An outline of polar spaces: basics and advances Part 1

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Organization

Part 1. Background.

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Part 3.

- Classical Dual Polar Spaces and their embeddings

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References:

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(2) J. Tits. *Buildings of Spherical type and Finite BN-pairs*. Lecture Notes in Mathematics 386. Springer, Berlin, 1974.

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(b) there exist isotropic points not contained in $Rad(\phi)$.

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Rmk: Given a (not null) sesquilinear form ϕ which is not trace-valued, it is always possible to consider the associated non-degenerate trace-valued sesquilinear form ϕ_0.

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Corollary

Suppose ϕ is a trace-valued sesquilinear form of finite Witt index n. Then $2n \leq \dim(\mathbb{V})$.

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Non-deg. alternating forms of Witt index n exist only in vector spaces of dimension $2n$.

Examples of non-deg. trace-valued (σ, ε)-sesquilinear forms

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The canonical matrix of a non-deg. alternating form is $\begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}$.

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where $-1 = \not\square$ and M_0 is the identity matrix of rank $\dim(\mathbb{V}) - 2n$.

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A function $f : \mathbb{V} \to \mathbb{K}/\mathbb{K}_{\sigma,\varepsilon}$ is a (σ, ε)-pseudoquadratic form if

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If $\mathbb{K}_{\sigma,\varepsilon} \neq \mathbb{K}$ then the sesquilinear form ϕ is uniquely determined by the pseudoquadratic form f.

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$$f(x) = \phi(x,x)/(1 + (t^\sigma/t)^2) + \mathbb{K}_{\sigma,\varepsilon}, \quad \forall x \in \mathbb{V}$$

where $t \in Z(\mathbb{K})$ *with* $t^\sigma \neq t$.

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* Many results proved for non-degenerate sesquilinear forms have an analogue for non singular quadratic forms.

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Orthogonal polar space $\leftrightarrow$ symmetric bilinear form $\leftrightarrow$ $O(N, \mathbb{K})$

Definition

A polar space which can be obtained from either a non-degenerate reflexive trace-valued sesquilinear or a non-singular pseudo-quadratic form is called a classical polar space.

Symplectic polar space $\leftrightarrow$ alternating bilinear form $\leftrightarrow$ $Sp(2n, \mathbb{K})$

Orthogonal polar space $\leftrightarrow$ symmetric bilinear form $\leftrightarrow$ $O(N, \mathbb{K})$

Hermitian polar space $\leftrightarrow$ hermitian sesquilinear form $\leftrightarrow$ $U(N, \mathbb{K})$

Are there any non-degenerate ordinary polar spaces of finite rank which are not classical?

Definition

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Theorem

Any embeddable polar space of rank $n \geq 2$ is classical.

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An ordinary polar space of rank n is thick if every singular subspace of dimension $n-2$ is contained in at least three maximal singular subspaces.

A polar space of rank n is top-thin if every singular subspace of dimension $n-2$ is contained in exactly two maximal singular subspaces.

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(5) Any ordinary, top-thin polar space of rank 3 is obtained as the Grassmannian of lines of a projective space $\mathrm{PG}(3, \mathbb{K})$.

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Corollary

Every ordinary and finite polar space of rank at least 3 is classical.