1 = \frac{n+1}{2}$ in this case by looking at the action of i.

Also, for $r > 1$, looking at the orbit $\mathcal{O} = \{a_r\}$, the above necessary condition gives by induction that $a_i = \frac{ia_r}{r}$ for all i.

Taking $\mathcal{O}' = \{a_{r-1}\}$, again by induction one gets

$$a_r - a_{r-1} = n + 1 - a_r.$$

Therefore, we get $a_i = \frac{i(n+1)}{r+1}$ for $1 \leq i \leq r$.

Classical groups

We first describe in detail the indices coming from types A, B, C, D and how Galois cohomology is being used to derive them.

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It will turn out that the (simply connected) groups themselves can be described as special unitary groups of some Hermitian or skew-Hermitian forms over a division algebra with an involution of first or second kind.

Digression on involutions

For any field K, the following are examples of involutive antiautomorphisms:

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$M_n(K) \times M_n(K) \to M_n(K) \times M_n(K); (X, Y) \mapsto (Y^t, X^t)$.

Proposition. Let σ, τ be involutive antiautomorphisms on a central simple algebra over a field $L = Z(A)$. Assume that σ and τ agree on L. Then, there exists $g \in A^*$ such that $\sigma = Int(g) \circ \tau$ where $\tau(g) = \pm g$ when τ is of the first kind (that is, τ is identity on L) and where $\tau(g) = g$ when τ is of the 2nd kind (that is, τ is the nontrivial automoprphism of L over $K := L^\tau$).
Conversely, given τ on A and g satisfying the above property, the σ defined as $\sigma = Int(g) \circ \tau$ is another involutive antiautomoprphism agreeing with τ on L.

Proof. Now $\sigma\tau^{-1}$ is an automorphism of A which is trivial on the center L. Thus, by the Skolem-Noether theorem, there is $g \in A^*$ such that $\sigma(x) = g\tau(x)g^{-1}$ for all $x \in A$.
Applying σ again to the above equality, we get $g\tau(g)^{-1} \in L$.
If τ is of the first kind, then $\tau(g\tau(g)^{-1}) = g\tau(g)^{-1}$ which gives $\tau(g) = \pm g$.
If τ is of the second kind, then
$N_{L/K}(g\tau(g)^{-1}) = g\tau(g)^{-1}g^{-1}\tau(g) = 1$.
By Hilbert 90, pick $a \in L$ with $g\tau(g)^{-1} = a\tau(a)^{-1}$ which means $a^{-1}g$ in place of g fulfils the properties asserted.
The converse is clear.

For any central simple algebra A over $L := Z(A)$, look at any involutive antiautomoprphism τ. We put $K = L^{\tau}$.

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We will choose the above isomorphism π in such a way that the $\bar{K}$-linear extension of τ to $M_n(\bar{K})$ is either of first type or second type above.

Start with any isomorphism $\pi : A \otimes_K \bar{K}$ to $M_n(\bar{K})$ and apply the proposition to the involutive antiautomorphisms $\sigma := \pi\tau\pi^{-1}$ and 'transpose' on $M_n(\bar{K})$.

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We obtain $F \in GL_n(\bar{K})$ such that $\sigma(x) = Fx^tF^{-1}$ for all $x \in M_n(\bar{K})$ where $F^t = \pm F$.

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We obtain $F \in GL_n(\bar{K})$ such that $\sigma(x) = Fx^tF^{-1}$ for all $x \in M_n(\bar{K})$ where $F^t = \pm F$.
Now, if $F^t = F$, choose $g \in GL_n(\bar{K})$ with $F = gg^t$;
if $F^t = -F$ (then n is even), choose g with $F = gJg^t$.

In either case, it can be verified that the changed isomorphism $Int(g)^{-1} \circ \pi$ does the asserted job.

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If τ is now of the second kind on A with center of A equal to L, say, then

$$A \otimes_K \bar{K} = (A \otimes_K L) \otimes_L \bar{K} \cong M_n(\bar{K}) \oplus M_n(\bar{K})$$

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Once again, an application of the proposition helps us choose an isomorphism under which the $\bar{K}$-linear extension of τ acts on $M_n(\bar{K}) \oplus M_n(\bar{K})$ as $(x, y) \mapsto (y^t, x^t)$.

(Skew/)Hermitian forms on D

Let (D, τ) be a division algebra with an involution; put $L = Z(D)$ and $K = L^\tau$ and $[D : L] = d^2$.

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Let (D, τ) be a division algebra with an involution; put $L = Z(D)$ and $K = L^\tau$ and $[D : L] = d^2$.
Look at the (left) D-vector space $V = D^m$.
Now τ induces an involutive antiautomorphism on $M_m(D)$ (again denoted τ) as:

$$(g^{tau})_{ij} := \tau(g_{ji}) \ \forall \ g \in M_m(D)$$

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For a basis $\{e_1, \cdots, e_m\}$ of V, the matrix $F = (f(e_i, e_j)_{ij})$ satisfies $F^\tau = \pm F$.
Consider the unitary group

$$U_m(D, f) := \{g \in GL(m, D) : g^\tau F g = F\}$$

and the special unitary group

$$SU_m(D, f) := \{g \in GL(m, D) : g^\tau F g = F, N_{red}(g) = 1\}$$

Looking at the regular representation $D \to M_n(\bar{K})$ where $n = d^2$ or $2d^2$, one may view $U_m(D, f)$ and $SU_m(D, f)$ as K-rational points of algebraic groups $\mathcal{U}_{m,D,f}$ and $\mathcal{SU}_{m,D,f}$ defined over K.

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For the new involution σ on $M_m(D)$ defined by $\sigma(x) = F^{-1}x^{\tau}F$, we have

$$\mathcal{U}_{m,D,f} = \{g \in M_m(D) \otimes_K \bar{K} : \sigma(g) = g^{-1}\}$$

$$\mathcal{SU}_{m,D,f} = \{g \in (M_m(D) \otimes_K \bar{K})^* : \sigma(g) = g^{-1}, N_{red}(g) = 1\}$$

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If τ is of the second kind, then $M_m(D) \otimes_K \bar{K} \cong M_{md}(\bar{K}) \oplus M_{md}(\bar{K})$ and by the proposition, τ gives the involution $(x, y) \mapsto (y^t, x^t)$ which means:

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$$\mathcal{U}_{m,D,f} = \{(x, (x^t)^{-1}) : x \in GL_{md}(\bar{K})\}$$

$$\mathcal{SU}_{m,D,f} = \{(x, (x^t)^{-1}) : x \in SL_{md}(\bar{K})\}$$

We summarize as follows. If (D, τ) is a division algebra with an involution τ and $f : D^m \times D^m \to D$ is a non-degenerate τ-Hermitian/skew-Hermitian form, then the algebraic group $\mathcal{SU}_{m,D,f}$ defined over K is identified over $\bar{K}$ with:

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Sp_{md} if τ is of first kind, has type II when f is Hermitian and type I when f is skew-Hermitian;
SO_{md} if τ is of first kind, has type I when f is Hermitian or type II when f is skew-Hermitian;
SL_{md} if τ is of the second kind.

Type ${}^1A_{n-1}$

The inner K-forms of SL_n are its twists given by cocycles with the cohomology class $c \in H^1(K, PSL_n)$ as $Int(SL_n) \cong PSL_n$.

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If $\pi : B \otimes_K \bar{K} \to M_{md}(\bar{K})$ is the isomorphism, then the cocycle is $c_\gamma = \pi(\pi^\gamma)^{-1}$ which gives (as N_{red} maps to det under π) the inner type A_{n-1} group we get is $SL_{m,D}$ where $n/m = deg(D)$.

As we saw, the adjoint group G can be identified with $GL_{m,D}$ mod center. Write $[D : K] = d^2$, then $dm = n$.

As we saw, the adjoint group G can be identified with $GL_{m,D}$ mod center. Write $[D : K] = d^2$, then $dm = n$.
The maximal torus T of $GL_{m,D}$ over K_{sep} is such that $T(K_{sep})$ is the diagonal torus of $GL_n(K_{sep})$ and we know that $\{\alpha_i = e_i - e_{i+1} : 1 \leq i \leq n-1\}$ is a basis for the absolute root system corresponding to the maximal torus T (and the usual upper triangular Borel).

If $S = \bigcap\{Ker(\alpha_i) : d \not| i\}$, then S consists of elements of the form

$diag(a_1, \cdots, a_{dm})$ such that

$a_1 = a_2 = \cdots = a_d, a_{d+1} = a_{d+2} = \cdots = a_{2d}$ etc.

So, S is a maximal K-split torus of G and $Z_G(S) \cong (GL_{1,D})^m$.

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So, S is a maximal K-split torus of G and $Z_G(S) \cong (GL_{1,D})^m$.

For G adjoint, and of inner type A_{n-1}, there exists $d|n$ and a division algebra over K, of degree d, such that G is isogenous to $SL_{\frac{n}{d},D}$.

The distinguished vertices are $\alpha_d, \alpha_{2d}, \cdots$ and the K-root system is of type $A_{\frac{n}{d}-1}$.

Such an index arises if and only if, there exists a division algebra over K, of degree dividing n.

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Consider $A := M_n(\bar{K}) \oplus M_n(\bar{K})$ with $(x, y) \mapsto (y^t, x^t)$; we saw $\mathcal{SU} = \{a \in A : \tau(a) = a^{-1}, N(a) = 1\}$ as the image of SL_n in A by $x \mapsto (x, (x^t)^{-1})$.

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The cocycle c takes values in automorphisms of A which commute with $(x, y) \mapsto (y^t, x^t)$.

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Hence, $E = B^\Gamma$ has an involution θ which is of the second kind because it is the 'exchange' on $E \otimes_K \bar{K} \cong M_n(\bar{K}) \oplus M_n(\bar{K})$.

Therefore, the twisted algebra $B = A_c$ has an involution which commutes with the Galois action because the cocycle commutes with the Galois action.
Hence, $E = B^{\Gamma}$ has an involution θ which is of the second kind because it is the 'exchange' on $E \otimes_K \bar{K} \cong M_n(\bar{K}) \oplus M_n(\bar{K})$.
Write $L = Z(E)$ and $K = L^{\theta}$; $E \cong M_m(D)$ where D is a division algebra over L.

By a theorem of Albert, the involution θ on $M_m(D)$ comes from one on D; call it δ again.

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Once again, by the proposition, there is $F \in GL_m(D)$ such that $\theta(x)_{ij} = F\delta(x_{ji})F^{-1}$ with $F^{\theta} = F$.

By a theorem of Albert, the involution θ on $M_m(D)$ comes from one on D; call it δ again.
Once again, by the proposition, there is $F \in GL_m(D)$ such that $\theta(x)_{ij} = F\delta(x_{ji})F^{-1}$ with $F^\theta = F$.
Finally, this means that the group is
$\{g \in SL_m(D) : \theta(g) = g^{-1}\} = SU_m(D, F)$.

For the outer Type ${}^2A_{n-1}$, we saw that there is a separable, quadratic extension L, a division algebra D over L with an L/K-involution δ, and a δ-Hermitian form h over D^m such that G is $SU_m(D, h)$.

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Let S_1 denote the torus we had in the above inner type case, viz., the diagonals $diag(x_1,\cdots,x_n)$ where

$x_1=\cdots=x_d, x_{d+1}=\cdots=x_{2d},\cdots$

This is a torus defined over the center L of D.

Its subtorus S consisting of those diagonals for which

$$x_{rd+1}=x_{rd+2}=\cdots=x_{n-rd}=1,$$

$$x_{n-rd+1}=x_{rd}^{-1}, x_{n-rd+2}=x_{rd-1}^{-1},\cdots,x_n=x_1^{-1}$$

is defined over K and is K-split and contained in G.

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D possesses an L/K-involution δ and
where h is a nonsingular δ-hermitian form on $D^{\frac{n}{d}}$ of Witt index $r \leq n/2d$.

The distinguished vertices are in the orbits $(\alpha_d, \alpha_{n-d}), (\alpha_{2d}, \alpha_{n-2d}), \cdots (\alpha_{rd}, \alpha_{n-2rd})$ where α_{n-2rd} is omitted if $2rd = n$.

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If a division algebra with involution of the second kind and a hermitian form of dimension $\frac{n}{d}$ and Witt index r exists as above, then the corresponding K-root system is of type BC_r (respectively, C_r) if $2rd < n$ (respectively, $2rd = n$).

Exceptional indices of K-rank two

There are precisely 8 admissible indices of exceptional groups (and of ${}^3D_4$) which have K-rank 2.
Four of them have absolute type E_6, one of E_7, two of E_8 and one of ${}^3D_4$.

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We shall describe the anisotropic kernel in each case. All these indices exist over number fields and the real field but not over finite fields or p-adic fields. We will need some information about quadratic spaces of exceptional types.

Quadratic spaces, defect, norm splitting

A quadratic space over a field K, is a triple (K, V, q) where V is a finite-dimensional K-vector space equipped with a map $q : V \longrightarrow K$ which is homogeneous of degree 2 and its associated form $f : V \times V \longrightarrow K$;
$(x, y) \mapsto q(x + y) - q(x) - q(y)$ is K-bilinear.

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Isomorphism of two quadratic spaces over K, direct sum of two quadratic spaces, translate of a quadratic space etc. are defined in an obvious manner.

Definition:
A quadratic space (K, V, q) is said to admit a *norm splitting* if there is a separable quadratic extension L/K, an extension of the K-linear structure of V to an L-linear structure and an L-basis $\{v_1, \cdots, v_d\}$ of V for this structure so that

$$q(\sum_{i=1}^{d} t_i v_i) = \sum_{i=1}^{d} q(v_i) N_{L/K}(t_i)$$

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The *Clifford algebra* $C(V, q)$ of a quadratic space (K, V, q) is the quotient of the tensor algebra $T(V)$ by the two-sided ideal generated by all $(v \otimes v) - q(v).1$.

It is easy to see that $C(q)$ satisfies the universal property with respect to all K-linear maps ϕ from V to associative K-algebras A such that $\phi(x)^2 = q(x).1$ for all $x \in V$.

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The filtration on the tensor algebra gives the even and odd Clifford algebras in the obvious manner.

If L/K is a separable, quadratic extension with norm form N and if $s \in K^*$, then (K, L, sN) is a quadratic space and its Clifford algebra $C(sN)$ is isomorphic to the quaternion algebra $(L/K, s)$ - this is the subring of matrices in $M_2(L)$ consisting of all $\begin{pmatrix} a & s\bar{b} \\ b & \bar{a} \end{pmatrix}$ with $a, b \in L$.

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Here, '$-$' is the nontrivial automorphism of L over K. The quaternion algebra $(L/K, s)$ is a matrix algebra (respectively, division algebra) if and only if s is (respectively, is not) a norm from L.

Generalizing the above, if L/K is as above, and if $s_1, \cdots, s_d \in K^*$, then

$$C(s_1N \perp \cdots \perp s_dN) \cong M(2^{d-1}, K) \otimes (L/K, (-1)^{[d/2]}s_1 \cdots s_d)$$

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Thus, the dimension is 4^d.
If (K, V, q) is a non-defective quadratic space admitting a norm splitting $(L, v_1, \cdots, v_d)$, and if $s := (-1)^{[d/2]}q(v_1)\cdots q(v_d)$ is a norm (respectively, not a norm) from L, then $C(q) \cong M(2^d, K)$ (respectively, $C(q) \cong M(2^{d-1}, D)$ where D is the quaternion algebra $(L/K, (-1)^{[d/2]}s_1 \cdots s_d)$).

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of type E_8, if q is anisotropic, dim $V = 12$ and q admits a norm splitting $(L, v_1, \cdots, v_6)$ with $-\prod_{i=1}^{6} q(v_i) \in N(L)$.

The List of Eight

The FOLLOWING INDEX is of a group of type E_6 and K-rank two. Recall that we computed the K-root system which was the non-reduced system of type BC_2. The anisotropic kernel H is of type ${}^2D_3$. The Dynkin diagram is:

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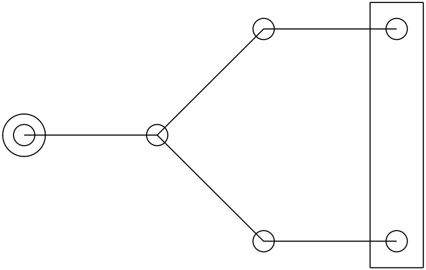

If L/K is a separable, quadratic extension such that H becomes inner, then there is an anisotropic quadratic form q in 6 variables over K with discriminant field L such that the even Clifford algebra $C_0(q)$ is isomorphic to a matrix algebra over L. In fact, L is the splitting field of $x^2 - disc(q)$ over K, if char $K \neq 2$ and, is the splitting field of $x^2 + x + \delta$ over K when char $K = 2$, where δ is the Arf invariant of q. Thus, the quadratic space q is precisely what was defined as one of type E_6. The anisotropic kernel H is the spin group of q.

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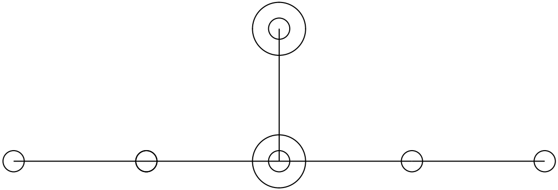

Thus, H is of type $A_2 \times A_2$. There exists a degree 3 division algebra D over K such that H is K-isomorphic to $D^* \otimes (D^{op})^*$.

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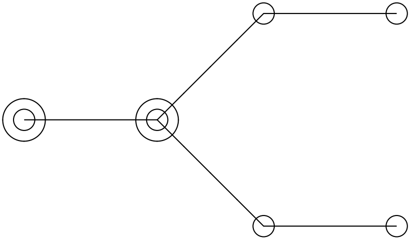

Here, there is a separable, quadratic extension L/K over which H becomes an inner form of $A_2 \times A_2$. One can identify H with the restriction of scalars $R_{L/K}$ of a division algebra D of degree 3 over L whose Brauer class has trivial corestriction in $Br(K)$. By a theorem of Albert and Riehm, this is equivalent to D possessing a L/K-involution.

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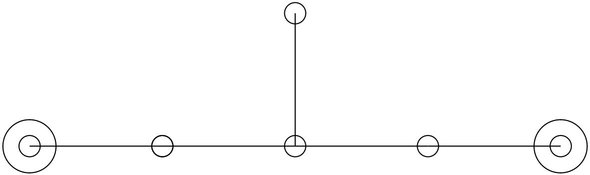

H is of type D_4; it is the spin group of a quadratic form q in 8 variables over K and having trivial discriminant. As the even Clifford algebra $C_0(q)$ is isomorphic to $M_8(K) \times M_8(K)$, the form q is the norm form of a Cayley (octonion/nonassociative alternative) division algebra.

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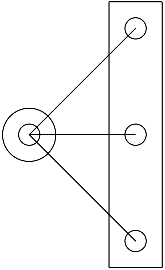

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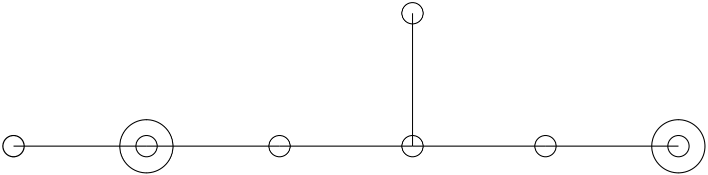

Unwinding the admissibility criterion on Brauer invariants, it turns out that there is an anisotropic quadratic form q in 8 variables over K such that $C_0(q)$ is isomorphic to the direct product of two isomorphic central simple algebras whose Brauer class is that of a quaternion division algebra. These forms q were what we called as the forms of type E_7.

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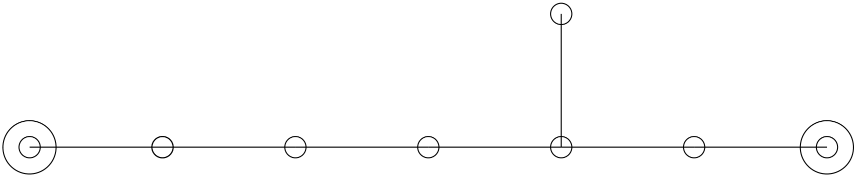

The anisotropic kernel H is the spin group of a quadratic form in 12 variables over K with trivial discriminant such that $C_0(q)$ is isomorphic to the direct product of two matrix algebras over K. Such q were what we called the forms of type E_8.

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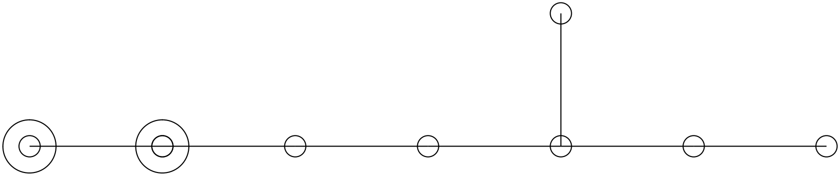

The admissibility criterion implies that H is the structure group of a 27-dimensional Jordan algebra over K.