On the isomorphism problem for Coxeter groups and related topics

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Contents of This Talk

A survey of results on **the isomorphism problem for Coxeter groups**

- "forgotten" subject until about 15 years ago
- active subject in recent years

Some relevant results on "group theory on Coxeter groups"

- finitely or non-finitely generated cases

Outline

Preliminaries

Isomorphism Problem

Given presentations of two mathematical objects in a class,
"decide" whether these are isomorphic or not

- Here we do not concern computability, especially when studying infinite presentations

Isomorphism Problem for Groups

- General groups (of finite presentations): Uncomputable
- Finitely generated abelian groups: Textbook
- Free groups
- ...
- **Coxeter groups**

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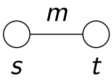
- Generators: $s \in S$ (**possibly** $|S| = \infty$)
- Fundamental relations:
  - $s^2 = 1$ ($\forall s \in S$)
  - $(st)^{m(s,t)} = 1$ ($\forall s \neq t \in S$), where
    - $2 \leq m(s,t) = m(t,s) \leq \infty$
    - relations with $m(s,t) = \infty$ are ignored

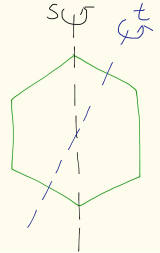
Such a presentation is expressed by a *Coxeter graph*

Examples

$W = D_m = W(I_2(m))$ (symmetry group of regular m-gon)

- $S = \{s, t\}$, $m(s, t) = m$ with $3 \leq m < \infty$

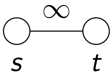


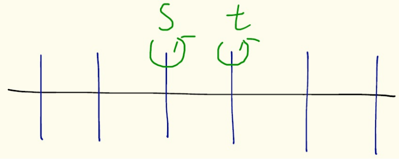


Examples

$W = D_\infty = W(\widetilde{A}_1)$ (infinite dihedral group)

- $S = \{s, t\}$, $m(s, t) = \infty$

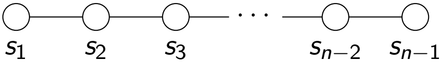




Examples

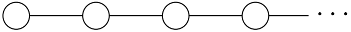
$W = S_n = W(A_{n-1})$ (symmetric group on n letters)

- $S = \{s_1, s_2, \dots, s_{n-1}\}$, $s_i = (i \;\; i+1)$
- $m(s_i, s_{i+1}) = 3$, $m(s_i, s_j) = 2$ (if $|i-j| \geq 2$)



Examples

$W =$ limit of $S_2 \hookrightarrow S_3 \hookrightarrow S_4 \hookrightarrow \cdots$
(Here we call it $W(A_\infty)$)



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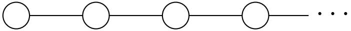
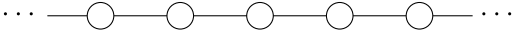


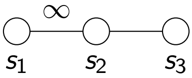
There are another embeddings $S_2 \hookrightarrow S_3 \hookrightarrow S_4 \hookrightarrow \cdots$
(Here we call the limit $W(A_{\infty,\infty})$)



Examples

$$W = \mathrm{PGL}(2,\mathbb{Z}) = \mathrm{GL}(2,\mathbb{Z})/\{\pm 1\}$$

- $S = \{s_1, s_2, s_3\} = \left\{ \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$
- $m(s_1, s_2) = \infty,\ m(s_1, s_3) = 2,\ m(s_2, s_3) = 3$



Direct/Free Product Decompositions

If $S = S_1 \cup S_2$ (disjoint) and

- $m(s_1, s_2) = 2$ ($\forall s_1 \in S_1, s_2 \in S_2$), then $W = \langle S_1 \rangle \times \langle S_2 \rangle$;
- $m(s_1, s_2) = \infty$ ($\forall s_1 \in S_1, s_2 \in S_2$), then $W = \langle S_1 \rangle * \langle S_2 \rangle$

The Problem

Isomorphism Problem for Coxeter Groups

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Problem Given two Coxeter graphs Γ_1, Γ_2, decide whether $W(\Gamma_1) \simeq W(\Gamma_2)$ (as abstract groups) or not

or: Given a Coxeter group W, determine the possible Coxeter generating sets S for W (or the types of (W, S))

Rigid Coxeter Groups

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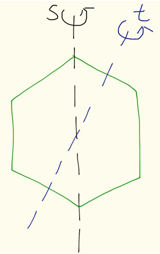
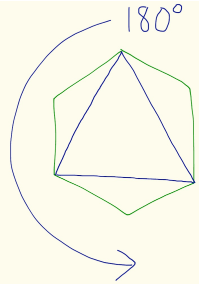
Notes:

- strongly rigid $\Rightarrow$ rigid
- difference of two properties $\approx$ outer automorphisms of W

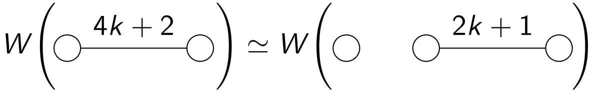
"Folklore" Non-Rigid Example

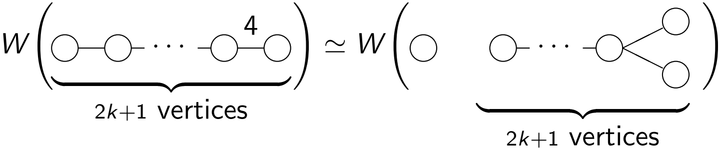
$$W\left(\circ\overset{6}{\text{———}}\circ\right) \simeq W\left(\circ \qquad \circ\text{———}\circ\right)$$





Non-Rigid Examples (Finite Cases)





These are only non-rigid finite irreducible Coxeter groups

"Krull–Remak–Schmidt-like" Property

Write $W = W_{\text{fin}} \times W_{\text{inf}}$, where

- W_{fin}: Product of finite irreducible components
- W_{inf}: Product of other components

Fact [N. 2006] The subset W_{fin} is uniquely determined by W, independent of S (**possibly when** $|S| = \infty$)

K. Nuida, On the direct indecomposability of infinite irreducible Coxeter groups and the isomorphism problem of Coxeter groups, *Comm. Algebra* **34**(7) (2006) pp.2559–2595

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(**Note:** [Mihalik–Ratcliffe–Tschantz 2005 (preprint)] showed “free product decomposition” version of the fact)

L. Paris, Irreducible Coxeter groups, *Int. J. Algebra Comput.* **17**(3) (2007) 427–447

M. Mihalik, J. Ratcliffe, S. Tschantz, On the isomorphism problem for finitely generated Coxeter groups. I: Basic matching, arXiv:math.GR/0501075v1 (2005)

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Tool in [N. 2006]: **centralizers of normal subgroups generated by involutions in W**

1. If $W = H_1 \times H_2$, then H_j are generated by involutions
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1. If $W = H_1 \times H_2$, then H_j are generated by involutions
2. **Key Fact** If $H \trianglelefteq W$ is generated by involutions, then its centralizer $Z_W(H)$ is either W or "too small"
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2. **Key Fact** If $H \trianglelefteq W$ is generated by involutions, then its centralizer $Z_W(H)$ is either W or “too small”
3. Since $H_j \subset Z_W(H_{3-j})$, at least one of H_j should be W

"Krull–Remak–Schmidt-like" Property

Theorem [N. 2006] (informal) Any $f \colon W \xrightarrow{\sim} W'$ is "approximately decomposed" into

- isomorphisms between infinite irreducible components of W and W', and
- an isomorphism $W_{\mathrm{fin}} \xrightarrow{\sim} W'_{\mathrm{fin}}$

Hence the isomorphism problem (**including the case** $|S| = \infty$) is "essentially" **reduced to infinite irreducible cases**

A Natural Question

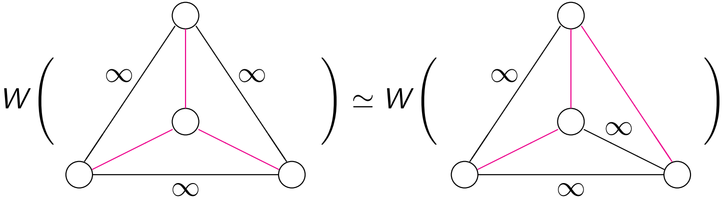
[Cohen 1991] Is the isomorphism problem for **irreducible, finite-rank** Coxeter groups trivial?

I.e., does $W(\Gamma) \simeq W(\Gamma')$, with Γ, Γ' **finite** and **connected**, imply $\Gamma \simeq \Gamma'$?

A. M. Cohen, Coxeter groups and three related topics, in: *Generators and Relations in Groups and Geometries* (A. Barlotti et al., eds.), NATO ASI Series, Kluwer Acad. Publ. (1991) pp.235–278

A Counterexample

[Mühlherr 2000] (1-page paper!)



B. Mühlherr, On isomorphisms between Coxeter groups, *Des. Codes Cryptogr.* **21** (2000) p.189

Rigidity and Reflections

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$w \in W$ is a *reflection* $\overset{\text{def}}{\iff}$ conjugate to an $s \in S$
$\iff$ in the standard geometric representation of W, w acts as a reflection w.r.t. a hyperplane

$\mathrm{Ref}(W) = \mathrm{Ref}_S(W) := \{s^w := wsw^{-1} \mid s \in S, w \in W\}$: set of reflections

Rigidity and Reflections

$W(\Gamma)$ is *reflection rigid* $\overset{\text{def}}{\Longleftrightarrow}$ $f \colon W(\Gamma') \xrightarrow{\sim} W(\Gamma)$ **and** $f(\mathrm{Ref}(W(\Gamma'))) = \mathrm{Ref}(W(\Gamma))$ implies $\Gamma' \simeq \Gamma$

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W is *strongly reflection rigid* $\overset{\text{def}}{\Longleftrightarrow}$ all Coxeter generating sets for W **defining the same set of reflections** are conjugate with each other

Note:

strongly rigid	$\Rightarrow$	rigid
$\Downarrow$		$\Downarrow$
strongly reflection rigid	$\Rightarrow$	reflection rigid

Rigidity and Reflections

Rigidity and strong reflection rigidity are incomparable

- $D_6 = W(I_2(6))$ is strongly reflection rigid, but not rigid
- $D_5 = W(I_2(5))$ is rigid, but not strongly reflection rigid

[Brady–McCammond–Mühlherr–Neumann 2002]

N. Brady, J. P. McCammond, B. Mühlherr, W. D. Neumann, Rigidity of Coxeter groups and Artin groups, *Geom. Dedicata* **94** (2002) pp.91–109

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W is *reflection independent* $\overset{\text{def}}{\Longleftrightarrow}$ $\mathrm{Ref}_S(W)$ is uniquely determined by W (independent of S)

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Intuitively, the isomorphism problem can be divided into two parts

- Given W, how "far" from being reflection independent?
- Given W, how "far" from being reflection rigid?

Rigidity and Reflections

[Bahls–Mihalik 2005] ($|S| < \infty$) Reflection independent & *even* ($\overset{\text{def}}{\iff}$ $m(s,t)$ is not odd ($\forall s \neq t$)) $\Rightarrow$ rigid

Rigidity and Reflections

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Counterexample for non-even case: Mühlherr's example

- [Bahls 2003] W is **reflection independent** if $\neg\exists s, t$ s.t., $m(s,t) \equiv 2 \pmod 4$

P. Bahls, M. Mihalik, Reflection independence in even Coxeter groups, *Geom. Dedicata* **110** (2005) pp.63–80

P. Bahls, A new class of rigid Coxeter groups, *Int. J. Algebra Comput.* **13**(1) (2003) pp.87–94

Finite Rank Cases ($|S| < \infty$)

Strongly Rigidity: Geometric Arguments

[Charney–Davis 2000] W is **strongly rigid** if W is capable of acting effectively, properly and cocompactly on some contractible manifold

- Affine Weyl groups
- Cocompact hyperbolic reflection groups
- ...

Tool: **complex, cohomology, CAT(0) space, etc.**
(Detail omitted (due to lack of my geometric knowledge ...))

R. Charney, M. Davis, When is a Coxeter system determined by its Coxeter group?, *J. London Math. Soc.* (2) **61** (2000) pp.441–461

Strongly Rigidity: Geometric Arguments

State-of-the-art result in this direction:

Theorem [Caprace–Przytycki 2011] “Bipolar” Coxeter groups are **strongly rigid**, where

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- (W, S) is *bipolar* $\overset{\text{def}}{\Longleftrightarrow}$ in the Cayley graph X of (W, S), $\forall s \in S$, any tubular neighbourhood of the s-invariant wall separates X into exactly two connected components
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- (W, S) is *bipolar* $\overset{\text{def}}{\Longleftrightarrow}$ in the Cayley graph X of (W, S), $\forall s \in S$, any tubular neighbourhood of the s-invariant wall separates X into exactly two connected components
- Precise definition and Coxeter graph characterization for bipolar Coxeter groups are also given

P.-E. Caprace, P. Przytycki, Bipolar Coxeter groups, *J. Algebra* **338** (2011) pp.35–55

Strongly Rigidity: Geometric Arguments

Bipolar Coxeter groups include the following cases

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Bipolar Coxeter groups include the following cases

- [Charney–Davis 2000]
- virtual Poincaré duality groups
- infinite irreducible *2-spherical* ($\overset{\text{def}}{\Longleftrightarrow}$ $m(s,t) < \infty$ $(\forall s,t)$) [Franzsen–Howlett–Mühlherr 2006; Caprace–Mühlherr 2007]
    - a subclass appeared in [Kaul 2002]

W. N. Franzsen, R. B. Howlett, B. Mühlherr, Reflections in abstract Coxeter groups, *Comment. Math. Helv.* **81** (2006) pp.665–697

P.-E. Caprace, B. Mühlherr, Reflection rigidity of 2-spherical Coxeter groups, *Proc. London Math. Soc.* (3) **94** (2007) pp.520–542

A. Kaul, A class of rigid Coxeter groups, *J. London Math. Soc.* (2) **66** (2002) pp.592–604

Maximal Finite Subgroups

$H \le W$ is *standard parabolic* $\overset{\text{def}}{\Longleftrightarrow}$ $H = W_I := \langle I \rangle$ for some $I \subset S$

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- Any maximal finite standard parabolic subgroup is a maximal finite subgroup

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1. Find a finite number of $I_j \subset S$ s.t. W_{I_j} are maximal finite standard parabolic and $\bigcap_j I_j = \{s\}$
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3. so is $\langle f(s) \rangle = \bigcap_j f(W_{I_j})$
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4. each $f(W_{I_j})$ is parabolic, so is $\langle f(s) \rangle$
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2. $\langle s \rangle = \bigcap_j W_{I_j}$ is the intersection of maximal finite subgroups
3. so is $\langle f(s) \rangle = \bigcap_j f(W_{I_j})$
4. each $f(W_{I_j})$ is parabolic, so is $\langle f(s) \rangle$
5. hence $f(s)$ is conjugate to some $s' \in S'$, i.e., $f(s) \in \mathrm{Ref}(W')$

Maximal Finite Subgroups

Some results using maximal finite subgroups:

- [Radcliffe 2001] W is **rigid** if W is *right-angled*
  ($\overset{\text{def}}{\iff} m(s,t) \in \{2,\infty\}$ $(\forall s \neq t)$)
- [Hosaka 2006] generalized to a wider class

D. G. Radcliffe, Unique presentation of Coxeter groups and related groups, Ph.D. thesis, Univ. Wisconsin-Milwaukee (2001)

T. Hosaka, A class of rigid Coxeter groups, *Houston J. Math.* **32**(4) (2006) pp.1029–1036

Maximal Finite Subgroups

Notes:

- Radcliffe mentioned (without proof) that the rank may be infinite
- [Radcliffe 2003] extended to **graph products** of directly indecomposable groups
- [Castella 2006] gave a new proof, with structural result on $\mathrm{Aut}(W)$

D. G. Radcliffe, Rigidity of graph products of groups, *Algebraic & Geom. Top.* **3** (2003) pp.1079–1088

A. Castella, Sur les automorphismes et la rigidité des groupes de Coxeter à angles droits, *J. Algebra* **301** (2006) pp.642–669

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    - hence $\Rightarrow$ W is **reflection independent**
    - [Caprace–Mühlherr 2007] W is **strongly reflection rigid**, hence **strongly rigid**

Relations with Even Orders

[Radcliffe 2001] W is **rigid** if $m(s,t) \in \{2,\infty\} \cup 4\mathbb{Z}$ $(\forall s \neq t)$

- Tool: **projection to abelianization of W**

[Brady et al. 2002] $W(\Gamma)$ is **reflection rigid** if it is even

D. G. Radcliffe, Unique presentation of Coxeter groups and related groups, Ph.D. thesis, Univ. Wisconsin-Milwaukee (2001)

Relations with Even Orders

[Bahls–Mihalik 2005] gave characterizations of

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- reflection independent cases **among even Coxeter systems**
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[Mihalik 2007] gave an algorithm to determine possible types of generating sets for W, when W is even

- Hence **the isomorphism problem is solved for even Coxeter systems**

M. Mihalik, The even isomorphism theorem for Coxeter groups, *Trans. AMS* **359**(9) (2007) pp.4297–4324

Diagram Twisting

Recall the example in [Mühlherr 2000]

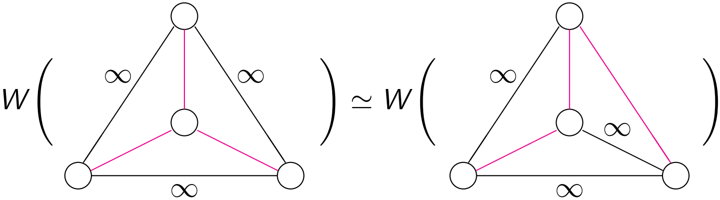

Diagram Twisting

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Let $U, V \subset S$ be disjoint subsets with the conditions:

- W_V is finite, with longest element $w = w_V$
- if $s_1 \in S \setminus (U \cup V)$ is adjacent to V, then $m(s_1, s_2) = \infty$ $(\forall s_2 \in U)$

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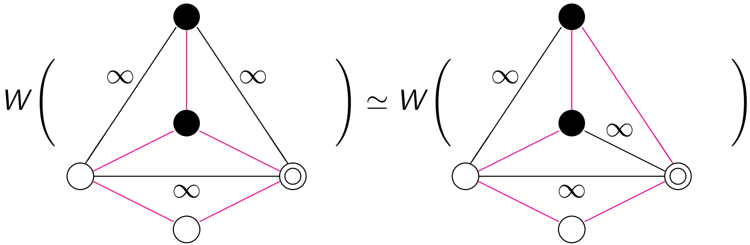
Then (W, S') is a Coxeter system with Coxeter graph Γ', where

- $S' := (S \setminus U) \cup U^w \subset \mathrm{Ref}_S(W)$, $U^w := \{u^w \mid u \in U\}$
- Γ' is obtained from Γ by replacing each edge $U \ni u — v \in V$ with an edge from $u^w \in U^w$ to $v^w \in V$

Diagram Twisting

Example:



● : Elements of V ◎ : Element of U (or U^w)

Diagram Twisting

Conjecture [Brady et al. 2002] Coxeter systems are "**reflection rigid up to diagram twistings**",

i.e., if $\mathrm{Ref}_S(W) = \mathrm{Ref}_{S'}(W)$, then $\Gamma(W, S)$ is converted to $\Gamma(W, S')$ by consecutive diagram twistings

Diagram Twisting

Positive results on the conjecture:

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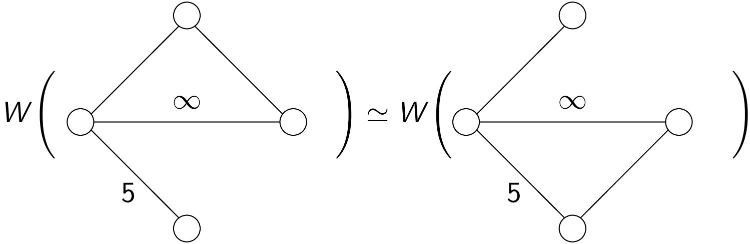
- They proved the conjecture when the *presentation graph* (i.e., $s, t \in S$ are joined when $m(s,t) < \infty$) is a tree
- [Mühlherr–Weidmann 2002] proved the conjecture for *skew-angled* cases ($\overset{\text{def}}{\Longleftrightarrow}$ $m(s,t) \geq 3$ $(\forall s, t)$)
    - They also characterized reflection independent skew-angled cases, and gave a sufficient condition for strongly rigid skew-angled cases

B. Mühlherr, R. Weidmann, Rigidity of skew-angled Coxeter groups, *Adv. Geom.* **2** (2002) pp.391–415

Diagram Twisting

But there is a counterexample! [Ratcliffe–Tschantz 2008]



by another kind of transformation, called "5-edge angle deformation"

J. G. Ratcliffe, S. T. Tschantz, Chordal Coxeter groups, *Geom. Dedicata* **136** (2008) pp.57–77

Some More Solved Cases

(W, S) is *chordal* $\overset{\text{def}}{\iff}$ every cycle of length ≥ 4 in the presentation graph of (W, S) has a shortcutting edge

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Results by [Ratcliffe–Tschantz 2008]

- The chordal property is independent of the choice of S
- An algorithm to decide whether or not two Chordal $W(\Gamma), W(\Gamma')$ are isomorphic; hence **the isomorphism problem is solved for chordal cases**

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(W, S) is *twist-rigid* $\overset{\text{def}}{\Longleftrightarrow}$ it admits no diagram twists

Some More Solved Cases

(W, S) is *twist-rigid* $\overset{\text{def}}{\iff}$ it admits no diagram twists

Results by [Caprace–Przytycki 2010]

- The twist-rigidity is independent of the choice of S
- An algorithm to output all possible Γ' with $W(\Gamma') \simeq W(\Gamma)$ from given twist-rigid $W(\Gamma)$; hence **the isomorphism problem is solved for twist-rigid cases**

P.-E. Caprace, P. Przytycki, Twist-rigid Coxeter groups, *Geom. Topology* **14** (2010) pp.2243–2275

Reduction to Reflection-Preserving Cases

[Hosaka 2005] studied *2-dimensional* (W, S) ($\overset{\text{def}}{\Longleftrightarrow}$ $|W_I| = \infty$ for every $I \subset S$ with $|I| > 2$)

- $\Leftrightarrow$ the **Davis–Vinberg complex** $\Sigma(W, S)$ has dimension ≤ 2

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- $\Leftrightarrow$ the **Davis–Vinberg complex** $\Sigma(W, S)$ has dimension ≤ 2

Theorem If (W, S) and (W, S') are 2-dimensional, then (W, S) can be converted to (W, S'') s.t. $\Gamma(W, S) \simeq \Gamma(W, S'')$ and $\mathrm{Ref}_{S'}(W) = \mathrm{Ref}_{S''}(W)$

- Hence **the isomorphism problem for 2-dimensional (W, S) is reduced to “reflection-preserving” cases**

T. Hosaka, Coxeter systems with two-dimensional Davis–Vinberg complexes, *J. Pure Appl. Algebra* **197** (2005) pp.159–170

Reduction to Reflection-Preserving Cases

Results on general cases by [Howlett–Mühlherr 2004 (preprint)]; cf. [Mühlherr 2006]

Reduction to Reflection-Preserving Cases

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$s \in S$ is a *pseudo-transposition* $\overset{\text{def}}{\Longleftrightarrow}$ $s \in \exists J \subset S$ s.t.

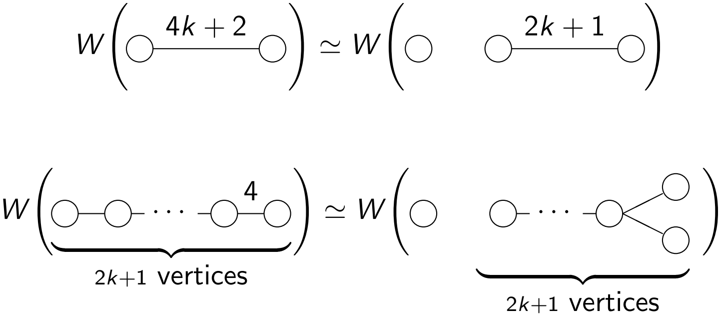
- for each $t \in S \setminus J$, either $m(s,t) = \infty$ or $t \in Z_W(J)$
- either $\Gamma_J = \Gamma|_J = \Gamma(I_2(4k+2))$, or $\Gamma_J = \Gamma(B_{2k+1})$ and s is the end vertex of Γ_J adjacent to the 4-edge

R. B. Howlett, B. Mühlherr, Isomorphisms of Coxeter groups which do not preserve reflections, preprint (2004)

B. Mühlherr, The isomorphism problem for Coxeter groups, in: *The Coxeter legacy* (C. Davis, E. W. Ellers, eds.), AMS (2006) pp.1–15

Reduction to Reflection-Preserving Cases

Recall the following relations:



Reduction to Reflection-Preserving Cases

Then the pseudo-transposition can be removed by "locally" applying the relations $W(I_2(4k+2)) \simeq W(A_1 \times I_2(2k+1))$ and $W(B_{2k+1}) \times W(A_1 \times D_{2k+1})$

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Iterating the process, we can convert (W, S) into (W, S') having no pseudo-transpositions (called *reduced* Coxeter system)

Reduction to Reflection-Preserving Cases

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Reduction to Reflection-Preserving Cases

Theorem For a reduced (W, S), there is a finite $\Sigma \le \mathrm{Aut}(W)$ (determined by using finite continuations) s.t. **if (W', S') is reduced and $f \colon W \xrightarrow{\sim} W'$, then $f(\sigma(S)) \subset \mathrm{Ref}_{S'}(W')$ for some $\sigma \in \Sigma$**

Hence **the isomorphism problem is reduced to "reflection-preserving" cases**: Given $W(\Gamma)$ and $W(\Gamma')$,

1. first convert them into reduced $W(\Gamma_*)$ and $W(\Gamma'_*)$
2. then for all (finitely many) $\sigma \in \Sigma$, decide whether or not $\exists \varphi \colon \sigma(W(\Gamma_*)) \xrightarrow{\sim} W(\Gamma'_*)$ with $\varphi(\mathrm{Ref}(\sigma(W(\Gamma_*)))) = \mathrm{Ref}(W(\Gamma'_*))$

Further Reduction

Theorem [Marquis–Mühlherr 2008] The isomorphism problem is reduced to the following problem: Given (W, S), find all $S' \subset \mathrm{Ref}_S(W)$ s.t. (W, S') is a Coxeter system **and S' is sharp-angled w.r.t. S**

- S' is *sharp-angled* w.r.t. S $\overset{\text{def}}{\Longleftrightarrow}$ for all $s, t \in S$ with $m(s, t) < \infty$, $\{s, t\}$ is conjugate to a subset of S'

Note: This is used by [Caprace–Przytycki 2010] to give the complete solution for twist-rigid cases

T. Marquis, B. Mühlherr, Angle-deformations in Coxeter groups, *Algebraic & Geom. Top.* **8** (2008) pp.2175–2208

Arbitrary Rank Cases

When $|S| = \infty$

Several key properties in finite rank cases do not hold when $|S| = \infty$!

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Several key properties in finite rank cases do not hold when $|S| = \infty$!

- Maximal finite (standard parabolic) subgroups do not necessarily exist
- Finite continuation is not well-defined in general
- The intersection of infinitely many parabolic subgroups is not necessarily parabolic

Centralizers and Reflection Independence

[N. 2007] An alternative strategy to show that $f \colon W \xrightarrow{\sim} W'$ maps $s \in S$ into $\mathrm{Ref}(W')$

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2. f induces $Z_W(s) \xrightarrow{\sim} Z_{W'}(f(s)) = N_{W'}(W'_J)$
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3. This implies $f(\langle s \rangle \times (W^{\perp s})_{\mathrm{fin}}) \supset W'_J$, where $W^{\perp s}$ is the subgroup generated by reflections orthogonal to s
4. If $(W^{\perp s})_{\mathrm{fin}} = 1$, then $|J| = 1$, i.e., $f(s) \in \mathrm{Ref}(W')$
    - The same conclusion holds when $(W^{\perp s})_{\mathrm{fin}} = \langle s^w \rangle$, $w \in W$

K. Nuida, Almost central involutions in split extensions of Coxeter groups by graph automorphisms, *J. Group Theory* **10** (2) (2007) pp.139–166

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  - Intuitively, $(W^{\perp s})_{\mathrm{fin}} = 1$ in "generic" cases
- In particular, W is **reflection independent** for the following cases:
  - Infinite irreducible 2-spherical
  - All reflections in W are conjugate

K. Nuida, Centralizers of reflections and reflection-independence of Coxeter groups, arXiv:math/0602165v1 (2006)

Centralizers and Reflection Independence

Note: $W(A_\infty) \simeq W(A_{\infty,\infty})$, i.e., **infinite irreducible 2-spherical (W, S) is not necessarily reflection rigid when $|S| = \infty$**

- $W(A_\infty)$ is the group of permutations on $\mathbb{N}$ fixing all but finitely many letters
- $W(A_{\infty,\infty})$ is the group of permutations on $\mathbb{Z}$ fixing all but finitely many letters
- A bijection $\mathbb{N} \simeq \mathbb{Z}$ induces the desired $W(A_\infty) \xrightarrow{\sim} W(A_{\infty,\infty})$

On Finite Continuation

Key properties in finite rank cases for using $\mathrm{FC}(w)$:

- Any finite intersection of parabolic subgroups is parabolic
- Any finite $H \leq W$ is contained in a maximal finite $G \leq W$, which is parabolic
- Any maximal finite standard parabolic subgroup is a maximal finite subgroup

How to generalize these properties to infinite rank cases?

On Finite Continuation

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Note: When $|S| < \infty$, locally parabolic subgroups and parabolic subgroups coincide

K. Nuida, Locally parabolic subgroups in Coxeter groups of arbitrary ranks, *J. Algebra* **350** (2012) pp.207–217

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- The intersection of an **arbitrary** family of locally parabolic subgroups is locally parabolic, hence *locally parabolic closure* is well-defined
- When $|S| < \infty$, the locally parabolic closure coincides with the parabolic closure
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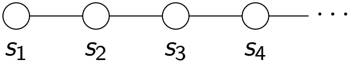
On Finite Continuation

Note: (for intersection of parabolic subgroups)



Let G be s.t. $S(G) = \{s_1 s_2 s_1, s_3 s_4 s_3, s_5 s_6 s_5, \dots\}$

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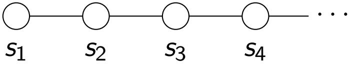

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- G is locally parabolic, but not parabolic

Let $G_i := \langle S(G) \cup \{s_{2i+1}, s_{2i+2}, s_{2i+3}, \dots\} \rangle$ $(1 \leq i \in \mathbb{Z})$

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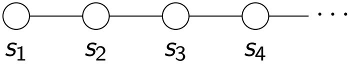

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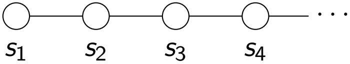

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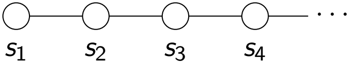

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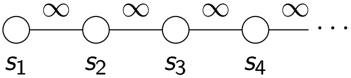
- Each G_i is parabolic and $G_1 \supsetneq G_2 \supsetneq G_3 \supsetneq \cdots$
- $\bigcap_{i=1}^{\infty} G_i = G$
- **Hence $\bigcap_i G_i$ is not parabolic, though G_i are parabolic**

On Finite Continuation

Note: **The locally parabolic closure is not equal to the parabolic closure in general**



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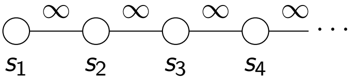

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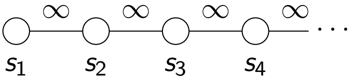

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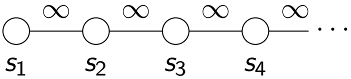

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- G is locally parabolic, hence the locally parabolic closure of G is G itself
- The parabolic closure of G is W
- $s_1 \notin G$, hence $G \neq W$

On Finite Continuation

[Mühlherr–N. (preprint)] introduced *locally finite continuation* of $X \subset W$; $\mathrm{LFC}(X) := \bigcap\{H \mid X \subset H \leq W \text{ maximal locally finite}\}$

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Theorem For any reflection r, $\mathrm{LFC}(r)$ is completely determined

- $\mathrm{LFC}(r)$ is a parabolic subgroup for any reflection r

B. Mühlherr, K. Nuida, Reflection independent Coxeter groups of arbitrary ranks, in preparation

On Finite Continuation

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- Hence **the isomorphism problem is reduced to the class of reduced Coxeter systems**

On Finite Continuation

Theorem

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Theorem

- Characterization of reduced (W, S) which is **reflection independent among reduced Coxeter systems** (by using locally finite continuations)
- (W, S) is reflection independent, if infinite irreducible and 2-spherical, or all reflections are conjugate
- **Characterization of reflection independent 2-dimensional Coxeter systems**
  - including skew-angled cases and cases with tree presentation graphs

Conclusion

Isomorphism problem for Coxeter groups of finite ranks

- has been solved in some special cases
- has been reduced to reflection-preserving cases in general
    - We have two kinds of "elementary transformations"; are these enough?

Isomorphism problem for Coxeter groups of infinite ranks

- has been "almost" reduced to reflection-preserving cases
    - A new transformation: $W(A_\infty) \simeq W(A_{\infty,\infty})$
    - How to proceed? Geometry? Combinatorics?