The archaeology of unipotent radicals

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Theorem (Liebeck–Seitz, 96)

(Paraphrased) For a simple root system X and exceptional group G defined over $k = \bar{k}$ with char $k = p$, let $N(X, G)$ be defined by the following table:

	$G = E_8$	E_7	E_6	F_4	G_2
$X = A_1$	7	7	5	3	3
A_2	5	5	3	3	
B_2	5	3	3	2	
G_2	7	7	3	2	
A_3	2	2	2		
B_3	2	2	2	2	
C_3	3	2	2	2	
B_4, C_4, D_4	2	2	2		

Then if $p > N(X, G)$ any closed, connected simple subgroup H of G with root system X is G-cr.

Theorem (S, 2012)

For a simple root system X and exceptional group G defined over $k = \bar{k}$ with char $k = p$, let $\mathcal{N}(X, G)$ be a set of prime numbers defined by the following table:

	$G = E_8$	E_7	E_6	F_4	G_2
$X = A_1$	≤ 7	≤ 7	≤ 5	≤ 3	$\cancel{3}\ 2$
A_2	$\cancel{5}\ 3\ 2$	$\cancel{5}\ 3\ 2$	$3\ 2$	$3\ \cancel{2}$	
B_2	$5\ \cancel{3}\ 2$	$\cancel{3}\ 2$	$\cancel{3}\ 2$	2	
G_2	$7\ \cancel{5}\ 3\ 2$	$7\ \cancel{5}\ \cancel{3}\ 2$	$\cancel{3}\ 2$	2	
A_3	2	$\cancel{2}$	$\cancel{2}$		
B_3	2	2	2	2	
C_3	$3\ \cancel{2}$	$\cancel{2}$	$\cancel{2}$	$\cancel{2}$	
B_4	2	$\cancel{2}$	$\cancel{2}$		
C_4, D_4	2	2	$\cancel{2}$		

Then all closed, connected subgroups H of G with root system X are G-cr, if and only if $p \notin \mathcal{N}(X, G)$.

The connected, reductive subgroups of G_2

Theorem (S, 07)

Let X denote a closed, connected semisimple subgroup of $G = G_2(K)$ where G is defined over an algebraically closed field K of characteristic $p > 0$. Then up to conjugacy in G, $(X, p, V_7 \downarrow X)$ is precisely one entry in the following table where $V_7 \downarrow X$ denotes the restriction of the seven-dimensional Weyl module $W_G(\lambda_1)$ to X.

X	p	$V_7 \downarrow X$
A_2	any	$10 \oplus 01 \oplus 0$
$\tilde{A}_2$	$p = 3$	11
$A_1\tilde{A}_1$	any	$1 \otimes \tilde{1} \oplus 0 \otimes \tilde{W}(2)$
$\bar{L}_0$	any	$1 \oplus 1 \oplus 0^3$
$\tilde{L}_0$	any	$1 \oplus 1 \oplus W(2)$
Z_1	$p = 2$	$T(2) \oplus W(2)$
Z_2	$p = 2$	$W(2) \oplus W(2)^* \oplus 0$
$A_1 \hookrightarrow A_1\tilde{A}_1; x \mapsto (x^{(p^r)}, x^{(p^s)})\ r \neq s$	any	$(1^{(p^r)} \otimes 1^{(p^s)}) \oplus W(2)^{(p^s)}$
$A_1 \hookrightarrow A_2$, *irred*	$p > 2$	$2 \oplus 2 \oplus 0$
A_1, *max*	$p \geq 7$	6

H^2 for SL_2

Theorem (S, 09)

Let $V = L(r)^{[d]}$ be any Frobenius twist (possibly trivial) of the irreducible G-module $L(r)$ with highest weight r where r is one of

$$\begin{gathered} 2p \\ 2p^2 - 2p - 2 \ (p > 2) \\ 2p - 2 + (2p - 2)p^e \ (e > 1) \end{gathered}$$

Then $H^2(G, V) = K$. For all other irreducible G-modules V, $H^2(G, V) = 0$.

(Similar information is known for $G = SL_3$; [S, 09].)

and F_4, rank at least 2

Theorem (S, 09)

Let X be a closed, connected, simple subgroup of $G = F_4(K)$ of rank at least 2. Suppose X is non-G-cr. Then $p = 2$ or 3 and X is in a subsystem subgroup of G. Moreover X is conjugate to precisely one of the nine subgroups in the table below,

X of type	p	Description
A_2	$p=3$	$X \leq B_4 \leq GL_9$ via $V_9 \downarrow X = (10, 01)$
A_2	$p=3$	$X \leq A_2\tilde{A}_2$ by $(V_3, V_3) \downarrow X = (10, 10)$
B_2	$p=2$	$X \leq D_3$
B_2	$p=2$	$X \leq \tilde{D}_3$
B_2	$p=2$	$X \leq B_2B_2$ by $(V_5, V_5) \downarrow X = (10, 10)$
G_2	$p=2$	$X \leq D_4$
G_2	$p=2$	$X \leq \tilde{D}_4$
B_3	$p=2$	$X \leq D_4$
B_3	$p=2$	$X \leq \tilde{D}_4$.

A_1s, $p = 3$

Let X be a closed, connected subgroup of type A_1. Suppose X is non-G-cr. Then $p = 2$ or 3.
If $p = 3$, then X is conjugate to just one of the following subgroups:

1. $X \hookrightarrow B_3$ by $V_7 \downarrow X = T(4) + 0$;
2. $X \hookrightarrow C_3$ by $V_6 \downarrow X = T(3)$;
3. $X \hookrightarrow B_4$ by $V_9 \downarrow X = T(4)^{[r]} + 2^{[s]}$, $rs = 0$;
4. $X \hookrightarrow A_2\tilde{A}_2$ by $(V_3, \tilde{V}_3) \downarrow X = (2, \tilde{2})$;
5. $X \hookrightarrow A_1 C_3$ by $(1^{[r]}, T(3)^{[s]})$, $rs = 0$; or
6. X in a one-dimensional variety $\Omega \cong \mathbb{A}^1(K) \setminus \{0\}$ parameterising subgroups of type A_1 in no maximal reductive subgroup of G. These subgroups are described explicitly by generators.

A_2s, $p = 2$

If $p = 2$ then X is conjugate to just one of the following subgroups:

1. $X \hookrightarrow A_2$ or $\tilde{A}_2$ by $V_3 \downarrow X = W(2)$;
2. $X \hookrightarrow \tilde{A}_1^2 \leq B_2$ or $A_1^2 \leq B_2$ by $x \mapsto (x, x)$;
3. $X \hookrightarrow A_3$ or $\tilde{A}_3$ by $V_4 \downarrow X = T(2)$;
4. $X \hookrightarrow A_1^4$ or $\tilde{A}_1^4$ by $x \mapsto (x, x, x, x)$;
5. $X \hookrightarrow A_2\tilde{A}_2$ by $(V_3, V_3) \downarrow X = (2/0, 0/2)$ or $(2/0, 2/0)$.
6. X, not one of the above, in a 2-dimensional variety parameterising subgroups of type A_1 in a B_4 or C_4 subsystem subgroup of G, explicitly known;
7. X in a 2-dimensional variety parameterising subgroups of type A_1 not in any reductive maximal subgroup of G, explicitly described by generators.

Corollaries

Corollary (S, 09; Amende, 05)

All conjugacy classes of simple subgroups of F_4 are known.

Corollary (S, 10)

Let X be a connected, closed, simple subgroup of $G = F_4(K)$. Then X is generated by short root elements if and only if exactly one of the following holds;

1. *X is a subsystem subgroup of type B_4, C_4 $(p = 2)$, D_4 $(p > 2)$, $\tilde{D}_4$ $(p = 2)$, A_3 $(p > 2)$, $\tilde{A}_3$ $(p = 2)$, B_3, C_3, $\tilde{A}_2$, B_2 or $\tilde{A}_1$;*
2. *$p = 2$ and X is non-G-cr, conjugate to the one of subgroups $A_1 \hookrightarrow \tilde{A}_2$ by $V_3 \downarrow X = W(2)$, $B_2 \leq \tilde{D}_3$, $B_3 \leq \tilde{D}_4$ or $G_2 \leq \tilde{D}_4$.*
3. *$p > 2$ and X is a subgroup of type A_1 in a subsystem subgroup of type A_1^2 corresponding to long roots projecting non-trivially to each factor.*

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2. *$p = 2$ and X is non-G-cr, conjugate to the one of subgroups $A_1 \hookrightarrow \tilde{A}_2$ by $V_3 \downarrow X = W(2)$, $B_2 \le \tilde{D}_3$, $B_3 \le \tilde{D}_4$ or $G_2 \le \tilde{D}_4$.*
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Somewhat eccentric corollary

Corollary (S, 10)

The following statements are equivalent:

1. *All closed subgroups of $F_4(K)$ of type A_1 are contained in closed subgroups of type A_1^2;*
2. *The characteristic p of K is 2 or 5.*