An outline of polar spaces: basics and advances Part 2

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Workshop and Conference on Groups and Geometries
Indian Statistical Institute, Bangalore, December 10-21, 2012

- Part 2a: Polar Spaces of Infinite Rank

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- Part 2b: Embeddings of polar spaces in groups

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- Part 2b: Embeddings of polar spaces in groups

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$$Rank(\mathcal{S}) := min(\mathbf{n} \colon \mathbf{n} \geq rank(M),\ M : max\ singular\ subspace\ of\ \mathcal{S})$$

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Then $Rank(\mathcal{S})$ is finite and $|C| = 1 + Rank(\mathcal{S})$ for every maximal chain C of singular subspaces, i.e. $Rank(\mathcal{S}) = Rank(M)$ for every maximal singular subspace M of $\mathcal{S}$.

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$\hookrightarrow$ There are examples of non-degenerate polar spaces of infinite rank that admit no Witt index

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f induces an injective morphism $\bar{f} : M/X \to (M'/X)^*$

$$X = \cap_{x \in M} f(x)$$

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Remarks:

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- Differences between polar spaces of finite and infinite rank depend on the fact that sesquilinear or pseudo-quadratic forms of infinite rank can behave rather differently from forms of finite rank.

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(SS) (Strong Separation Property) For every maximal singular subspace M, there exists a maximal singular subspace M' such that $M \cap M' = \emptyset$.

(WS) (Weak Separation Property) There exists at least one pair of mutually disjoint maximal singular subspaces.

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(O2) Prove the validity of either the strong or weak separation properties for non-degenerate polar spaces of infinite not-countable rank.

$\hookrightarrow$ So far no example is known where the strong separation property is not fullfilled.

Example of polar space of infinite rank

$\mathbb{K}$: division ring; σ: anti-automorphism of $\mathbb{K}$; $\varepsilon \in \mathbb{K} \setminus \{0\}$ such that $\varepsilon^\sigma \varepsilon = 1$ and $t^{\sigma^2} = \varepsilon t \varepsilon^{-1}$ for every t in $\mathbb{K}$.

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Define the *natural scalar product $\langle ., . \rangle$ of the pair $(\mathbb{V}, \mathbb{V}^*)$* as

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$$\dim(\mathbb{V}^*) \geq 2^{\dim(\mathbb{V})} > \dim(\mathbb{V}).$$

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Note that $A = A^{\perp\perp}$ for any subspace A of $\mathbb{V}$.

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Then

$$M_{A,B,f} = \{x \oplus \xi | x \in A, \xi \in f(x + A_0)\}.$$

is a maximal Φ-singular subspace of $\overline{\mathbb{V}}$ and every maximal Φ-singular subspace of $\overline{\mathbb{V}}$ can be obtained as above, for suitable choices of A, B and f.

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is a maximal Φ-singular subspace of $\overline{\mathbb{V}}$ and every maximal Φ-singular subspace of $\overline{\mathbb{V}}$ can be obtained as above, for suitable choices of A, B and f.

$$\dim(\mathbb{V}) \leq \dim(M_{A,B,f}) \leq \dim(\mathbb{V}^*).$$

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If G is commutative then ε is called abelian.

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Incidence relation is inclusion between cosets and between elements and cosets.

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The universal cover of a geometry is its $(n-1)$-universal cover.

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Definition

An embedding is dominant if it is its own hull.

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$Far_{\Pi}(p_0)$: geometry far from a point p_0 of a polar space Π of rank $n+1$

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Theorem

The embedding ε is dominant if and only if $\varepsilon(\Gamma)$ is a quadric.